Chapter 2
Measure on the Real Line

2.1 Introduction

There are many examples of functions that associate a nonnegative real number or $+\infty$ with a set. There is, for example, the number of members forming the set. Given a finite probability experiment, probabilities are associated with outcomes. Riemann integration associates with each finite interval in the real line, the length of that interval. These are all examples of a "finitely additive measure." Recall that an **algebra** $\mathscr{A}$ of subsets of a set X is a collection that contains the set X together with the complement in X of each of its members; it is also stable under the operation of taking finite unions and, therefore, finite intersections. Also recall that a collection of sets is pairwise disjoint if for any two sets A and B in the collection, $A \cap B = \varnothing$.

Definition 2.1.1. A **finitely additive measure** m is a function from an algebra $\mathscr{A}$ of subsets of a set X into the extended nonnegative real line, $\mathbb{R} \cup \{+\infty\}$, such that $m(\varnothing) = 0$ and for any finite collection $\{A_i : i = 1, 2, \ldots, n\}$ of pairwise disjoint sets in $\mathscr{A}$,

$$m\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(A_i).$$

Such a function m is **countably additive** if for any pairwise disjoint sequence $\{A_i : i \in \mathbb{N}\}$ in $\mathscr{A}$ with union also in $\mathscr{A}$,

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i).$$

Remark 2.1.1. If the summation condition for countable additivity holds and $m(\varnothing) = 0$, then the summation condition for finite additivity also holds.

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Definition 2.1.2. Given a set $A \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, the **translate** of A by r, denoted by $A+r$, is the set $\{a+r : a \in A\}$. A finitely additive measure m defined on an algebra $\mathscr{A}$ of subsets of $\mathbb{R}$ is **translation invariant** if for each $A \in \mathscr{A}$ and each $r \in \mathbb{R}$, $A+r$ is in $\mathscr{A}$ and

$$m(A+r) = m(A).$$

The translation invariant, finitely additive measure m that associates to each subinterval of $\mathbb{R}$ the length of that interval is defined on the algebra consisting of all finite unions of subintervals of $\mathbb{R}$. Any such union can be written as a finite union of pairwise disjoint intervals. The sum of the lengths of those intervals is the value, independent of the decomposition, that is taken by m. We want to extend Riemann integration. We need, therefore, to extend the function m to a larger class of sets. We would like the extension to be countably additive and translation invariant. It turns out that an extension with these properties cannot be defined for all subsets of $\mathbb{R}$. There is, however, an important translation invariant extension that is defined for all subsets of $\mathbb{R}$.

2.2 Lebesgue Outer Measure

For each interval $I \subseteq \mathbb{R}$, we write $l(I)$ for the length of I. For example, if $I = (a,b)$, then $l(I) = b-a$. If I is an infinitely long interval, then $l(I) = +\infty$. Given a set $A \subseteq \mathbb{R}$, we let $\mathscr{C}(A)$ denote the family of all collections of open intervals such that the intervals in the collection cover A. That is, $\mathscr{I}$ is a member of $\mathscr{C}(A)$ if and only if $\mathscr{I}$ is a set of open intervals in $\mathbb{R}$ and the union of the intervals in $\mathscr{I}$ contains the set A. By $\sum_{I\in\mathscr{I}} l(I)$ we mean the unordered sum of the length of the intervals in $\mathscr{I}$. Recall that this is the supremum of the sums obtained by adding the length of intervals in finite subsets of $\mathscr{I}$. If $\mathscr{I}$ is an uncountable collection of intervals, then by the Lindelöf theorem, a finite or countably infinite subfamily of $\mathscr{I}$ also covers A and has a sum of lengths that is no greater than the sum for the whole family. Therefore, in applying the following definition, we usually consider just finite and countably infinite families of open intervals that cover A. Every enumeration of a countably infinite family of intervals will produce the same sum of lengths, which is the usual limit of partial sums.

Definition 2.2.1 (Lebesgue outer measure). For each subset $A \subseteq \mathbb{R}$, the **Lebesgue outer measure**, denoted by $\lambda^*(A)$, is obtained as follows:

$$\lambda^*(A) = \inf_{\mathscr{I}\in\mathscr{C}(\mathscr{A})} \left(\sum_{I\in\mathscr{I}} l(I) \right).$$

Lebesgue outer measure is defined on the power set of $\mathbb{R}$, that is, the algebra comprised of all subsets of $\mathbb{R}$. We will show that Lebesgue outer measure is translation invariant and extends the notion of interval length. To obtain finite additivity, however, we will need to restrict λ^* to a proper subfamily of the algebra of all subsets of $\mathbb{R}$.

Proposition 2.2.1. *For each $A \subseteq \mathbb{R}$, $\lambda^*(A) \geq 0$, $\lambda^*(\varnothing) = 0$, $\lambda^*(\mathbb{R}) = +\infty$, and if $A \subseteq B \subseteq \mathbb{R}$, then $\lambda^*(A) \leq \lambda^*(B)$.*

Proof. Since every open interval contains the empty set, $\lambda^*(\varnothing) = 0$. The rest is clear from the definition.

Theorem 2.2.1. *Lebesgue outer measure is translation invariant. That is, for any $A \subseteq \mathbb{R}$ and each $r \in \mathbb{R}$, $\lambda^*(A + r) = \lambda^*(A)$.*

Proof. Exercise 2.4(A).

Definition 2.2.2. Given a closed and bounded interval $[a,b]$ with $a < b$, let $\mathscr{BP}[a,b]$ be the sequence of **bisection partitions** $\langle P_n : n \in \mathbb{N}\rangle$ of $[a,b]$. That is, P_1 is the pair $\left\{[a, a+\frac{b-a}{2}], [a+\frac{b-a}{2}, b]\right\}$, and for each $n \in \mathbb{N}$, P_{n+1} is the set of closed intervals obtained by cutting each interval in P_n in half, thus forming closed intervals of length $(b-a)/2^{n+1}$.

Proposition 2.2.2. *Fix an interval $[a,b]$ and a finite collection of open intervals $\mathscr{I} = \{(a_k, b_k) : k = 1, \cdots, k_0\}$ covering $[a,b]$. There is a $j \in \mathbb{N}$ such that every interval in the bisection partition $P_j \in \mathscr{BP}[a,b]$ is contained in at least one of the open intervals (a_k, b_k) from $\mathscr{I}$.*

Proof. Since a is contained in an open interval from $\mathscr{I}$, there is a first $m \in \mathbb{N}$ such that $[a, a+\frac{b-a}{2^m}]$ is contained in an open interval from $\mathscr{I}$. For any $n < m$, let $x_n = a$. For each $n > m$, let x_n be the largest right endpoint of the intervals in P_n such that each of the intervals in P_n below x_n is contained in an open interval from $\mathscr{I}$. The increasing sequence $\langle x_n\rangle$ has a limit x_0 in $[a,b]$. Since x_0 is contained in an open interval from $\mathscr{I}$, that limit is b, and $b = x_j$ for some $j \in \mathbb{N}$.

Theorem 2.2.2. *The Lebesgue outer measure of an interval is its length.*

Proof. For any $x \in \mathbb{R}$, $\lambda^*([x,x]) = \lambda^*(\{x\}) = 0$. Now assume the interval is $[a,b]$ with $a < b$. For each $\varepsilon > 0$, $[a,b] \subset (a-\varepsilon, b+\varepsilon)$, so $\lambda^*([a,b]) \leq b - a + 2\varepsilon$, and since ε is arbitrary, $\lambda^*([a,b]) \leq l([a,b])$. Note that this proof works for any finite interval. To show the reverse inequality, we must show that whatever the finite or countably infinite covering of $[a,b]$ by open intervals, the sum of their lengths is no less than $b-a$. Fix such a covering, and let $\{(a_k, b_k) : k = 1, \cdots, n_0\}$ be a finite subcovering. We need only show that $\sum_{k=1}^{n_0} b_k - a_k \geq b - a$. By Proposition 2.2.2, we may fix a bisection partition P_n of $[a,b]$ so that each member of P_n, which is a subinterval of $[a,b]$ of length $(b-a)/2^n$, is contained in at least one of the open intervals (a_k, b_k). For each $k \leq n_0$, $b_k - a_k$ is greater than the sum of the lengths of the closed intervals from P_n that are contained in (a_k, b_k). Since $b-a$ is the sum of the lengths of the intervals in P_n and every one of those intervals is in at least one of the intervals (a_k, b_k), it follows that $\sum_{k=1}^{n_0} b_k - a_k \geq b - a$.

We have already shown that for an arbitrary, not necessarily closed, finite interval I of positive length, the Lebesgue outer measure of I is less than or equal to the length of I. On the other hand, the length is less than or equal to the outer measure since there are closed intervals $J_n \subset I$ with $l(J_n) \uparrow l(I)$. That is, for each

$\varepsilon > 0$, there is an $n \in \mathbb{N}$ so that $l(I) - \varepsilon \le l(J_n) = \lambda^*(J_n) \le \lambda^*(I)$. It follows that $\lambda^*(I) = l(I)$. Finally, an infinite interval contains arbitrarily large closed subintervals, so the outer measure of an infinite interval is $+\infty$.

Theorem 2.2.3. *Lebesgue outer measure is finitely and countably subadditive. That is, for any finite or infinite sequence $\langle A_n \rangle$ of subsets of $\mathbb{R}$,*

$$\lambda^*\left(\bigcup_n A_n\right) \le \sum_n \lambda^*(A_n).$$

Proof. If for some n we have $\lambda^*(A_n) = +\infty$, then the inequality is clear. If not, we fix $\varepsilon > 0$ and for each n find a countable family of intervals covering A_n with the sum of the length of those intervals less than $\lambda^*(A_n) + \varepsilon/2^n$. The union of these families of intervals forms a countable interval covering of $\cup A_n$, and the sum of the lengths is less than $\sum_n \lambda^*(A_n) + \varepsilon$. Since ε is arbitrary, the result follows.

Corollary 2.2.1. *A countable set has Lebesgue outer measure* 0.

Corollary 2.2.2. *Any interval of positive length is uncountable.*

Example 2.2.1. The set of integers has Lebesgue outer measure 0, and the set of rational numbers has Lebesgue outer measure 0.

Recall that a G_δ set is a set that is the countable intersection of open sets. An F_σ set is a set that is the countable union of closed sets. A $G_{\delta\sigma}$ set is a countable union of G_δ sets. An $F_{\sigma\delta}$ set is a countable intersection of F_σ sets, etc.

Proposition 2.2.3. *Given $A \subseteq \mathbb{R}$ and $\varepsilon > 0$, there is an open set O with $A \subseteq O$ and $\lambda^*(O) \le \lambda^*(A) + \varepsilon$. Moreover, there is a G_δ set $S \supseteq A$ with $\lambda^*(S) = \lambda^*(A)$.*

Proof. The first and second part are clear if $\lambda^*(A) = +\infty$. For example, let $O = S = \mathbb{R}$. Otherwise, take open intervals that cover A with total length at most ε, and let O be the union. For the second part, let O_n be an open set given in the first part that works when $\varepsilon = 1/n$. Now the desired set is $S = \bigcap_n O_n$.

To obtain Lebesgue measure, we will restrict λ^* to a family of sets on which it is finitely additive. The restriction will then, in fact, be countably additive. We will call the reduced family of sets "the Lebesgue measurable sets", and the restriction of λ^* will be Lebesgue measure λ.

2.3 General Outer Measures

Lebesgue outer measure generalizes the length of finite open intervals. The length of a finite interval is the change on the interval of the function $F(x) = x$. More general outer measures are constructed using the changes of more general increasing functions. Such functions will have discontinuities at points where the limit from the right and the limit from the left are not equal. Our more general outer measures will be formed using increasing functions, called "integrators", that are continuous from the right.

Definition 2.3.1. An increasing real-valued function F is an **integrator** if for each x in the domain of F, $F(x) = \lim_{y \to x+} F(y)$.

We are only interested in the changes of an integrator, so when we restrict work to a finite interval in $\mathbb{R}$ on which the integrator is bounded below, we may add a constant so that the integrator is nonnegative. The integral that results from general integrators relates to what is called "the Riemann–Stieltjes integral" in the same way that the Lebesgue integral relates to the Riemann integral. This generalization is very important in probability theory. It will cost us essentially nothing to work with results for which a more general integrator can be used. This generalization of the approach to Lebesgue integration also simplifies later material on measure differentiation. It will be clear which results hold only for Lebesgue outer measure and the corresponding Lebesgue measure.

As noted, the construction of Lebesgue outer measure employs the change of the integrator $F(x) = x$ on open intervals. For a general integrator F, however, we use the change $F(b) - F(a)$ on intervals of the form $(a,b]$. In this way, the value of any jump of F is associated with the interval on which it occurs. If we already have a measure taking only finite values, then we may set $F(x)$ equal to the measure of $(-\infty, x]$. If F is only defined on a finite interval $[a,b]$, then we can extend F with the value $F(a)$ to points below a and $F(b)$ to points above b. Then the change of F will be 0 on any interval that does not intersect $[a,b]$. If an integrator is continuous, such as the integrator $F(x) = x$ for Lebesgue outer measure, then the same outer measure is obtained using open intervals or intervals of the form $(a,b]$ (Exercise 2.13). We have shown in Corollary 1.7.1 that any collection of intervals of the form $(a,b]$ has a finite or countably infinite subcollection with the same union.

Definition 2.3.2. Let F be an integrator, that is, an increasing real-valued function, continuous from the right at each point of $\mathbb{R}$. For each subset $A \subseteq \mathbb{R}$, let $m^*(A)$ be defined in a way similar to Lebesgue outer measure, but using finite intervals of the form $(a,b]$ and the change $F(b) - F(a)$.

When we used length, we used compactness and open coverings to show that the outer measure of an interval is its length. The analogous result for a general integrator F is still true.

Proposition 2.3.1. *Let F be an integrator on $\mathbb{R}$. Then $m^*(\varnothing) = 0$. If $A \subseteq B \subseteq \mathbb{R}$, then $m^*(A) \le m^*(B)$, and for any interval $(a,b]$, $m^*((a,b]) = F(b) - F(a) < +\infty$.*

Proof. Every interval $(a,b]$ contains the empty set, and for every $\varepsilon > 0$ there is such an interval for which $F(b) - F(a) < \varepsilon$ (Problem 2.14). Therefore, $m^*(\varnothing) = 0$. It is clear that a more general outer measure is still an increasing function; that is, the bigger the set, the bigger the outer measure. It is also clear that $m^*((a,b]) \le F(b) - F(a)$

since $(a,b]$ covers itself. To show the reverse inequality for F, we fix $\varepsilon > 0$. Fix a countable covering of $(a,b]$ by intervals of the form $(c_n,d_n]$. Since F is continuous from the right, we may replace each interval $(c_n,d_n]$ with an open interval (c_n,e_n) where $e_n > d_n$, but $F(e_n) - F(c_n) \leq F(d_n) - F(c_n) + \varepsilon/2^{n+1}$. Fix δ with $0 < \delta < b-a$ and $F(a+\delta) < F(a) + \varepsilon/2$. The intervals (c_n,e_n) form an open interval covering of $[a+\delta,b]$, and so we may assume it is a finite covering of that interval. By Proposition 2.2.2, we may fix a bisection partition P_n of $[a+\delta,b]$ so that each member of P_n, which is a subinterval of $[a+\delta,b]$ of length $(b-a-\delta)/2^n$, is contained in at least one of the open intervals (c_k,e_k). For each of the intervals (c_k,e_k), $F(e_k) - F(c_k)$ is greater than or equal to the sum of the changes of F on the closed intervals of P_n contained in (c_k,e_k). Moreover, $F(b) - F(a+\delta)$ is equal to the sum of the changes of F on the intervals of P_n. Since each interval of P_n is contained in at least one of the intervals (c_k,e_k), it follows that

$$\begin{aligned}
&\sum_{k=1}^{\infty} [F(d_k) - F(c_k)] + \frac{\varepsilon}{2} \\
&\geq \sum_{k=1}^{n} F(e_k) - F(c_k) \geq F(b) - F(a+\delta) \geq F(b) - F(a) - \frac{\varepsilon}{2}.
\end{aligned}$$

Since ε is arbitrary, the result follows.

Remark 2.3.1. It is no longer necessarily true for a more general integrator F that points have 0 outer measure. If F jumps at a point x, then the outer measure of $\{x\}$ is the size of the jump.

Example 2.3.1. If we defined m^* using open intervals, then it would no longer be always true that the outer measure of an open interval would equal the change of the integrator at the endpoints of the interval. For example, if $F(x) = 0$ for $x < 1$ and $F(x) = 1$ for $x \geq 1$, then the change of F for $(0,1)$ is 1, but the outer measure using countable coverings by small open intervals would be 0.

Remark 2.3.2. In what follows, results and proofs that hold for general integrators will be stated using m^* and m for the corresponding outer measure and measure. We will use λ^* and λ when the result is special for the Lebesgue case. Since most results use only the common properties of outer measures, in only a few instances, such as Proposition 2.3.1 above, is there a difference in wording of proofs for the Lebesgue and the general case. For the next result, already established for the Lebesgue case, one can also use the fact that there is an integrator $F(x) = x$ for the Lebesgue case.

Theorem 2.3.1. *Outer measure is finitely and countably subadditive. That is, for any sequence $\langle A_n : n \in \mathbb{N}\rangle$ of subsets of $\mathbb{R}$, where some sets may be empty,*

$$m^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} m^*(A_n).$$

Proof. If for some n, $m^*(A_n) = +\infty$, then the inequality is clear. If not, fix $\varepsilon > 0$, and for each $n \in \mathbb{N}$ find a countable family $\mathscr{F}_n$ of appropriate intervals covering A_n

such that the sum of the changes of the integrator F is less than $m^*(A_n)+\varepsilon/2^n$. The union $\cup_{n\in\mathbb{N}}\mathscr{F}_n$ is a countable interval covering of $\cup_{n\in\mathbb{N}}A_n$ such that the sum of the changes of F is less than $\sum_{n\in\mathbb{N}} m^*(A_n)+\varepsilon$. Since ε is arbitrary, the result follows.

2.4 Measure from Outer Measure

As we shall see, Lebesgue outer measure is not even finitely additive on the family of all subsets of $\mathbb{R}$. There is, however, a finite additivity condition that yields not just finite additivity, but also countable additivity on an appropriate family of subsets of $\mathbb{R}$. It is a condition, due to Carathéodory, that is applicable to all outer measures.

Definition 2.4.1 (Carathéodory). A set $E \subseteq \mathbb{R}$ is called **measurable** if for all subsets $A \subseteq \mathbb{R}$,

$$m^*(A) = m^*(A\cap E) + m^*(A\cap \widetilde{E}).$$

We denote the family of measurable sets by $\mathscr{M}$. If the outer measure extends length, we may say “Lebesgue measurable.”

The idea is that a set E is in $\mathscr{M}$ if and only if E splits any set in an additive fashion. Since outer measure is subadditive, we always have

$$m^*(A) \le m^*(A\cap E) + m^*(A\cap \widetilde{E}).$$

We also have the reverse inequality if $m^*(A) = +\infty$. Therefore, to show E is measurable, we need only show that for any set $A \subseteq \mathbb{R}$ with $m^*(A) < +\infty$,

$$m^*(A) \ge m^*(A\cap E) + m^*(A\cap \widetilde{E}).$$

Proposition 2.4.1. *Any set of outer measure* 0 *is measurable.*

Proof. If $m^*(E) = 0$, then for any $A \subseteq \mathbb{R}$,

$$m^*(A) \ge m^*(A\cap \widetilde{E}) = m^*(A\cap \widetilde{E}) + m^*(A\cap E),$$

since $m^*(A\cap E) \le m^*(E) = 0$.

Lemma 2.4.1. *The family $\mathscr{M}$ of measurable sets is an algebra of sets.*

Proof. By symmetry, a set E is in $\mathscr{M}$ if and only if the complement $\widetilde{E} = \complement E$ is in $\mathscr{M}$. Moreover, $\mathbb{R}$ and $\varnothing$ are clearly measurable. We need to show that $\mathscr{M}$ is stable under the operation of taking finite unions. For this we need only consider two measurable sets E_1 and E_2. Fix $A \subseteq \mathbb{R}$. We will use the fact that since E_1 and E_2 are measurable,

$$\begin{aligned} m^*(A) &= m^*(A\cap E_1) + m^*(A\cap \widetilde{E}_1), \\ m^*(A\cap \widetilde{E}_1) &= m^*(A\cap \widetilde{E}_1\cap E_2) + m^*(A\cap \widetilde{E}_1\cap \widetilde{E}_2). \end{aligned}$$

We will also use the following consequence of subadditivity:

$$m^*(A\cap[E_1\cup E_2])\leq m^*(A\cap E_1)+m^*(A\cap E_2\cap\widetilde{E}_1).$$

Now,

$$\begin{aligned}m^*(A)&\leq m^*(A\cap[E_1\cup E_2])+m^*(A\cap\complement[E_1\cup E_2])\\&=m^*(A\cap[E_1\cup E_2])+m^*(A\cap[\widetilde{E}_1\cap\widetilde{E}_2])\\&\leq m^*(A\cap E_1)+m^*(A\cap E_2\cap\widetilde{E}_1)+m^*(A\cap[\widetilde{E}_1\cap\widetilde{E}_2])\\&=m^*(A\cap E_1)+m^*(A\cap\widetilde{E}_1)=m^*(A).\end{aligned}$$

Therefore, $E_1\cup E_2\in\mathscr{M}$.

Lemma 2.4.2. *For any finite, pairwise disjoint sequence of measurable sets E_i, $1\leq i\leq n$, and any $A\subseteq\mathbb{R}$,*

$$m^*(A\cap[\cup_1^n E_i])=\sum_1^n m^*(A\cap E_i).$$

Proof. The proof is by induction. The equality is clear for $n=1$. Assuming it holds for $n-1$, that is,

$$\sum_{i=1}^{n-1}m^*(A\cap E_i)=m^*(A\cap[\cup_1^{n-1}E_i])=m^*(A\cap[\cup_1^n E_i]\cap\widetilde{E}_n),$$

we also have

$$m^*(A\cap E_n)=m^*(A\cap[\cup_1^n E_i]\cap E_n).$$

Therefore, equality holds for n since E_n is measurable and

$$\begin{aligned}\sum_1^n m^*(A\cap E_i)&=m^*(A\cap[\cup_1^n E_i]\cap\widetilde{E}_n)+m^*(A\cap[\cup_1^n E_i]\cap E_n)\\&=m^*(A\cap[\cup_1^n E_i]).\end{aligned}$$

Recall that an algebra of sets is called a σ-algebra if it is stable with respect to the operation of taking countable unions.

Definition 2.4.2. A nonnegative function μ defined on a σ-algebra $\mathscr{A}$ is a **measure** on $\mathscr{A}$ if $\mu(\varnothing)=0$ and μ is **countably additive**; that is, given a countable, pairwise disjoint sequence $\langle A_n:n\in\mathbb{N}\rangle$ of sets in $\mathscr{A}$, where some sets may be empty,

$$\mu\left(\bigcup_{n\in\mathbb{N}}A_n\right)=\sum_{n\in\mathbb{N}}\mu(A_n).$$

The pair $(\mathscr{A},\mu)$ is called a **measure space**.

Theorem 2.4.1. *The family $\mathscr{M}$ is a σ-algebra containing all sets of outer measure* 0*, and the restriction of m^* to $\mathscr{M}$ is a measure on $\mathscr{M}$.*

Proof. We have already noted that $\mathscr{M}$ contains all sets of outer measure 0. Let B_i, $i \in \mathbb{N}$, be a countable family of sets in $\mathscr{M}$, and let E be the union. We must show that $E \in \mathscr{M}$. Since $\mathscr{M}$ is an algebra, it follows from Proposition 1.3.1 that we may replace each set B_i with a subset $E_i \in \mathscr{M}$ so that the E_i's are pairwise disjoint but have the same union E. For each finite n, let $F_n = \bigcup_{i=1}^n E_i$. Then, because $\mathscr{M}$ is an algebra and $\widetilde{F_n} \supseteq \widetilde{E}$, for each $A \subseteq \mathbb{R}$ we have

$$\begin{aligned} m^*(A) &= m^*(A \cap F_n) + m^*(A \cap \widetilde{F_n}) \\ &= \sum_{i=1}^n m^*(A \cap E_i) + m^*(A \cap \widetilde{F_n}) \\ &\geq \sum_{i=1}^n m^*(A \cap E_i) + m^*(A \cap \widetilde{E}). \end{aligned}$$

Since this is true for all $n \in \mathbb{N}$, we have by subadditivity

$$\begin{aligned} m^*(A) &\geq \sum_{i=1}^\infty m^*(A \cap E_i) + m^*(A \cap \widetilde{E}) \qquad (2.4.1) \\ &\geq m^*(A \cap E) + m^*(A \cap \widetilde{E}). \end{aligned}$$

Therefore, $E \in \mathscr{M}$. It now follows from Inequality (2.4.1) applied to any pairwise disjoint sequence $\langle E_i : i \in \mathbb{N} \rangle$ in $\mathscr{M}$ and the set $A = E = \cup_{i=1}^\infty E_i$ that the restriction of m^* to $\mathscr{M}$ is countably additive.

Definition 2.4.3. Lebesgue measure λ is λ^* restricted to the σ-algebra $\mathscr{M}$ of sets measurable with respect to λ^*. For a general outer measure m^*, including Lebesgue outer measure, we let m denote the measure obtained by restricting m^* to the corresponding collection of measurable sets.

Recall that the intersection of all σ-algebras in $\mathbb{R}$ containing a family of sets $\mathscr{S}$ is again a σ-algebra; it is the smallest σ-algebra in $\mathbb{R}$ containing the family $\mathscr{S}$.

Definition 2.4.4. The family of **Borel sets** in $\mathbb{R}$ is the smallest σ-algebra containing the open subsets of $\mathbb{R}$.

We have seen that every open subset of $\mathbb{R}$ is a finite or countably infinite union of pairwise disjoint open intervals. To show, therefore, that a σ-algebra on $\mathbb{R}$, such as $\mathscr{M}$, contains the Borel sets, it is enough to show that it contains every open interval. Indeed, this need only be shown for certain open intervals.

Lemma 2.4.3. *The interval $I = (a, +\infty)$ is measurable.*

Proof. Fix $A \subseteq \mathbb{R}$ with $m^*(A) < +\infty$. Let $A_1 = A \setminus I = \{x \in A : x \leq a\}$ and $A_2 = A \cap I = \{x \in A : x > a\}$. We must show that $m^*(A) \geq m^*(A_1) + m^*(A_2)$. Fix $\varepsilon > 0$.

For the Lebesgue case, find a countable family of open intervals I_n that cover A with total length less than $\lambda^*(A)+\varepsilon$. Let $I'_n = I_n \setminus I$ and $I''_n = I_n \cap I$. For each n, I'_n is either empty or it is an interval, and I''_n is either empty or an interval. Moreover, the nonempty intervals I'_n cover A_1, so by subadditivity their total length, which is the same as their total outer measure, is greater than or equal to $\lambda^*(A_1)$. Similarly, the nonempty intervals I''_n cover A_2, so their total length, which is the same as their total outer measure, is greater than or equal to $\lambda^*(A_2)$. Note that for each n, the length of I_n is the length of I'_n added to the length of I''_n. Therefore,

$$\begin{aligned}\lambda^*(A_1)+\lambda^*(A_2) &\le \lambda^*(\cup_n I'_n)+\lambda^*(\cup_n I''_n)\\ &\le \sum \lambda^*(I'_n)+\sum \lambda^*(I''_n)=\sum l(I_n)\le \lambda^*(A)+\varepsilon,\end{aligned}$$

and since ε is arbitrary, the result is established for the Lebesgue case.

For more general outer measures, we modify the above proof using the fact that if an interval $(\alpha,\beta]$ is cut by an interval $(a,+\infty)$, that is, if $\alpha < a < \beta$, then $(\alpha,\beta]$ will be cut into two intervals of the same kind: $(\alpha,a]$ and $(a,\beta]$. In this case, the sum of the changes on the two intervals of an integrator F will be the total change on $(\alpha,\beta]$.

Proposition 2.4.2. *The family of measurable sets $\mathscr{M}$ contains the Borel sets. In particular, $\mathscr{M}$ contains every open set and every closed set.*

Proof. We have shown that every open interval of the form $(a,+\infty)$ is in $\mathscr{M}$. Therefore, intervals of the form $(-\infty,a]$ are in $\mathscr{M}$. Since $(-\infty,a) = \bigcup_{n\in\mathbb{N}}(-\infty, a-\frac{1}{n}] \in \mathscr{M}$, and for each $a,b\in\mathbb{R}$, $(a,b) = (-\infty,b)\cap(a,+\infty)$, every open interval is in $\mathscr{M}$. Thus every open set is in $\mathscr{M}$. Since $\mathscr{M}$ is a σ-algebra containing the open sets, $\mathscr{M}$ contains the smallest σ-algebra containing the open sets, namely, the Borel sets.

Remark 2.4.1. The collection $\mathscr{M}$ of measurable sets changes with changes in the integrator, but $\mathscr{M}$ always contains the Borel sets. The collection of sets of measure 0 will, in general, be different. For example, suppose an integrator F is constant on the interval $I=(0,1)$. Then every subset of I will be measurable and have m-measure 0. As shown in Problem 2.32, however, there are non-Lebesgue measurable subsets of I.

Proposition 2.4.3. *If E and F are measurable sets such that $F\subseteq E$ and F has finite measure, then $m(E\setminus F) = m(E)-m(F)$.*

Proof. This follows from the fact that $m(E\setminus F)+m(F) = m(E)$.

Definition 2.4.5. We will use the notation $E_n \nearrow E$ to indicate a sequence of sets such that $E_n \subseteq E_{n+1}$ for all n and $\cup_n E_n = E$. Similarly, $E_n \searrow E$ indicates a sequence of sets such that $E_n \supseteq E_{n+1}$ for all n and $\cap_n E_n = E$.

Proposition 2.4.4. *Let $\langle E_n : n\in\mathbb{N}\rangle$ be a sequence of measurable sets. If $E_n \nearrow E$, then $m(E) = \lim m(E_n)$. If $E_n \searrow E$, and for some k, $m(E_k)$ is finite, then $m(E) = \lim m(E_n)$.*

Proof. Fix a sequence $E_1 \subseteq E_2 \subseteq \cdots$ with union E, and set $E_0 = \varnothing$. Form the disjoint sequence $F_k = E_k \setminus E_{k-1}$ in $\mathscr{M}$ with union E. Now, for each n, the set E_n is the disjoint union $\bigcup_{k=1}^{n} F_k$. Moreover, $E = \bigcup_{k=1}^{\infty} F_k$, and so $m(E) = \sum_{k=1}^{\infty} m(F_k)$. The last equality means $m(E) = \lim_n \sum_{k=1}^{n} m(F_k) = \lim_n m(E_n)$.

Now assume that $E_n \searrow E$ and for some k, which we may assume is 1, $m(E_1) < +\infty$. Let $H_n = E_1 \setminus E_n$ and $H = E_1 \setminus E$. Then $H_n \nearrow H$, so

$$m(H_n) = m(E_1) - m(E_n) \nearrow m(E_1) - m(E) = m(H),$$

whence $m(E_n) - m(E_1) \searrow m(E) - m(E_1)$. Since $m(E_1) < +\infty$, it follows that $m(E_n) \searrow m(E)$.

Example 2.4.2. An example showing that the finiteness condition cannot be dropped is given by Lebesgue measure and the sequence $[n, +\infty) \searrow \varnothing$.

2.5 Approximation of Measurable Sets

Results for measurable sets are often obtained using results for a smaller class of approximating sets. In this section we have examples of such approximations.

Lemma 2.5.1. *If $E \in \mathscr{M}$ and $m(E) < +\infty$, then for any $\varepsilon > 0$, there is an open set $O \supseteq E$ with $m(O \setminus E) < \varepsilon$.*

Proof. If $E = \varnothing$, set $O = \varnothing$. Otherwise, for Lebesgue measure $\lambda = m$, we take a covering of E by a countable number of open intervals so that the sum of their lengths is less than $\lambda(E) + \varepsilon$. The open set O is the union of the intervals. By subadditivity, $\lambda(E) \le \lambda(O) < \lambda(E) + \varepsilon$. Since $O = (O \setminus E) \cup E$, $\lambda(O \setminus E) = \lambda(O) - \lambda(E) < \varepsilon$.

For a general integrator F, we can take a covering by intervals of the form $(a_n, b_n]$ such that the sum of the changes in F is smaller than $m(E) + \varepsilon/2$. Since F is continuous from the right, we may replace each interval $(a_n, b_n]$ with an open interval (a_n, c_n) where $c_n > b_n$, but $F(c_n) - F(a_n) \le F(b_n) - F(a_n) + \varepsilon/2^{n+1}$. Now by Proposition 2.3.1,

$$m((a_n, c_n)) \le m((a_n, c_n]) = F(c_n) - F(a_n) \le F(b_n) - F(a_n) + \varepsilon/2^{n+1}.$$

Let $O = \cup_{n=1}^{\infty} (a_n, c_n)$. Then $O \supseteq E$, and by subadditivity,

$$m(O) \le \sum_{n=1}^{\infty} m((a_n, c_n)) \le \sum_{n=1}^{\infty} \left(F(b_n) - F(a_n) + \frac{\varepsilon}{2^{n+1}} \right) < m(E) + \varepsilon,$$

whence $m(O \setminus E) = m(O) - m(E) < \varepsilon$.

Theorem 2.5.1. *Fix $E \subseteq \mathbb{R}$. Then the following are equivalent:*

1) *$E \in \mathscr{M}$.*
2) *$\forall \varepsilon > 0$, $\exists$ an open set $O \supseteq E$ with $m^*(O \setminus E) < \varepsilon$.*

3) *$\forall \varepsilon > 0$, $\exists$ a closed set $F \subseteq E$ with $m^*(E \setminus F) < \varepsilon$.*
4) *$\exists$ a G_δ set G with $E \subseteq G$ such that $m^*(G \setminus E) = 0$.*
5) *$\exists$ an F_σ set S with $S \subseteq E$ such that $m^*(E \setminus S) = 0$.*
6) *$\exists$ a G_δ set G and a set A of outer measure 0 such that $E = G \backslash A = G \cap \widetilde{A}$.*
7) *$\exists$ an F_σ set S and a set A of outer measure 0 such that $E = S \cup A$.*

Proof. $(0 \Rightarrow 1)$ Assume E is measurable. Let $I_1 = [-1,1]$ and $E_1 = E \cap I_1$. For each integer $n > 1$, let $I_n = [-n, -n+1) \cup (n-1, n]$ and $E_n = E \cap I_n$. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, there is by Lemma 2.5.1 an open set $O_n \supseteq E_n$ such that

$$m(O_n \setminus E_n) < \varepsilon/2^n.$$

Now, $O := \cup_n O_n$ contains E, and since

$$O \setminus E = O \cap \widetilde{E} = \cup_n (O_n \cap \widetilde{E}) = \cup_n (O_n \setminus E) \subseteq \cup_n (O_n \setminus E_n),$$

$m^*(O \setminus E) < \varepsilon$ by subadditivity.

$(1 \Rightarrow 3)$ By taking the intersection over a countable sequence of open sets O_n given by Condition 1 with $\varepsilon_n = 1/n$, we find a G_δ set $G \supseteq E$ with $m^*(G \setminus E) = 0$.

$(3 \Rightarrow 5)$ Given a G_δ set $G \supseteq E$ with $m^*(G \setminus E) = 0$, we set $A = G \setminus E$. Then $E = G \backslash A = G \cap \widetilde{A}$.

$(5 \Rightarrow 0)$ Any set $E \subseteq \mathbb{R}$ for which there is a G_δ set $G \supseteq E$ such that $A := G \setminus E$ has outer measure 0 is measurable since $E = G \cap \widetilde{A}$.

We have shown that measurability, Condition 1, Condition 3, and Condition 5 are equivalent. It follows that the following are equivalent statements with respect to an arbitrary set $E \subseteq \mathbb{R}$:

i) E is measurable.
ii) $\mathbb{R} \setminus E$ is measurable.
iii) $\forall \varepsilon > 0$, $\exists$ an open $O \supseteq \mathbb{R} \setminus E$, whence $\mathbb{R} \setminus O \subseteq E$, such that

$$m^*(O \setminus (\mathbb{R} \setminus E)) = m^*(O \cap E) = m^*(E \setminus (\mathbb{R} \setminus O)) < \varepsilon.$$

iv) $\forall \varepsilon > 0$, $\exists$ a closed $F \subseteq E$ with $m^*(E \setminus F) < \varepsilon$.
v) $\exists$ an F_σ set $S \subseteq E$ such that $m^*(E \setminus S) = 0$.
vi) $\exists$ an F_σ set S and a set A of measure 0 such that $E = S \cup A$.

Thus, measurability, Condition 2, Condition 4, and Condition 6 are equivalent.

Corollary 2.5.1. *A set $E \subseteq \mathbb{R}$ is measurable if and only if E is a Borel set, in fact an F_σ set, to which a set of outer measure 0 has been adjoined.*

We will see that very nice properties hold for sets of finite measure from which appropriate sets of small measure have been removed. The following is an example of such a result.

Corollary 2.5.2. *Given a measurable set $A \subseteq \mathbb{R}$ with $m(A) < +\infty$, and given $\varepsilon > 0$, there is a compact set $K \subseteq A$ with $m(A \setminus K) < \varepsilon$.*

Proof. Since A is measurable, there is a closed subset F of A with $m(A \setminus F) < \varepsilon/2$. Since the sequence

$$F \cap [-n,n] \nearrow F,$$

and $m(F) < +\infty$, there is an n_0 such that $m(F \setminus [-n_0,n_0]) < \varepsilon/2$. The desired compact set is $F \cap [-n_0,n_0]$.

Proposition 2.5.1. *If $A \notin \mathscr{M}$, then there is a G_δ set S containing A such that $m^*(S \cap A) + m^*(S \cap \widetilde{A}) \neq m(S)$. Therefore, there is no collection larger than $\mathscr{M}$ on which the restriction of m^* is even finitely additive.*

Proof. Exercise 2.24.

2.6 LimSup and LimInf of a Sequence of Sets

Recall that for a sequence $\langle x_n : n \in \mathbb{N}\rangle$ in $\mathbb{R}$, $\limsup x_n := \inf_{n\in\mathbb{N}}(\sup_{k\geq n} x_k) = \bigwedge_{n\in\mathbb{N}}\left(\bigvee_{k\geq n} x_k\right)$, and $\liminf x_n := \sup_{n\in\mathbb{N}}(\inf_{k\geq n} x_k) = \bigvee_{n\in\mathbb{N}}\left(\bigwedge_{k\geq n} x_k\right)$. Here are analogous operations on sets.

Definition 2.6.1. Let $\langle A_n : n \in \mathbb{N}\rangle$ be an infinite sequence of subsets of a set X.

$$\limsup A_n := \bigcap_{n\in\mathbb{N}}\left(\bigcup_{k\geq n} A_k\right)$$

$$\liminf A_n := \bigcup_{n\in\mathbb{N}}\left(\bigcap_{k\geq n} A_k\right).$$

Theorem 2.6.1. *Let $\langle A_n : n \in \mathbb{N}\rangle$ be an infinite sequence of subsets of a set X. Then* $\limsup A_n$ *is the set of points in an infinite number of the sets A_n, while* $\liminf A_n$ *is the set of points in all but a finite number of the sets A_n.*

Proof. Exercise 2.25.

Theorem 2.6.2 (Borel-Cantelli Lemma). *Let $\langle E_n : n \in \mathbb{N}\rangle$ be an infinite sequence of measurable subsets of $\mathbb{R}$ such that $\sum_{n=1}^\infty m(E_n) < +\infty$. Then* $\limsup E_n$ *is a set of measure* 0. *That is, outside of a set of measure* 0, *all points are in at most a finite number of the sets E_n.*

Proof. Let $S_k = \cup_{n=k}^\infty E_n$. Since $m(S_1) = m\left(\cup_{n=1}^\infty E_n\right) \leq \sum_{n=1}^\infty m(E_n) < +\infty$ and $S_k \searrow \limsup E_n$,

$$m(\limsup E_n) = \lim_{k\to\infty} m(S_k) \leq \lim_{k\to\infty} \sum_{n=k}^\infty m(E_n) = 0.$$

2.7 The Existence of a Non-measurable Set

In this section and the next, we work just with Lebesgue outer measure and Lebesgue measure. Using the Axiom of Choice (see the Appendix), we will show that there are subsets of $[0,1]$ that are not Lebesgue measurable. Robert Solovay [47] showed in 1970 that there exist models of set theory in which the Axiom of Choice does not hold and every subset of the real line is Lebesgue measurable. We will say "measurable" when we mean Lebesgue measurable.

For the construction of a non-measurable set, we work with $[0,1)$ and addition modulo 1. That is, for x, $y \in [0,1)$ we set $x +' y = x + y$ if $x + y < 1$, and we set $x +' y = x + y - 1$ if $x + y \geq 1$. By associating 0 with 1, one can think of $[0,1)$ with addition modulo 1 as the circle of circumference 1 centered at the origin in the plane. The operation $+'$ corresponds to rotation or addition of angles. It is easy, therefore, to see that the operation $+'$ is commutative and associative.

Lemma 2.7.1. *Given $y \in [0,1)$ and $E \subseteq [0,1)$, $\lambda^*(E +' y) = \lambda^*(E)$. If E is Lebesgue measurable, then so is $E +' y$.*

Proof. Set $E_1 = E \cap [0,\ 1-y)$ and $E_2 = E \cap [1-y, 1)$. If E is measurable, so is $E +' y = (E_1 + y) \cup (E_2 + y - 1)$. If E is any subset of $[0,1)$, then since $[0,\ 1-y)$ is measurable and Lebesgue outer measure is translation invariant,

$$\begin{aligned}
\lambda^*(E) &= \lambda^*(E_1) + \lambda^*(E_2) = \lambda^*(E_1 + y) + \lambda^*(E_2 + y - 1) \\
&\geq \lambda^*((E_1 + y) \cup (E_2 + y - 1)) = \lambda^*(E +' y) \\
&\geq \lambda^*\big((E +' y) +' (1-y)\big) = \lambda^*(E).
\end{aligned}$$

The last equality follows since if $x \in E$ and $x + y < 1$, then $(x +' y) +' (1-y) = x$, and the same is true if $x + y \geq 1$.

We now define an equivalence relation $\sim$ in $[0,1)$ by setting $x \sim y$ if x and y differ by a rational number. By the Axiom of Choice, there is a set $P \subseteq [0,1)$ containing exactly one element from each equivalence class. Let $\langle r_i : i \in \mathbb{N} \cup \{0\} \rangle$ be an enumeration of the rational numbers in $[0,1)$ with $r_0 = 0$. Let $P_i = P +' r_i$, so $P_0 = P$. If $i \neq j$, then $P_i \cap P_j = \varnothing$. To see this, assume $x \in P_i \cap P_j$. Then for elements p_i and p_j in P, we have

$$x = p_i +' r_i = p_j +' r_j.$$

It follows that $\left|p_i - p_j\right|$ is a rational number, i.e., $p_i \sim p_j$. Since P contains only one element from each equivalence class, $p_i = p_j$ and so $r_i = r_j$. That is, $P_i = P_j$. On the other hand, for each $x \in [0,1)$, x is in some equivalence class, so for some $p \in P$ and some r_i, $x = p +' r_i$. Therefore, the collection $\{P_i\}$ is a countable, pairwise disjoint collection of sets with union $[0,1)$.

Proposition 2.7.1. *The set P is not Lebesgue measurable.*

Proof. Assume that P is measurable. Then $\lambda(P)$ is defined, and by Lemma 2.7.1, $\lambda(P) = \lambda(P_i)$ for all i, whence $\lambda([0,1)) = \sum \lambda(P_i) = \sum \lambda(P)$. Since the sum is finite, $\lambda(P) = 0$. But then $\lambda([0,1)) = 0$. Since this is not true, we conclude that P is not measurable.

We have actually shown that the following is true.

Proposition 2.7.2. *If μ is a σ-additive,* ***translation invariant*** *measure defined on a σ-algebra containing P, then $\mu([0,1))$ is either* 0 *or* $+\infty$.

2.8 Cantor Set

The **Cantor set** C, also called the Cantor ternary set, is the closed subset of $[0,1]$ obtained by removing the following open set:

$$\left(\tfrac{1}{3}, \tfrac{2}{3}\right) \cup \left[\left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right)\right] \cup \left[\left(\tfrac{1}{27}, \tfrac{2}{27}\right) \cup \left(\tfrac{7}{27}, \tfrac{8}{27}\right) \cup \left(\tfrac{19}{27}, \tfrac{20}{27}\right) \cup \left(\tfrac{25}{27}, \tfrac{26}{27}\right)\right] \cdots$$

That is, remove the open middle third from $[0,1]$, and at each successive step, remove the open middle third of each of the remaining closed intervals. It consists of all numbers in $[0,1]$ that have a ternary expansion (i.e., an expansion base 3) that does not use the digit 1. If there are two ternary expansions of a point in C, one of them satisfies this property. The set C is the intersection of closed subsets of $[0,1]$ such that no finite subcollection of these closed sets has an empty intersection. Since $[0,1]$ is compact, the intersection C of all of the closed subsets is nonempty.

The set C is, in fact, uncountable. To see this, assume $\langle c_n : n \in \mathbb{N} \rangle$ is a sequence of points in C. Let F_1 be the closed interval remaining after removing $\left(\frac{1}{3}, \frac{2}{3}\right)$ from $[0,1]$ that does not contain c_1. At stage $n-1$, we have a closed interval that does not contain the points $c_1, c_2, \cdots, c_{n-1}$. We remove the middle third, and let F_n be the one of the two remaining closed intervals that does not contain c_n. For each $n \in \mathbb{N}$, $\cap_{i=1}^{n} F_i \neq \varnothing$. Therefore, the set $\cap_{i=1}^{\infty} F_i$ is a nonempty subset of C, and it contains no point of the enumeration. This shows that we cannot exhaust C with an enumeration; that is, C is uncountable. Working Exercise 2.33, one shows that the Lebesgue measure of the removed open set is 1, so $\lambda(C) = 0$. Note that C is an example of an uncountable set of measure 0.

One can form a **generalized Cantor set** with positive measure by scaling each of the removed intervals by α where $0 < \alpha < 1$ and removing the scaled intervals from the centers of the intervals left in the previous stage of the construction. The removed set is an open set O of measure α, and the complement F has measure $1-\alpha$. A generalized Cantor set with positive measure is also called a **fat Cantor set**.

For the Cantor set and any generalized Cantor set, the removed open set O is dense in $[0,1]$; that is, its closure is $[0,1]$. To see this, note that for any $x \in C$, there is a point y_1 removed at the first stage so that $|x - y_1| \leq 1/2$. Similarly, at the n^{th} stage, there is a point y_n removed at that stage such that $|x - y_n| \leq 1/2^n$. The extreme case would be realized if we removed first the singleton set $\{1/2\}$, then the set

$\{1/4, 3/4\}$, etc. It would still be true that this removed set (no longer open) would be dense in $[0,1]$. Note that if we take the union of the generalized Cantor sets for each $\alpha = 1/n$, we get an F_σ set with total measure 1.

Along with the Cantor set, there is a continuous increasing function g called the **Cantor-Lebesgue function** mapping $[0,1]$ onto $[0,1]$ taking all of its increase on the Cantor set, that is, on a set of measure 0. The function g is identically equal to $1/2$ on the removed middle third; it is identically equal to $1/4$ and $3/4$, respectively, on the next two removed open intervals, etc. The value at points of the Cantor set is the limit of values on the removed open intervals.

2.9 Problems

Problem 2.1. Let $\mathscr{B}$ be the collection of all subsets $A \subseteq \mathbb{R}$ such that either A or $\mathbb{R} \setminus A$ is finite or countably infinite. For each $A \in \mathscr{B}$, let $\mu(A) = 0$ if A is finite or countably infinite, and let $\mu(A) = 1$ otherwise. Show that $\mathscr{B}$ is a σ-algebra and μ is a measure on $\mathscr{B}$; that is, μ is a nonnegative, countably additive function on $\mathscr{B}$ with $\mu(\emptyset) = 0$.

Problem 2.2. Recall that a measure on a set E is a mapping μ from a σ-algebra $\mathscr{A}$ of subsets of E into $[0, +\infty]$ such that $\mu(\emptyset) = 0$ and μ is countably additive, whence μ is also finitely additive. Show that such a **general measure** is subadditive. That is, the measure of the union of a countable number of not necessarily disjoint sets in $\mathscr{A}$ is less than or equal to the sum of the measures of the sets forming the union.

Problem 2.3. Let ν be a finitely additive measure on a σ-algebra $\mathscr{A}$ of sets in a set X.

a) Suppose that for any sequence $\langle E_n \rangle$ of sets in $\mathscr{A}$, if $E_n \nearrow E$, then $\nu(E) = \lim_n \nu(E_n)$. Show that in fact ν is countably additive.
b) Suppose that for any sequence $\langle E_n \rangle$ of sets in $\mathscr{A}$, if $E_n \searrow \varnothing$, then $\lim_n \nu(E_n) = 0$. Show that in fact ν is countably additive.

Problem 2.4 (A). Prove Proposition 2.2.1.

Problem 2.5 (A). Recall that the set A consisting of the rationals between 0 and 1 is countable, and so it has Lebesgue outer measure 0. Show that any *finite* collection of open intervals covering A has total length ≥ 1.

Problem 2.6. Fix nonempty sets A and $B \subseteq \mathbb{R}$ such that

$$d(A,B) := \inf\{|x-y| : x \in A, y \in B\} = a > 0.$$

Show that Lebesgue outer measure $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$. **Hint:** Show that for any $\varepsilon > 0$, there is a countable covering of $A \cup B$ by open intervals I_k, each having length strictly less than a, such that $\sum_{k=1}^{\infty} \ell(I_k) \leq \lambda^*(A \cup B) + \varepsilon$.

Problem 2.7. Let $E \subseteq \mathbb{R}$ have finite Lebesgue outer measure. Show that E is Lebesgue measurable if and only if for any open, bounded interval (a,b) we have $b-a = \lambda^*((a,b)\cap E) + \lambda^*((a,b)\setminus E)$.

Problem 2.8. Suppose $A \subseteq \mathbb{R}$ is a Lebesgue measurable set with $\lambda(A) > 0$. Show that for any δ with $0 < \delta < 1$, there is a bounded interval $I_\delta = [a,b]$, with $a < b$, such that $\lambda(A \cap I_\delta) \geq \delta \cdot \lambda(I_\delta)$. That is, A occupies a large part of I_δ.

Problem 2.9. Suppose that $A \subseteq [0,1]$ is a Lebesgue measurable set with $\lambda(A) = 1$. Show that A is dense in $[0,1]$; that is, the closure $\overline{A} = [0,1]$.

Problem 2.10. For this problem, let $\mathscr{M}$ be the Lebesgue measurable sets in $[0,1]$, and let ν be a nonnegative, real-valued function on $\mathscr{M}$ such that for disjoint sets A and B in $\mathscr{M}$, $\nu(A \cup B) = \nu(A) + \nu(B)$. Also assume that for any $\varepsilon > 0$ there is a $\delta > 0$ so that if $A \in \mathscr{M}$ and its Lebesgue measure $\lambda(A) < \delta$, then $\nu(A) < \varepsilon$. Prove that ν is a measure. **Hint:** If $\langle A_i \rangle$ is a sequence of pairwise disjoint sets in $\mathscr{M}$ with union A, what can you say about $A \setminus \cup_{i=1}^n A_i$?

Problem 2.11. Let $\mathscr{M}$ be the collection of Lebesgue measurable sets in $\mathbb{R}$, and let λ be Lebesgue measure on $\mathbb{R}$. Let f be a real-valued function defined on $\mathbb{R}$. Let $\mathscr{A}$ be the collection of subsets of $\mathbb{R}$ with inverse image in $\mathscr{M}$. That is, $\mathscr{A} := \left\{S \subseteq \mathbb{R} : f^{-1}[S] \in \mathscr{M}\right\}$. Show that $\mathscr{A}$ is a σ-algebra of sets in $\mathbb{R}$. Then for each $S \in \mathscr{A}$, let $\mu(S) := \lambda\left(f^{-1}[S]\right)$. Show that μ is a measure on $\mathscr{A}$; that is, μ is countably additive with $\mu(\varnothing) = 0$.

Problem 2.12. Let f be an increasing function on $[0,1]$; that is, for $x < y$, $f(x) \leq f(y)$. The jump of f at a point x is $\lim_{y\to x+} f(y) - \lim_{y\to x-} f(y)$, with the obvious modification at endpoints of $[0,1]$. Show that if the jump of f is 0 at every point of $[0,1]$, then f is continuous on $[0,1]$.

Problem 2.13. Show that if an integrator is continuous, such as the integrator $F(x) = x$ for Lebesgue outer measure, then the same outer measure is obtained using open intervals and intervals of the form $(a,b]$.

Problem 2.14. Let F be an integrator on $\mathbb{R}$. Show that for any $\varepsilon > 0$, there is an interval $(a,b]$ such that $F(b) - F(a) < \varepsilon$.

Problem 2.15. Consider the integrator F on $\mathbb{R}$ given by $F(x) = 0$ for $x < 0$, and $F(x) = x^2$ for $x \geq 0$. Let m^* be the outer measure generated by the integrator F.

a) Given $M \in \mathbb{N}$, suppose $\langle I_n \rangle$ is a sequence of intervals contained in $[0,M]$. Show that if $l(I_n) \to 0$, then $m^*(I_n) \to 0$.
b) Construct a sequence of intervals $\langle J_n \rangle$ contained in $\mathbb{R}$ such that $l(J_n) \to 0$, but $m^*(J_n) \to \infty$.
c) Construct a sequence of intervals $\langle K_n \rangle$ contained in $\mathbb{R}$ such that $l(K_n) \to 0$, but $m^*(K_n) = 1$ for all n.

Problem 2.16. Show that an outer measure is translation invariant if and only if the integrator is $F(x) = cx + d$ for some constants $c \geq 0$, and d.

Problem 2.17. Prove or disprove: All subsets of $\mathbb{R}$ having 0 Lebesgue measure also have 0 measure with respect to the measure generated by any continuous, increasing integrator.

Problem 2.18. Let $F : (0,+\infty) \mapsto \mathbb{R}$ be given by setting $F(x) = 0$ for $x < 1$, and $F(x) = n$ for $n \in \mathbb{N}$ and $n \leq x < n+1$. Let m^* be the outer measure generated by the integrator F.

a) For each set $A \subseteq (0,+\infty)$, what is the value of $m^*(A)$?
b) Prove that every subset of $(0,\infty)$ is measurable with respect to m^*.
c) Give an example of a Lebesgue measurable set $E \subseteq (0,+\infty)$ such that $m(E) = \infty$, but the Lebesgue measure $\lambda(E) = 0$.
d) Give an example of a Lebesgue measurable set $F \subseteq (0,+\infty)$ such that $\lambda(F) = \infty$, but $m(F) = 0$.

Problem 2.19. Give an example or disprove the following statement: There exists an integrator $F : \mathbb{R} \mapsto \mathbb{R}$ such that for some set A of strictly positive Lebesgue measure, the outer measure m^* generated by F has value $m^*(\{x\}) > 0$ for each point $x \in A$.

Problem 2.20. Suppose A is a measurable subset of $\mathbb{R}$ such that $m(A \cap (a,b)) \leq \frac{1}{2}(b-a)$ for any $a,b \in \mathbb{R}$, where $a < b$. Show that $m(A) = 0$.

Problem 2.21. a) The **Heaviside step function** is $H = \chi_{[0,\infty)}$. That is, $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geq 1$. Show that the resulting outer measure is in fact a measure on the σ-algebra consisting of all subsets of $\mathbb{R}$. It is called a **Dirac measure** or **unit mass** at 0, and denoted by δ_0. Show that for each set $E \subseteq \mathbb{R}$, we have $\delta_0(E) = 1$ if $0 \in E$ and $\delta_0(E) = 0$ if $0 \notin E$. A similar unit mass δ_a can exist at any point $a \in \mathbb{R}$.
b) Define an integrator F such that the corresponding measure on $\mathbb{R}$ is Lebesgue measure to which is added a unit mass at 0, at 1, and at 2.

Problem 2.22 (A). Prove the following result, which is valid for Lebesgue measure, and show that it is not valid for general measures: If $E \in \mathscr{M}$, then $\forall r \in \mathbb{R}$, $E + r \in \mathscr{M}$.

Problem 2.23. Let $\langle \mu_n \rangle$ be a sequence of finite measures on a σ-algebra $\mathscr{A}$ of subsets of $\mathbb{R}$; that is, $\mu_n(\mathbb{R}) < \infty$ for all $n \in \mathbb{N}$. Let $\langle a_n \rangle$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n \mu_n(\mathbb{R}) < \infty$. Let $\mu(A) = \sum_{n=1}^{\infty} a_n \mu_n(A)$ for each $A \in \mathscr{A}$. Show that μ is a finite measure on $\mathscr{A}$.

Problem 2.24 (A). Prove Proposition 2.5.1.

Problem 2.25. Prove Theorem 2.6.1.

Problem 2.26. Let μ be a measure defined on the Borel subsets of $J := [-1,1]$ such that $\mu(J) = 17$. Assume that any Borel set of Lebesgue measure 0 in J is a set of μ-measure 0. Show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if E is a Borel set in J and $\lambda(E) < \delta$, then $\mu(E) < \varepsilon$. **Hint:** Suppose there is a sequence of Borel sets E_n contained in J with $\lambda(E_n) < 2^{-n}$ and yet $\mu(E_n) \geq \varepsilon$ for each n. Let $E = \limsup_n E_n$. What is $\lambda(E)$? What is $\mu(E)$?

Problem 2.27. Let f be a real-valued, continuous function defined on $\mathbb{R}$. Show that for each Borel set $E \subseteq \mathbb{R}$, $f^{-1}[E]$ is a Borel set.

Problem 2.28. Let m^* be the outer measure on $\mathbb{R}$ generated by an integrator F.

a) Show that for any $E \subseteq \mathbb{R}$, there is a Borel set B with $E \subseteq B$ and $m(B) = m^*(E)$.
b) Let $\langle E_n \rangle$ be a sequence of sets in $\mathbb{R}$ and E a subset of $\mathbb{R}$ such that $E_n \nearrow E$. Show that $\lim_n m^*(E_n) = m^*(E)$. **Hint:** For each n, let $B_n \supseteq E_n$ be the Borel set given by Part a. Let $C_n = \bigcap_{k=n}^{\infty} B_k$.

Problem 2.29. Let m be a measure on $\mathbb{R}$ generated by an integrator F. Let $\langle A_n \rangle$ be a sequence of measurable subsets of $\mathbb{R}$. Show that $m(\liminf A_n) \le \liminf m(A_n)$. Assume that m is a finite measure, and show that $m(\limsup A_n) \ge \limsup m(A_n)$.

Problem 2.30. Let m be a measure on $\mathbb{R}$ generated by an integrator F. Let K be a compact set such that $m(K) < +\infty$. For each $x \in K$, let $B_1(x)$ be the interval $(x-1, x+1)$, and define $f : K \mapsto \mathbb{R}$ by setting $f(x) = m(B_1(x))$. Show that for some $x_0 \in K$, $f(x_0) = \alpha := \inf_{x \in K} f(x)$. **Hint:** Show that there is a convergent sequence $\langle x_n \rangle$ in K such that $f(x_n) \searrow \alpha$, and use Problem 2.29.

Problem 2.31 (A). Show that if E is a Lebesgue measurable subset of the non-measurable set P constructed in Section 2.7, then $\lambda(E) = 0$.

Problem 2.32 (A). Show that if A is any set with Lebesgue outer measure $\lambda^*(A) > 0$, then there is a non-measurable set $E \subseteq A$.

Problem 2.33. Show that the Cantor set has Lebesgue measure 0.

Problem 2.34 (A). Use a generalized Cantor set of positive Lebesgue measure to show there is an open subset of $[0,1]$ having a boundary (i.e., the closure of the set from which the open set has been removed) such that the boundary has positive measure.

Problem 2.35. A nonempty set S is **perfect** if it is closed and each element of S is an accumulation point of S. Prove that the Cantor set is perfect and has no interior points.

Problem 2.36. How is the Cantor set changed if closed middle third intervals are removed at each step?

Problem 2.37. Show the Cantor-Lebesgue function g is continuous on $[0,1]$ and has derivative g' equal to 0 outside of a set of Lebesgue measure 0 in $[0,1]$. **Hint:** How much does g increase on the part of the Cantor set C between two successive open intervals that have been removed at the k^{th} step of the removal process?

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