Bivariate Right Fractional Pseudo-Polynomial Monotone Approximation

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Abstract In this article we deal with the following general two-dimensional problem: Let f be a two variable continuously differentiable real valued function of a given order, let $\overline{L}$ be a linear right fractional mixed partial differential operator and suppose that $\overline{L}(f) \geq 0$ on a critical region. Then for sufficiently large $n, m \in \mathbb{N}$, we can find a sequence of pseudo-polynomials $Q^*_{n,m}$ in two variables with the property $\overline{L}\left(Q^*_{n,m}\right) \geq 0$ on this critical region such that f is approximated with rates right fractionally and simultaneously by $Q^*_{n,m}$ in the uniform norm on the whole domain of f. This restricted approximation is given via inequalities involving the mixed modulus of smoothness $\omega_{s,q}$, $s, q \in \mathbb{N}$, of highest order integer partial derivative of f.

1 Introduction

The topic of monotone approximation started in [10] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [3] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1 *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1, 1]$ with modulus of continuity $\omega_1\left(f^{(p)}, x\right)$ there. Let $a_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1, 1]$ and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1, 1]$. Consider the operator*

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j}\right] \tag{1}$$

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and suppose, throughout $[-1, 1]$,

$$L(f) \geq 0. \tag{2}$$

Then, for every integer $n \geq 1$, *there is a real polynomial* $Q_n(x)$ *of degree* $\leq n$ *such that*

$$L(Q_n) \geq 0 \text{ throughout } [-1, 1] \tag{3}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \tag{4}$$

where C *is independent of* n *or* f.

Next let $n, m \in \mathbb{Z}_+$, P_θ denote the space of algebraic polynomials of degree $\leq \theta$. Consider the tensor product spaces $P_n \otimes C([-1, 1])$, $C([-1, 1]) \otimes P_m$ and their sum $P_n \otimes C([-1, 1]) + C([-1, 1]) \otimes P_m$, that is

$$\begin{aligned} & P_n \otimes C([-1, 1]) + C([-1, 1]) \otimes P_m \\ = & \left\{ \sum_{i=0}^{n} x^i A_i(y) + \sum_{j=0}^{m} B_j(x) y^j ;\ A_i, B_j \in C([-1, 1]),\ x, y \in [-1, 1] \right\}. \end{aligned} \tag{5}$$

This is the space of pseudo-polynomials of degree $\leq (n, m)$, first introduced by A. Marchaud in 1924–1927 (see [7, 8]). Here $f^{(k,l)}$ denotes $\frac{\partial^{k+l} f}{\partial x^k \partial y^l}$, the (k, l)-partial derivative of f.

In this section we consider the space $C^{r,p}\left([-1, 1]^2\right) = \{f : [-1, 1]^2 \to \mathbb{R};\ f^{(k,l)}$ is continuous for $0 \leq k \leq r, 0 \leq l \leq p\}$. Let $f \in C\left([-1, 1]^2\right)$; for $\delta_1, \delta_2 \geq 0$, define the mixed modulus of smoothness of order (s, q), $s, q \in \mathbb{N}$ (see [9], pp. 516–517) by

$$\begin{aligned} \omega_{s,q}(f; \delta_1, \delta_2) \equiv \sup \Big\{ \left| {}_x\Delta_{h_1}^s \circ_y \Delta_{h_2}^q f(x, y) \right| : (x, y), \\ (x + sh_1, y + qh_2) \in [-1, 1]^2, |h_i| \leq \delta_i, i = 1, 2 \Big\}. \end{aligned} \tag{6}$$

Here

$$\begin{aligned} & {}_x\Delta_{h_1}^s \circ_y \Delta_{h_2}^q f(x, y) \\ \equiv & \sum_{\sigma=0}^{s} \sum_{\mu=0}^{q} (-1)^{s+q-\sigma-\mu} \binom{s}{\sigma} \binom{q}{\mu} f(x + \sigma h_1, y + \mu h_2) \end{aligned} \tag{7}$$

is a mixed difference of order (s, q).

We mention

Theorem 2 (see Gonska [4]) *Let* $r, p \in \mathbb{Z}_+$, $s, q \in \mathbb{N}$, *and* $f \in C^{r,p}\left([-1,1]^2\right)$. *Let* $n, m \in \mathbb{N}$ *with* $n \geq \max\{4(r+1), r+s\}$ *and* $m \geq \max\{4(p+1), p+q\}$. *Then there exists a linear operator* $Q_{n,m}$ *from* $C^{r,p}\left([-1,1]^2\right)$ *into the space of pseudopolynomials* $(P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$ *such that*

$$\left|\left(f - Q_{n,m}(f)\right)^{(k,l)}(x,y)\right| \tag{8}$$
$$\leq M_{r,s} \cdot M_{p,q} \left(\Delta_n(x)\right)^{r-k} \cdot \left(\Delta_m(y)\right)^{p-l} \cdot \omega_{s,q}\left(f^{(r,p)}; \Delta_n(x), \Delta_m(y)\right),$$

for all $(0,0) \leq (k,l) \leq (r,p)$, $x, y \in [-1,1]$, *where*

$$\Delta_\theta(z) = \frac{\sqrt{1-z^2}}{\theta} + \frac{1}{\theta^2}, \quad \theta = n, m; \ z = x, y \in [-1,1].$$

The constants $M_{r,s}$, $M_{p,q}$, *are independent of* f, (x,y) *and* (n,m); *they depend only on* (r,s), (p,q), *respectively.*

See also [5], saying that $Q_{n,m}^{(r,p)}(f)$ is continuous on $[-1,1]^2$.

The need following result which is an easy consequence of the last theorem (see [9, p. 517]).

Corollary 3 *Let* $r, p \in \mathbb{Z}_+$, $s, q \in \mathbb{N}$, *and* $f \in C^{r,p}\left([-1,1]^2\right)$. *Let* $n, m \in \mathbb{N}$ *with* $n \geq \max\{4(r+1), r+s\}$ *and* $m \geq \max\{4(p+1), p+q\}$. *Then there exists a pseudopolynomial*

$$Q_{n,m} \equiv Q_{n,m}(f) \in (P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$$

such that

$$\left\| f^{(k,l)} - Q_{n,m}^{(k,l)} \right\|_\infty \leq \frac{\dot{C}}{n^{r-k} m^{p-l}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{9}$$

for all $(0,0) \leq (k,l) \leq (r,p)$. *Here the constant* $\dot{C}$ *depends only on* r, p, s, q.

Corollary 3 was used in the proof of the main motivational result that follows.

Theorem 4 (see [1]) *Let* h_1, h_2, v_1, v_2, r, p *be integers,* $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$ *and let* $f \in C^{r,p}\left([-1,1]^2\right)$, *with* $f^{(r,p)}$ *having a mixed modulus of smoothness* $\omega_{s,q}\left(f^{(r,p)}; x, y\right)$ *there,* $s, q \in \mathbb{N}$. *Let* $\alpha_{i,j}(x,y)$, $i = h_1, h_1+1, ..., v_1$; $j = h_2, h_2+1, ..., v_2$ *be real-valued functions, defined and bounded in* $[-1,1]^2$ *and suppose* $\alpha_{h_1 h_2}$ *is either* $\geq \alpha > 0$ *or* $\leq \beta < 0$ *throughout* $[-1,1]^2$. *Take the operator*

$$L = \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y) \frac{\partial^{i+j}}{\partial x^i \partial y^j} \tag{10}$$

and assume, throughout $[-1,1]^2$ *that*

$$L(f) \geq 0. \tag{11}$$

Then for any integers n, m *with* $n \geq \max\{4(r+1), r+s\}$, $m \geq \max\{4(p+1), p+q\}$, *there exists a pseudopolynomial*

$$Q_{n,m} \in (P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$$

such that $L(Q_{m,n}) \geq 0$ *throughout* $[-1,1]^2$ *and*

$$\left\| f^{(k,l)} - Q_{n,m}^{(k,l)} \right\|_\infty \leq \frac{C}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{12}$$

for all $(0,0) \leq (k,l) \leq (h_1, h_2)$. *Moreover we get*

$$\left\| f^{(k,l)} - Q_{n,m}^{(k,l)} \right\|_\infty \leq \frac{C}{n^{r-k} m^{p-l}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{13}$$

for all $(h_1+1, h_2+1) \leq (k,l) \leq (r,p)$. *Also* (13) *is valid whenever* $0 \leq k \leq h_1$, $h_2 + 1 \leq l \leq p$ *or* $h_1 + 1 \leq k \leq r$, $0 \leq l \leq h_2$. *Here* C *is a constant independent of* f *and* n, m. *It depends only on* r, p, s, q, L.

We are also motivated by [2].

We need

Definition 5 (*see* [6]) Let $[-1,1]^2$; $\alpha_1, \alpha_2 > 0$; $\alpha = (\alpha_1, \alpha_2)$, $f \in C([-1,1]^2)$, $x = (x_1, x_2)$, $t = (t_1, t_2) \in [-1,1]^2$. We define the right mixed Riemann-Liouville fractional two dimensional integral of order α

$$\left(I_{1-}^{\alpha} f\right)(x) \tag{14}$$
$$:= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^{1} \int_{x_2}^{1} (t_1 - x_1)^{\alpha_1 - 1} (t_2 - x_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2,$$

with $x_1, x_2 < 1$. Notice here that $I_{1-}^{\alpha}(|f|) < \infty$.

Definition 6 Let $\alpha_1, \alpha_2 > 0$ with $\lceil \alpha_1 \rceil = m_1$, $\lceil \alpha_2 \rceil = m_2$, ($\lceil \cdot \rceil$ ceiling of the number). Let here $f \in C^{m_1, m_2}([-1,1]^2)$. We consider the right Caputo type fractional partial derivative:

$$D_{1-}^{(\alpha_1,\alpha_2)} f(x)$$
$$:= \frac{(-1)^{m_1+m_2}}{\Gamma(m_1-\alpha_1)\Gamma(m_2-\alpha_2)} \tag{15}$$

$$\cdot \int_{x_1}^{1} \int_{x_2}^{1} (t_1 - x_1)^{m_1-\alpha_1-1} (t_2 - x_2)^{m_2-\alpha_2-1} \frac{\partial^{m_1+m_2} f(t_1, t_2)}{\partial t_1^{m_1} \partial t_2^{m_2}} dt_1 dt_2,$$

$\forall\, x = (x_1, x_2) \in [-1, 1]^2$, where Γ is the gamma function

$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt, \quad \nu > 0. \tag{16}$$

We set

$$D_{1-}^{(0,0)} f(x) := f(x), \quad \forall\, x \in [-1, 1]^2; \tag{17}$$

$$D_{1-}^{(m_1,m_2)} f(x) := (-1)^{m_1+m_2} \frac{\partial^{m_1+m_2} f(x)}{\partial x_1^{m_1} \partial x_2^{m_2}}, \quad \forall\, x \in [-1, 1]^2. \tag{18}$$

Definition 7 We also set

$$D_{1-}^{(0,\alpha_2)} f(x) := \frac{(-1)^{m_2}}{\Gamma(m_2-\alpha_2)} \int_{x_2}^{1} (t_2 - x_2)^{m_2-\alpha_2-1} \frac{\partial^{m_2} f(x_1,t_2)}{\partial t_2^{m_2}} dt_2, \tag{19}$$

$$D_{1-}^{(\alpha_1,0)} f(x) := \frac{(-1)^{m_1}}{\Gamma(m_1-\alpha_1)} \int_{x_1}^{1} (t_1 - x_1)^{m_1-\alpha_1-1} \frac{\partial^{m_1} f(t_1,x_2)}{\partial t_1^{m_1}} dt_1, \tag{20}$$

and

$$D_{1-}^{(m_1,\alpha_2)} f(x) := \frac{(-1)^{m_2}}{\Gamma(m_2-\alpha_2)} \int_{x_2}^{1} (t_2 - x_2)^{m_2-\alpha_2-1} \frac{\partial^{m_1+m_2} f(x_1,t_2)}{\partial x_1^{m_1} \partial t_2^{m_2}} dt_2, \tag{21}$$

$$D_{1-}^{(\alpha_1,m_2)} f(x) := \frac{(-1)^{m_1}}{\Gamma(m_1-\alpha_1)} \int_{x_1}^{1} (t_1 - x_1)^{m_1-\alpha_1-1} \frac{\partial^{m_1+m_2} f(t_1,x_2)}{\partial t_1^{m_1} \partial x_2^{m_2}} dt_1. \tag{22}$$

In this article we extend Theorem 4 to the fractional level. Indeed here L is replaced by $\overline{L}$, a linear right Caputo fractional mixed partial differential operator. Now the monotonicity property holds true only on the critical square of $[-1, 0]^2$. Simultaneously fractional convergence remains true on all of $[-1, 1]^2$.

2 Main Result

We present

Theorem 8 *Let h_1, h_2, v_1, v_2, r, p be integers, $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$ and let $f \in C^{r,p}\left([-1,1]^2\right)$, with $f^{(r,p)}$ having a mixed modulus of smoothness $\omega_{s,q}\left(f^{(r,p)}; x, y\right)$ there, $s, q \in \mathbb{N}$. Let $\alpha_{ij}(x,y)$, $i = h_1, h_1+1, ..., v_1$; $j = h_2, h_2 + 1, ..., v_2$ be real valued functions, defined and bounded in $[-1,1]^2$ and suppose $\alpha_{h_1 h_2}$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ throughout $[-1,0]^2$. Assume that $h_1 + h_2 = 2\gamma$, $\gamma \in \mathbb{Z}_+$. Here $n, m \in \mathbb{N} : n \geq \max\{4(r+1), r+s\}$, $m \geq \max\{4(p+1, p+q)\}$. Set*

$$l_{ij} := \sup_{(x,y)\in[-1,1]^2} \left|\alpha_{h_1 h_2}^{-1}(x,y)\,\alpha_{ij}(x,y)\right| < \infty, \tag{23}$$

for all $h_1 \leq i \leq v_1$, $h_2 \leq j \leq v_2$. Let $\alpha_{1i}, \alpha_{2j} > 0$, $\alpha_{1i}, \alpha_{2j} \notin \mathbb{N}$, with $\lceil \alpha_{1i} \rceil = i$, $\lceil \alpha_{2j} \rceil = j$, $i = 0, 1, ..., r$; $j = 0, 1, ..., p$, ($\lceil \cdot \rceil$ ceiling of the number), $\alpha_{10} = 0$, $\alpha_{20} = 0$.

Consider the right fractional bivariate differential operator

$$\overline{L} := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y)\, D_{1-}^{(\alpha_{1i}, \alpha_{2j})}. \tag{24}$$

Assume $\overline{L} f(x,y) \geq 0$, on $[-1,0]^2$. Then there exists

$$Q_{n,m}^* \equiv Q_{n,m}^*(f) \in \left(P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m\right)$$

such that $\overline{L} Q_{n,m}^*(x,y) \geq 0$, on $[-1,0]^2$. Furthermore it holds:

1.

$$\left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})}(f) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty,[-1,1]^2}$$
$$\leq \frac{\dot{C}\, 2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}}{\Gamma(i - \alpha_{1i} + 1)\, \Gamma\left(j - \alpha_{2j} + 1\right) n^{r-i} m^{p-j}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{25}$$

where $\dot{C}$ is a constant that depends only on r, p, s, q; $(h_1 + 1, h_2 + 1) \leq (i, j) \leq (r, p)$, or $0 \leq i \leq h_1$, $h_2 + 1 \leq j \leq p$, or $h_1 + 1 \leq i \leq r$, $0 \leq j \leq h_2$,

2.

$$\left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})}(f) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty,[-1,1]^2}$$
$$\leq \frac{c_{ij}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{26}$$

for $(1, 1) \leq (i, j) \leq (h_1, h_2)$, *where* $c_{ij} = \dot{C} A_{ij}$, *with*

$$\begin{aligned} &A_{ij} \\ &:= \left\{ \left[\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} \frac{l_{\tau\mu} 2^{(\tau+\mu)-\left(\alpha_{1\tau}+\alpha_{2\mu}\right)}}{\Gamma\left(\tau - a_{1\tau} + 1\right) \Gamma\left(\mu - \alpha_{2\mu} + 1\right)} \right] \right. \\ &\cdot \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k! \Gamma\left(h_1 - \alpha_{1i} - k + 1\right)} \right) \\ &\left. \cdot \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma\left(h_2 - \alpha_{2j} - \lambda + 1\right)} \right) + \frac{2^{(i+j)-\left(\alpha_{1i}+\alpha_{2j}\right)}}{\Gamma\left(i - \alpha_{1i} + 1\right) \Gamma\left(j - \alpha_{2j} + 1\right)} \right\}, \end{aligned} \tag{27}$$

3.

$$\left\| f - Q_{n,m}^{*} \right\|_{\infty,[-1,1]^2} \leq \frac{c_{00}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \tag{28}$$

where $c_{00} := \dot{C} A_{00}$, *with*

$$A_{00} := \frac{1}{h_1! h_2!} \left(\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} l_{\tau\mu} \frac{2^{(\tau+\mu)-\left(\alpha_{1\tau}+\alpha_{2\mu}\right)}}{\Gamma\left(\tau - a_{1\tau} + 1\right) \Gamma\left(\mu - \alpha_{2\mu} + 1\right)} \right) + 1,$$

4.

$$\begin{aligned} &\left\| D_{1-}^{\left(0,\alpha_{2j}\right)}(f) - D_{1-}^{\left(0,\alpha_{2j}\right)} Q_{n,m}^{*} \right\|_{\infty,[-1,1]^2} \\ &\leq \frac{c_{0j}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \end{aligned} \tag{29}$$

where $c_{0j} = \dot{C} A_{0j}$, $j = 1, \ldots, h_2$, *with*

$$\begin{aligned} &A_{0j} \\ &:= \left[\frac{1}{h_1!} \left(\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} l_{\tau\mu} \frac{2^{(\tau+\mu)-\left(\alpha_{1\tau}+\alpha_{2\mu}\right)}}{\Gamma\left(\tau - a_{1\tau} + 1\right) \Gamma\left(\mu - \alpha_{2\mu} + 1\right)} \right) \right. \\ &\left. \cdot \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma\left(h_2 - \alpha_{2j} - \lambda + 1\right)} \right) + \frac{2^{j-\alpha_{2j}}}{\Gamma\left(j - \alpha_{2j} + 1\right)} \right], \end{aligned} \tag{30}$$

5.

$$\left\| D_{1-}^{(\alpha_{1i},0)}(f) - D_{1-}^{(\alpha_{1i},0)} Q_{n,m}^{*} \right\|_{\infty,[-1,1]^2} \leq \frac{c_{i0}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{31}$$

where $c_{i0} = \dot{C} A_{i0}$, $i = i, ..., h_1$, *with*

$$A_{i0} := \left[\frac{1}{h_2!} \left(\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} l_{\tau\mu} \frac{2^{(\tau+\mu)-(\alpha_{1\tau}+\alpha_{2\mu})}}{\Gamma(\tau - a_{1\tau} + 1)\, \Gamma(\mu - \alpha_{2\mu} + 1)} \right) \cdot \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k!\,\Gamma(h_1 - \alpha_{1i} - k + 1)} \right) + \frac{2^{i-\alpha_{1i}}}{\Gamma(i - \alpha_{1i} + 1)} \right]. \tag{32}$$

Proof By Corollary 3 there exists

$$Q_{n,m} \equiv Q_{n,m}(f) \in (P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$$

such that

$$\left\| f^{(i,j)} - Q_{n,m}^{(i,j)} \right\|_{\infty} \leq \frac{\dot{C}}{n^{r-i} m^{p-j}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{33}$$

for all $(0,0) \leq (i,j) \leq (r,p)$, while $Q_{n,m} \in C^{r,p}([-1,1])^2$. Here $\dot{C}$ depends only on $r, p, s, q,$ where $n \geq \max\{4(r+1), r+s\}$ and $m \geq \max\{4(p+1), p+q\}$, with $r, p \in \mathbb{Z}_+$, $s, q \in \mathbb{N}$, $f \in C^{r,p}([-1,1]^2)$. Indeed by [5] we have that $Q_{n,m}^{(r,p)}$ is continuous on $[-1,1]^2$. We observe the following ($i = 1, ..., r;\ j = 1, ..., p$)

$$\left| D_{1-}^{(\alpha_{1i},\alpha_{2j})} f(x_1,x_2) - D_{1-}^{(\alpha_{1i},\alpha_{2j})} Q_{n,m}(x_1,x_2) \right| = \frac{1}{\Gamma(i-\alpha_{1i})\,\Gamma(j-\alpha_{2j})} \left| \int_{x_1}^{1} \int_{x_2}^{1} (t_1 - x_1)^{i-\alpha_{1i}-1} (t_2 - x_2)^{j-\alpha_{2j}-1} \cdot \left(\frac{\partial^{i+j} f(t_1,t_2)}{\partial t_1^i \partial t_2^j} - \frac{\partial^{i+j} Q_{n,m}(t_1,t_2)}{\partial t_1^i \partial t_2^j} \right) dt_1 dt_2 \right| \tag{34}$$

$$\leq \frac{1}{\Gamma(i-\alpha_{1i})\,\Gamma(j-\alpha_{2j})} \int_{x_1}^{1} \int_{x_2}^{1} (t_1 - x_1)^{i-\alpha_{1i}-1} (t_2 - x_2)^{j-\alpha_{2j}-1} \tag{35}$$

$$\cdot \left| \frac{\partial^{i+j} f\left(t_1, t_2\right)}{\partial t_1^i \partial t_2^j} - \frac{\partial^{i+j} Q_{n,m}\left(t_1, t_2\right)}{\partial t_1^i \partial t_2^j} \right| dt_1 dt_2$$

$$\overset{(9)}{\leq} \frac{1}{\Gamma\left(i-\alpha_{1i}\right) \Gamma\left(j-\alpha_{2j}\right)} \tag{36}$$
$$\cdot \left(\int_{x_1}^{1} \int_{x_2}^{1} \left(t_1 - x_1\right)^{i-\alpha_{1i}-1} \left(t_2 - x_2\right)^{j-\alpha_{2j}-1} \right)$$
$$\cdot \frac{\dot{C}}{n^{r-i} m^{p-j}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right)$$

$$= \frac{1}{\Gamma\left(i-\alpha_{1i}\right) \Gamma\left(j-\alpha_{2j}\right)} \frac{\left(1-x_1\right)^{i-\alpha_{1i}}}{i-\alpha_{1i}} \frac{\left(1-x_2\right)^{j-\alpha_{2j}}}{j-\alpha_{2j}} \frac{\dot{C}}{n^{r-i} m^{p-j}} \tag{37}$$
$$\cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right)$$
$$= \frac{\left(1-x_1\right)^{i-\alpha_{1i}}}{\Gamma\left(i-\alpha_{1i}+1\right)} \frac{\left(1-x_2\right)^{j-\alpha_{2j}}}{\Gamma\left(j-\alpha_{2j}+1\right)} \frac{\dot{C}}{n^{r-i} m^{p-j}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right).$$

That is there exists $Q_{n,m}$:

$$\left| D_{1-}^{\left(\alpha_{1i}, \alpha_{2j}\right)} f\left(x_1, x_2\right) - D_{1-}^{\left(\alpha_{1i}, \alpha_{2j}\right)} Q_{n,m}\left(x_1, x_2\right) \right| \tag{38}$$
$$\leq \frac{\left(1-x_1\right)^{i-\alpha_{1i}} \left(1-x_2\right)^{j-\alpha_{2j}}}{\Gamma\left(i-\alpha_{1i}+1\right) \Gamma\left(j-\alpha_{2j}+1\right)} \frac{\dot{C}}{n^{r-i} m^{p-j}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right),$$

$i = 1, ..., r,\ j = 1, ..., p,\ \forall \left(x_1, x_2\right) \in [-1, 1]^2$.

We proved there exists $Q_{n,m}$ such that

$$\left\| D_{1-}^{\left(\alpha_{1i}, \alpha_{2j}\right)}(f) - D_{1-}^{\left(\alpha_{1i}, \alpha_{2j}\right)} Q_{n,m}^{*} \right\|_{\infty} \tag{39}$$
$$\leq \frac{2^{(i+j)-\left(\alpha_{1i}+\alpha_{2j}\right)} \dot{C}}{\Gamma\left(i-\alpha_{1i}+1\right) \Gamma\left(j-\alpha_{2j}+1\right) n^{r-i} m^{p-j}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right),$$

$i = 0, 1, ..., r,\ j = 0, 1, ..., p$. Define

$$\rho_{n,m} \equiv \dot{C} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right) \tag{40}$$

$$\cdot\left[\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}\left(l_{ij}\frac{2^{(i+j)-\left(\alpha_{1i}+\alpha_{2j}\right)}}{\Gamma\left(i-a_{1i}+1\right)\Gamma\left(j-\alpha_{2j}+1\right)}n^{i-r}m^{j-p}\right)\right].$$

I. Suppose, throughout $[-1,0]^2$, $\alpha_{h_1h_2}(x,y)\geq\alpha>0$. Let $Q^*_{n,m}(x,y)$, $(x,y)\in[-1,1]^2$, as in (39), so that

$$\begin{aligned}&\left\|D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)}\left(f(x,y)+\rho_{n,m}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right)-D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)}Q^*_{n,m}(x,y)\right\|_\infty\\ \leq\;&\frac{2^{(i+j)-\left(\alpha_{1i}+\alpha_{2j}\right)}\dot{C}}{\Gamma\left(i-\alpha_{1i}+1\right)\Gamma\left(j-\alpha_{2j}+1\right)n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\\ =:\;&T_{ij},\end{aligned}\tag{41}$$

$i=0,1,...,r;\ j=0,1,...,p.$

If $(h_1+1,h_2+1)\leq(i,j)\leq(r,p)$, or $0<i\leq h_1$, $h_2+1\leq j\leq p$, or $h_1+1\leq i\leq r$, $0<j\leq h_2$ we get from the last

$$\left\|D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)}(f)-D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)}Q^*_{n,m}\right\|_\infty\tag{42}$$
$$\leq\frac{2^{(i+j)-\left(\alpha_{1i}+\alpha_{2j}\right)}\dot{C}}{\Gamma\left(i-\alpha_{1i}+1\right)\Gamma\left(j-\alpha_{2j}+1\right)n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right),$$

proving (25).

If $(0,0)<(i,j)\leq(h_1,h_2)$, we get

$$\begin{aligned}&\left\|D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)}f(x,y)+\rho_{n,m}D_{1-}^{\alpha_{1i}}\left(\frac{x^{h_1}}{h_1!}\right)D_{1-}^{\alpha_{2j}}\left(\frac{y^{h_2}}{h_2!}\right)\right.\\ &\left.-D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)}Q^*_{n,m}(x,y)\right\|_\infty\leq T_{ij}.\end{aligned}\tag{43}$$

That is for $i=1,...,h_1;\ j=1,...,h_2$, we obtain

$$\begin{aligned}&\left\|D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)}f(x,y)+\rho_{n,m}\left((-1)^{h_1}\sum_{k=0}^{h_1-i}\frac{(-1)^k(1-x)^{h_1-\alpha_{1i}-k}}{k!\Gamma\left(h_1-\alpha_{1i}-k+1\right)}\right)\right.\\ &\left.\cdot\left((-1)^{h_2}\sum_{\lambda=0}^{h_2-j}\frac{(-1)^\lambda(1-y)^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)}\right)-D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)}Q^*_{n,m}(x,y)\right\|_\infty\\ &\leq T_{ij}.\end{aligned}\tag{44}$$

Hence for $(1,1) \leq (i,j) \leq (h_1,h_2)$, we have

$$\left\| D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)} f - D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)} Q_{n,m}^{*} \right\|_{\infty} \leq \rho_{n,m} \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k!\Gamma\left(h_1-\alpha_{1i}-k+1\right)} \right) \cdot \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)} \right) + T_{ij} \tag{45}$$

$$= \dot{C}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right) \cdot \left[\sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}} \frac{2^{\left(\bar{i}+\bar{j}\right)-\left(\alpha_{1\bar{i}}+\alpha_{2\bar{j}}\right)}}{\Gamma\left(\bar{i}-\alpha_{1\bar{i}}+1\right)\Gamma\left(\bar{j}-\alpha_{2\bar{j}}+1\right)} \frac{1}{n^{r-\bar{i}}} \frac{1}{m^{p-\bar{j}}} \right] \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k!\Gamma\left(h_1-\alpha_{1i}-k+1\right)} \right) \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)} \right) + \frac{2^{(i+j)-\left(\alpha_{1i}+\alpha_{2j}\right)}\dot{C}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right)}{\Gamma\left(i-\alpha_{1i}+1\right)\Gamma\left(j-\alpha_{2j}+1\right) n^{r-i} m^{p-j}} \tag{46}$$

$$\leq \dot{C}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right) \frac{1}{n^{r-v_1} m^{p-v_2}} A_{ij}, \tag{47}$$

where

$$A_{ij} := \left\{ \left[\sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}} \frac{2^{\left(\bar{i}+\bar{j}\right)-\left(\alpha_{1\bar{i}}+\alpha_{2\bar{j}}\right)}}{\Gamma\left(\bar{i}-\alpha_{1\bar{i}}+1\right)\Gamma\left(\bar{j}-\alpha_{2\bar{j}}+1\right)} \right] \cdot \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k!\Gamma\left(h_1-\alpha_{1i}-k+1\right)} \right) \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)} \right) + \frac{2^{(i+j)-\left(\alpha_{1i}+\alpha_{2j}\right)}}{\Gamma\left(i-\alpha_{1i}+1\right)\Gamma\left(j-\alpha_{2j}+1\right)} \right\}. \tag{48}$$

(Set $c_{ij} := \dot{C}A_{ij}$)

We proved, for $(1,1) \leq (i,j) \leq (h_1,h_2)$, that

$$\left\| D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)} f - D_{1-}^{\left(\alpha_{1i},\alpha_{2j}\right)} Q_{n,m}^{*} \right\|_{\infty} \leq \frac{c_{ij}}{n^{r-v_1} m^{p-v_2}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right). \tag{49}$$

So that (26) is established.

When $i = j = 0$ from (41) we obtain

$$\left\| f(x,y) + \rho_{n,m} \frac{x^{h_1}}{h_1!} \frac{y^{h_2}}{h_2!} - Q^*_{n,m}(x,y) \right\|_\infty \leq \frac{\dot{C}}{n^r m^p} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right). \quad (50)$$

Hence

$$\left\| f - Q^*_{n,m} \right\|_\infty \leq \frac{\rho_{n,m}}{h_1! h_2!} + \frac{\dot{C}}{n^r m^p} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right) \quad (51)$$

$$\begin{aligned} =& \frac{\dot{C}}{h_1! h_2!} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right) \quad (52) \\ & \cdot \left[\sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}} \frac{2^{(\bar{i}+\bar{j})-(\alpha_{1\bar{i}}+\alpha_{2\bar{j}})}}{\Gamma\left(\bar{i}-\alpha_{1\bar{i}}+1\right)\Gamma\left(\bar{j}-\alpha_{2\bar{j}}+1\right)} \frac{1}{n^{r-\bar{i}}} \frac{1}{m^{p-\bar{j}}} \right] \\ & + \frac{\dot{C}}{n^r m^p} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right) \end{aligned}$$

$$\leq \frac{\dot{C}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right)}{n^{r-v_1} m^{p-v_2}} A_{00}, \quad (53)$$

where

$$A_{00} := \left[\frac{1}{h_1! h_2!} \sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} \frac{l_{\bar{i}\bar{j}} 2^{(\bar{i}+\bar{j})-(\alpha_{1\bar{i}}+\alpha_{2\bar{j}})}}{\Gamma\left(\bar{i}-\alpha_{1\bar{i}}+1\right)\Gamma\left(\bar{j}-\alpha_{2\bar{j}}+1\right)} + 1 \right]. \quad (54)$$

(Set $c_{00} = \dot{C} A_{00}$).

Then

$$\left\| f - Q^*_{n,m} \right\|_\infty \leq \frac{c_{00}}{n^{r-v_1} m^{p-v_2}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right). \quad (55)$$

So that (28) is established.

Next case of $i = 0$, $j = 1, ..., h_2$, from (41) we get

$$\left\| D_{1-}^{\left(0,\alpha_{2j}\right)} f(x, y) + \rho_{n,m} \frac{x^{h_1}}{h_1!} \left((-1)^{h_2} \sum_{\lambda=0}^{h_2-j} \frac{(-1)^{\lambda} (1-y)^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma\left(h_2 - \alpha_{2j} - \lambda + 1\right)} \right) \right. \\ \left. - D_{1-}^{\left(0,\alpha_{2j}\right)} Q_{n,m}^{*}(x, y) \right\|_{\infty} \leq T_{0j}. \tag{56}$$

Then

$$\left\| D_{1-}^{\left(0,\alpha_{2j}\right)} f - D_{1-}^{\left(0,\alpha_{2j}\right)} Q_{n,m}^{*} \right\|_{\infty} \\ \leq \frac{\rho_{n,m}}{h_1!} \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma\left(h_2 - \alpha_{2j} - \lambda + 1\right)} \right) + T_{0j} \tag{57}$$

$$= \frac{\dot{C}}{h_1!} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right) \tag{58}$$

$$\cdot \left[\sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}} \frac{2^{\left(\bar{i}+\bar{j}\right)-\left(\alpha_{1\bar{i}}+\alpha_{2\bar{j}}\right)}}{\Gamma\left(\bar{i} - \alpha_{1\bar{i}} + 1\right) \Gamma\left(\bar{j} - \alpha_{2\bar{j}} + 1\right)} \frac{1}{n^{r-\bar{i}}} \frac{1}{m^{p-\bar{j}}} \right]$$

$$\cdot \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma\left(h_2 - \alpha_{2j} - \lambda + 1\right)} \right)$$

$$+ \frac{2^{j-\alpha_{2j}} \dot{C}}{\Gamma\left(j - \alpha_{2j} + 1\right) n^r m^{p-j}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right)$$

$$\leq \frac{\dot{C} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right)}{n^{r-v_1} m^{p-v_2}} A_{0j}, \tag{59}$$

where

$$A_{0j} \\ := \left[\frac{1}{h_1!} \left(\sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}} \frac{2^{\left(\bar{i}+\bar{j}\right)-\left(\alpha_{1\bar{i}}+\alpha_{2\bar{j}}\right)}}{\Gamma\left(\bar{i} - \alpha_{1\bar{i}} + 1\right) \Gamma\left(\bar{j} - \alpha_{2\bar{j}} + 1\right)} \right) \right. \\ \left. \cdot \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma\left(h_2 - \alpha_{2j} - \lambda + 1\right)} \right) + \frac{2^{j-\alpha_{2j}}}{\Gamma\left(j - \alpha_{2j} + 1\right)} \right]. \tag{60}$$

(Set $c_{0j} := \dot{C} A_{0j}$)

We proved that (case of $i = 0$, $j = 1, ..., h_2$)

$$\left\| D_{1-}^{\left(0,\alpha_{2j}\right)} f - D_{1-}^{\left(0,\alpha_{2j}\right)} Q_{n,m}^{*} \right\|_{\infty} \leq \frac{c_{0j}}{n^{r-v_1} m^{p-v_2}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right). \tag{61}$$

establishing (29).

Similarly we get for $i = 1, ..., h_1,\ j = 0$, that

$$\left\| D_{1-}^{(\alpha_{1i},0)} f - D_{1-}^{(\alpha_{1i},0)} Q_{n,m}^{*} \right\|_{\infty} \leq \frac{c_{i0}}{n^{r-v_1} m^{p-v_2}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{62}$$

where $c_{i0} := \dot{C} A_{i0}$, with

$$\begin{aligned} &A_{i0} \\ &:= \left[\frac{1}{h_2!} \left(\sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}} \frac{2^{(\bar{i}+\bar{j})-(\alpha_{1\bar{i}}+\alpha_{2\bar{j}})}}{\Gamma\left(\bar{i}-\alpha_{1\bar{i}}+1\right)\Gamma\left(\bar{j}-\alpha_{2\bar{j}}+1\right)} \right) \right. \\ &\left. \cdot \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k!\Gamma\left(h_1-\alpha_{1i}-k+1\right)} \right) + \frac{2^{i-\alpha_{1i}}}{\Gamma\left(i-\alpha_{1i}+1\right)} \right], \end{aligned} \tag{63}$$

deriving (31). So if $(x, y) \in [-1, 0]^2$, then

$$\alpha_{h_1h_2}^{-1}(x, y)\, \overline{L}\left(Q_{n,m}^{*}(x, y)\right) \tag{64}$$

$$\begin{aligned} =&\alpha_{h_1h_2}^{-1}(x, y)\, \overline{L}\left(f(x, y)\right) + \rho_{n,m} \frac{(1-x)^{h_1-\alpha_{1i}}}{\Gamma\left(h_1-\alpha_{1i}+1\right)} \frac{(1-y)^{h_2-\alpha_{2j}}}{\Gamma\left(h_2-\alpha_{2j}+1\right)} \\ &+ \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{h_1h_2}^{-1}(x, y)\, \alpha_{ij}(x, y) \\ &\cdot \left[D_{1-}^{(\alpha_{1i},\alpha_{2j})} Q_{n,m}^{*}(x, y) - D_{1-}^{(\alpha_{1i},\alpha_{2j})} f(x, y) - \rho_{n,m} D_{1-}^{(\alpha_{1i},\alpha_{2j})} \left(\frac{x^{h_1}}{h_1!} \frac{y^{h_2}}{h_2!} \right) \right] \end{aligned} \tag{65}$$

$$\begin{aligned} &\overset{(41)}{\geq} \rho_{n,m} \frac{(1-x)^{h_1-\alpha_{1i}}}{\Gamma\left(h_1-\alpha_{1i}+1\right)} \frac{(1-y)^{h_2-\alpha_{2j}}}{\Gamma\left(h_2-\alpha_{2j}+1\right)} \\ &- \left[\sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} \frac{2^{(i+j)-\left(\alpha_{1i}+\alpha_{2j}\right)}}{\Gamma(i-\alpha_{1i}+1)\Gamma\left(j-\alpha_{2j}+1\right)} \frac{\dot{C}}{n^{r-i} m^{p-j}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right) \right] \end{aligned} \tag{66}$$

$$= \rho_{n,m} \left[\frac{(1-x)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)} \frac{(1-y)^{h_2-\alpha_{2j}}}{\Gamma\left(h_2-\alpha_{2j}+1\right)} - 1 \right] \tag{67}$$

$$\geq \rho_{n,m} \left[\frac{1}{\Gamma(h_1-\alpha_{1i}+1)\,\Gamma\left(h_2-\alpha_{2j}+1\right)} - 1 \right]$$

$$= \rho_{n,m} \left[\frac{1-\Gamma(h_1-\alpha_{1i}+1)\,\Gamma\left(h_2-\alpha_{2j}+1\right)}{\Gamma(h_1-\alpha_{1i}+1)\,\Gamma\left(h_2-\alpha_{2j}+1\right)} \right] \geq 0. \tag{68}$$

Explanation: we have that $\Gamma(1)=1$, $\Gamma(2)=1$, and Γ is convex on $(0,\infty)$ and positive there, here $0 \leq h_1-\alpha_{1h_1}, h_2-\alpha_{2h_2} < 1$ and $1 \leq h_1-\alpha_{1h_1}+1, h_2-\alpha_{2h_2}+1 < 2$. Thus $0 < \Gamma\left(h_1-\alpha_{1h_1}+1\right), \Gamma\left(h_2-\alpha_{2h_2}+1\right) \leq 1$, and

$$0 \leq \Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right) \leq 1. \tag{69}$$

And

$$1-\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right) \geq 0. \tag{70}$$

Therefore it holds

$$\overline{L}\left(Q^*_{n,m}(x,y)\right) \geq 0, \ \forall \ (x,y) \in [-1,0]^2. \tag{71}$$

II. Suppose, throughout $[-1,0]^2$, $\alpha_{h_1h_2}(x,y) \leq \beta < 0$. Let $Q^{**}_{n,m}(x,y)$, $(x,y) \in [-1,1]^2$, as in (39), so that

$$\left\| D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(f(x,y)-\rho_{n,m}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right) - D_{1-}^{(\alpha_{1i},\alpha_{2j})} Q^{**}_{n,m}(x,y) \right\|_\infty$$

$$\leq \frac{2^{(i+j)-\left(\alpha_{1i}+\alpha_{2j}\right)}\dot{C}}{\Gamma(i-\alpha_{1i}+1)\,\Gamma\left(j-\alpha_{2j}+1\right) n^{r-i} m^{p-j}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{72}$$

$i=0,1,...,r,\ j=0,1,...,p$.

As earlier we produce the same convergence inequalities (25), (26), (28), (29), and (31). So for $(x,y) \in [-1,0]^2$ we get

$$\alpha_{h_1h_2}^{-1}(x,y)\,\overline{L}\left(Q_{n,m}^{**}(x,y)\right)$$
$$= \alpha_{h_1h_2}^{-1}(x,y)\,\overline{L}(f(x,y)) - \rho_{n,m}\frac{(1-x)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)}\frac{(1-y)^{h_2-\alpha_{2j}}}{\Gamma\left(h_2-\alpha_{2j}+1\right)} \tag{73}$$
$$+\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}\alpha_{h_1h_2}^{-1}(x,y)\,\alpha_{ij}(x,y)$$
$$\cdot\left[D_{1-}^{(\alpha_{1i},\alpha_{2j})}Q_{n,m}^{**}(x,y) - D_{1-}^{(\alpha_{1i},\alpha_{2j})}f(x,y) + \rho_{n,m}D_{1-}^{(\alpha_{1i},\alpha_{2j})}\left(\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right)\right]$$

$$\overset{(71)}{\leq} -\rho_{n,m}\frac{(1-x)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)}\frac{(1-y)^{h_2-\alpha_{2j}}}{\Gamma\left(h_2-\alpha_{2j}+1\right)} \tag{74}$$
$$+\left[\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}l_{ij}\frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}}{\Gamma(i-\alpha_{1i}+1)\,\Gamma\left(j-\alpha_{2j}+1\right)}\frac{\dot{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\right]$$
$$= \rho_{n,m}\left[1-\frac{(1-x)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)}\frac{(1-y)^{h_2-\alpha_{2j}}}{\Gamma\left(h_2-\alpha_{2j}+1\right)}\right]$$
$$= \rho_{n,m}\left[\frac{\Gamma(h_1-\alpha_{1i}+1)\,\Gamma\left(h_2-\alpha_{2j}+1\right)-(1-x)^{h_1-\alpha_{1i}}(1-y)^{h_2-\alpha_{2j}}}{\Gamma(h_1-\alpha_{1i}+1)\,\Gamma\left(h_2-\alpha_{2j}+1\right)}\right]$$
$$\leq \rho_{n,m}\left[\frac{1-(1-x)^{h_1-\alpha_{1i}}(1-y)^{h_2-\alpha_{2j}}}{\Gamma(h_1-\alpha_{1i}+1)\,\Gamma\left(h_2-\alpha_{2j}+1\right)}\right]\leq 0. \tag{75}$$

Explanation: for $x, y \in [-1, 0]$ we get that $1-x,\ 1-y \geq 1$, and $0 \leq h_1 - \alpha_{1h_1}$, $h_2 - \alpha_{2h_2} < 1$. Hence $(1-x)^{h_1-\alpha_{1h_1}}$, $(1-y)^{h_2-\alpha_{2h_2}} \geq 1$, and then

$$(1-x)^{h_1-\alpha_{1h_1}}(1-y)^{h_2-\alpha_{2h_2}} \geq 1,$$

so that

$$1-(1-x)^{h_1-\alpha_{1h_1}}(1-y)^{h_2-\alpha_{2h_2}} \leq 0. \tag{76}$$

Hence again

$$\overline{L}\left(Q_{n,m}^{**}(x,y)\right) \geq 0,\ \text{ for }\ (x,y)\in[-1,0]^2\,. \tag{77}$$

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