Generalized Fuzzy Sets and Fuzzy Relations

Yan-cai Zhao, Zu-hua Liao, Teng Lu and Juan Tong

Abstract In classical fuzzy set theory, a fuzzy set is a membership function which associates with each element a real number in $[0, 1]$, a fuzzy relation is a function which associates with each pair of elements a real number in $[0, 1]$. In the present paper, we generalize the above two concepts by associating with each set a real number in $[0, 1]$, and associating with each pair of sets a real number in $[0, 1]$, respectively. We then give a series of properties for these two types of generalized concepts. We also show that a generalized fuzzy relation can be induced by a classical fuzzy relation, which shows the communication of our generalized fuzzy concepts with the classical fuzzy theory.

Keywords Fuzzy set · Fuzzy relation · Generalized fuzzy set · Generalized fuzzy relation · Power set

1 Introduction

The concept of a fuzzy set was introduced by Zadeh [10]. A *fuzzy set* in a referential (universe of discourse) X is characterized by a membership function A which associates with each element $x \in X$ a real number $A(x) \in [0, 1]$, having the interpretation that $A(x)$ is the membership degree of x in the fuzzy set A. For convenience, we also call the set X in above definition the *base set* of a the fuzzy set A.

Let X, Y be two sets. A mapping $R : X \times Y \to [0, 1]$ is called a *fuzzy relation* [10]. The number $R(x, y) \in [0, 1]$ can be interpreted as the degree of relationship between x and y.

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B.-Y. Cao et al. (eds.), *International Conference on Oriental Thinking and Fuzzy Logic*, Advances in Intelligent Systems and Computing 443,
DOI 10.1007/978-3-319-30874-6_2

Since fuzzy set was introduced by Zadeh in 1965 [10], many extensions have been developed, such as intutionistic fuzzy set [1], type-2 fuzzy set [2, 4], type-n fuzzy set [2], fuzzy multiset [5, 9] and hesitant fuzzy set [6–8, 11]. So far as we know, all types of fuzzy set assign a value or a set of values to an element of a classical set. However, there are many situations in our real lives in which one has to make decision on a whole set. For example, a patient usually has several symptoms. His/Her doctor has to make a decision by whole of his/her symptoms, which means that the doctor assigns a value to the set of all the patient's symptoms. We will give the details of this example later.

In this paper, we first generalize the classical fuzzy set on a base set X to one on a collection of sets. Then, we concentrate on the study of the generalized fuzzy sets on base set $\mathcal{P}(X)$, the power set of X, and obtain a series of properties; We also generalize the classical fuzzy relation between two elements to several types of relations between two sets, and obtain a series of properties for these generalized fuzzy relations. Among them, we construct a equivalence fuzzy relation at the end of the paper.

2 Generalized Fuzzy Sets and Their Operations

In our real lives, there exist many situations in which we should make decisions on a collection of sets.

Example 1 A doctor usually judge that if a patient has caught a cold by the following symptoms of this patient: Fever (F for short), Headache (H for short) and Cough (C for short). The following table give the corresponding numbers to different sets of symptoms. Each number means a degree of a patient catch a cold.

In the following table, the collection D of decisions is a fuzzy set on the collection of the set of different symptoms of a patient (Fig. 1).

$$D = \frac{0}{\emptyset} + \frac{0.2}{\{F\}} + \frac{0.1}{\{H\}} + \frac{0.2}{\{C\}} + \frac{0.3}{\{F,H\}} + \frac{0.5}{\{F,C\}} + \frac{0.4}{\{H,C\}} + \frac{0.9}{\{F,H,C\}}.$$

A classical fuzzy set assigns a value to an element. We now give a generalized type of fuzzy set which assigns a value to a set.

Definition 1 Let $\mathcal{S} = \{S_i | \ i \in I\}$ be a collection of sets. Then a mapping A such that $A : \mathcal{S} \to [0, 1], S \mapsto A(S), \forall S \in \mathcal{S}$ is called a fuzzy set on $\mathcal{S}$. $A(S)$ is called the *membership degree* of S in $\mathcal{S}$. Let the collection of all the fuzzy sets on $\mathcal{S}$ be $\mathcal{F}(\mathcal{S})$.

Fig. 1 Fuzzy decisions on the power set of a set

symptoms	$\emptyset$	$\{F\}$	$\{H\}$	$\{C\}$	$\{F,H\}$	$\{F,C\}$	$\{H,C\}$	$\{F,H,C\}$
decision	0	0.2	0.1	0.2	0.3	0.5	0.4	0.9

As an special collection of sets, the power set of a set is interesting. So we will concentrate our studies on the fuzzy sets on a power set.

Definition 2 Let X be a set. Then a mapping $A_{\mathcal{P}}$ such that

$$A_{\mathcal{P}} : \mathcal{P}(X) \to [0,1], P \mapsto A_{\mathcal{P}}(P), \forall P \in \mathcal{P}(X)$$

is called a fuzzy set on $\mathcal{P}(X)$. $A_{\mathcal{P}}(P)$ is called the *membership degree* of P in $\mathcal{P}(X)$. Let the collection of all the fuzzy sets on $\mathcal{P}(X)$ be $\mathcal{F}(\mathcal{P}(X))$.

Give a fuzzy set on X, we can deduce a fuzzy set on $\mathcal{P}(X)$ as follows.

Definition 3 Given a fuzzy set A on X, the mapping such that

$$A_{max} : \mathcal{P}(X) \to [0,1], P \mapsto A_{max}(P) = \vee_{x \in P} A(x)$$

is called the *max-type induced fuzzy set* by A. Denote $\mathcal{F}_{max}(\mathcal{P}(X)) = \{A_{max} \mid A \in \mathcal{F}(X)\}$.

Definition 4 Given a fuzzy set A on X, the mapping such that

$$A_{min} : \mathcal{P}(X) \to [0,1], P \mapsto A_{min}(P) = \vee_{x \in P} A(x)$$

is called the *min-type induced fuzzy set* by A. Denote $\mathcal{F}_{min}(\mathcal{P}(X)) = \{A_{min} \mid A \in \mathcal{F}(X)\}$.

Given any fuzzy set A on X, then A induces a relation $\sim$ on X by defining that $x \sim y \Leftrightarrow A(x) = A(y)$. It is easy to see that $\sim$ is an equivalence relation on X. Therefore, there exists the quotient set $X/\sim$. Similarly, A_{max} or A_{min} induces a relation $\sim'$ on $\mathcal{P}(X)$ such that $P_1 \sim' P_2 \Leftrightarrow A_{\mathcal{P}}(x) = A_{\mathcal{P}}(y)$ for any $P_1, P_2 \in \mathcal{P}(X)$. $\sim'$ is an equivalence relation on $\mathcal{P}(X)$ and thus there exists the quotient set $\mathcal{P}(X)/\sim'$.

The order relation of elements in a set is usually defined as $x \leq y \Leftrightarrow A(x) \leq A(y)$. Now we define the order relation of two sets as follows.

Definition 5 Let A be a fuzzy set on X. For any $P_1, P_2 \in \mathcal{P}(X), P_1 \leq P_2 \Leftrightarrow A_{\mathcal{P}}(P_1) \leq A_{\mathcal{P}}(P_2)$.

We further give a more general order relation on $\mathcal{P}(X)$ as follows.

Definition 6 Let $\widetilde{A}, \widetilde{B} \in \mathcal{F}(\mathcal{P}(X))$. If $\forall P \in \mathcal{P}(X), \widetilde{A}(P) \leq \widetilde{B}(P)$, then we say $\widetilde{A} \subseteq \widetilde{B}$. If $\forall P \in \mathcal{P}(X), \widetilde{A}(P) = \widetilde{B}(P)$, then we say that $\widetilde{A} = \widetilde{B}$.

Theorem 1 *Let $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \mathcal{F}(\mathcal{P}(X))$, then we have the follows.*

(1) Self-reflexivity. $\widetilde{A} \subseteq \widetilde{A}$.
(2) Anti-symmetry. $\widetilde{A} \subseteq \widetilde{B}, \widetilde{B} \subseteq \widetilde{A} \Rightarrow \widetilde{A} = \widetilde{B}$.
(3) Transitivity. $\widetilde{A} \subseteq \widetilde{B}, \widetilde{B} \subseteq \widetilde{C} \Rightarrow \widetilde{A} \subseteq \widetilde{C}$.

From Theorem 1 we know that $(\mathcal{F}(\mathcal{P}(X)), \subseteq)$ is a partially ordered set.

3 Generalized Fuzzy Relations

The classical fuzzy set theory defined the relations between two elements as follows. Let X, Y be two classical sets. A mapping $R : X \times Y \to [0, 1]$ is called a *fuzzy relation* [10]. The number $R(x, y) \in [0, 1]$ can be interpreted as the degree of relationship between x and y.

Now we define the fuzzy relation between two sets as follows.

Definition 7 Let X, Y be two classical sets. A mapping $R : \mathcal{P}(X) \times \mathcal{P}(Y) \to [0, 1]$ is called a *fuzzy relation*. The number $R(A, B) \in [0, 1]$ can be interpreted as the degree of relationship between A and B.

Further, we give two types of induced relations between two sets as follows.

Definition 8 Let $R : X \times X \to [0, 1]$ be a fuzzy relation on X. The mapping $\widetilde{R} : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, 1]$ such that $\widetilde{R}(A, B) = \bigvee_{a \in A, b \in B} R(a, b)$ for $A, B \in \mathcal{P}(X)$ is called *the max-type induced fuzzy relation between A and B*.

Definition 9 Let $R : X \times X \to [0, 1]$ be a fuzzy relation on X. The mapping $\widetilde{R} : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, 1]$ such that $\widetilde{R}(A, B) = \bigwedge_{a \in A, b \in B} R(a, b)$ for $A, B \in \mathcal{P}(X)$ is called *the min-type induced fuzzy relation between A and B*.

Theorem 2 *If R is a fuzzy relation on X, then the max-type induced fuzzy relation* $\widetilde{R}$ *on* $\mathcal{P}(X)$ *has the following properties.*

(1) $A \subseteq C, B \subseteq D \Rightarrow \widetilde{R}(A, B) \leq \widetilde{R}(C, D)$*;*
(2) If R is self-reflexive and $A \cap B \neq \emptyset$*, then* $\widetilde{R}(A, B) = 1$*;*
(3) $\widetilde{R}(A, B \cup C) = \widetilde{R}(A, B) \vee \widetilde{R}(A, C)$*;*
(4) $\widetilde{R}(A, B \cap C) \leq \widetilde{R}(A, B) \wedge \widetilde{R}(A, C)$.

Proof (1). By Definition 8, it is easy to see.

(2). Since R is self-reflexive, $R(x, x) = 1$ for any $x \in X$. Choose an element $Y \in A \cap B$. Then by (1) $\widetilde{R}(A, B) \geq \widetilde{R}(\{x\}, \{x\}) = R(x, x) = 1$. So $\widetilde{R}(A, B) = 1$.

(3).

$$\begin{aligned}
\widetilde{R}(A, B \cup C) &= \bigvee_{\substack{a \in A \\ u \in B \cup C}} R(a, u) \\
&= \bigvee_{\substack{a \in A \\ u \in B}} R(a, u) \overset{or}{\bigvee_{\substack{a \in A \\ u \in C}}} R(a, u) \\
&= \left\{ \bigvee_{\substack{a \in A \\ u \in B}} R(a, u) \right\} \vee \left\{ \bigvee_{\substack{a \in A \\ u \in C}} R(a, u) \right\} \\
&= \widetilde{R}(A, B) \vee \widetilde{R}(A, C).
\end{aligned}$$

(4). It is easy to be deduced from (1).

Theorem 3 *If R is a fuzzy relation on X, then the min-type induced fuzzy relation $\widetilde{R}$ on $\mathcal{P}(X)$ has the following properties.*

(1) $A \subseteq C, B \subseteq D \Rightarrow \widetilde{R}(A,B) \geq \widetilde{R}(C,D)$;
(2) *If R is self-reflexive and $A - B \neq \emptyset$ (Res. $B - A \neq \emptyset$), then* $\widetilde{R}(A,B) = \widetilde{R}(A-B,B)$ *(Res.* $\widetilde{R}(A,B) = \widetilde{R}(A,B-A)$*);*
(3) $\widetilde{R}(A,B \cup C) = \widetilde{R}(A,B) \wedge \widetilde{R}(A,C)$;
(4) $\widetilde{R}(A,B \cap C) \geq \widetilde{R}(A,B) \vee \widetilde{R}(A,C)$.

Proof The proofs are similar to those in Theorem 2, and thus we omit.

Now we give two types of definitions of transitivity of fuzzy relations on $\mathcal{P}(X)$, and will provide an equivalence fuzzy relation on $\mathcal{P}(X)$.

Definition 10 A fuzzy relation R on $\mathcal{P}(X)$ is called *I-type transitive*, if $R^2 = R \circ R \subseteq R$, that is, $\forall (U,V) \in \mathcal{P}(X) \times \mathcal{P}(X), \bigvee_{W \in \mathcal{P}(X)} R(U,W) \bigwedge R(W,V) \leq R(U,V)$.

Definition 11 A fuzzy relation R on $\mathcal{P}(X)$ is called *II-type transitive*, if $R^2 = R \circ R \subseteq R$, that is, $\forall (U,V) \in \mathcal{P}(X) \times \mathcal{P}(X), \bigvee_{\{x\} \in \mathcal{P}(X)} R(U,\{x\}) \bigwedge R(\{x\},V) \leq R(U,V)$.

Definition 12 Let $R : X \times X \to [0,1]$ be a fuzzy relation on X. For any two different elements $x, y \in X$, a path from x to y, denoted $P(x,y)$, is a set of continuous pairs $(x,a_1),(a_1,a_2),(a_2,a_3),\ldots,(a_n,y)$, where, each pair is called an *edge* of the path. Let $E(P(x,y))$ be the set of all edges in a path $P(x,y)$. The *degree* of a path $P(x,y)$ is $S(P(x,y)) = \bigwedge_{e \in E(P(x,y))} R(e)$. Suppose that there are l paths $P_1, P_2, \ldots, P_l$ from x to y. Then the *connective degree* between x and y is

$$S(x,y) == \begin{cases} \bigvee_{i=1}^{l} S(P_i), & \text{if } x \neq y; \\ 1, & \text{if } x = y. \end{cases}$$

The set of paths from A to B is $P(A,B) = \{P(a,b) | a \in A, b \in B\}$. The *connective degree* between A and B is $S(A,B) = \bigvee_{a \in A, b \in B} S(a,b)$.

The induced fuzzy relation S on $\mathcal{P}(X)$ has the following properties.

Theorem 4 *If R is a fuzzy relation on X, then the fuzzy relation S on $\mathcal{P}(X)$ has the following properties.*

(1) $A \subseteq C, B \subseteq D \Rightarrow S(A,B) \leq S(C,D)$;
(2) *If $A \cap B \neq \emptyset$, then $S(A,B) = 1$;*
(3) $S(A,B \cup C) = S(A,B) \vee S(A,C)$;
(4) $S(A,B \cap C) \leq S(A,B) \wedge S(A,C)$.

Proof The proof is similar to that of Theorem 2, and thus we omit.

Theorem 5 *If R is a symmetric fuzzy relation on X, then S is an equivalence relation on $\mathcal{P}(X)$, under the meaning of the II-type transitivity.*

Proof By (2) of Theorem 4, $S(A, A) = 1$, and thus S is self-reflexive. Since R is symmetric, it is easy to see that S is symmetric. It remains to prove that S satisfies transitivity. Note that

$$S^2(A, B) = \bigvee_{w \in X} (S(A, \{w\}) \wedge S(\{w\}, B)), \forall A, B \in \mathcal{P}(X).$$

If $A \cap B \neq \emptyset$, then $S(A, B) = 1$, we have $S(A, B) \geq S^2(A, B)$. Now assume that $A \cap B = \emptyset$. Choose an arbitrary element $w \in X$. If $w \in A$, then $S(A, \{w\}) = 1$. It follows that $S(A, B) \geq S(\{w\}, B) = S(A, \{w\}) \wedge S(\{w\}, B)$. If $w \in B$, then $S(\{w\}, B) = 1$. It follows that $S(A, B) \geq S(A, \{w\}) = S(A, \{w\}) \wedge S(\{w\}, B)$. When $w \notin A \cup B$, let P_1 and P_2 are the optimal paths of $(A, \{w\})$ and $(\{w\}, B)$, respectively. Then $P_1 \cup P_2$ contains a path P of (A, B), and thus $E(P) \subseteq E(P_1) \cup E(P_2)$. Therefore, $S(A, B) \geq S(P) \geq S(P_1) \wedge S(P_2) = S(A, \{w\}) \wedge S(\{w\}, B)$. In either case, $S(A, B) \geq S(A, \{w\}) \wedge S(\{w\}, B)$. So $S(A, B) \geq S^2(A, B)$, which means the transitivity of $S(A, B)$.

Acknowledgments This article is completed during the first author being a visiting scholar in Jiangnan University. Research was partially supported by the domestic senior visiting scholar program in higher occupation colleges in Jiangsu province (2014FX075).

The authors are grateful to the referees for their valuable suggestions, which result in the present version of the paper. This paper was also supported by the Natural Science Foundation of Jiangsu Province (No. BK20151117).

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http://www.springer.com/978-3-319-30873-9

International Conference on Oriental Thinking and Fuzzy Logic
Celebration of the 50th Anniversary in the era of Complex Systems and Big Data
Cao, B.-Y.; Wang, P.-Z.; Liu, Z.-L.; Zhong, Y.-B. (Eds.)
2016, XVI, 707 p. 169 illus., 101 illus. in color.,
Softcover
ISBN: 978-3-319-30873-9