Preface

This book presents the work of many mathematicians. It is the product of the two-semester course I have taught at the University of Illinois since 1968, a span of over 1% of recorded western history. The development of the book's diverse topics has often been improved with input from current research, including my own and joint research with coauthors. In particular, the most difficult material of the first semester, appearing in Chapter 5, has been considerably simplified and shortened by application of work with potential theorist Jürgen Bliedtner.

As is common practice for a book on measure theory, the first five chapters are devoted to measures on the real numbers. This of course includes the generalization of the Riemann integral using Lebesgue measure. With essentially no more work for the student, however, it also includes more general measures on $\mathbb{R}$. The generalization is important for many applications, such as in statistics and probability theory. It is also helpful in developing the connection between integration and differentiation. That topic is considerably simplified in Chapter 5 with the use of a local maximal function and a simple, optimal covering theorem.

From there, the book continues with an introduction to general measure, metric, and normed spaces. This includes the Baire Category Theorem and classical L^p spaces. It is the material needed as a foundation for Chapter 8 on Hilbert spaces, Fourier series, and the proof of the important Radon-Nikodým Derivative Theorem.

Chapter 9 gives a parallel treatment of spaces with a distance function, i.e., metric spaces, and more general topological spaces. Open balls are used in a metric space to determine how close a point x is to a point z: The more balls centered at z that contain x, the closer x is to z. More general topological spaces replace open balls with sets not defined by a distance function. The chapter includes a development of the infinite product of such spaces and an elementary proof that if the spaces are compact, so is the product.

Chapter 10 is devoted to the construction of general measures including measures on product spaces. Chapter 11 develops properties of infinite-dimensional vector spaces with a norm, i.e., a function specifying the distance to $\mathbf{0}$. The chapter includes further information about classical L^p spaces and measures associated with linear maps on spaces of continuous functions.

The Axiom of Choice is assumed when needed throughout the book. It allows a point to be chosen from each set in an infinite collection of nonempty sets even when no rule can be given. The equivalence of several forms of the axiom is proved in the book's first appendix. The second appendix simplifies what is needed to show that a Radon-Nikodým derivative is the result of a limit process. That appendix also provides two powerful covering theorems for finite-dimensional spaces.

The third appendix introduces the reader to the rigorous use of infinitely large and infinitely small numbers in subjects such as calculus and measure theory. In calculus, for example, the treatment of the chain rule and applications of the integral can inform instruction in an ordinary calculus course. The measure spaces developed in the appendix have had a number of important applications over several decades. With the work of Yeneng Sun, for example, there is now a rigorous way of treating a continuum of independent random variables and traders in an economy.

Answers are provided at the end of the book for many of the problems; these are marked with an "A". By putting the period outside the quotes, I have just used the British rule allowing context to be considered for such punctuation. I have read that the American rule was set by typesetters to protect delicate type for commas and periods. Typesetters, however, have not always understood the needs of mathematicians. A famous example is the proof sheet returned to an author with a minute speck that magnification showed to be an epsilon. The text read "Let epsilon be as small as possible."

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