Chapter 2
Linear Systems and the Fourier Transform

In Chap. 1 we saw that an ultrasonic system has many components. Those components individually can be complex electromechanical systems such as, for example, the ultrasonic transducers. To model each of the elements that go into an ultrasonic system and how they work together to produce a measured response is a challenging task, indeed. In this Chapter we will present a very general modeling framework of linear time-shift invariant (LTI) systems which we will use in order to describe a complete ultrasonic NDE measurement system. Many of the remaining chapters will fill in the details of this general framework and ultimately produce an explicit LTI models of the entire ultrasonic measurement process.

2.1 Linear Time-Shift Invariant Systems

Figure 2.1 shows the general schematic of a system which takes some input, $i(t)$,as a function of time, t, and produces an output $o(t)$. For example, the system might represent the entire ultrasonic measurement process itself, where $i(t)$ is the driving voltage pulse generated from the pulser and $o(t)$ is the voltage versus time trace on the oscilloscope screen. Alternatively, the system may be only a particular component of the entire measurement process, such as a transducer, where $i(t)$ is the voltage driving the transducer and $o(t)$ is the resulting mechanical velocity or force that is used to launch waves into the surrounding medium. In all such cases, we will assume that the system being described can be modeled as a linear time-shift invariant (LTI) system where

$$o(t) = L[i(t)]. \tag{2.1}$$

L.W. Schmerr, Jr., *Fundamentals of Ultrasonic Nondestructive Evaluation*, Springer Series in Measurement Science and Technology, DOI 10.1007/978-3-319-30463-2_2

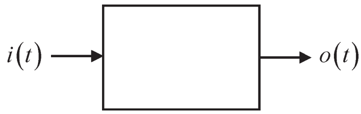

Fig. 2.1 A general input–output system

The linearity requirement means that L is a linear operator, i.e.

$$\begin{aligned} o(t) &= L[c_1 i_1(t) + c_2 i_2(t)] \\ &= c_1 L[i_1(t)] + c_2 L[i_2(t)] \ , \end{aligned} \tag{2.2}$$

where, i_1 and i_2 are two arbitrary inputs and c_1 and c_2 are constants. Thus, LTI systems obey the principle of superposition. The time-shift invariance property of LTI systems requires

$$o(t - t_0) = L[i(t - t_0)], \tag{2.3}$$

which says that a delay in the input produces an identical delay in the output. Most ultrasonic NDE systems can be characterized as LTI systems. However, in some other ultrasonic applications where extremely high power is used, such as in ultrasonic cutting, for example, nonlinear behavior may invalidate the use of Eq. (2.1).

There are several concepts that play key roles in LTI systems. One of these concepts is convolution, where by definition the *convolution integral* of two functions, $f(t)$ and $g(t)$, $f * g$, is given by

$$f * g = \int_{-\infty}^{+\infty} f(t - \tau) g(\tau)\, d\tau = \int_{-\infty}^{+\infty} g(t - \tau) f(\tau)\, d\tau. \tag{2.4}$$

We will see shortly where convolution appears in LTI systems. Another important concept for LTI systems is that of the Fourier transform.

2.2 The Fourier Transform

Ultrasonic NDE deals primarily with pulses of various types: voltage pulses, pressure pulses in fluids, elastic wave pulses in solids, etc. These pulses are transient time disturbances that characterize the behavior of an ultrasonic system component in the time domain. It is often desirable, however, to consider other domains for describing the response of the components. One domain that is particularly useful for LTI systems is the frequency domain. In the frequency domain, one describes responses in terms of the decomposition of a pulse into a distribution of sinusoids of different frequencies and amplitudes. A time domain pulse is transformed into the frequency domain through the *Fourier transform*, defined as

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\exp(i\omega t)\,dt, \tag{2.5}$$

where $F(\omega)$ is the Fourier transform of $f(t)$ and $i = \sqrt{-1}$. This transformation is reversible so that given $F(\omega)$ we can recover $f(t)$ through the *inverse Fourier transform*, defined by:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\exp(-i\omega t)\,d\omega. \tag{2.6}$$

Note that while $f(t)$ is a real function, $F(\omega)$ is complex. Thus we can write F in general in terms of its magnitude, $|F(\omega)|$, and phase, $\phi(\omega)$, as

$$F(\omega) = |F(\omega)|\exp[i\phi(\omega)] \tag{2.7}$$

or, in terms of its real and imaginary parts, as

$$F(\omega) = R(\omega) + iI(\omega). \tag{2.8}$$

The frequency variable, ω, in the Fourier transform is a circular frequency, i.e. it is measured in rad/s. Alternately, we can describe the frequency domain components in terms of a frequency, f, measured in cycles/s or Hertz (Hz) given by

$$f = \omega/2\pi, \tag{2.9}$$

where the 2π factor appears since there are 2π radians in one cycle. Note that in recovering $f(t)$ one needs to integrate over both positive and negative frequency components of $F(\omega)$ as shown in Eq. (2.6). However, these negative frequency components are present only as a mathematical requirement to guarantee that the function $f(t)$ that is recovered from Eq. (2.6) is indeed real. In fact, all the information actually needed in the frequency domain is contained in the behavior of $F(\omega)$ for the positive frequency components only. This can be seen from the definition of $F(\omega)$ in Eq. (2.5), where if $f(t)$ is real we can easily show that

$$F(-\omega) = F^*(\omega), \tag{2.10}$$

where ( )* indicates complex conjugation, i.e. $(a + i\,b)^* = a - i\,b$. Sections A.1 and A.2 in Appendix A list some of the other important properties of the Fourier transform and give some example transforms for specific functions.

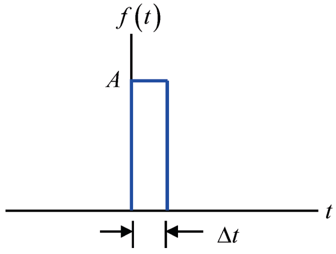

Fig. 2.2 Rectangular pulse wave form

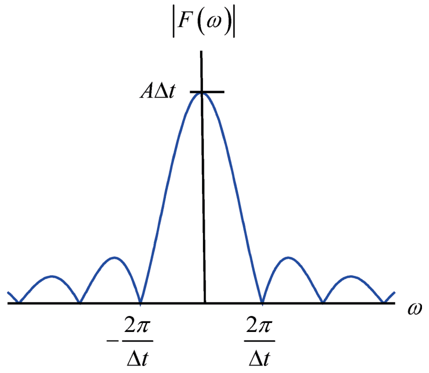

Fig. 2.3 Magnitude of the Fourier transform of the rectangular pulse of Fig. 2.2

As an example of a Fourier transform calculation, consider the rectangular-shaped pulse of Fig. 2.2 of amplitude A and duration Δt. The Fourier transform of this function is easily calculated as

$$F(\omega) = A\Delta t \exp(i\omega\Delta t/2)\sin(\omega\Delta t/2)/(\omega\Delta t/2). \tag{2.11}$$

The magnitude of F is plotted in Fig. 2.3. As can be seen from Fig. 2.3, most of the frequency content of F is contained in the main "lobe" of frequencies between $-2\pi/\Delta t$ and $2\pi/\Delta t$. If Δt is small, this main lobe will be very wide and we say the pulse has a very broad band response in the frequency domain. Conversely, when Δt is large, the pulse has a narrow band response, mostly centered in this case about its zero frequency (d.c.) component.

An important limiting case of this function is obtained if we set $A\Delta t = 1$ and consider $f(t)$ and $F(\omega)$ as $\Delta t \to 0$. In this case, $f(t)$ becomes a spike of infinite amplitude (but containing unit area) at $t = 0$ while $F(\omega) \to 1$ for all frequencies (Fig. 2.4a, b). This limiting "function" is called a delta function, $\delta(t)$. As can be seen from Fig. 2.4b, the delta function has infinite bandwidth and excites all frequency components in the frequency domain equally. Appendix B lists some of the important properties of delta functions.

In practice, Fourier transforms often cannot be obtained analytically in this fashion. However, the numerical calculation of Fourier transforms is now

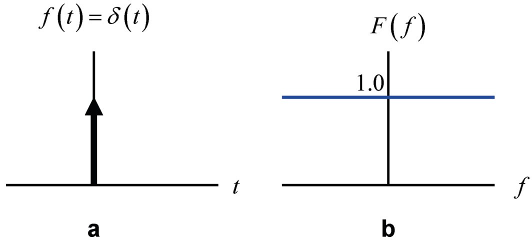

Fig. 2.4 (**a**) A delta function and (**b**) its Fourier transform

commonplace, through the use of the concepts of the *discrete Fourier transform* and its efficient calculation by the *Fast Fourier Transform* (FFT) algorithm, as discussed in Sects. A.3 and A.4 in Appendix A.

One property of Fourier transforms which is very useful for LTI systems is embodied in the following theorem:

Let $F[f(t)] = F(\omega)$ be the Fourier Transform of $f(t)$
and $F[g(t)] = G(\omega)$ be the Fourier Transform of $g(t)$.
Then $F[f*g] = F(\omega)G(\omega)$.

Thus, in the frequency domain the frequency components of two convolved functions is just the product of their individual frequency components. Note that the multiplication involved here is a complex multiplication since in general both F and G are themselves complex.

Proof of this theorem is not difficult. From the definition of the Fourier transform and convolution we have

$$F(f*g) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t-\tau)g(\tau)\exp(i\omega t)\,d\tau dt. \tag{2.12}$$

Letting $t - \tau = u$, Eq. (2.12) becomes

$$F(f*g) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(u)\exp(i\omega u)g(\tau)\exp(i\omega\tau)\,d\tau\,du \tag{2.13}$$

which, using the definitions of the Fourier transforms of f and g, gives

$$F(f*g) = F(\omega)G(\omega) \tag{2.14}$$

and thus proves the theorem.

2.3 LTI Systems and the Impulse Response Function

In this section, we will show that by combining the concepts of LTI systems, convolution, Fourier transforms, and the delta function, one can arrive at a result which will allow us to model complex systems such as an ultrasonic measurement system.

Consider a LTI system as shown in Fig. 2.5 where the input is a delta function, i.e. $i(t) = \delta(t)$. The output, $g(t)$, of this system is called the *unit impulse response function*. As we saw in the last section, $g(t)$ is the response to an ideal infinitely wide band input. The impulse response function is important for LTI systems because the response of such a system to an arbitrary input, $i(t)$, is given by the convolution integral of $g(t)$ with that input:

$$o(t) = \int_{-\infty}^{+\infty} g(t-\tau)\, i(\tau)\, d\tau. \tag{2.15}$$

Thus, knowing the impulse response function of a LTI system completely characterizes the output of that system to any input.

To prove this result, we first break $i(t)$ into small rectangles (Fig. 2.6). At $t = \tau$ consider the highlighted rectangle shown in Fig. 2.6. This rectangle approximates a delta function input at time τ of strength (area) $i(\tau)\Delta\tau$ so that the output, $\Delta o(t)$, at time t, using the time-shift invariance property of an LTI, is $\Delta o(t) \cong i(\tau)\Delta\tau g(t-\tau)$. By superposition we can then add up all the contributions from all the rectangular areas of the input to obtain the total output, $o(t)$, as

$$o(t) \cong \sum i(\tau)\Delta\tau g(t-\tau)$$

Fig. 2.5 An LTI input–output system excited by a delta function

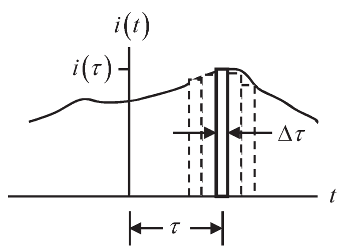

Fig. 2.6 Decomposition of a general input into rectangular "delta-function-like" components

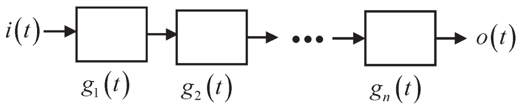

Fig. 2.7 A series of LTI systems and their impulse response functions

or, in the limit as $\Delta t \to 0$

$$o(t) = \int_{-\infty}^{+\infty} g(t-\tau)\, i(\tau)\, d\tau. \tag{2.16}$$

Ultrasonic systems are examples of causal systems [1], i.e. we can assume $g(t) = 0$ and $i(t) = 0$ for $t < 0$. In this case Eq. (2.16) can also be written equivalently as

$$o(t) = \int_{\tau=0}^{\tau=t} g(t-\tau)\, i(\tau)\, d\tau. \tag{2.17}$$

From the theorem of the last section on the relationship between the convolution integral and Fourier transforms, it follows from Eq. (2.16) that

$$O(\omega) = I(\omega)G(\omega), \tag{2.18}$$

where O, I, and G are the Fourier transforms of o, i, and g, respectively. As a generalization of Eqs. (2.16) and (2.18) we can consider a series of n LTI systems in a cascade (Fig. 2.7). In this case the output, $o(t)$, for a given input, $i(t)$, is obtained merely by applying Eq. (2.16) n times, assuming we know the impulse response functions, $g_i(t)$ ($i = 1, n$) for all these systems:

$$o(t) = g_1(t) * g_2(t) * \ldots * g_n(t) * i(t). \tag{2.19}$$

In a similar fashion, the frequency components of these functions are related through

$$O(\omega) = G_1(\omega)G_2(\omega)\ldots G_n(\omega)I(\omega). \tag{2.20}$$

Working with the frequency domain components of impulse response functions in product fashion in Eq. (2.20) is much more convenient than with the impulse response functions themselves in Eq. (2.19) because of the multiple convolution integrals present in the time domain. The frequency domain functions $G_i(\omega)$ are also called the *transfer functions* for these LTI systems. From Eq. (2.18) these transfer functions can formally be obtained by *deconvolution* if the frequency spectra of the input and output are known or measured, i.e.

$$G_i(\omega) = \frac{O_i(\omega)}{I_i(\omega)}, \tag{2.21}$$

where $(O_i(\omega), I_i(\omega))$ are the frequency domain values of the output and input, respectively, of the ith LTI system. Unlike the convolution relationship of Eq. (2.18), however, deconvolution by the direct division of Eq. (2.21) is unstable in the presence of noise. This problem is inherently present in ultrasonic systems when the input and output signals are measured experimentally since those signals are typically weak at frequencies outside the bandwidth of the transducer(s) present where an application of Eq. (2.21) would simply be dividing noise by noise. In ultrasonic NDE a Wiener filter is typically used to replace Eq. (2.21) by a form that makes the deconvolution process stable without significantly affecting the end result. Details of that replacement are given in problems discussed in Chaps. 9 and 14.

2.4 An Ultrasonic NDE Measurement System as an LTI System

Figure 2.8 shows the components of an ultrasonic NDE testing situation where a flawed part is being interrogated in a pitch-catch immersion arrangement. This type of setup will be used here, and in later Chapters, to represent a generic NDE measurement system. Contact systems, however, can be treated in a very similar manner, as we will see in Chap. 12. First, we will consider the ultrasonic system in three parts (Fig. 2.9). During sound generation a driving voltage pulse, $v_i(t)$, generated by the pulser travels to the transmitting transducer and is transformed to a velocity on the face of the transducer. We can represent this part of the system by an impulse response function, $t_g(t)$, whose Fourier transform is the sound generation transfer function, $T_g(\omega)$. The motion of the sending transducer face produces a sound beam that travels through the water and is transmitted into the part where it interacts with a flaw. The waves that are incident on this flaw generate scattered waves, some of which propagate to the receiving transducer, generating a

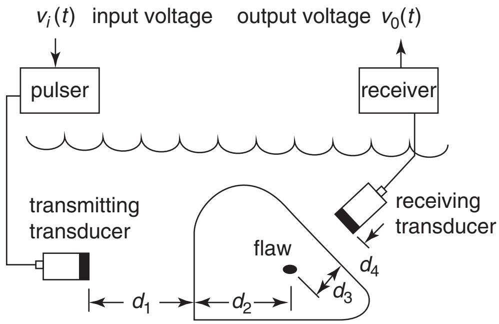

Fig. 2.8 An ultrasonic NDE immersion testing configuration

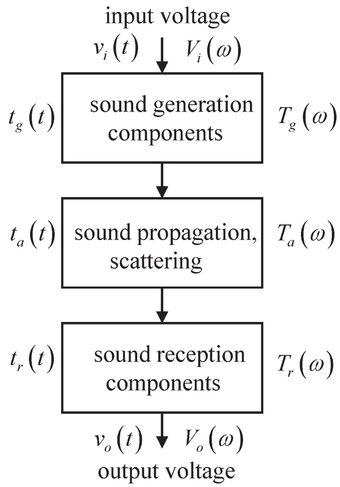

Fig. 2.9 An ultrasonic NDE system as a series of LTI systems

force, on the face of that transducer. This part of the system, which involves the acoustic and elastic ultrasonic waves present, we will characterize by an impulse response, $t_a(t)$ whose corresponding acoustic/elastic transfer function is $T_a(\omega)$. The force on the receiving transducer is transformed into an electrical pulse that travels to the receiver, where it is amplified and produces an output voltage, $v_0(t)$. This reception part of the system can be characterized by the impulse response function, $t_r(t)$, and its corresponding sound reception transfer function, $T_r(\omega)$. Thus, the entire system can be characterized by the three LTI systems shown in Fig. 2.9, where $V_i(\omega)$ is the Fourier transform of the driving voltage and $V_0(\omega)$ is the Fourier transform of the output voltage. Obviously, for the entire ultrasonic system we have

$$V_0(\omega) = T_g(\omega)T_a(\omega)T_r(\omega)V_i(\omega). \tag{2.22}$$

The sound generation transfer function, $T_g(\omega)$, is a function of the electrical properties of the pulser and cabling between the pulser and sending transducer and the electromechanical properties of the transducer itself while the sound reception transfer function, $T_r(\omega)$, is a function of the electromechanical properties of the receiving transducer and the electrical properties of the receiver and cabling between the receiving transducer and the output of the receiver. Both of these transfer functions can be modeled in terms of their components and the model parameters of those components can be directly obtained with purely electrical measurements [2]. The acoustic/elastic transfer function, $T_a(\omega)$, however, is a function of the 3-D propagating and scattered waves present. The properties of these acoustic and elastic waves cannot be measured directly so this transfer function must be obtained with the use of ultrasonic beam propagation and scattering models. Thus, in an ultrasonic measurement system if we measure $V_i(\omega)$ and the elements of the sound generation and reception transfer functions and model the acoustic/elastic transfer function, with the use of Eq. (2.22) we can predict the

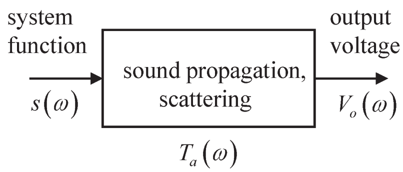

Fig. 2.10 Representation of an ultrasonic system as a system function and an acoustic/elastic transfer function

measured output voltage spectrum, $V_0(\omega)$, which can then be transformed back into the time domain to obtain the A-scan time domain flaw response measured by the system, $V_0(t)$. Since there are many elements that need to be measured in the sound generation and reception transfer functions, it would be advantageous to also have a simpler approach to characterize the ultrasonic system. Fortunately, this is possible by combining the sound generation and reception transfer functions and the input voltage spectrum into a single function, $s(\omega)$, that we will call the *system function*, i.e.

$$s(\omega) = T_g(\omega)T_r(\omega)V_i(\omega). \tag{2.23}$$

Then an LTI model of the entire ultrasonic system becomes simply (Fig. 2.10)

$$V_0(\omega) = s(\omega)T_a(\omega). \tag{2.24}$$

This representation is useful, since, as shown in Chap. 9, we can model the acoustic/elastic transfer function, $T_a^{ref}(\omega)$, explicitly in a simple calibration setup and measure the output voltage spectrum, $V_0^{ref}(\omega)$, in that same setup. Then by deconvolution we have formally

$$s(\omega) = \frac{V_0^{ref}(\omega)}{T_a^{ref}(\omega)}. \tag{2.25}$$

In practice, a Wiener filter is used to stabilize this deconvolution process in the presence of noise, as discussed briefly in the last section, and which is described in more detail in Chap. 9. Once the system function is measured then this system function can be used in Eq. (2.24) for a flaw measurement provided the same system components (pulser-receiver, cabling, transducers) and equipment settings that were present in the calibration setup are also present in the flaw measurement setup. In the first edition of this book we used a function defined in a slightly different manner, called the *system efficiency factor*, $\beta(\omega)$, in place of the system function. But, as shown in Chap. 9, these two functions are just proportional to one another so that there is no significant difference in the use of either. We have chosen to emphasize the system function in many of our discussions in this second edition since it is the function which appears naturally when one explicitly models all the elements of the measurement system, as done in Schmerr and Song [2] and Schmerr [3].

If we obtain the system function by a single calibration setup measurement, then to characterize the ultrasonic flaw measurement system of Fig. 2.8 we must also be able to model the acoustic/elastic transfer function, $T_a(\omega)$. In Chap. 12, we will obtain a very general model of this transfer function that uses a reciprocity relationship originally developed by Auld [4]. Thus, with this transfer function expression we will call Eq. (2.24) *Auld's ultrasonic measurement model*. We will also show in Chap. 12 that with some additional assumptions we can decompose the acoustic/elastic transfer function into a product of simpler terms where the flaw response appears separately from the beam propagation terms in a form similar to that originally developed by Bruce Thompson and Tim Gray [5]. This separation of the beam propagation and flaw response terms is extremely valuable, since in a flaw measurement one is primarily interested in the type of flaw present and its properties. In the Thompson-Gray model of the entire ultrasonic system, we can write the acoustic/elastic transfer function as a product of LTI components in the form:

$$T_a(\omega) = P(\omega)M(\omega)T_1(\omega)C_1(\omega)T_2(\omega)C_2(\omega)A(\omega), \tag{2.26}$$

where $P(\omega)$ and $M(\omega)$ account for the propagation time delays and attenuation, respectively, of the ultrasound in going from the sending transducer to the flaw and back to the receiving transducer, $T_1(\omega)$ and $T_2(\omega)$ are plane wave transmission coefficients associated with the propagation of the sound beam through the fluid-solid interfaces for the incident and scattered waves. The $C_1(\omega)$ term is a diffraction correction that characterizes the sound beam incident on the flaw, and $C_2(\omega)$ is a diffraction correction that accounts for the sound beam of the waves scattered from the flaw and the averaging of those waves (over the face of the receiving transducer) during sound reception. The $A(\omega)$ term is the far field scattering amplitude of the flaw, a quantity that is discussed extensively in Chap. 10. If we place Eq. (2.26) into Eq. (2.24) we then have a model of the entire ultrasonic system in the form

$$V_0(\omega) = s(\omega)P(\omega)M(\omega)T_1(\omega)C_1(\omega)T_2(\omega)C_2(\omega)A(\omega). \tag{2.27}$$

This model is in a form we will call the *Thompson-Gray ultrasonic measurement model*. In the following chapters we will give expressions and procedures for obtaining all the LTI system responses listed in Eq. (2.27). Summary boxes which describe those expressions and procedures can be found near the end of the following Chapters:

$$\begin{array}{ll}
\text{Pr}\mathit{opagation} : P(\omega) & \mathit{Chapter}\ 4 \\
\mathit{Transmission} : T_1(\omega), T_2(\omega) & \mathit{Chapter}\ 6 \\
\mathit{Diffraction}\,(\mathit{on\ transmission}) : C_1(\omega) & \mathit{Chapter}\ 8 \\
\mathit{Attenuation} : M(\omega) & \mathit{Chapter}\ 9 \\
\mathit{System\ function} : s(\omega) & \mathit{Chapter}\ 9 \\
\mathit{Flaw\ scattering} : A(\omega) & \mathit{Chapter}\ 10 \\
\mathit{Diffraction\ and\ averaging}\,(\mathit{on\ reception}) : C_2(\omega) & \mathit{Chapter}\ 11
\end{array}$$

2.5 About the Literature

Linear time-shift invariant systems are often used to describe some closely related optical problems of light propagation and transmission so that the books by Gaskill [6] and Papoulis [1, 7] are particularly good references. However, LTI systems are also frequently used elsewhere, particularly in the field of electrical engineering so that many electrical engineering texts can also provide substantial background information on LTI systems. The Fourier integral also is widely used in many fields and there are numerous references available. The books by Sneddon [8], Bracewell [9], and Papoulis [10] contain most of the results needed for our ultrasonic NDE applications. As mentioned previously, calculating Fourier integrals numerically typically involves the use of Fast Fourier Transform (FFT) routines. Burrus and Parks [11] describe the basic FFT algorithm and give some explicit implementations in software.

One of the earliest uses of a combination of LTI system and Fourier concepts specifically for ultrasonic NDE problems is the work of Frederick and Seydel [12]. That same theme also appears in applications of "ultrasonic spectral analysis" considered by a number of authors, as described, for example, in Fitting and Adler [13]. The most complete representation of an entire ultrasonic system with LTI system concepts is in the book of Schmerr and Song [2] which showed how all of the electrical and electro-mechanical elements (pulser/receiver, cabling, transducer (s)) of an ultrasonic measurement system can be explicitly modeled and also measured with purely electrical measurements.

2.6 Problems

2.1. Which of the following systems are linear time-shift invariant systems?

$$S[i(t)] = \left\{ a\frac{d^2}{dt^2} + b\frac{d}{dt} + c \right\} i(t)$$

$$S[i(t)] = \int_{-\infty}^{t} i(t)\,dt$$

$$S[i(t)] = a\{i(t)\}^2 + b\{i(t)\}$$

$$S[i(t)] = \int_{-\infty}^{+\infty} i(t)\,exp(2\pi i \tau t)\,dt$$

2.2. The cross correlation of two functions, f and g, is defined as

$$f \circ g = \int_{-\infty}^{+\infty} f(t)\, g(t-\tau)\, dt = \int_{-\infty}^{+\infty} f(t+\tau)\, g(t)\, dt,$$

where note that $f \circ g \neq g \circ f$. Show that the Fourier transform of the cross correlation of f and g, $F[f \circ g]$, is related to the Fourier transforms of the functions themselves through

$$F[f \circ g] = F[f] G^*[g],$$

where the asterisk superscript indicates complex conjugation.

2.3. Given two functions f and g (possibly complex) and their Fourier transforms, F, and G, respectively, prove the "power" theorem

$$\int_{-\infty}^{+\infty} f(\tau)\, g^*(\tau)\, d\tau = \int_{-\infty}^{+\infty} F(\omega)\, G^*(\omega)\, d\omega.$$

Note that when $f = g$ this reduces to Rayleigh's theorem (also called the energy theorem, Parseval's theorem, or Plancherel's theorem):

$$\int_{-\infty}^{+\infty} f(\tau)\, g^*(\tau)\, d\tau = \int_{-\infty}^{+\infty} F(\omega)\, G^*(\omega)\, d\omega.$$

2.4. Show that the area under the convolution is equal to the product of the areas under the functions being convolved, i.e. if $h = f * g$ then

$$\int_{-\infty}^{+\infty} h(\tau)\, d\tau = \int_{-\infty}^{+\infty} f(\tau)\, d\tau \int_{-\infty}^{+\infty} g(\tau)\, d\tau.$$

2.5. Show that the convolution $h = f * g$ of two functions f and g has the following properties

(a) scaling

$$f(\tau/b) * g(\tau/b) = |b|\, h(\tau/b)$$

(b) derivatives

$$f^{(m)}(\tau) * g^{(n)}(\tau) = h^{(m+n)}(\tau)$$

(c) commutative property

$$f(\tau) * g(\tau) = g(\tau) * f(\tau)$$

(d) distributive property

$$[av(\tau) + bw(\tau)] * g(\tau) = a[v(\tau)] * g(\tau) + b[w(\tau)] * g(\tau)$$

(e) associative property

$$[v(\tau) * w(\tau)] * g(\tau) = v(\tau) * [w(\tau) * g(\tau)]$$

(f) shift invariance

$$h(\tau - \tau_0) = f(\tau - \tau_0) * g(\tau) = f(\tau) * g(\tau - \tau_0)$$

2.6. If we define the kth moment of a function f, m_k, as

$$m_k = \int_{-\infty}^{+\infty} \tau^k f(\tau)\, d\tau,$$

prove the "moment" theorem, i.e. m_k is related to the derivatives of the Fourier transform, F, of f via

$$m_k = \frac{F^{(k)}(0)}{(2\pi i)^k}$$
$$where \quad F^{(k)}(0) = \left.\frac{d^k F(\omega)}{d\omega^k}\right|_{\omega=0}.$$

The centroid (mean abscissa) of f, $\bar{f}$, is then given by

$$\bar{f} = \frac{m_1}{m_0}.$$

Similarly, the mean-square abscissa, $\overline{f^2}$, and variance, σ^2 are defined as

$$\overline{f^2} = \frac{m_2}{m_0}$$
$$\sigma^2 = \overline{f^2} - (\bar{f})^2.$$

For the Gaussian function $f(t) = \exp\left[-\pi(t - t_0)^2/b\right]$ determine its centroid, mean-square abscissa, and variance.

2.7. Consider the LTI system shown in Fig. P2.1. If we give this *RC* circuit an input voltage shown in Fig. P2.2, determine by any means (except the use of convolution) the output voltage, $V_0(t)$. Taking the appropriate limit of this answer, find the impulse response function of this system. Using this impulse response function and the convolution theorem, show that the output produced by the input function of Fig. P2.2 agrees with your original answer.

2.8. All ultrasonic systems are band limited due to the limited frequency response range of real transducers, attenuation effects, etc. Thus, all measured ultrasonic responses are to some extent filtered responses. If we represent a component of our ultrasonic system as a black box LTI system with a band limited frequency response given by Fig. P2.3, determine

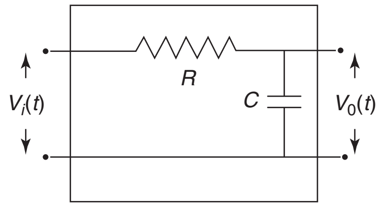

Fig. P2.1 An *RC* circuit LTI system

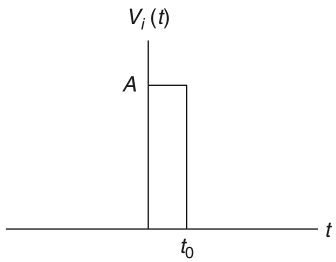

Fig. P2.2 Input wave form for the *RC* circuit of Fig. P2.1

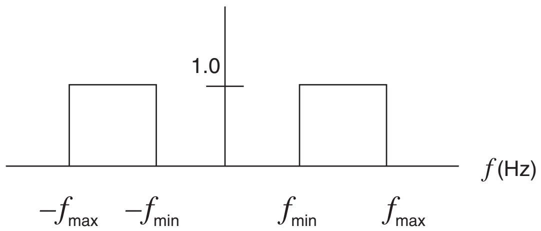

Fig. P2.3 Frequency response of a bandlimited LTI system

(a) the response of the system to an ideal delta function input.
(b) the response of the system due to a derivative of a delta function (doublet) input

Sketch your results for both (a) and (b).

2.9. Take the low frequency value $f_{\min} = 0$ in Fig. P2.3 and plot an expression for the maximum peak-to-peak amplitude for these responses as a function of $f_{\max}$.

2.10. In this problem we want to investigate the consequences of throwing away the negative frequencies of the Fourier transform of a real function, $f(t)$. Thus, determine the function $u(t)$ that is recovered by calculating

$$u(t) = \frac{1}{2\pi} \int_0^\infty F(\omega) \exp(-i\omega t)\, d\omega,$$

where F is the Fourier transform of f. Hint: consider u as the limit

$$u(t) = \lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{-\infty}^{+\infty} \{1 + \operatorname{sgn} \omega \exp(-\varepsilon|\omega|)\} \exp(-i\omega t) F(\omega)\, d\omega.$$

2.11. Consider a spring-mass system subjected to a force $F(t)$ as shown in Fig. P2.4. If the force is given as a unit step function $F(t) = H(t)$, where

$$H(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases},$$

what is the resulting unit step response, $x_H(t)$, of the system? How is the unit step response related to the impulse response of the system? Using convolution, write the response of the system, $x(t)$, to an arbitrary force input $F(t)$ in terms of the unit step response.

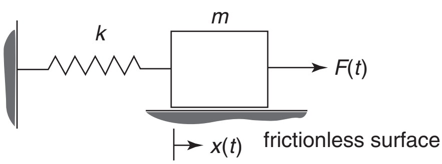

Fig. P2.4 A spring-mass system subjected to a transient driving force $F(t)$

References

1. A. Papoulis, *Signal Analysis* (McGraw-Hill, New York, 1977)
2. L.W. Schmerr, S.-J. Song, *Ultrasonic Nondestructive Evaluation Systems—Models and Measurements* (Springer, New York, 2007)
3. L.W. Schmerr, *Fundamentals of Ultrasonic Phased Arrays* (Springer, New York, 2014)
4. B.A. Auld, General electromechanical reciprocity relations applied to the calculation of elastic wave scattering coefficients. Wave Motion **1**, 3–10 (1979)
5. R.B. Thompson, T.A. Gray, A model relating ultrasonic scattering measurements through liquid-solid interfaces to unbounded medium scattering amplitudes. J. Acoust. Soc. Am. **74**, 1279–1290 (1983)
6. J.D. Gaskill, *Linear Systems, Fourier Transforms, and Optics* (Wiley, New York, 1978)
7. A. Papoulis, *Systems and Transforms with Applications in Optics* (McGraw-Hill, New York, 1968)
8. I.N. Sneddon, *The Use of Integral Transforms* (McGraw-Hill, New York, 1972)
9. R. Bracewell, *The Fourier Transform and its Applications* (McGraw-Hill, New York, 1956)
10. A. Papoulis, *The Fourier Integral and its Applications* (McGraw-Hill, New York, 1962)
11. C.S. Burrus, T.W. Parks, *DFT/FFT and Convolution Algorithms* (Wiley, New York, 1985)
12. J.R. Frederick, J.A. Seydel, *Improved Discontinuity Detection Using Computer-Aided Ultrasonic Pulse-Echo Techniques* (*Welding Research Council Bulletin,* No. 185, 1973)
13. D.W. Fitting, L. Adler, *Ultrasonic Spectral Analysis for Nondestructive Evaluation* (Plenum Press, New York, 1981)

http://www.springer.com/978-3-319-30461-8

Fundamentals of Ultrasonic Nondestructive Evaluation
A Modeling Approach
Schmerr Jr, L.W.
2016, XVIII, 758 p. 370 illus., 62 illus. in color.,
Hardcover
ISBN: 978-3-319-30461-8