

Chapter 2

Potential Games

Abstract This chapter deals with theories related to the class of games known as potential games. What make potential games attractive are their useful properties concerning the existence and attainability of their Nash equilibria. These properties have direct consequences: the straightforward applications of equilibrium-seeking dynamics to the gameplay of potential games can lead to a Nash equilibrium solution. However, before being able to successfully apply such rewarding results to practical problems, one needs to answer the important question of *how to identify if a game is a potential game*. To put the question in another way is *how to formulate a problem so that the resulting game is a potential game*. We hope that this chapter can successfully address these questions and generalize techniques in identifying and formulating a potential game. Through a systematic examination over existing work, we are able to account for the methodologies involved, and provide readers with useful and unified insights. In the identification problems, we examine the structures of the game's strategy space and utility functions, and their properties upon which the conditions of potential games are satisfied. Meanwhile in the formulation problem, we suggest two distinct approaches. For the forward approach, we examine the methods to design utility functions with certain properties so that a potential function can be derived, and hence a potential game is formulated. In the reverse approach, we begin by defining a potential function whereby the utility functions of players can later be obtained. We will also discuss practical examples in the context of wireless communications and networking in order to illustrate the ideas.

2.1 Definition

The seminal paper by Monderer and Shapley in 1996 [31] coined the term “potential games”. It presented the first systematic investigation and fundamental results for a certain type of games for which potential functions exist. However, the first concept of potential games can actually be traced back to the work by Rosenthal in 1973 [40], about games having pure-strategy Nash equilibria known as “congestion games”. Today, the theory of potential games has been further developed by many authors. They have also grown out of their pure mathematical realm and found successful applications in solving engineering problems.

Mathematically, there can be various types of potential games. In all these types of games, however, the common thread is the existence of an associated function—the *potential function*—that maps the game’s strategy space $\mathbb{S}$ to the space of real number $\mathbb{R}$. Their classifications depend on the specific relationship between the potential function and the utility functions of players. The potential function is therefore the most important element in the studies of potential games. The origin of the term “potential” was drawn from analogies to the similarly-named concept of potential in vector field analysis, whose leading examples include gravitational potential and electric potential in physics.

Monderer and Shapley [31] listed four types of potential games: *ordinal potential games*, *weighted potential games*, *exact potential games*, and *generalized ordinal potential games*. Other extensions also exist in the literature. Of interest in this monograph are *best-response potential games* proposed by Voorneveld [49], and *pseudo-potential games* proposed by Dubey et al. [13]. For completeness, we will present here the definitions for all these types of potential games.

2.1.1 Exact Potential Games

Definition 2.1 (Exact Potential Game). The game $\mathcal{G}^1$ is an exact potential game if and only if a potential function $F(S) : \mathbb{S} \mapsto \mathbb{R}$ exists such that, $\forall i \in \mathcal{N}$:

$$\begin{aligned} U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) &= F(T_i, S_{-i}) - F(S_i, S_{-i}), \\ \forall S_i, T_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i}. \end{aligned} \quad (2.1)$$

In exact potential games, the change in a single player’s utility due to his/her own strategy deviation results in exactly the same amount of change in the potential function.

Assuming that each strategy set $\mathbf{S}_i$ is a continuous interval of $\mathbb{R}$ and each utility function U_i is everywhere continuous and differentiable, we say $\mathcal{G}$ is a continuous game. For such a game to be an exact potential game, an equivalent definition to (2.1) states that, $\forall i \in \mathcal{N}$:

$$\frac{\partial U_i(S_i, S_{-i})}{\partial S_i} = \frac{\partial F(S_i, S_{-i})}{\partial S_i}, \quad \forall S_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i}. \quad (2.2)$$

Among the various types of potential games, exact potential games are those whose definition requires the strictest condition of exact equality. Other types of potential games are defined by loosening this condition. Exact potential games are however the most important and have received the highest level of interest in both theoretical research and practical applications.

¹In this chapter, a game $\mathcal{G}$ will be understood as $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i \in \mathcal{N}}]$, unless otherwise stated.

Fig. 2.1 Prisoners' dilemma with payoff matrix in (a) and a potential function in (b)

a

	C	D
C	-1, -1	-6, 0
D	0, -6	-4, -4

b

	C	D
C	1	2
D	2	4

Example 2.1. The prisoner's dilemma in Sect. 1.2.7 is an exact potential game. The payoff table is reproduced in Fig. 2.1a and a corresponding potential function is given alongside in Fig. 2.1b. It is not difficult to verify this by simply stepping through all possible unilateral strategy changes. For instance, for the switch from (C, C) to (D, C) due to player 1, $U_1(C, C) - U_1(D, C) = (-1) - 0 = -1$. Correspondingly, $F(C, C) - F(D, C) = 1 - 2 = -1$. The other strategy changes can be verified similarly.

Please note that the potential function is not unique. How to obtain such a function is one of the objectives of this chapter and will become clearer after more contents are introduced. The actual procedures will be discussed in Sect. 2.3.2.

2.1.2 Weighted Potential Games

Definition 2.2 (Weighted Potential Game). The game $\mathcal{G}$ is a weighted potential game if and only if a potential function $F(S) : \mathbb{S} \mapsto \mathbb{R}$ exists such that, $\forall i \in \mathcal{N}$:

$$\begin{aligned} U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) &= w_i (F(T_i, S_{-i}) - F(S_i, S_{-i})), \\ \forall S_i, T_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i} \end{aligned} \quad (2.3)$$

where $(w_i)_{i \in \mathcal{N}}$ constitutes a vector of positive numbers, known as the weights.

In weighted potential games, a player's change in payoff due to his/her unilateral strategy deviation is equal to the change in the potential function (also known as w -potential function in [31]) but scaled by a weight factor. Clearly, all exact potential games are weighted potential games with all players having identical weights of 1.

Similarly, (2.3) is equivalent to the following condition for continuous games. That is, $\forall i \in \mathcal{N}$:

$$\frac{\partial U_i(S_i, S_{-i})}{\partial S_i} = w_i \frac{\partial F(S_i, S_{-i})}{\partial S_i}, \quad \forall S_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i}. \quad (2.4)$$

Although defined separately, weighted potential games and exact potential games can be made equivalent by scaling the utility functions appropriately.

Fig. 2.2 A weighted potential game with payoff matrix in (a) and a potential function in (b)

a

	<i>C</i>	<i>D</i>
<i>C</i>	-2, -3	-12, 0
<i>D</i>	0, -18	-8, -12

b

	<i>C</i>	<i>D</i>
<i>C</i>	1	2
<i>D</i>	2	4

Lemma 2.1. $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i \in \mathcal{N}}]$ is a weighted potential game with potential function $F(S)$ and weights $(w_i)_{i \in \mathcal{N}}$ if and only if $\mathcal{G}' = [\mathcal{N}, \mathbb{S}, \{V_i = \frac{1}{w_i} U_i\}_{i \in \mathcal{N}}]$ is an exact potential game with potential function $F(S)$.

Proof. Clearly, the following conditions

$$U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) = w_i (F(T_i, S_{-i}) - F(S_i, S_{-i}))$$

and

$$V_i(T_i, S_{-i}) - V_i(S_i, S_{-i}) = F(T_i, S_{-i}) - F(S_i, S_{-i})$$

are equivalence. Thus, necessity and sufficiency are apparent. $\square$

Due to their equivalence, in our subsequent discussion, we will focus our discussion on exact potential games. However, equivalent results should be equally available for weighted potential games as well.

Example 2.2. The prisoner's dilemma in Example 2.1 with scaled utility functions, $(w_1, w_2) = (2, 3)$, is a weighted potential game (Fig. 2.2). Note that these two games can have the same potential function. The validation for the potential function is also very straightforward.

2.1.3 Ordinal Potential Games

Definition 2.3 (Ordinal Potential Game). The game $\mathcal{G}$ is an ordinal potential game if and only if a potential function $F(S) : \mathbb{S} \mapsto \mathbb{R}$ exists such that, $\forall i \in \mathcal{N}$:

$$\begin{aligned} U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) &> 0 \Leftrightarrow F(T_i, S_{-i}) - F(S_i, S_{-i}) > 0, \\ \forall S_i, T_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i}. \end{aligned} \quad (2.5)$$

Note that (2.5) can be equivalently rewritten as follows. $\forall i \in \mathcal{N}$:

$$\begin{aligned} \text{sgn}[U_i(T_i, S_{-i}) - U_i(S_i, S_{-i})] &= \text{sgn}[F(T_i, S_{-i}) - F(S_i, S_{-i})], \\ \forall S_i, T_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i} \end{aligned} \quad (2.6)$$

where $\text{sgn}()$ is the signum function.

Fig. 2.3 An ordinal potential game with payoff matrix in (a) and a potential function in (b)

a

	C	D
C	$-1, -2$	$-9, 0$
D	$0, -8$	$-4, -5$

b

	C	D
C	0	1
D	1	2

Unlike in exact potential games, ordinal potential games only require that the change in the potential function due to a unilateral strategy deviation only needs to be of the same sign as the change in the player's utility function. In other words, if player i gains a better (worse) utility from switching his/her strategy, this should lead to an increase (decline) in the potential function F , and vice versa.

For continuous games, $\forall i \in \mathcal{N}$:

$$\text{sgn} \left[\frac{\partial U_i(S_i, S_{-i})}{\partial S_i} \right] = \text{sgn} \left[\frac{\partial F(S_i, S_{-i})}{\partial S_i} \right], \quad \forall S_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i}. \quad (2.7)$$

Example 2.3. The following ordinal potential game (Fig. 2.3) is a variant of the prisoner's dilemma (see Example 2.1) with modified payoffs. The procedures for obtaining the associated potential function will be discussed in Sect. 2.3.1.

2.1.4 Generalized Ordinal Potential Games

Generalized ordinal potential games are an extension from ordinal potential games, as defined in [31]. We include their definition for completeness.

Definition 2.4 (Generalized Ordinal Potential Game). The game $\mathcal{G}$ is a generalized ordinal potential game if and only if a potential function $F(S) : \mathbb{S} \mapsto \mathbb{R}$ exists such that, $\forall i \in \mathcal{N}$:

$$U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) > 0 \Rightarrow F(T_i, S_{-i}) - F(S_i, S_{-i}) > 0, \\ \forall S_i, T_i \in \mathbf{S}_i; \forall S_{-i} \in \mathbf{S}_{-i}. \quad (2.8)$$

Basically, an increase (decrease) in a player's utility due to his/her unilateral strategy deviation implies an increase (decrease) in the potential function. But the reverse is not true, unlike in ordinal potential games.

Example 2.4. The game presented in Fig. 2.4 is a generalized ordinal potential game. Note that $F(1A, 2A) - F(1A, 2B) > 0$ does not imply $U_2(1A, 2A) - U_2(1A, 2B) > 0$. Hence, the game is not an ordinal potential game and the example indicates that ordinal potential games are a subset of generalized ordinal potential games.

Fig. 2.4 A generalized ordinal potential game with payoff matrix in (a) and a potential function in (b)

a			
		2A	2B
1A		4, 3	3, 3
1B		3, 4	4, 3

b			
		2A	2B
1A		3	0
1B		2	1

2.1.5 Best-Response Potential Games

Best-response potential games were introduced by Voorneveld [49]. We include their definition for completeness.

Definition 2.5 (Best-Response Potential Game). The game $\mathcal{G}$ is a best-response potential game if and only if a potential function $F(S) : \mathbb{S} \mapsto \mathbb{R}$ exists such that, $\forall i \in \mathcal{N}$:

$$\mathcal{B}_i(S_{-i}) = \arg \max_{S_i \in \mathbf{S}_i} F(S_i, S_{-i}), \quad \forall S_{-i} \in \mathbf{S}_{-i} \quad (2.9)$$

where $\mathcal{B}_i(S_{-i})$ is player i 's best-response correspondence which is defined in (1.8).

Note that the equality in (2.9) should be interpreted as the two sets are equal. The notion of best-response potential games deviates considerably from the previous notions in Sects. 2.1.1–2.1.4. It requires that all strategies that maximize player i 's utility must also maximize the potential function, and vice versa.

The next lemma discusses the relationship between best-response potential games and ordinal potential games.

Lemma 2.2. *Every ordinal potential game is also a best-response potential game.*

Proof. We shall prove by contradiction. Let us consider an ordinal potential game $\mathcal{G}$ with potential function F . Then, for an arbitrary player i , we consider some of his/her best responses $\hat{S}_i \in \mathcal{B}_i(S_{-i})$. Now, assuming $\mathcal{G}$ is not a best-response potential game, then there exists at least one $\hat{S}_i$ which does not maximize $F(\hat{S}_i, S_{-i})$. Consequently, it implies that there exists S'_i such that $F(S'_i, S_{-i}) > F(\hat{S}_i, S_{-i})$ which results in $U(S'_i, S_{-i}) > U(\hat{S}_i, S_{-i})$ from the definition given in (2.5). This contradicts the fact that $\hat{S}_i$ is a best-response strategy. Hence, the set of maximizers for every player's utility function should also be identical to the set of maximizers for the potential function and the lemma holds. $\square$

On the other hand, a best-response potential game may not necessarily be an ordinal or generalized ordinal potential game as the following example shows.

Example 2.5. The following game by Voorneveld [49] shown in Fig. 2.5 is a best-response potential game. However, since $F(1A, 2B) - F(1A, 2C) > 0$ while $U_2(1A, 2B) - U_2(1A, 2C) < 0$, the game cannot be an ordinal or generalized ordinal potential game.

Fig. 2.5 A best-response potential game with payoff matrix in (a) and a potential function in (b)

a

	2A	2B	2C
1A	2,2	1,0	0,1
1B	0,0	0,1	1,0

b

	2A	2B	2C
1A	4	3	0
1B	0	2	1

2.1.6 Pseudo-Potential Games

The concept of pseudo-potential games was introduced by Dubey et al. [13].

Definition 2.6 (Pseudo-Potential Game). The game $\mathcal{G}$ is a pseudo-potential game if and only if a continuous function $F : \mathbb{S} \mapsto \mathbb{R}$ exists such that, $\forall i \in \mathcal{N}$:

$$\mathcal{B}_i(S_{-i}) \supset \arg \max_{S_i \in \mathbb{S}_i} F(S_i, S_{-i}), \quad \forall S_{-i} \in \mathbb{S}_{-i}. \quad (2.10)$$

The above implies that the set of maximizers of the function F with respect to the strategy of player i , while keeping opponents' strategies constant, is *included* in player i 's best-response correspondence. It suffices to say that in order for player i to obtain one of his/her best responses, he/she might do so by maximizing the pseudo-potential function F .

Example 2.6. Consider again the game in Example 2.4 (see Fig. 2.4a). The game is also a pseudo-potential game. Its potential function is also given by Fig. 2.4b. We note that it is not a best-response potential game as $\arg \max_{S_2} F(1A, S_2) = 2A$ while $\arg \max_{S_2} U_2(1A, S_2) = \{2A, 2B\}$.

Pseudo-potential games are included because they have applications in distributed power control for wireless networks. Specifically, two special classes of pseudo-potential games known as games of weak strategic substitutes and/or weak strategic complements with aggregation (WSC-A/WSS-A) are applied to analyze power control problems. We will return to these applications in Sect. 5.2.

2.1.7 Relations Among Classes of Potential Games

Several classes of potential games have been defined. The following theorem sums up their inter-relationships.

Theorem 2.1. Let E , W , O , G , B and P denote the classes of finite exact, weighted, ordinal, generalized ordinal, best-response, and pseudo-potential games, respectively. Then

- (i) $E \subset W \subset O \subset G \subset P$
- (ii) $E \subset W \subset O \subset B \subset P$
- (iii) $G \cap B \neq \emptyset$, $G \setminus B \neq \emptyset$, and $B \setminus G \neq \emptyset$.

Proof. The results were concluded due to several works such as [13, 31, 43, 49].

From their definitions and the results presented in (2.1), (2.3), (2.5), and (2.8), it is obvious that $E \subset W \subset O \subset G$. To see that $G \subset P$, Schipper [43] argued that if $\mathcal{G} \in G$, then $\mathcal{G}$ has no strict improvement cycle which means it also has no strict best-response cycle; and hence, we also have $\mathcal{G} \in P$. However, $\mathcal{G} \in P$ does not imply $\mathcal{G} \in G$, which means $P \not\subset G$. Hence, (i) is proven. The concept of improvement cycles and strict best-response cycles will be defined in details in Sect. 2.3.1.

In Sect. 2.1.5, Lemma 2.2 shows that $O \subset B$, while Example 2.5 shows that $B \not\subset O$. Also by definition, $B \subset P$. Meanwhile, Example 2.6 shows that $P \not\subset B$. Hence, (ii) is proven.

To establish (iii), [49] gave examples of games that are both in G and B , in G but not in B , and in B but not in G . $\square$

2.2 Fundamental Properties of Potential Games

In this section, we discuss the properties possessed by potential games. These include two results of paramount importance, which are the *existence* of pure-strategy Nash equilibria and the *convergence* to these equilibria in potential games. In the literature, Monderer and Shapley [31] established the key existence and convergence results for ordinal potential games. According to Theorem 2.1, these results should also apply to exact and weighted potential games. Later works [13, 43] extended these existence and convergence properties to pseudo-potential games; however, the results are more restrictive than those of ordinal potential games. Again, from Theorem 2.1, results for pseudo-potential games directly apply to generalized ordinal and best-response potential games, both of which are subsets of pseudo-potential games. Thus, we will present our discussion according to two main types of games, i.e., ordinal potential games and pseudo-potential games, separately.

2.2.1 Nash Equilibrium Existence

The key idea to show that a Nash equilibrium exists in potential games is the observation that *the set of equilibria in such a game is tied to that of an identical-interest game*, where every player maximizes the common potential function. We begin our discussion with the case of ordinal potential games.

Theorem 2.2 (Monderer and Shapley). *If F is a potential function for the ordinal potential game $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i \in \mathcal{N}}]$, then the set of Nash equilibria of $\mathcal{G}$ coincides with the set of Nash equilibria for the identical interest game $\mathcal{G}^\dagger = [\mathcal{N}, \mathbb{S}, \{F\}_{i \in \mathcal{N}}]$. That is,*

$$\text{NESet}(\mathcal{G}) \equiv \text{NESet}(\mathcal{G}^\dagger) \quad (2.11)$$

where NESet denotes the set of Nash equilibria of a game.

Proof. First, assume that S^* is a Nash equilibrium for $\mathcal{G}$. Then, $\forall i$:

$$U_i(S_i^*, S_{-i}^*) - U_i(S_i, S_{-i}^*) \geq 0 \quad \forall S_i \in \mathbb{S}_i. \quad (2.12)$$

Also by the definition of ordinal potential game (2.5), this leads to, $\forall i$:

$$F(S_i^*, S_{-i}^*) - F(S_i, S_{-i}^*) \geq 0 \quad \forall S_i \in \mathbb{S}_i. \quad (2.13)$$

Hence, S^* is also a Nash equilibrium for $\mathcal{G}^\dagger$. Thus, $\text{NESet}(\mathcal{G}) \subseteq \text{NESet}(\mathcal{G}^\dagger)$.

Similarly, we can show that $\text{NESet}(\mathcal{G}^\dagger) \subseteq \text{NESet}(\mathcal{G})$. Thus, $\text{NESet}(\mathcal{G}^\dagger) \equiv \text{NESet}(\mathcal{G})$. $\square$

Corollary 2.1. *If F has a maximum point in $\mathbb{S}$ then $\mathcal{G}$ has a pure-strategy Nash equilibrium.*

Clearly, every maximum point S^* for F has to satisfy (2.13) and thus coincides with a (pure-strategy) Nash equilibrium for $\mathcal{G}$. Note also that S^* can either be a local or a global optimum. The set of global maximizers for F therefore is a subset of $\text{NESet}(\mathcal{G})$. However, one may only consider these global maxima more “desirable” in terms of social optimality if the potential function itself represents a meaningful measure of such optimality, such as the utilitarian welfare function (1.18). In Chaps. 3 and 4, we will discuss applications where the potential function in fact coincides with the utilitarian welfare function.

The next two theorems characterize Nash equilibrium existence for ordinal and pseudo-potential games, according to the properties of their strategy spaces and potential functions.

Theorem 2.3. *The following statements are true.*

- Every finite (ordinal) potential game admits at least one pure-strategy Nash equilibrium.
- Every continuous (ordinal) potential game whose strategy space $\mathbb{S}$ is compact (i.e., closed and bounded) and potential function F is continuous admits at least one pure-strategy Nash equilibrium. Moreover, if F is strictly concave, the Nash equilibrium is unique.

Proof. For finite games, $\mathbb{S}$ is bounded and a maximum of $F(S)$ always exists. Hence, a Nash equilibrium exists.

For continuous games with compact $\mathbb{S}$ and continuous F , the same argument holds. If F is also strictly concave, it has a unique, global maximum.

Note that all the results apply to all exact potential games equally well. Several engineering applications of potential games only need to invoke those results for exact potential games. $\square$

For pseudo-potential games, similar, albeit weaker, results are also obtained.

Theorem 2.4 (Dubey and Schipper). *Consider the pseudo-potential game $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i \in \mathcal{N}}]$ with potential function F , and $\mathcal{G}^\dagger = [\mathcal{N}, \mathbb{S}, \{F\}_{i \in \mathcal{N}}]$. If F has a maximum, then $\mathcal{G}$ has a pure-strategy Nash equilibrium; and*

$$\text{NESet}(\mathcal{G}^\dagger) \subseteq \text{NESet}(\mathcal{G}). \quad (2.14)$$

Proof. Every $S^* \in \arg \max F(S)$ is, by definition, a Nash equilibrium of $\mathcal{G}^\dagger$. From the definition of pseudo-potential games in (2.10), we derive that $S^* \in \text{NESet}(\mathcal{G})$. Therefore, $\text{NESet}(\mathcal{G}^\dagger) \subseteq \text{NESet}(\mathcal{G})$ and $\mathcal{G}$ has at least one (pure-strategy) Nash equilibrium as long as F has a maximum. $\square$

As a direct consequence, we have:

Corollary 2.2. *Any pseudo-potential game which is either finite, or has a compact strategy space and a continuous potential function, possesses a pure-strategy Nash equilibrium.*

2.2.2 Nash Equilibrium Convergence

Previously, we have established the existence of at least one pure-strategy Nash equilibrium. This section looks at how the players can achieve a Nash equilibrium in potential games. The main idea is via sequential decision dynamics in which *players take turn to act in sequence or in a round-robin manner*. Each player in turn selects a new strategy based on a certain decision rule, thus creating a unilateral strategy deviation and inducing a corresponding change in the potential function. If the change represents an improvement in the value of the function, one expects a *series of improvement that drives the game toward one of its equilibria*.

We can formalize the aforementioned idea by introducing the concept of improvement path.

Definition 2.7. A sequence of strategy profile $\rho = (S^0, S^1, S^2, \dots)$ such that for every index $k \geq 0$, S^{k+1} is obtained from S^k by allowing a player $i(k)$ (the single deviator in step k) to change his/her strategy, is called a *path*. A path ρ is an *improvement path* if in each step k , the deviator $i(k)$ experiences a gain in his/her utility, i.e., $U_{i(k)}(S^{k+1}) > U_{i(k)}(S^k)$. Moreover, ρ is called a *cycle* if ρ is of finite length and its terminal element S^K coincides with its initial element S^0 .

An improvement path ρ is allowed to terminate if no further possible improvement can be obtained. Some paths might not terminate (i.e., are infinite or become a cycle). We are interested in *finite* improvement paths.

Theorem 2.5. *For any strategic game $\mathcal{G}$, if a finite improvement path exists, its end point corresponds to a Nash equilibrium.*

Proof. We prove by contradiction. Let ρ be a finite improvement path whose end point is S^K . If we assume that S^K is not a Nash equilibrium, then there exists a player

$i(K)$ who can deviate from his/her current strategy $S_{i(K)}^K$ to a new strategy $T_{i(K)}$ in order to improve his/her utility. We could now add $S^{K+1} = (T_{i(K)}, S_{-i(K)}^K)$ to ρ to extend this path. This contradicts the initial assumption that ρ must terminate at S^K . $\square$

The above result implies that any decision dynamic that can generate a *finite* improvement path will eventually end up at a Nash equilibrium. It has an immense consequence—*most practical applications of game theoretic formulations in general, and potential games in particular, apply this principle in finding a Nash equilibrium*. This result does not require a game to be a potential game; but a potential game will guarantee this, which we will show shortly.

One might now ask, what kinds of decision dynamics, among those we introduced in Sect. 1.3.2, will generate finite improvement paths for potential games?

Clearly, myopic best-response and better-response (random or deterministic) dynamics create improvement paths, by their definitions in (1.16) and (1.17). Thus, they are prime candidates for further investigation.

We will discuss the answer for different types of potential games subsequently.

2.2.2.1 Finite Ordinal Potential Games

We look at an important theorem by Monderer and Shapley [31].

Theorem 2.6 (Monderer and Shapley). *For finite ordinal potential games, every improvement path is finite. This is known as the finite improvement path property.*

Proof. It is obvious that every improvement path must terminate as the increment of the potential function is finite and bounded. $\square$

Corollary 2.3. *For finite ordinal potential games, every sequence of better and best responses converges to a Nash equilibrium, regardless of its starting point.*

2.2.2.2 Continuous Ordinal Potential Games

In continuous context, absolute convergence may or may not be realized in a finite number of steps. A classic example is the convergent sequence $\{1 - \frac{1}{2^n}\}_{n=1,2,\dots}$ which ultimately converges but goes on infinitely. This is due to the sequence's infinitesimal stepsizes as $n \rightarrow \infty$.

One however can control the stepsize by defining the concept of ϵ -improvement path.

Definition 2.8. A path $\rho = (S^0, S^1, S^2, \dots)$ is an ϵ -improvement path if in each step k , the deviating player $i(k)$ experiences $U_{i(k)}(S^{k+1}) > U_{i(k)}(S^k) + \epsilon$, for some $\epsilon \in \mathbb{R}_+$.

This also facilitates the concept of ϵ -equilibrium, which is a strategy profile that is approximately close to an actual Nash equilibrium.

Definition 2.9. The strategy profile $\tilde{S} \in \mathbb{S}$ is an ϵ -equilibrium if and only if $\exists \epsilon \in \mathbb{R}_+$ such that, $\forall i \in \mathcal{N}$:

$$U_i(\tilde{S}_i, \tilde{S}_{-i}) \geq U_i(S_i, \tilde{S}_{-i}) - \epsilon, \quad \forall S_i \in \mathbf{S}_i. \quad (2.15)$$

The ϵ -equilibrium is a refinement of the original Nash equilibrium and is sometimes preferred as a solution concept especially in situations which require less computational complexity.

Theorem 2.7 (Monderer and Shapley). *For continuous ordinal potential games with bounded utility functions, every ϵ -improvement path is finite. This is known as the approximate finite improvement path property.*

Proof. For ordinal potential games whose utility functions are bounded, their potential functions must also be bounded. That is, $\exists L \in \mathbb{R}, L < \infty$ such that $L = \sup_{S \in \mathbb{S}} F(S)$.

Now suppose that $\rho = (S^0, S^1, \dots, S^k, \dots)$ is an ϵ -improvement path which is also infinite. By definition, $U_{i(k-1)}(S^k) - U_{i(k-1)}(S^{k-1}) > \epsilon, \forall k$. As the game is an ordinal potential game, there exists a sufficiently small constant ϵ' such that $F(S^k) - F(S^{k-1}) > \epsilon', \forall k$. This implies $F(S^k) - F(S^0) > k\epsilon'$ or

$$F(S^k) > F(S^0) + k\epsilon', \forall k. \quad (2.16)$$

Clearly, $\lim_{k \rightarrow \infty} F(S^k) = \infty$ which is a contradiction. $\square$

Thus, any ϵ -improvement path ρ must terminate after a certain K steps, at which point $F(S^k) \leq L < F(S^k) + \epsilon'$ or $F(S^k) > L - \epsilon'$. This suggests that the end point of such a path is an ϵ -equilibrium, which we state in the following corollary.

Corollary 2.4. *For continuous ordinal potential games, every better-response sequence that is compatible with ϵ -improvement converges to an ϵ -equilibrium in a finite number of steps.*

Note that although traditional best-response and better-response dynamics still advance towards a Nash equilibrium in continuous games, whether they will terminate in a finite number of steps is not guaranteed. However, in case this happens, ϵ -improvement path can be used to approximate the solution.

2.2.2.3 Pseudo-Potential Games

For pseudo-potential games, the following results hold.

Theorem 2.8 (Dubey and Schipper). *For finite pseudo-potential games, sequential best-response dynamics converges to a Nash equilibrium in a finite number of steps.*

Proof. Convergence in finite games is established from Proposition 2 of [43]. We omit the details. $\square$

In summary, convergence to a Nash equilibrium is guaranteed in all finite ordinal potential games by using best-response and better-response dynamics. For continuous ordinal potential games, these dynamics are able to converge to an ϵ -Nash equilibrium. On the other hand, convergence results in pseudo-potential games are only guaranteed for best-response dynamics, as the definition of pseudo-potential games is strongly tied to best responses.

2.3 Identification of Potential Games

In this section and Sect. 2.4 that follows, we present our studies to address the challenges when applying potential games to practical problems. We have seen that being able to know if a game is a potential game is important as it guarantees that at least one equilibrium solution exists. This section specifically provides an answer to this crucial question of *how to identify that a game is a potential game*. We hope to develop a set of rules which allow us to achieve this purpose. Specifically, we look into characterizing the necessary and sufficient conditions in the strategy space and utility functions of players, for a game to be a potential game. We will first present the results for ordinal and pseudo-potential games in Sect. 2.3.1. Exact potential games, both continuous and finite, will be tackled in Sect. 2.3.2.

2.3.1 Ordinal and Pseudo- Potential Game Identification

2.3.1.1 Ordinal Potential Game Identification

For ordinal potential games, Voorneveld et al. [50] derived two necessary and sufficient conditions which will be stated in Theorem 2.9 (Theorem 3.1 of [50]). Before introducing the theorem, a few concepts need to be defined. Recall the definitions of paths and cycles previously (Definition 2.7).

Definition 2.10. A path $\rho = (S^0, S^1, S^2, \dots, S^K)$ is *non-deteriorating* if $U_{i(k)}(S^k) \leq U_{i(k)}(S^{k+1})$, $\forall k$ where $i(k)$ is the deviating player in step k . For two arbitrary strategy profiles S and T , we write $S \rightarrow T$ if there exists a non-deteriorating path from S to T .

Definition 2.11. The cycle $\rho = (S^0, S^1, S^2, \dots, S^K = S^0)$ is called a *weak improvement cycle* if it is non-deteriorating; and $U_{i(k)}(S^k) < U_{i(k)}(S^{k+1})$ at some k .

We now define a binary relation $\approx$ on $\mathbb{S}$ based on the $\rightarrow$ relation, such that for $S, T \in \mathbb{S}$, $S \approx T$ if both $S \rightarrow T$ and $T \rightarrow S$ hold. Then, this binary relation $\approx$ can

easily be verified to be an equivalence relation.² The equivalent class of an element $S \in \mathbb{S}$ is now defined as the set $[S]$ where $[S] = \{T \in \mathbb{S} \mid S \approx T\}$. Subsequently, one can define the set of equivalence classes on $\mathbb{S}$ induced by $\approx$, denoted by

$$\mathbb{S}_{\approx} = \{[S], \forall S \in \mathbb{S}\}. \quad (2.17)$$

On $\mathbb{S}_{\approx}$, we then define another preference relation $<$ such that $[S] < [T]$ if $[S] \neq [T]$ and $S \rightarrow T$. Moreover, $<$ can be shown to be irreflexive and transitive.

Finally, we define the proper order as follows.

Definition 2.12. The tuple $(\mathbb{S}_{\approx}, <)$ is said to be *properly ordered* if there exists a function $F : \mathbb{S}_{\approx} \mapsto \mathbb{R}$ such that $\forall S, T \in \mathbb{S}, [S] < [T] \Rightarrow F([S]) < F([T])$.

For a better understanding of various concepts here, we refer the readers to some textbooks in abstract algebra, such as [14].

The following theorem provides characterization of ordinal potential games. We omit the proof due to its technicalities which are out of the scope of our book.

Theorem 2.9 (Voorneveld). *The game $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i \in \mathcal{N}}]$ is an ordinal potential game if and only if the two following conditions hold:*

- (1) $\mathbb{S}$ has no weak improvement cycle.
- (2) $(\mathbb{S}_{\approx}, <)$ is properly ordered.

Corollary 2.5 (Voorneveld). *If $\mathbb{S}$ is finite (or countably infinite), condition (2) in Theorem 2.9 can be omitted.*

Remark 2.1. For finite/countable games, a crude method of identifying ordinal potential games is to *exhaustively check for lack of weak improvement cycles*.

The following example involves verifying whether a finite game is an ordinal potential game using this method.

Example 2.7. Consider the 3-player game in Fig. 2.6.

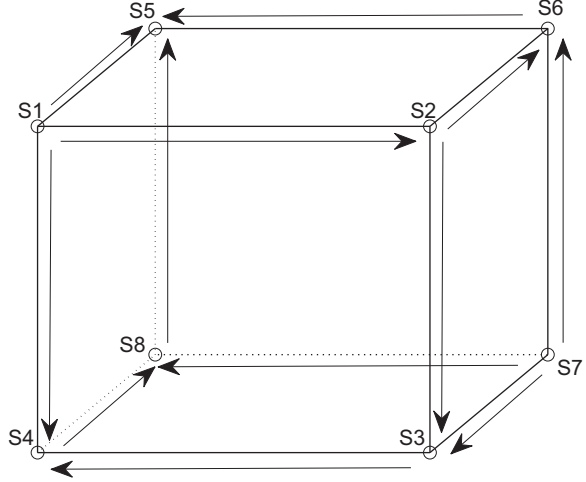
In this game, strategy space is finite and can be represented by a *graph*. Vertices of the graph correspond to $S1, S2, \dots, S8$. An edge connects two vertices if a single

		2A		2B				2A		2B			
1A	(S1)	1, 0, 1	(S2)	1, 1, 1		1A	(S5)	6, 6, 6	(S6)	6, 4, 6			
1B	(S4)	6, 6, 4	(S3)	4, 4, 6		1B	(S8)	4, 6, 6	(S7)	4, 4, 4			
		3A						3B					

Fig. 2.6 A 3-player strategic game. Strategy profiles are labeled S1–S8

²In mathematics, a binary relation between two members of a set is an equivalence relation if and only if it is reflexive, symmetric and transitive [14].

Fig. 2.7 Graphical representation of the game in Fig. 2.6



player's deviation causes a switch between the two corresponding profiles. The full graph is depicted in Fig. 2.7.

Moreover, we use an *arrow* superimposed on an edge to indicate that a non-deteriorating path exists from one vertex to another. Any resulting *directed cycle* is a weak improvement cycle. Clearly, by searching within this graph, we find no such cycle. Therefore, this is an ordinal potential game according to Corollary 2.5.

2.3.1.2 Pseudo-Potential Game Identification

For pseudo-potential games, similar results were provided by Schipper [43]. Note that concurrently, [49] also presented the characterization for best-response potential games but these results are not repeated here.

Analogous to ordinal potential games, the conditions involve a lack of *strict best-response cycles*, and *proper order on* $(\mathbb{S}_{\approx}, <)$. Note that the requirement of path improvement is now restricted to best-response moves only.

Definition 2.13. A path $\rho = (S^0, S^1, S^2, \dots, S^K)$ is *strict best-response compatible* (SBRC) if $\forall k = 0, \dots, K$:

$$S_{i(k)}^{k+1} = \begin{cases} S_{i(k)}^k & \text{if } S_{i(k)}^k \in \mathcal{B}_{i(k)}(S_{-i(k)}^k) \\ \hat{S}_{i(k)} \in \mathcal{B}_{i(k)}(S_{-i(k)}^k) & \text{otherwise,} \end{cases} \quad (2.18)$$

where $i(k)$ is the deviating player in step k . That is, the deviator either deviates to his/her best-response strategy, or stays at the current strategy if it is already a best response. For two arbitrary profiles S and T , we write $S \triangleright T$ if there exists a SBRC path from S to T .

Definition 2.14. The cycle $\rho = (S^0, S^1, S^2, \dots, S^K = S^0)$ is called a *strict best-response cycle* if it is SBRC; and $U_{i(k)}(S^k) < U_{i(k)}(S^{k+1})$ for at some k .

Similarly, the binary equivalence relation $S \approx T$ is then defined on $\mathbb{S}$ if $S \triangleright T$ and $T \triangleright S$. The corresponding set of equivalence classes is denoted by $\mathbb{S}_{\approx}$ and the preference relation $<$ is similarly defined on $\mathbb{S}_{\approx}$.

Main results are stated in the next theorem and corollary (Theorems 1–4 from [43]). Once again we omit the proofs.

Theorem 2.10 (Schipper). Consider a game $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i \in \mathcal{N}}]$ and the following two conditions:

- (1) $\mathbb{S}$ has no strict best-response cycle,
- (2) $(\mathbb{S}_{\approx}, <)$ is properly ordered.

Then,

- **Sufficiency:** $\mathcal{G}$ is a pseudo-potential game if (1) and (2) hold.
- **Necessity:** If $\mathcal{G}$ is a pseudo-potential game with potential function F , then (1) and (2) hold for the game $\mathcal{G}^\dagger = [\mathcal{N}, \mathbb{S}, \{F\}_{i \in \mathcal{N}}]$.

Corollary 2.6 (Schipper). If either $\mathbb{S}$ is countable or each $\mathbf{S}_i \subseteq \mathbb{R}$, Theorem 2.10 is valid without considering condition (2).

Finally, a method of knowing if a finite game is *not* a pseudo-potential game is by *exhaustively checking for existence of strict best-response cycles*. This is a weaker result than the case of ordinal potential games.

The following example shows a game which is not a pseudo-potential game.

Example 2.8. Consider the 2-player game in Fig. 2.8.

It is easy to see that $(S1, S2, S5, S4, S1)$ is a strict best-response cycle. Thus, this is not a pseudo-potential game (as well as ordinal potential game).

2.3.1.3 Construction of Potential Functions

For finite ordinal and pseudo-potential games, we observe that there is a simple method to construct the ordinal and pseudo-potential functions. The method was first proposed by Schipper [43] for finite pseudo-potential games. We find that it can be extended to finite ordinal potential games. The following discussion for ordinal potential games may be an unreported result as the authors are not aware of any existing relevant articles in the literature.

Fig. 2.8 A 2-player 3×3 strategic game

		Player 2		
		2A	2B	2C
Player 1	1A	(S1) 3,6	(S2) 0,10	(S3) 1,7
	1B	(S4) 1,4	(S5) 2,0	(S6) 5,2
	1C	(S7) 2,2	(S8) 1,5	(S9) 3,3

The idea is to define a potential function by assigning a strength ranking to each strategy profile S in $\mathbb{S}$. This rank is measured by *counting the number of other strategy profiles $S' \neq S$ such that from S' there exists a path leading to S* . We require this path to be a non-deteriorating path for ordinal potential games.

Recall the notation $S' \rightarrow S$ when there is a non-deteriorating path from S' to S . We denote $r : \mathbb{S} \mapsto \mathbb{N}$ the rank function of a strategy profile $S \in \mathbb{S}$ (where $\mathbb{N}$ is the set of natural numbers). The rank function is thereby defined as follows

$$r(S) = \sum_{S' \in \mathbb{S}} \mathbf{1}(S' \rightarrow S) \quad (2.19)$$

where $\mathbf{1}(S' \rightarrow S)$ is the indicator function for the relation $\rightarrow$, i.e.,

$$\mathbf{1}(S' \rightarrow S) = \begin{cases} 1 & \text{if } S' \rightarrow S, \\ 0 & \text{otherwise.} \end{cases} \quad (2.20)$$

Theorem 2.11. *If the game $\mathcal{G}$ is a finite ordinal potential game, then the rank function $r(S)$ defines a potential function for $\mathcal{G}$.*

Proof. We prove by contradiction.

Assume that $\mathcal{G}$ is an ordinal potential game. Along any $S' \rightarrow S$ path, the stepwise change in the utility function of any deviating player must be non-negative. Hence, if $F()$ is a potential function it must satisfy $F(S') \leq F(S)$. Now supposing that $r(S)$ is not a potential function, then there exists a $S' \rightarrow S$ path such that $r(S') > r(S)$. By definition of $r()$, S' has more strategy profiles that can be led to it via a non-deteriorating path than S does. This means,

$$\exists S'' \in \mathbb{S} : S'' \rightarrow S', \text{ but } S'' \not\rightarrow S. \quad (2.21)$$

However, since $\rightarrow$ is transitive, $S'' \rightarrow S'$ and $S' \rightarrow S$ imply $S'' \rightarrow S$. This contradicts (2.21) which requires $S'' \not\rightarrow S$. Hence, $r(S)$ must be a potential function. $\square$

An intuitive interpretation of the above theorem is that, given any finite ordinal potential game, the number of strategy profiles that have a non-deteriorating path leading to S can be set as the potential value of S . We further observe that if $S \rightarrow S'$ and $S' \rightarrow S$, or S and S' belong to the same equivalence class $[S]$, i.e., $[S] = [S']$, then $r(S) = r(S')$. In addition, if $[S] < [S']$ then $r(S) < r(S')$. Thus, the function $r()$ ranks all equivalence classes of $\mathcal{G}$ in accordance with the relation $<$. This is essentially an implication of the *proper order* on $(\mathbb{S}_{\approx}, <)$ as stated in Theorem 2.9.

Our next theorem shows that there can be infinitely many ways of constructing potential functions, as long as all the strategy profiles are assigned with numbers which preserve the order as that given by $r()$.

Theorem 2.12. *Let $\mathcal{G}$ be a finite ordinal potential game and assume that the rank function $r()$ assigns values $\{0, 1, \dots, M\}$ to its strategy profiles. Also, let*

$(x_0, x_1, \dots, x_M)$ be an arbitrary $(M + 1)$ -tuple of real numbers such that $x_0 < x_1 < \dots < x_M$. Then $\mathcal{G}$ admits any function F as its potential function where

$$F(S) = x_i \quad \text{if} \quad r(S) = i, \quad \forall i = 0, 1, \dots, M. \quad (2.22)$$

Proof. The proof is left as an exercise. $\square$

Example 2.9. Consider the ordinal potential game in Example 2.7. For this game, the possible non-deteriorating paths can be deduced from Fig. 2.7. A potential function therefore can be computed from the rank function as follows.

Profile	Non-deteriorating paths from	Rank $r(S)$
S1	None	0
S2	S1	1
S3	S1, S2, S7	3
S4	S1, S2, S3, S7	4
S5	S1, S2, S3, S4, S5, S6, S7	7
S6	S1, S2, S7	3
S7	None	0
S8	S1, S2, S3, S4, S7	5

One can verify that this function satisfy the ordinal potential function definition.

Exercise 2.1. Use the rank function to compute a potential function for the ordinal potential game in Example 2.3.

For pseudo-potential games, by replacing the requirement of non-deteriorating paths by strict best-response compatible (SBRC) paths, we can have similar results. Recall the notation $S' \triangleright S$ if there is a SBRC path from S' to S . The indicator function $\mathbf{1}(S' \triangleright S)$ is also defined similarly.

Theorem 2.13 (Schipper). *Let $\mathcal{G}$ be a finite pseudo-potential game. Then, $\mathcal{G}$ admits the following rank function $r : \mathbb{S} \mapsto \mathbb{N}$ as its potential function:*

$$r(S) = \sum_{S' \in \mathbb{S}} \mathbf{1}(S' \triangleright S). \quad (2.23)$$

Proof. Similar to the proof of Theorem 2.11. $\square$

Analogous to ordinal potential games, more generalized constructions can be realized.

Corollary 2.7. *Let $\mathcal{G}$ be a finite pseudo-potential game and the rank function $r()$ assigns values $\{0, 1, \dots, M\}$ to its strategy profiles. Also, let $(x_0, x_1, \dots, x_M)$ be an arbitrary $(M + 1)$ -tuple of real numbers such that $x_0 < x_1 < \dots < x_M$. Then $\mathcal{G}$ admits any function F as its potential function where*

$$F(S) = x_i \quad \text{if} \quad r(S) = i, \quad \forall i = 0, 1, \dots, M. \quad (2.24)$$

Exercise 2.2. Use the rank function in (2.24) to compute a potential function for the pseudo-potential game in Example 2.6.

2.3.2 Exact Potential Game Identification

2.3.2.1 Continuous Exact Potential Game Identification

This section looks into the issue of identifying continuous exact potential games. In our discussion, we will assume that each strategy set of the game $\mathcal{G}$ is a continuous interval of real numbers, i.e., $S_i \subseteq \mathbb{R}$; and that the utility function $U_i : S_i \mapsto \mathbb{R}$ is continuous and differentiable everywhere on S_i . In accordance with previous notations, we denote $F : \mathbb{S} \mapsto \mathbb{R}$ the possible potential function.

The condition for continuous potential games is relatively straightforward.

Theorem 2.14 (Monderer and Shapley). *The game $\mathcal{G}$ is a continuous exact potential game with potential function F if and only if*

$$\frac{\partial^2 U_i}{\partial S_i \partial S_j} = \frac{\partial^2 U_j}{\partial S_i \partial S_j}, \quad \forall i, j \in \mathcal{N}. \quad (2.25)$$

Proof. Equation (2.25) follows directly from our alternative definition (2.2). $\square$

A benefit of the condition (2.25) is that it allows us to identify a continuous exact potential game without knowing its potential function.

The next question is how to find the potential function, assuming that the game is known to be a potential game. Fortunately, Monderer and Shapley [31] also provided us with a useful formula. Their result is restated as follows.

Theorem 2.15 (Monderer and Shapley). *If $\mathcal{G}$ is a continuous exact potential game then its potential function F satisfies*

$$F(S) - F(T) = \sum_{i \in \mathcal{N}} \int_0^1 \left(\gamma'(z) \frac{\partial U_i}{\partial S_i}(\gamma(z)) \right) dz \quad (2.26)$$

where $\gamma(z) : [0, 1] \mapsto \mathbb{S}$ is a continuously differentiable path in $\mathbb{S}$ that connects two strategy profiles S and T ; such that $\gamma(0) = T$ and $\gamma(1) = S$.

Proof. The proof comes directly from the gradient theorem in vector calculus [51]. For any smooth curve C from T to S in $\mathbb{S} \subseteq \mathbb{R}^{|\mathcal{N}|}$ and any function F whose gradient vector ∇F is continuous on $\mathbb{S}$, the gradient theorem allows us to evaluate the line integral along $\gamma(z)$ as

$$F(S) - F(T) = \int_{C[T \rightarrow S]} \nabla F(\mathbf{s}) \cdot d\mathbf{s}, \quad (2.27)$$

where $\mathbf{s}$ is a vector variable representing points along C .

After introducing $\mathbf{s} = \gamma(z)$ such that $\mathbf{s} = T$ when $z = 0$ and $\mathbf{s} = S$ when $z = 1$, by chain rule, $d\mathbf{s} = \gamma'(z)dz$ and therefore

$$\begin{aligned} F(S) - F(T) &= \int_0^1 (\gamma'(z) \cdot \nabla F(\gamma(z))) dz \\ &= \sum_{i=1}^{|\mathcal{N}|} \int_0^1 \left(\gamma'_i(z) \frac{\partial F_i}{\partial S_i}(\gamma(z)) \right) dz \end{aligned} \quad (2.28)$$

Then, since F is a potential function, $\frac{\partial F_i}{\partial S_i} = \frac{\partial U_i}{\partial S_i}$ and (2.26) follows. $\square$

In conclusion, for a given continuous game, we can theoretically verify its exact potential property and evaluate its potential function. However, (2.26) is often too general and tedious to evaluate. In most practical applications of potential games, derivation of potential functions may be obtained through much simpler procedures. For example, Sect. 2.4 discusses potential games with utility functions having certain properties where the potential functions can be automatically derived.

We give an example of finding the potential function using (2.26). This example is introduced in [31].

Example 2.10 (A Cournot Competition). In a Cournot competition, players are the N firms, which compete in the market for a certain product.

Player i 's strategy is to produce q_i products. The strategy space is $\mathbb{S} = \mathbb{R}_{\geq 0}^N$. If player i produces q_i products, it bears a cost $c_i(q_i)$. We can assume the function c_i is differentiable and $c_i(0) = 0$.

All generated products are sold at a common price p , determined by the total supplies $Q = \sum_{i \in \mathcal{N}} q_i$ via an inverse demand function $p = f(Q)$. We assume a linear inverse demand $p = f(Q) = a - bQ$, where $a, b > 0$.

Each player's utility function equals its profit given by

$$U_i = pq_i - c_i(q_i) = \left(a - b \sum_{j=1}^N q_j \right) q_i - c_i(q_i). \quad (2.29)$$

We can check the utility function against (2.25). Our game is a continuous exact potential game as

$$\frac{\partial^2 U_i}{\partial q_i \partial q_j} = \frac{\partial^2 U_j}{\partial q_i \partial q_j} = -b. \quad (2.30)$$

To find the potential function using (2.26), we select $S = (q_1, q_2, \dots, q_N)$ and $T = \mathbf{0}$, the origin. Take the path γ to be the straight line from T to S .

Then, at point $\gamma(z)$ along the path, its projection on the i th-axis is zq_i ; and its gradient is $(q_1, q_2, \dots, q_N)$.

We assume further that $F(\mathbf{0}) = 0$. Then (2.26) becomes

$$F(S) = \sum_{i=1}^N \left(\int_0^1 q_i \frac{\partial U_i(zq_i, zq_{-i})}{\partial q_i} dz \right). \quad (2.31)$$

Here,

$$\frac{\partial U_i}{\partial q_i} = a - b \sum_{j \neq i} q_j - 2bq_i - c'_i(q_i) \quad (2.32)$$

so

$$\begin{aligned} \int_0^1 q_i \frac{\partial U_i(zq_i, zq_{-i})}{\partial q_i} dz &= q_i \int_0^1 \left[a - b \sum_{j \neq i} zq_j - 2bzq_i - c'_i(zq_i) \right] dz \\ &= q_i \left[a - \frac{1}{2}b \sum_{j \neq i} q_j - bq_i - \frac{1}{q_i} (c_i(q_i) - c_i(0)) \right] \\ &= aq_i - \frac{1}{2}b \sum_{j \neq i} q_i q_j - bq_i^2 - c_i(q_i). \end{aligned} \quad (2.33)$$

Finally, from (2.31) and (2.33),

$$F(S) = a \sum_{i=1}^N q_i - \frac{1}{2}b \sum_{i=1}^N \sum_{j \neq i} q_i q_j - b \sum_{i=1}^N q_i^2 - \sum_{i=1}^N c_i(q_i). \quad (2.34)$$

This is the potential function we obtain, which is identical to the one given in [31].

From the example above, one sees that the potential function depends on the assigned value of $F(T)$. The next result addresses its uniqueness.

Theorem 2.16 (Monderer and Shapley). *If F_1 and F_2 are two possible potential functions of an exact potential game $\mathcal{G}$ (both continuous or finite) then they differ only by a constant c , i.e.,*

$$F_1(S) - F_2(S) = c \quad \forall S \in \mathbb{S}. \quad (2.35)$$

Proof. See [31] (Lemma 2.7). $\square$

Thus, unlike ordinal and pseudo-potential games, the potential functions in exact potential games are unique up to addition of a constant.

2.3.2.2 Finite Exact Potential Game Identification

Previously, conditions for continuous games are established due to the smoothness property of the strategy space. Similarly, for finite games, we will also examine their strategy spaces. However, the property we look at now involves cycles, i.e., paths that start and end at the same point. Recall that identifying finite ordinal and pseudo-potential games involves checking for lack of cycles having weak improvement or strict best-response properties, respectively. For exact potential games, interestingly, it happens that the total change in the utility functions along every cycle is 0.

Theorem 2.17 (Monderer and Shapley). *Let $\rho = (S^0, S^1, S^2, \dots, S^K = S^0)$ be an arbitrary finite cycle in a finite game $\mathcal{G}$. Denote $i(k)$ the deviating player from S^k to S^{k+1} . Then, $\mathcal{G}$ is an exact potential game if and only if*

$$\sum_{k=0}^{K-1} [U_{i(k)}(S^{k+1}) - U_{i(k)}(S^k)] = 0. \quad (2.36)$$

Proof. See Appendix A of [31]. $\square$

Naturally, the above necessary and sufficient condition leads to the method of *exhaustively checking all finite cycles* to verify if a given game is also an exact potential game. The next corollary makes the task less tedious by restricting our search to *all cycles of length 4* only.

Corollary 2.8 (Monderer and Shapley). *Suppose that ρ is a cycle of length 4 in a game $\mathcal{G}$,³ as described below:*

$$\begin{array}{ccc} & A \longleftarrow D & \\ \rho = \downarrow & & \uparrow \\ & B \longrightarrow C & \end{array} \quad (2.37)$$

where $A = (S_i, S_j, S_{-\{i,j\}})$, $B = (T_i, S_j, S_{-\{i,j\}})$, $C = (T_i, T_j, S_{-\{i,j\}})$, and $D = (S_i, T_j, S_{-\{i,j\}})$ are the 4 strategy profiles forming the cycle, due to two deviating players i and j .

Then, $\mathcal{G}$ is an exact potential game if and only if, $\forall i, j \in \mathcal{N}$:

$$\underbrace{[U_i(B) - U_i(A)]}_{i \text{ from } A \text{ to } B} + \underbrace{[U_j(C) - U_j(B)]}_{j \text{ from } B \text{ to } C} + \underbrace{[U_i(D) - U_i(C)]}_{i \text{ from } D \text{ to } C} + \underbrace{[U_j(A) - U_j(D)]}_{j \text{ from } D \text{ to } A} = 0, \quad (2.38)$$

$$\forall S_i, T_i \in \mathbf{S}_i; \forall S_j, T_j \in \mathbf{S}_j \forall S_{-\{i,j\}}.$$

³Although this section deals with finite games, this corollary is valid also for continuous games.

Although this result has reduced the amount of computation significantly, the verification process is still very time consuming. For example, if $\mathcal{G}$ has N players and each player has $|\mathbf{S}_i| = M$ strategies, then the number of times (2.38) needs to be checked is

$$\binom{N}{2} \left[\binom{M}{2} \right]^2 = \frac{N(N-1)}{2} \left[\frac{M(M-1)}{2} \right]^2. \quad (2.39)$$

The complexity for this checking method is thus $O(N^2M^4)$. Recently, there have been efforts to simplify the procedure and reduce this computation complexity.

Remark 2.2. Hino [17] observed that instead of checking all 2×2 combinations of strategy profiles, we only need to check for those cycles consisting of adjacent rows and columns when the game is represented in payoff matrix. If there are N players and each has M strategies, then the number of equations to be checked is reduced to

$$\binom{N-1}{1} \left[\binom{M-1}{1} \right]^2 = (N-1)(M-1)^2. \quad (2.40)$$

This proposed method will reduce the checking time complexity from $O(N^2M^4)$ to $O(NM^2)$.

Remark 2.3. In a recent paper, Cheng [10] used the technique of semi-tensor product of matrices to verify a potential game by obtaining the *potential equation*. The game $\mathcal{G}$ is an exact potential game if and only if the potential equation has a solution. The potential function can be constructed from the solution of potential equation. Refer to [10] for more details.

We conclude this discussion with the construction of potential function for finite exact potential games. Two previous results come in handy. First, Theorem 2.17 shows that from one point S , any path that comes back to S gives zero change in the sum of utilities as well as potential function. In other words, any two possible paths from S to T yield *exactly the same sum*. Our second result is Theorem 2.16 allowing us to obtain a unique potential function up to addition of a constant. Thus, we present an algorithm which calculates the potential function for every strategy profile of a finite exact potential game $\mathcal{G}$, given a starting strategy profile S and an arbitrary initial potential value α assigned to S . The idea is to walk through all strategy profiles in $\mathcal{G}$ starting from S ; and at each point we accumulate to α the sum of utility changes and assign this value to the potential function at the current point. Algorithm 2.1 gives the procedures.

In Algorithm 2.1, walking through all points in strategy space $\mathbb{S}$ starting from S is performed using the procedure $\text{traverse}(\mathcal{G}, S)$. Meanwhile, $\text{traverse}(\mathcal{G}, S)$ itself returns a Boolean value, which is **true** if there are unvisited nodes, and **false** otherwise. Assuming $\mathbb{S}$ can be represented by a graph structure with vertices corresponding to strategy profiles and edges corresponding to possible strategy deviation by a single player, $\text{traverse}(\mathcal{G}, S)$ might make use of classic graph traversal

Algorithm 2.1 Exact potential function computation for finite games.**Require:** Finite exact potential game $\mathcal{G}$, initial profile S , constant α

```

1:  $F(S) \leftarrow \alpha$ 
2: while  $\text{traverse}(\mathcal{G}, S)$  do ▷ Visiting all profiles in  $\mathcal{G}$  starting from  $S$ 
3:    $i \leftarrow$  deviating player
4:    $A \leftarrow$  previous strategy profile
5:    $B \leftarrow$  new strategy profile
6:    $F(B) \leftarrow F(A) + U_i(B) - U_i(A)$ 
7: end while
8: return  $F$ 

```

Fig. 2.9 A generalized prisoner's dilemma game where $a > b > c > d$

		Player 2	
		C	D
Player 1	C	$(M) b, b$	$(N) d, a$
	D	$(Q) a, d$	$(P) c, c$

algorithms such as the *breadth-first search* or *depth-first search* algorithms [19] to exhaustively visit all nodes in an optimal manner. At each unvisited node, say B , it will identify its predecessor A and the deviating player i in the current move. Subsequently, it calculates the potential function for the new node by adding the utility change of that player to the predecessor's potential value. When $\text{traverse}(\mathcal{G}, S)$ ends, all nodes have been visited and the final potential function F is returned.

We shall go through a simple example where Algorithm 2.1 is used to compute the potential function for the Prisoner's dilemma.

Example 2.11. Consider the Prisoners' dilemma with a generalized payoff matrix in Fig. 2.9. Its graphical representation has a simple structure of 4 vertices (M, N, P, Q) and 4 edges (MN, NP, PQ, QM).

We assume that our algorithm starts at N and an initial value $\alpha = 1$ is used. We assume the particular graph traversal order $N \rightarrow M \rightarrow Q \rightarrow P$ (note that the order is not important). The computation is as follows.

Profile	Path	Deviating player	Potential
N	None	None	1
M	$N \rightarrow M$	2	$1 + b - a$
Q	$M \rightarrow Q$	1	$1 + b - a + a - b = 1$
P	$Q \rightarrow P$	2	$1 + c - d$

Exercise 2.3. Use Algorithm 2.1 to compute a potential function for Example 2.1.

2.4 Formulation of Exact Potential Games

In the previous section, we have tackled the question of how to identify whether a game is a potential game. We specified the conditions for a given game to be a certain type of potential games and how to compute the potential function. The reversed question, which is the second objective of this chapter, is *how to construct potential games*. This is of equal importance, because if one knows the techniques to formulate potential games, including how to define the utility functions of players and the potential functions, we believe that potential games can become a more effective problem-solving tool and can find a much wider range of applications.

For a given engineering problem, especially in the field of wireless communications, it may sometimes be more desirable to formulate this problem as a potential game in order to have solutions which are stable and achievable. However, how to formulate such a game is still a great challenge. In most existing problems, potential games are often established by introducing their associated potential functions and verifying them with the definition. Moreover, generally known models are limited to a few known forms of potential functions as well as players' utility functions. The unanswered questions are, is there an effective method to design utility functions and potential functions so that one is able to construct potential games? Are there any special properties or structures in these functions that can help us identify potential games? Some formulation principles may already exist or have implicitly been used in the literature, but there is still a lack of systematic studies to formalize the underlying rules. Thus, our studies aim to uncover them and hopefully inspire researchers to fully exploit the “potential” in using the potential game technique. In this section, our focus will therefore shift towards the *design* aspect—the generalized method which are useful in constructing new potential games. Our discussion focuses especially on *exact potential games* which have gained the most attention and practical applications.

We start our investigation by firstly identifying useful properties of the utility functions of players based on which the resulting games are potential games. A number of such properties are presented in Sect. 2.4.1. Subsequently, we propose two formal approaches of formulating potential games in Sect. 2.4.2 which are the forward and the backward methods. The forward method defines the utility functions of players such that they satisfy aforementioned properties, and how a potential function can be obtained from these utility functions efficiently. In the backward method, the potential function for the game is first defined and we will use it to derive the utility functions of players.

2.4.1 Utility Function Considerations

A few properties are identified to be useful in constructing the utility functions of games which turn out to be a potential game. They are *separability*, *symmetry of observations* as well as *linear combinations among utility functions*. Additionally, we discuss the impact of imposing *constraints* on a given potential game.

2.4.1.1 Linear Combination of Utility Functions

Our first observation is that we can derive new exact potential games from existing ones by using a new utility function which is a linear combination of existing utility functions. This property stems from a much more powerful result which states that the set of all potential games with the same player set and strategy space forms a *linear space*. Fachini et al. [15] mentioned this property and a proof was available in [32]. The following result holds.

Theorem 2.18. *Let $\mathcal{G}_1 = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i \in \mathcal{N}}]$ and $\mathcal{G}_2 = [\mathcal{N}, \mathbb{S}, \{V_i\}_{i \in \mathcal{N}}]$ be two exact potential games. Then $\mathcal{G}_3 = [\mathcal{N}, \mathbb{S}, \{\alpha U_i + \beta V_i\}_{i \in \mathcal{N}}]$ is also an exact potential game, $\forall \alpha, \beta \in \mathbb{R}$.*

Proof. Suppose that $F(S)$ and $G(S)$ are potential functions of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. It is straightforward to show that $\alpha F(S) + \beta G(S)$ is a potential function for $\mathcal{G}_3$. $\square$

This property is useful when we would like to jointly maximize two objectives via a weighted sum of the two. Knowing that using each objective separately as the utility function results in an exact potential game, we are guaranteed that the combined objective also leads to another exact potential game.

Remark 2.4. Only exact potential games and weighted potential games have the linear combination property [15]. For ordinal potential games, counter-examples were given in [32].

2.4.1.2 Separability

Next, we observe that if every player's utility function is separable into multiple terms with certain structures, the game can be shown to be an exact potential game.

Strategic Separability

The first notion of separability is what we term *strategic separability*, meaning that one's utility function can be decomposed into the summation of a term contributed solely by one's own strategy, and another term contributed solely by the opponents' joint strategy.

Definition 2.15. The game $\mathcal{G}$ is strategically separable if $\forall i, \exists P_i : \mathbf{S}_i \mapsto \mathbb{R}$ and $\exists Q_i : \mathbf{S}_{-i} \mapsto \mathbb{R}$ such that

$$U_i(S_i, S_{-i}) = P_i(S_i) + Q_i(S_{-i}). \quad (2.41)$$

Theorem 2.19. *If $\mathcal{G}$ is strategically separable, then it is also an exact potential game with the following potential function*

$$F(S) = \sum_{i \in \mathcal{N}} P_i(S_i). \quad (2.42)$$

Proof. For any unilateral strategy deviation of an arbitrary player i from S_i to T_i , we have

$$\begin{aligned} U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) &= P_i(T_i) + Q_i(S_{-i}) - P_i(S_i) - Q_i(S_{-i}) \\ &= P_i(T_i) - P_i(S_i). \end{aligned} \quad (2.43)$$

At the same time,

$$\begin{aligned} F_i(T_i, S_{-i}) - F_i(S_i, S_{-i}) &= P_i(T_i) + \sum_{j \neq i} P_j(S_j) - P_i(S_i) - \sum_{j \neq i} P_j(S_j) \\ &= P_i(T_i) - P_i(S_i). \end{aligned} \quad (2.44)$$

Hence, $F(S)$ is a potential function for $\mathcal{G}$. $\square$

The following example from wireless communications shows a formulation which, by making some approximations, exhibits strategic separability and thus is a potential game.

Example 2.12 (Potentialized Game for CDMA Power Control). In code-division multiple access (CDMA) networks, all mobile stations (MSs) share the wireless medium and the transmission power of any MS can cause interference to the other MSs. The CDMA power control game studies the power adaptation for distributed, selfish MSs under such conflict of interest.

The players are N MSs, each of which transmits to a pre-assigned base station (BS). Note that some BSs can be shared by multiple players. Player i 's strategy is its power level p_i which is bounded in a certain range, e.g., $[0, P_{\max}]$. We use (p_i, p_{-i}) to refer to strategies in this game.

The signal-to-noise-and-interference ratio (SINR) γ_i of player i is calculated as

$$\gamma_i(p_i, p_{-i}) = \frac{p_i g_{ii}}{\sum_{j=1, j \neq i}^N p_j g_{ji} + \sigma^2} \quad (2.45)$$

where g_{ji} is the channel gain between player j 's transmitter and player i 's BS; and σ^2 is the noise power.

The achievable rate for player i is given by

$$R_i(p_i, p_{-i}) = \log_2 (1 + G\gamma_i) \quad (2.46)$$

where G is the spreading gain of the CDMA system [7].

A feasible optimization objective often incorporates rate R_i as the reward obtained by player i , as well as a cost for spending power. Linear pricing is often used for CDMA systems [42] where the cost is expressed as $c_i p_i$ for player i . The positive constant c_i indicates the price per unit power used. Thus, the following utility function can be considered:

$$U_i(p_i, p_{-i}) = R_i(p_i, p_{-i}) - c_i p_i, \forall i \in \mathcal{N}. \quad (2.47)$$

The resulting game is $\mathcal{G} = [\mathcal{N}, [0, P_{\max}]^N, \{U_i\}_{i \in \mathcal{N}}]$. Candogan et al. [7] proposed the following approximation to the utility function, so that a “potentialized” game is obtained.

The approximated utility function at high SINR is proposed to be

$$\tilde{U}_i(p_i, p_{-i}) = \log_2 \left(G \frac{p_i g_{ii}}{\sum_{j=1, j \neq i}^N p_j g_{ji} + \sigma^2} \right) - c_i p_i, \forall i \in \mathcal{N} \quad (2.48)$$

where the term 1 has been dropped within rate calculation.

We can see that this function is strategically separable. In fact,

$$\tilde{U}_i(p_i, p_{-i}) = \log_2 (G p_i g_{ii}) - c_i p_i - \log_2 \left(\sum_{j=1, j \neq i}^N p_j g_{ji} + \sigma^2 \right), \quad (2.49)$$

where the first two terms only depend on player i 's strategy and the last term only depends on the opponents' strategies. According to Theorem 2.19, the game $\mathcal{G}' = [\mathcal{N}, [0, P_{\max}]^N, \{\tilde{U}_i\}_{i \in \mathcal{N}}]$ is an exact potential game.

Note that strategic separability is a sufficient condition for exact potential games. However, not all potential games are separable in this manner.

Coordination-Dummy Separability

The second notion of separability is known as *coordination-dummy separability*. It was first discussed by Slade [45], and was later linked to potential games by Fachini et al. [15] and Ui [46].

Definition 2.16. The game $\mathcal{G}$ is coordination-dummy separable if $\exists P : \mathbb{S} \mapsto \mathbb{R}$ and $\exists Q_i : \mathbf{S}_{-i} \mapsto \mathbb{R}, \forall i$ such that

$$U_i(S_i, S_{-i}) = P(S) + Q_i(S_{-i}). \quad (2.50)$$

Basically, one's utility function is a linear combination of two terms: $P(S)$ which is common and identical to all players, and $Q_i(S_{-i})$ which only depends on joint actions of one's opponents. With $P(S)$ alone, one effectively plays an

identical-interest game, also known as a perfect coordination game. On the other hand, $Q_i(S_{-i})$ is said to be a dummy function because it solely depends on other players' strategies. Altogether, such a utility function $U_i = P(S) + Q_i(S_{-i})$ is called coordination-dummy separable. The term coordination-dummy was suggested in [15].

Theorem 2.20 (Slade, Fachini, Ui). *$\mathcal{G}$ is coordination-dummy separable if and only if it is an exact potential game with potential function $P(S)$.*

Proof. To prove the sufficiency, suppose $\mathcal{G}$ is coordination-dummy separable. Then (2.50) holds. By checking the definition (2.1) on $P(S)$, clearly it is a potential function and $\mathcal{G}$ is an exact potential game.

To prove the necessity, we see that if $\mathcal{G}$ is an exact potential game with some potential function $P(S)$, then for each i we can let $Q_i(S) = U_i(S) - P(S)$. Then for any $S_i, T_i \in \mathbf{S}_i$, since $P(S)$ is a potential function, by definition,

$$U_i(S_i, S_{-i}) - P(S_i, S_{-i}) = U_i(T_i, S_{-i}) - P(T_i, S_{-i}), \quad (2.51)$$

which implies

$$Q_i(S_i, S_{-i}) = Q_i(T_i, S_{-i}), \quad \forall S_i, T_i \in \mathbf{S}_i. \quad (2.52)$$

Thus, Q_i depends only on S_{-i} . Thus, $\mathcal{G}$ is coordination-dummy separable. $\square$

Coordination-dummy separability provides both necessary and sufficient conditions for exact potential games. In constructing potential games, this notion and the previous strategic separability serve as useful rules—as long as we can design utility functions that are separable, our games will be exact potential games.

Remark 2.5. Separability notions give rise to special types of exact potential games. For example,

1. **Identical-interest games:** We encounter these games before (e.g., Theorem 2.2). It is a special case of coordination-dummy separability where all the dummy terms vanish.
2. **No-Conflict Games:** Here, $U_i \equiv U_i(S_i), \forall i$. As such, a player's utility function only depends on his/her own actions and is not affected by other players' actions.
3. **Dummy Games:** On the contrary, when the coordination term vanishes we have a dummy game.

We shall look at an example from wireless communications where an identical-interest game is considered.

Example 2.13 (Cooperative Players of Identical Interest in Wireless Mesh Networks). A wireless mesh network [1] is a self-configured wireless ad-hoc communications network where mobile nodes are organized in a mesh topology. There are two types of nodes: mesh routers (MR) and mesh clients (MC). The MRs form the backbone infrastructure of the network and are equipped with functionality to carry

out the tasks of resource allocation. In [12], Duarte et al. investigated the problem of decentralized channel assignment among players which are MRs in a wireless mesh network. Because the backbone networks of MRs are partially connected, [12] assumed that the MRs can play cooperatively; hence, all the MRs share an *identical objective function*.

In a wireless mesh network of arbitrary topology, there are N MRs which form the set of players $\mathcal{N}$ in the game, whose decisions are to assign channels to their associated MCs. There are K available channels. Let $\mathbf{A} \in \{0, 1\}^{N \times K}$ be the channel assignment matrix whose element a_{ik} equals 1 when channel k is assigned to one of player i 's MCs, and 0 otherwise. Hence, player i 's strategy is expressed by $S_i = \mathbf{a}_i^T$, the $1 \times K$ i^{th} row vector of $\mathbf{A}$. The game's strategy space is therefore $\mathbb{S} = \{0, 1\}^{N \times K}$. As usual, we denote a strategy profile by S .

The following metric is defined in [12] which characterizes the performance of a player in the game:

$$M_i = \frac{\alpha_i}{\beta_i} \sum_{k=1}^K a_{ik} \frac{R}{l_{ik}}, \quad \forall i \in \mathcal{N} \quad (2.53)$$

where

- α_i is a connectivity coefficient. If MR i can reach the network gateway, $\alpha_i = 1$. Otherwise, $\alpha_i = 0$.
- β_i is the hop count from MR i to the gateway.
- R is the link data rate, which is determined by the modulation and coding schemes.
- l_{ik} counts the number of interfering links sharing channel k with player i .

As the players are cooperative, a common network objective is defined which are jointly maximized among all players as follows.

$$U_i(S) = F(S) = \sum_{i=1}^N M_i, \quad \forall i \in \mathcal{N}. \quad (2.54)$$

The resulting strategic game is $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i \in \mathcal{N}}]$. As $\mathcal{G}$ is a game of identical interest, it is also an exact potential game with potential function $F(S)$.

An implication of this identical-interest game formulation is that cooperative players can play in a distributed manner via well-known methods such as best-response and better-response dynamics, at the same time ensuring convergence to a Nash equilibrium.

2.4.1.3 Symmetry of Observations

We next examine the situations where the utility functions of players exhibit symmetries across the variables (strategies). Potential games can arise in these circumstances.

Bilateral Symmetric Interaction

One straightforward notion of symmetric observations is that due to *bilateral* or *pairwise* strategic interactions. To be precise, for player i , the utility function U_i contains a term $w_{ij}(S_i, S_j)$ which takes place solely due to the pairwise interaction between him/her and another player j , and does not depend on the actions of the rest of the players. We interpret $w_{ij}(S_i, S_j)$ as the observation seen by i due to the strategy of j . Symmetry of observations occurs when, for all S_i and S_j , we have $w_{ij}(S_i, S_j) = w_{ji}(S_j, S_i)$. That is, the observations are said to be symmetric across the pair i and j .

Games where observations are symmetric across all pairs of players were termed *bilateral symmetric interaction* (BSI) games in Ui [46]. That is, $\forall i, j \in \mathcal{N}, i \neq j$, there exist functions $w_{ij} : \mathbf{S}_i \times \mathbf{S}_j \mapsto \mathbb{R}$ such that $w_{ij}(S_i, S_j) = w_{ji}(S_j, S_i)$ for all $S_i \in \mathbf{S}_i$ and $S_j \in \mathbf{S}_j$. Moreover, the utility function of player i is assumed to be of the form

$$U_i(S_i, S_{-i}) = \sum_{j \neq i} w_{ij}(S_i, S_j), \forall i \in \mathcal{N}. \quad (2.55)$$

The next theorem from [46] shows that BSI games are exact potential games.

Theorem 2.21 (Ui). *Assume $\mathcal{G}$ is a BSI game. Then it is also an exact potential game with the following potential function*

$$F(S) = \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}, j < i} w_{ij}(S_i, S_j). \quad (2.56)$$

Proof. For $S_i, T_i \in \mathbf{S}_i$, it is straightforward to see that

$$U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) = \sum_{j \in \mathcal{N}, j \neq i} w_{ij}(T_i, S_j) - \sum_{j \in \mathcal{N}, j \neq i} w_{ij}(S_i, S_j), \quad (2.57)$$

and at the same time,

$$F(T_i, S_{-i}) - F(S_i, S_{-i}) = \sum_{j \in \mathcal{N}, j \neq i} w_{ij}(T_i, S_j) - \sum_{j \in \mathcal{N}, j \neq i} w_{ij}(S_i, S_j). \quad (2.58)$$

□

Remark 2.6. Coupled with the linear combination property, we can combine BSI games with no-conflict games. The resulting exact potential game has utility functions of the form $U_i(S) = \sum_{j \neq i} w_{ij}(S_i, S_j) + P_i(S_i)$. This is the original utility function proposed in [46].

We shall look at an example from network engineering where one can enjoy a potential game formulation owing to BSI property.

Example 2.14 (Routing Game). In computer and telecommunications networks, the routing problem is where multiple players need to decide how to split their traffic loads from their sources to their destinations through various links in the network. The players, for example, can be various service providers sending data to their subscribers. Altman et al. [3] proposed a game-theoretic approach to this problem.

In the model, there are N players and a network of several nodes, connected by directed links. There are a total of L links. Each player can allocate a certain amount of their traffic loads to each link, i.e., λ_i^l is the load allocated to link l by player i . Thus, the strategy of player i is described by a L -tuple $S_i = (\lambda_i^l)_{l=1, \dots, L}$.

We denote the total traffic loads on a link l as $q_l = \sum_{i=1}^N \lambda_i^l$. It is assumed that for each link, a usage cost is present. The cost per unit traffic f_l is linearly proportional to the link usage and is given by

$$f_l(q_l) = a_l q_l + b_l, \quad a_l, b_l < 0 \quad (2.59)$$

where $a_l, b_l < 0$ are assumed so that players will maximize the negative sum of costs, equivalent to minimizing their actual costs.

Thus, the cost incurred by player i for link l is $\lambda_i^l f_l(q_l)$. Its utility function is thus given by

$$U_i(S_i, S_{-i}) = \sum_{l=1}^L \lambda_i^l (a_l q_l + b_l) \triangleq \sum_{l=1}^L u_i^l(S_i, S_{-i}). \quad (2.60)$$

Thus, we have a load allocation routing game. In [3], the authors first introduced a potential function and established the exact potential property through it. However, we can naturally deduce this property without knowing the existence of this function. We can verify using (2.25) but it can be tedious differentiating U_i with respect to a vector variable. Instead, we will check for BSI.

For an arbitrary link l and player i , we see that

$$u_i^l(S_i, S_{-i}) = a_l (\lambda_i^l)^2 + b_l \lambda_i^l + a_l \sum_{j \neq i} \lambda_i^l \lambda_j^l. \quad (2.61)$$

Thus, $a_l \lambda_i^l \lambda_j^l$ is a symmetric observation across player i and j on link l , for any j . Subsequently, one can appropriately express the overall utility into the form $U_i = \sum_{j \neq i} w_{ij}(S_i, S_j) + P_i(S_i)$ where

$$w_{ij}(S_i, S_j) = \sum_{l=1}^L a_l \lambda_i^l \lambda_j^l \quad (2.62)$$

and

$$P_i(S_i) = \sum_{l=1}^L (a_l (\lambda_i^l)^2 + b_l \lambda_i^l). \quad (2.63)$$

The game is an exact potential game as it is a linear combination of a BSI objective and a no-conflict term. Furthermore, by Theorems 2.21 and 2.18, a potential function can be automatically found, given by

$$\begin{aligned} F(S) &= \sum_{i=1}^N \sum_{j < i} w_{ij}(S_i, S_j) + \sum_{i=1}^N P_i(S_i) \\ &= \sum_{l=1}^L \left[a_l \sum_{i=1}^N \sum_{j < i} \lambda_i^l \lambda_j^l + \sum_{i=1}^N (a_l (\lambda_i^l)^2 + b_l \lambda_i^l) \right]. \end{aligned} \quad (2.64)$$

One can easily verify that this function coincides with the one introduced in [3].

General Symmetric Observations

Previously, we see that in practical scenarios with BSI structures, we can formulate a potential game. Next, we propose and investigate a more general notion of symmetric observations.

Definition 2.17. Consider a game $\mathcal{G}$. If for any pair of players $i \neq j \in \mathcal{N}$, there exist functions $g_{ij}: \mathbb{S} \mapsto \mathbb{R}$, $g_{ji}: \mathbb{S} \mapsto \mathbb{R}$, $Q_{ij}: \mathbf{S}_{-j} \mapsto \mathbb{R}$ and $Q_{ji}: \mathbf{S}_{-i} \mapsto \mathbb{R}$ such that $g_{ji}(S) = g_{ij}(S)$, $\forall S$ and

$$\begin{aligned} U_i(S) &= g_{ij}(S) + Q_{ij}(S_{-j}), \\ U_j(S) &= g_{ji}(S) + Q_{ji}(S_{-i}), \end{aligned} \quad (2.65)$$

then $\mathcal{G}$ is said to have *general symmetric observations* (GSO) across all players.⁴

Thus, in GSO games, any pair of players i and j share a common observation $g_{ij}(S)$ due to their strategic interactions. The utility function can be decomposed into $g_{ij}(S)$ and a second term $Q_{ij}(S_{-j})$ which represents the contribution to player i 's utility function due to all players except j .

⁴To the best of our knowledge, this GSO investigation has not been reported in the literature.

Games with BSI structures can be seen as a special case of GSO. In fact, for any $j \neq i$, one can rewrite the utility function of a BSI game (2.55) into

$$U_i(S_i, S_{-i}) = w_{ij}(S_i, S_j) + \underbrace{\sum_{k \neq i, j} w_{ik}(S_i, S_k)}_{Q_{ij}(S_{-j})}. \quad (2.66)$$

Meanwhile, unlike in BSI games, the symmetric observation in GSO games needs not involve only two players i and j . That is, in (2.65), g_{ij} can in fact be expressed as $g_{ij}(S_i, S_j, S_{-\{i, j\}})$. In addition, we do not require U_i to be the summation of all pairs of observations. A game is a GSO game as long as we can decompose the utility functions for any pair of players i and j according to our definition.

Our main result is as follows.

Theorem 2.22. *GSO games are exact potential games.*

Proof. We will make use of Corollary 2.8 and prove that (2.38) holds for any arbitrary cycle of length 4, say $\rho = (A, B, C, D, A)$ as in (2.37). Here, i and j are two active players. Moreover, $A = (S_i, S_j, R)$, $B = (T_i, S_j, R)$, $C = (T_i, T_j, R)$, and $D = (S_i, T_j, R)$ where $R = S_{-\{i, j\}}$.

In fact, using (2.65),

$$U_i(B) - U_i(A) = g_{ij}(B) - g_{ij}(A) + Q_{ij}(T_i, R) - Q_{ij}(S_i, R) \quad (2.67)$$

Similarly,

$$U_j(C) - U_j(B) = g_{ji}(C) - g_{ji}(B) + Q_{ji}(T_j, R) - Q_{ji}(S_j, R), \quad (2.68)$$

$$U_i(D) - U_i(C) = g_{ij}(D) - g_{ij}(C) + Q_{ij}(S_i, R) - Q_{ij}(T_i, R), \quad (2.69)$$

$$U_j(A) - U_j(D) = g_{ji}(A) - g_{ji}(D) + Q_{ji}(S_j, R) - Q_{ji}(T_j, R). \quad (2.70)$$

By summing up (2.67)–(2.70), we obtain

$$[U_i(B) - U_i(A)] + [U_j(C) - U_j(B)] + [U_i(D) - U_i(C)] + [U_j(A) - U_j(D)] = 0. \quad (2.71)$$

Thus, Corollary 2.8 guarantees that the defined GSO game is an exact potential game. $\square$

A well-known class of games in the literature, the *congestion games* proposed by Rosenthal [40], turn out to exhibit the GSO property.

Example 2.15 (Congestion Game). In a congestion game, there are N players. In addition, there are K resources which are indexed by $k \in \mathcal{K} = \{1, 2, \dots, K\}$. Each player is supposed to select a subset of the K available resources and their choices can be overlapped.

In practical scenarios, constraints may be imposed so that certain combinations of resources are infeasible or invalid for a particular player. For example, the resources can be a collection of roads where players need to take to get to a destination. Some road segments are physically separated and cannot be validly combined. Such constraints limit the strategy space of player i to only a subset of the complete strategy space $2^{\mathcal{K}}$, the power set of $\mathcal{K}$. Thus, we denote the strategy set of player i as $\mathbf{S}_i \subseteq 2^{\mathcal{K}}$, $\forall i$. Each strategy $S_i \in \mathbf{S}_i$ corresponds to a feasible *set of selected resources*.⁵

It is assumed that shared resources incur costs, which depend on the *number* of users occupying that resource. For resource k , denote this cost by $c_k(x_k)$ (assuming $c_k(x_k) < 0$), where x_k is the number of players choosing k . The same cost $c_k(x_k)$ is incurred on every sharing player. The utility function of player i is the total costs over all individual resources he/she selects, given by

$$U_i(S) = \sum_{k \in S_i} c_k(x_k(S)), \quad (2.72)$$

where the notion $x_k(S)$ indicates that the number of players choosing resource k can be determined from the joint strategies of all players.

In a congestion game, congestion occurs as multiple players simultaneously choose a resource. Thus, players try to minimize their total costs (or maximize the negative total costs). Congestion games are well-known to be potential games [31]. We will alternatively show that congestion games are indeed GSO games.

Between players i and j , let $\sigma_{ij} = S_i \cap S_j$ which represents the set of common resources shared between i and j , and define

$$g_{ij} = g_{ji} = \begin{cases} \sum_{k \in \sigma_{ij}} c_k(x_k(S)) & \sigma_{ij} \neq \emptyset \\ 0 & \sigma_{ij} = \emptyset. \end{cases} \quad (2.73)$$

Then, $\forall i$ and $\forall j \neq i$:

$$U_i(S) = g_{ij}(S) + \sum_{k \in S_i \setminus \sigma_{ij}} c_k(x_k(S_{-j})). \quad (2.74)$$

where $Q_{ij}(S_{-j}) \triangleq \sum_{k \in S_i \setminus \sigma_{ij}} c_k(x_k(S_{-j}))$ represents the remaining costs from player i 's resources that are not selected by player j . By Theorem 2.22, congestion games are exact potential games.

Due to their practical considerations, congestion games have found applications in several networking problems. One example is the network formation game introduced in Chap. 19 of Nisan et al. [39], in which players try to build a network

⁵The use of set-valued strategies here is originally considered in Rosenthal [40].

by forming edges across network nodes to connect their sources to destinations. The cost for an edge is evenly shared by players using it and thus is a function of the number of sharing players.

The readers are invited to work out the potential function for the above congestion game.

Exercise 2.4. Find a potential function for the congestion game.

Solution 2.4. The potential function for this game will be derived in the Appendix of this chapter.

In conclusion, symmetric observation across players' utility functions is another criterion to establish exact potential games.

2.4.1.4 External Constraints and Utility Functions

In the congestion games presented in Example 2.15, we see that imposing constraints can make some actions infeasible to some players and the question is how that affects the feasible strategy space. This happens commonly for practical systems. For example, in downlink cellular systems, two users within a cell might not be allocated the same frequency bands; and there may be a constraint for total transmitted power level. On the other hand, the system quality-of-service (QoS) requirements may specify some performance metrics such as minimum throughput or maximum delay, which similarly restricts the set of feasible actions. As such, these constraints reduce the feasible strategy space as compared to the game without any constraint.

The question we investigate here is how the constraints affect the formulation of potential games. In doing so, we look at our game-theoretic problem from the viewpoint of mathematical optimization. Suppose that the original game $\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i\}_{i \in \mathcal{N}}]$ is formulated before incorporating the constraints. This can be expressed as a collection of N optimization problems,

$$(\mathcal{G}) : \quad \forall i \in \mathcal{N} : \quad \max_{S_i \in \mathbb{S}_i} U_i(S_i, S_{-i}), \quad (2.75)$$

where the set of all strategy profiles feasible for (2.75) is the strategy space $\mathbb{S}$ of $\mathcal{G}$.

Now, let us assume that there are a number of constraints to be imposed on the game. Adopting the standard notations of a mathematical optimization problem [5], the new game with the imposed constraints can be written as

$$(\mathcal{G}') : \quad \forall i \in \mathcal{N} : \quad \max_{S_i \in \mathbb{S}_i} U_i(S_i, S_{-i}),$$

$$\text{subject to (s.t.)} \quad \begin{cases} g_k(S_i, S_{-i}) \leq 0, & k = 1, 2, \dots, K \\ h_m(S_i, S_{-i}) = 0, & m = 1, 2, \dots, M. \end{cases} \quad (2.76)$$

Here, we assume that there are K inequality constraints in the form $g_k(S) \leq 0$, as well as M equality constraints in the form $h_m(S) = 0$, where the strategy profile S is treated as a decision variable. The constraints commonly encountered in wireless communications problems can easily be expressed into one of the two forms above. For examples,

- In a wireless power control application, the players (wireless radios)' strategies are assumed to be power levels p_i , $i = 1, 2, \dots, N$. It is required that $0 \leq p_i \leq P_{\max}$, $\forall i \in \mathcal{N}$. Equivalently, we can express these constraints into $2N$ inequality constraints of the form $g_k(p) \leq 0$, $k = 1, 2, \dots, 2N$, where

$$g_k(p) \triangleq \begin{cases} -p_k, & k = 1, 2, \dots, N \\ p_{k-N} - P_{\max}, & k = N + 1, N + 2, \dots, 2N. \end{cases} \quad (2.77)$$

- In a wireless channel assignment problem, there are N players and M channels for data transmission. Here we define $a_{im} \in \{0, 1\}$ as the channel assignment indicator between player i and channel m . Player i 's strategy is represented by the $1 \times M$ vector $S_i = [a_{i1} \ a_{i2} \ \dots \ a_{iM}]$. Each of the M channels can be assigned to exactly one player. These constraints can then be expressed as

$$h_m(S) \triangleq \sum_{i=1}^N a_{im} - 1 = 0, \quad m = 1, 2, \dots, M. \quad (2.78)$$

We introduce some notations as follows. For each of the constraints, we can define the set of feasible strategy profiles that satisfy the particular constraint as follows.

$$\mathbb{G}_k = \{S \mid g_k(S) \leq 0\}, \quad \forall k = 1, 2, \dots, K \quad (2.79)$$

$$\mathbb{H}_m = \{S \mid h_m(S) = 0\}, \quad \forall m = 1, 2, \dots, M. \quad (2.80)$$

Thus, all feasible strategy profiles comprising the strategy space $\mathbb{S}'$ for the constrained game $\mathcal{G}'$ in (2.76) belong to the intersection of all the sets defined above. That is,

$$\mathbb{S}' \triangleq \left(\bigcap_{k=1}^K \mathbb{G}_k \right) \cap \left(\bigcap_{m=1}^M \mathbb{H}_m \right) \cap \mathbb{S}. \quad (2.81)$$

The set $\mathbb{S}'$ is assumed, for non-triviality, that $\mathbb{S}' \neq \emptyset$. This leads to a new game

$$\mathcal{G}' = [\mathcal{N}, \mathbb{S}', \{U_i\}_{i \in \mathcal{N}}]. \quad (2.82)$$

We claim the following result.

Theorem 2.23. *If $\mathcal{G}$ is an exact potential game then so is $\mathcal{G}'$.*

Proof. The equality $U_i(T_i, S_{-i}) - U_i(S_i, S_{-i}) = F(T_i, S_{-i}) - F(S_i, S_{-i})$ holds for any S_i, T_i such that $(S_i, S_{-i}), (T_i, S_{-i}) \in \mathbb{S}$, if $\mathcal{G}$ is an exact potential game. Thus, they remain valid if we restrict (S_i, S_{-i}) and (T_i, S_{-i}) to the new strategy space $\mathbb{S}'$, which is a subset of $\mathbb{S}$. $\square$

This theorem is also a useful property for deriving new potential games from existing ones. Suppose that practical considerations require additional constraints. If we are certain that such constraints only modify the strategy space, we can guarantee that the new game is still an exact potential game. In subsequent chapters (Chaps. 3 and 4), we will introduce applications where addition of constraints is encountered.

Remark 2.7. This result is based on the fact that the condition (2.1) for exact potential games holds universally across the strategy space. Then, it automatically holds within any of its subset. We note that sometimes this equality may even hold for a superset of the original strategy space. For example, in the Cournot competition (Example 2.10), we can hypothetically enlarge the strategy space by allowing it to take negative quantities (i.e., removing the constraints $q_i \geq 0, \forall i$) and still retain the equality relationship of the potential function, even if such a relaxation may not have practical meanings in this case. However, extreme care needs to be taken when considering an enlarged strategy space. It is always advisable to examine if the condition (2.1) still holds for supersets of the strategy space.

Remark 2.8. Note that there are other types of constraints that not only modify our strategy space but also require us to make more significant alterations, e.g., redefine our objective function or reformulate the problem. In such scenarios, this theorem may not apply.

To conclude this section, we look at an example from wireless communications.

Example 2.16 (Power Control with Coupled Constraints). Let us revisit the uplink of a CDMA wireless network similar to Example 2.12. A power control game for the N MSs, with power level p_i as the strategy, is considered. The SINR of player i is again given by $\gamma_i(p_i, p_{-i}) = \gamma_i(\mathbf{p})$ in (2.45). Scutari et al. [44] proposed a power minimization game where the objective is for each player to minimize its transmit power p_i , subject to a coupled constraint f_i on the SINR, given by

$$f_i(\gamma_i(\mathbf{p})) \geq \phi_i, \quad \forall i \quad (2.83)$$

where $f_i(\cdot)$ is a continuous function on $\mathbb{R}_+$, whose choice depends on the respective QoS requirements; and ϕ_i are real constants.

Here, we start with the following game

$$\mathcal{G} = [\mathcal{N}, \mathbb{S}, \{U_i = -\log(p_i)\}_{i \in \mathcal{N}}] \quad (2.84)$$

whose feasible strategy space is

$$\mathbb{S} = \{\mathbf{p} \mid p_i > 0, f_i(\gamma_i(\mathbf{p})) \geq \phi_i, \forall i\} \quad (2.85)$$

where $\mathbf{p}$ denotes the network power vector which also is a strategy profile. Note that 0 is not a feasible strategy for any player as the $\log()$ function is undefined at 0. It is straightforward to see that $\mathcal{G}$ is a game of no conflict; as such, it is an exact potential game and admits the following potential function $F(\mathbf{p}) = -\sum_{i \in \mathcal{N}} \log(p_i)$.

Now, let us consider a new game $\mathcal{G}'$ derived from $\mathcal{G}$ by imposing a maximum power constraint $p_i \leq P_{\max}, \forall i$. Its strategy space is therefore

$$\mathbb{S}' = \{\mathbf{p} \mid \mathbf{p} \in (0, P_{\max}]^N, f_i(\gamma_i(\mathbf{p})) \geq \phi_i, \forall i\}. \quad (2.86)$$

Clearly, we have $\mathbb{S}' \subset \mathbb{S}$. Thus, according to Theorem 2.23, $\mathcal{G}'$ is also an exact potential game with the same potential function $F(\mathbf{p})$.

In [44], it was also assumed that $\mathbb{S}' \neq \emptyset$ and $\mathbb{S}'$ is convex and compact. As a result, $F(\mathbf{p})$ has a unique maximum on $\mathbb{S}'$ and $\mathcal{G}'$ admits a unique Nash equilibrium.

The above is a simple example to illustrate the use of Theorem 2.23. In Chaps. 3 and 4, we will again encounter more non-trivial examples which demonstrate the usefulness of this result.

2.4.2 Game Formulation Principles

Having introduced several properties that can serve as useful guidelines in designing potential games, we now consolidate them into design principles. From our survey, we have observed that most of the formulations of exact potential games applied to wireless communications and networking can be classified under either one of two design principles: the *forward method* and the *backward method*.

Note that the described methods are primarily concerned with the design of utility functions and/or potential functions. Prior to formulating a game-theoretic problem for a practical scenario, one still needs to identify the players, their means of interactions and possible strategies, as well as the constraints, objectives and assumptions. There is no clear-cut process; and in this book we focus on generalizing methods for potential game and utility function design while assuming the parameters are already in place.

2.4.2.1 Forward Method

In the forward method, the utility functions are purposely designed to have one of the desired properties: separability, symmetric observations, or linear combination of utilities, which are known to lead to potential games. The potential function can then be associated with one of the known forms specified by these properties.

To be precise, we denote $\mathcal{D}$ as the set of utility functions having one or more of the above desirable properties. The forward method is as follows.

Algorithm 2.2 Forward method

- 1: $\forall i: U_i(S) \leftarrow \mathcal{D}$
 - 2: $F(S) \leftarrow$ associated potential function according to the structure of $U_i(S)$
-

Several authors have proposed potential game models with utility functions adhering to one of these properties.

- In [35, 36], Neel et al. discussed game models for cognitive radio networks. One of their models has the utility function of the form $U_i(S) = f_{i,1}(p_i) - f_{i,2}(I_i)$, where $f_{i,1}(p_i)$ is a function of the player i 's received signal strength which depends on its power level p_i . $f_{i,2}(I_i)$ is another function of the player i 's received interference which only depends on the opponents' transmit power p_{-i} . This design is based on the strategic separability.
- In [33, 34, 37], Neel et al. designed games that adheres to bilateral symmetric interference assumption. This approach leads to BSI games. Babidi [4], Wu [52] and a few others further extended this approach.
- The formulation by Nie et al. [38] on interference minimization for distributed radios as well as our works on OFDMA systems [20–23] used an interference sum minimization objective. Symmetric observations are present for all pairs of players.

In subsequent chapters, we will review some of the applications listed above in details. For now, let us look at one particular formulation to demonstrate this principle.

Example 2.17 (Cognitive Radio Interference Minimization Game). Cognitive radios [29] are smart radio devices that can learn their environment and optimize their performance by adjusting their transmission parameters. In the distributed spectrum access problem among multiple radios, their interactions can be modeled as a game. Nie et al. [38] studied a channel allocation game among N cognitive radio pairs of transmitter and receiver. The N pairs of nodes constitute the set of players $\mathcal{N}$. There are K frequency bands ($K < N$) which represent the available resources; and each player must select one frequency to transmit its data. Thus, for player i , its strategy is the channel it selects, i.e., $S_i = k \in \{1, 2, \dots, K\}$.

As the spectrum band is spatially reused by the distributed radios, co-channel interference is present which degrades the performance of the radios. We let p_i be the transmission power of pair i , and g_{ij} the channel gain between the transmitter of pair i and the receiver of pair j . We further define a variable δ_{ij} , which assumes a value 1 if players i and j are on the same channel, i.e., $S_i = S_j$, and 0 otherwise. As such, the possible co-channel interference that player i may experience from player j is given by $\delta_{ji}p_jg_{ji}$.

In this game, an interference minimization objective is adopted. The following utility function was firstly considered, i.e.,

$$U_i(S_i, S_{-i}) = - \sum_{j=1, j \neq i}^N \delta_{ji}p_jg_{ji} \quad (2.87)$$

in order for each player to minimize its received co-channel interference.

Naturally, the quantity $\delta_{ji}p_jg_{ji}$ perfectly represents the observation on player i 's utility function due to the action of player j . On the other hand, player j 's observation due to player i 's action is given by $\delta_{ij}p_ig_{ij}$. In general, $p_j \neq p_i$ and $g_{ij} \neq g_{ji}$. Thus, the observation is not bilateral symmetric for this formulation.

However, between players i and j , the observations become symmetric if the total $\delta_{ji}p_jg_{ji} + \delta_{ij}p_ig_{ij}$ is considered instead. This quantity represents the total interference that a player generates to another player and also experiences from the same player. This leads to a second proposed utility function

$$V_i(S_i, S_{-i}) = - \sum_{j=1, j \neq i}^N \delta_{ji}p_jg_{ji} - \sum_{j=1, j \neq i}^N \delta_{ij}p_ig_{ij}. \quad (2.88)$$

We can see that the second utility function satisfies the BSI property. For any two players i and j , we have

$$w_{ij}(S_i, S_j) = w_{ji}(S_j, S_i) = -(\delta_{ji}p_jg_{ji} + \delta_{ij}p_ig_{ij}). \quad (2.89)$$

Thus, according to Theorem 2.21, the game $[\mathcal{N}, \mathbb{S}, \{V_i\}_{i \in \mathcal{N}}]$ is an exact potential game. Its potential function is given by

$$F(S) = - \sum_{i=1}^N \sum_{j=1, j < i}^N (\delta_{ji}p_jg_{ji} + \delta_{ij}p_ig_{ij}). \quad (2.90)$$

This can be alternatively written in the form

$$F(S) = \sum_{i=1}^N \left(-\frac{1}{2} \sum_{j=1, j \neq i}^N \delta_{ji}p_jg_{ji} - \frac{1}{2} \sum_{j=1, j \neq i}^N \delta_{ij}p_ig_{ij} \right) \quad (2.91)$$

as seen in [38].

In summary, this example demonstrates the forward method which works by identifying a utility function in accordance with certain desirable properties.

2.4.2.2 Backward Method

The forward method works by defining utility functions that ensure potential game properties first before obtaining the potential function. On the other hand, the backward method first defines a network objective as the potential function and works backward to obtain individual utility functions. The steps are as follows.

In the backward method, one may first define a network function F as a global objective to maximize, which will also serve as the potential function. Next, in order

Algorithm 2.3 Backward method

- 1: Define $F(S)$
 2: Decompose $F(S)$, $\forall i$:

$$F(S) \rightarrow P_i(S_i, S_{-i}) + Q_i(S_{-i}) \quad (2.92)$$

- 3: Assign, $\forall i$:

$$U_i(S_i, S_{-i}) \leftarrow P_i(S_i, S_{-i}) \quad (2.93)$$

to define each player's utility function, a decomposition is applied to this global network function such that $F(S) = P_i(S_i, S_{-i}) + Q_i(S_{-i})$ where $Q_i(S_{-i})$ is a non-contributing term from the perspective of player i . Then, we can set $U_i = P_i(S_i, S_{-i})$.

Theorem 2.24. *Algorithm 2.3 results in an exact potential game.*

Proof. From (2.92),

$$U_i(S_i, S_{-i}) = F(S_i, S_{-i}) - Q_i(S_{-i}), \quad \forall i \quad (2.94)$$

By Definition 2.16, U_i is coordination-dummy separable. Therefore, the resulting game is an exact potential game. Its potential function is by default $F(S)$. $\square$

Remark 2.9. Note that such a decomposition does not always give non-trivial utility functions. It is possible that $Q_i(S_{-i}) = 0$ for all players and the process results in an identical-interest game.

In the literature, examples of the backward method are:

- Menon et al. [27, 28] as well as Buzzi et al. [6] which defined a sum of inverse SINRs as the potential function and subsequently defined players' utility through a similar decomposition.
- Xu et al. [53] also defined two network objectives which are total network throughput and total network collisions. They serve as the potential functions for the potential games that follow.

These approaches will be discussed in more details in Chap. 5.

2.5 Further Readings

This chapter has covered the theory of potential games, including fundamental results with accompanied mathematical analysis. We make an attempt to generalize and present a collective summary on the results available from the literature. Some of the mathematical proofs are omitted in the text. However, interested readers may

refer to the cited works. The authors are also aware that in the literature, more specialized topics are available for potential games. In what follows, we present a non-exhaustive list of related topics intended for further readings.

Although introduced, certain types of games such as generalized ordinal potential games and best-response potential games were not discussed in depth due to their rare appearances in communications applications. Readers who wish to explore their mathematical properties can refer to Monderer and Shapley [31] and Vorneveld [50].

There exist other notions of potential games in the literature. For example, in his paper [43], Schipper extended the class of pseudo-potential games to a broader class called *quasi-potential games*. However, the author did not elaborate further on the concept and their applications are not known. A few more generalizations from pseudo-potential games exist, such as *q-potential games* [30] and *nested potential games* [47]. One related concept to potential games is *near potential games* as proposed by Candogan et al. [8, 9]. The authors defined a notion of distance between games and those with a close distance to a potential game are called near-potential.

Extending potential properties of games outside the current static and complete information settings is a different line of literature. *Bayesian potential games* were studied by Facchini et al. [15] in which the games are set under incomplete information assumptions. Sandholm [41] presented an extension into potential games among *continuous populations*. On the other hand, Marden [24] defined *state-based potential games* in dynamic settings where there exists an underlying state space governing the system. González-Sánchez et al. [16] coupled potential games with dynamic stochastic control problems and characterized the conditions for the potential function in *dynamic stochastic potential games*. Another approach in the dynamic settings is to extend the potential game framework to continuous-time optimal control models, in which the concept of *Hamiltonian potential* and its use in characterization of open-loop equilibrium of differential games were proposed in Dragone et al. [11]. These recent interesting topics may attract further development.

There were also studies that linked potential games to the concept of Shapley value for coalitional games. Some fundamental results were presented by Monderer and Shapley [31], Ui [46], etc.

Our presented results on convergence of best/better-response dynamics give an elementary view of limiting behaviors of adaptive update rules for games. There is an extensive literature on this topic. Fictitious play and variants were discussed in Monderer and Shapley [31], Hofbauer and Sandholm [18], Marden et al. [26], and so on. Neel [32] defined some practical decision rules and demonstrated convergence properties. Logit-response dynamics are a different class considered in Alós-Ferrer et al. [2], Marden et al. [25], etc.

Regarding the identification of finite exact potential games, Cheng [10] gives some interesting results using the technique of semi-tensor product of matrices. His result may be of practical values, which due to the scope of this monograph we have omitted.

Figure 2.10b shows the state of each resource after player 2's selection. At player 2's selection, R_3 is unoccupied while R_2 already has one player. Hence, player 2's virtual cost is computed as $c_2(2) + c_3(1)$. The computation is similar for the next selecting players.

In general, at player i 's selection, resources have been occupied by the previous $i - 1$ players. For each of his/her selected resource $k \in S_i$, determining the current cost depends on the total number of users $\alpha_k(i)$ which includes him/herself and how many players have selected k previously. This number is estimated by

$$\alpha_k(i) = |\{j | k \in S_j, j \leq i\}| \quad (2.96)$$

Player i 's virtual cost is therefore given by

$$\gamma_i(S) = \sum_{k \in S_i} c_k(\alpha_k(i)). \quad (2.97)$$

The accumulated virtual costs for all players in this manner are given by

$$F'(S) = \sum_{i=1}^N \gamma_i(S) = \sum_{i=1}^N \sum_{k \in S_i} c_k(\alpha_k(i)). \quad (2.98)$$

This function is equal to the sum of values of cells that are filled in all the stacks (e.g., the sum of all values in Fig. 2.10d). By exchanging the order of summation, we can rewrite this into

$$F'(S) = \sum_{k=1}^K \sum_{j=1}^{x_j(S)} c_k(j) \quad (2.99)$$

which is exactly the Rosenthal's potential (2.95). That is, $F(S) = F'(S)$.

Now, for player N who is last to select resources, his/her virtual cost is exactly his/her real cost. That is, $\gamma_N(S) = U_N(S)$. We suppose that player N now wants to deviate to a new strategy unilaterally. By noting that

$$F(S) = \sum_{i=1}^{N-1} \gamma_i(S) + U_N(S) \quad (2.100)$$

where the first summation is unaffected by player N 's strategy, it is apparent that $F(S)$ will be changed by exactly the same amount as $U_N(S)$.

This property should hold for every permutation of selection orders, and any player can be equivalently considered to be the last selector. In short, $F(S)$ is a potential function.

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