

Chapter 2

First Definitions and Basic Properties

The object of this first chapter is the definition of an \mathcal{E}_K^\dagger -valued version of rigid cohomology, $H_{\text{rig}}^i(X/\mathcal{E}_K^\dagger)$, for schemes separated and of finite type over a Laurent series field $k((t))$ in positive characteristic. As explained in the introduction, once we have made the basic definition of what should constitute a ‘good’ frame in this situation, we will closely follow the original constructions of Berthelot. However, since we will need to work with adic spaces rather than Tate’s rigid analytic varieties, we also spend some time reproving some basic analogues of useful results concerning such spaces that we will need in order for many of Berthelot’s original proofs to carry over, as well as recasting Berthelot’s original definitions in the language of adic spaces. This aside, however, there should be nothing here that those familiar with the theory of rigid cohomology will find surprising.

As with ‘classical’ rigid cohomology, the \mathcal{E}_K^\dagger -valued rigid cohomology we construct will be defined as the hypercohomology of a suitable ‘overconvergent de Rham complex’ on a lift of a compactification of the variety in question to characteristic 0, there are therefore choices involved and the main result is that the resulting cohomology groups are independent of these choices. The proof, via the Strong Fibration Theorem (Proposition 2.45), is almost identical to Berthelot’s original proof. The other main result is then the interpretation of the category of ‘lisse’ coefficients for this theory in terms of modules with an overconvergent, integrable connection, exactly as in the classical case, and again, once everything has been set up correctly, the usual proofs carry over entirely straightforwardly.

Throughout this chapter k will be a characteristic p field, \mathcal{V} will be a complete discrete valuation ring with residue field k and fraction field K of characteristic 0. We will let π denote a uniformiser of \mathcal{V} , denote by $|\cdot|$ a norm on K such that $|p| = p^{-1}$, and write $r = |\pi^{-1}| > 1$. For any topological ring A and variables x_1, \dots, x_n we will denote by $A\langle x_1, \dots, x_n \rangle$ the Tate algebra over A in the variables x_i , that is the ring of series

$$\sum_{i_1, \dots, i_n=0}^{\infty} a_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n}$$

such that $a_{i_1, \dots, i_n} \rightarrow 0$ as $i_1 + \dots + i_n \rightarrow \infty$. For any scheme $X/\mathrm{Spec}(\mathcal{V})$, separated and of finite type, we will denote by \widehat{X} its π -adic completion, considered as a formal scheme over $\mathrm{Spf}(\mathcal{V})$.

2.1 Rigid Cohomology and Adic Spaces

Berthelot's theory of rigid cohomology

$$X \mapsto H_{\mathrm{rig}}^i(X/K)$$

is a p -adic cohomology theory for algebraic varieties over k , whose construction we give a detailed review of in Appendix A, and here we very briefly recall. To define $H_{\mathrm{rig}}^i(X/K)$ one first compactifies X into a proper scheme Y/k , and then embeds Y into a formal scheme \mathfrak{P}/\mathcal{V} , which is smooth over \mathcal{V} in a neighbourhood of X . One then considers the generic fibre \mathfrak{P}_K of \mathfrak{P} , which is a rigid analytic variety over K , and one has a specialisation map

$$\mathrm{sp} : \mathfrak{P}_K \rightarrow P$$

where P is the special fibre of \mathfrak{P} , that is its mod- π reduction. Associated to the subschemes $X, Y \subset P$ one has the tubes

$$]X[_{\mathfrak{P}} := \mathrm{sp}^{-1}(X), \quad]Y[_{\mathfrak{P}} = \mathrm{sp}^{-1}(Y),$$

let $j :]X[_{\mathfrak{P}} \rightarrow]Y[_{\mathfrak{P}}$ denote the inclusion. One then takes $j_X^\dagger \Omega_{]Y[_{\mathfrak{P}}}^*$ to be the subsheaf of $j_* \Omega_{]X[_{\mathfrak{P}}}^*$ consisting of *overconvergent* differential forms, that is those that converge on some strict neighbourhood of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$. The rigid cohomology of X is then

$$H_{\mathrm{rig}}^i(X/K) := H^i(]Y[_{\mathfrak{P}}, j_X^\dagger \Omega_{]Y[_{\mathfrak{P}}}^*),$$

this is independent of both Y and \mathfrak{P} . Actually, this does not always work: we may not be able to find a smooth formal \mathcal{V} scheme containing Y , however, we may always do so locally. We may then ‘glue’ together these local cohomology groups to define rigid cohomology in general.

Example 2.1 1. Let $X = \mathbb{A}_k^1 \setminus \{0\}$ be the affine line minus the origin. Then we may take $Y = \mathbb{P}_k^1$ to be the projective line, and $\mathfrak{P} = \widehat{\mathbb{P}}_{\mathcal{V}}^1$ to be formal projective space. Then the tube of Y is the rigid analytic projective line $\mathbb{P}_K^{1, \mathrm{an}}$ and the tube $]X[_{\mathfrak{P}}$ is the annulus

$$\mathbb{A}_K(0, 1, 1) = \left\{ x \in \mathbb{P}_K^{1, \mathrm{an}} \mid |x| = 1 \right\}.$$

A cofinal system of strict neighbourhoods is given by the annuli

$$\mathbb{A}_K(0, \lambda^{-1}, \lambda) = \left\{ x \in \mathbb{P}_K^{1, \text{an}} \mid \lambda^{-1} \leq |x| \leq \lambda \right\}.$$

for $\lambda > 1$, all of which are affinoid, hence we may calculate $H_{\text{rig}}^i(X/K)$ as the cohomology of the complex

$$0 \rightarrow \lim_{\lambda > 1} K \langle \lambda^{-1}t, \lambda t^{-1} \rangle \rightarrow \lim_{\lambda > 1} K \langle \lambda^{-1}t, \lambda t^{-1} \rangle \cdot dt \rightarrow 0$$

This is fairly easily seen to give $H_{\text{rig}}^0(X/K) = H_{\text{rig}}^1(X/K) = K$ and $H_{\text{rig}}^i(X/K) = 0$ for $i \geq 2$.

2. Let $X = Y = \mathbb{P}_K^1$ be the projective line, and take $\mathfrak{Y} = \widehat{\mathbb{P}}_{\mathcal{V}}^2$, with Y embedded in the special fibre P as some line. The inclusion $\mathbb{P}_K^{1, \text{an}} \rightarrow]X[_{\mathfrak{Y}}$, is, locally for the analytic topology on $\mathbb{P}_K^{1, \text{an}}$, isomorphic to $U \rightarrow U \times \mathbb{D}_K^{1, \circ}$, where $\mathbb{D}_K^{1, \circ}$ is the open unit disc. Since this has trivial de Rham cohomology, one easily verifies the isomorphism

$$H^i(]X[_{\mathfrak{Y}}, \Omega_{]X[_{\mathfrak{Y}}/K}^*) \cong H^i(\mathbb{P}_K^{1, \text{an}}, \Omega_{\mathbb{P}_K^{1, \text{an}}/K}^*).$$

In other words, computing the rigid cohomology of \mathbb{P}^1 using the ‘silly’ frame $(\mathbb{P}_K^1, \mathbb{P}_K^1, \widehat{\mathbb{P}}_{\mathcal{V}}^2)$, gives the same answer as using the more sensible frame $(\mathbb{P}_K^1, \mathbb{P}_K^1, \widehat{\mathbb{P}}_{\mathcal{V}}^1)$. Using GAGA, this may be checked to be simply the algebraic de Rham cohomology of \mathbb{P}_K^1 , i.e. K in degrees 0 and 2, and 0 otherwise.

In the theory that we wish to construct, we will want to consider the ‘generic fibres’ of more general formal schemes, namely π -adic formal schemes topologically of finite type over $\mathcal{V}[[t]]$, and as such this falls somewhat outside the scope of Tate’s theory of rigid analytic varieties. Luckily, this is nicely covered by Huber’s theory of adic spaces, or equivalently, Fujiwara–Kato’s theory of Zariski–Riemann spaces (the equivalence of these two perspectives, at least in all the cases we will need in this book, is Theorem II.A.5.2 of [6]). In Appendix B we give a gentle introduction to this theory, and as a warm-up for the rest of the chapter, as well as to ensure ‘compatibility’ of our new theory with traditional rigid cohomology, in this opening section we show that rigid cohomology can be computed using adic spaces.

This is rather straightforward, and is achieved more or less by showing that certain cofinal systems of strict neighbourhoods for $]X[_{\mathfrak{Y}}$ in $]Y[_{\mathfrak{Y}}$ inside the rigid analytic variety \mathfrak{X}_K are also cofinal systems of neighbourhoods of $]X[_{\mathfrak{Y}}$ in $]Y[_{\mathfrak{Y}}$ inside the corresponding adic space. We can then use the fact that corresponding rigid and adic spaces have the same underlying topoi to conclude that the two different constructions of $j_X^{\dagger} \Omega_{]Y[_{\mathfrak{Y}}}^*$ (in the rigid analytic and adic worlds) give *the same* object in the appropriate topos, and hence have the same cohomology.

So let (X, Y, \mathfrak{P}) be a smooth frame, as appearing in Berthelot's construction, that is:

- $X \rightarrow Y$ is an open immersion of k -varieties;
- $Y \rightarrow \mathfrak{P}$ is a closed immersion of formal \mathcal{V} -schemes;
- \mathfrak{P} flat over \mathcal{V} and formally smooth over \mathcal{V} in some neighbourhood of X .

Let P denote the special fibre of \mathfrak{P} , so that there is a homeomorphism of topological spaces $P \simeq \mathfrak{P}$, and let $Z = Y \setminus X$, with some closed subscheme structure.

Note 2.2 A *variety* will always mean a separated scheme of finite type, and formal schemes over \mathcal{V} will always be assumed to be separated, π -adic and topologically of finite type.

In this situation, we will want to consider three different sorts of generic fibre of \mathfrak{P} , rigid, Berkovich, and adic. To describe them, we work locally on \mathfrak{P} , and assume it to be of the form $\mathrm{Spf}(A)$ for some topologically finite type \mathcal{V} -algebra A . For any such A , we let A^+ denote the integral closure of A inside $A_K := A \otimes_{\mathcal{V}} K$.

- The rigid generic fibre $\mathfrak{P}^{\mathrm{rig}}$. This is the set $\mathrm{Sp}(A_K)$ of maximal ideals of A_K , considered as a locally G -ringed space in the usual way (see for example Chap. 4 of [5]). Alternatively, this is the collection of (equivalence classes of) *discrete* continuous valuations $v : A_K \rightarrow \{0\} \cup \mathbb{R}^{>0}$.
- The Berkovich generic fibre $\mathfrak{P}^{\mathrm{Ber}}$. This is the set $\mathcal{M}(A_K)$ of (equivalence classes of) continuous rank 1 valuations $v : A_K \rightarrow \{0\} \cup \mathbb{R}^{>0}$, considered as a topological space as in Chap. 1 of [1].
- The adic generic fibre. This is the set $\mathrm{Spa}(A_K, A^+)$ of (equivalence classes of) continuous valuations $v : A_K \rightarrow \{0\} \cup \Gamma$ into some totally ordered abelian group Γ (possibly of rank > 1), satisfying $v(A^+) \leq 1$. It is considered as a locally ringed space as in [7].

Remark 2.3 Whenever B is a topologically finite type K -algebra, that is a quotient of some Tate algebra $K \langle x_1, \dots, x_n \rangle$, we will also write B^+ for the integral closure of the image of $\mathcal{V} \langle x_1, \dots, x_n \rangle$ inside B , and $\mathrm{Spa}(B)$ instead of $\mathrm{Spa}(B, B^+)$. Note that B^+ does not depend on the choice of presentation of B .

Remark 2.4 It is generally conventional when working with higher rank valuations for them to be written multiplicatively, and we will do so throughout this book. Hence the slightly strange looking definition of the Berkovich space $\mathcal{M}(A_K)$.

There are several relations among these spaces, for example, there is an obvious inclusion $\mathfrak{P}^{\mathrm{Ber}} \rightarrow \mathfrak{P}^{\mathrm{ad}}$ which is *not* continuous, but there *is* a continuous map $[\cdot] : \mathfrak{P}^{\mathrm{ad}} \rightarrow \mathfrak{P}^{\mathrm{Ber}}$ which exhibits $\mathfrak{P}^{\mathrm{Ber}}$ as the maximal separated (Hausdorff) quotient of $\mathfrak{P}^{\mathrm{ad}}$ (as follows from Proposition II.C.1.8 of [6]). There is also an inclusion $\mathfrak{P}^{\mathrm{rig}} \rightarrow \mathfrak{P}^{\mathrm{ad}}$ whose image by definition consists of the subset of *rigid* points, this factors throughout $\mathfrak{P}^{\mathrm{Ber}}$. For x a point of any of these spaces, we will write $v_x(\cdot)$ for the corresponding valuation, note this is compatible with the embeddings $\mathfrak{P}^{\mathrm{rig}} \rightarrow \mathfrak{P}^{\mathrm{Ber}} \rightarrow \mathfrak{P}^{\mathrm{ad}}$ but *not* with the map $[\cdot] : \mathfrak{P}^{\mathrm{ad}} \rightarrow \mathfrak{P}^{\mathrm{Ber}}$.

Example 2.5 Let $\mathfrak{P} = \widehat{\mathbb{A}}_{\mathcal{V}}^1 = \mathrm{Spf}(\mathcal{V}\langle T \rangle)$ be the formal affine line over \mathcal{V} . Then the generic fibre of \mathfrak{P} is the closed unit disc over K , and the points of this space can be described in the three cases as follows.

- The rigid closed unit disc $\mathfrak{P}_K^{\mathrm{rig}} = \mathbb{D}_K^{1,\mathrm{rig}}$ consists of Galois orbits of points $\alpha \in \mathcal{O}_{\overline{K}}$ in the ‘naive’ closed unit disc $\mathcal{O}_{\overline{K}}$ over \overline{K} . The corresponding valuation is the composition of the evaluation map

$$\mathrm{ev}_{\alpha} : K\langle T \rangle \rightarrow \overline{K}$$

with the natural valuation on \overline{K} .

- The points of the Berkovich closed unit disc $\mathfrak{P}_K^{\mathrm{Ber}} = \mathbb{D}_K^{1,\mathrm{Ber}}$ naturally fall into 4 types.
 1. Type I points are exactly the points of the rigid closed unit disc $\mathfrak{P}_K^{\mathrm{rig}}$.
 2. Type II and III points are the Galois orbits of the valuations corresponding to supremum norms on closed discs $\mathbb{D}(a, r) := \{x \in \mathcal{O}_{\overline{K}} \mid v(x - a) \leq r\}$. Type II points correspond to those discs with radius r in the value group $v(\overline{K})$ of \overline{K} , and Type III points to those with radius not in $v(\overline{K})$.
 3. Type IV points occur when $\widehat{\overline{K}}$ is not spherically complete, and are associated to decreasing sequences $D_1 \supset D_2 \supset \dots$ of discs inside $\mathcal{O}_{\overline{K}}$ with $\cap_i D_i = \emptyset$. The valuation is the infimum of the supremum norms on each D_i .
- The adic closed unit $\mathfrak{P}_K^{\mathrm{ad}} = \mathbb{D}_K^{1,\mathrm{ad}}$ consists of all the points of the Berkovich closed unit disc together with a 5th type of point coming from each rank 2 valuation on $K\langle x \rangle$, these are described as follows. Take some $a \in \mathcal{O}_{\overline{K}}$, some real number $r \in v(\overline{K}) \cap \mathbb{R}^{<1}$, and define a totally ordered group $\Gamma = \mathbb{R}^{>0} \times \gamma^{\mathbb{Z}}$ by requiring that $r' < \gamma < r$ for all $r' < r$. Then we obtain a rank 2 valuation $v : K\langle T \rangle \rightarrow \{0\} \cup \Gamma$ by setting

$$v\left(\sum_i a_i (T - a)^i\right) = \sup_i |a_i| \gamma^i.$$

The map $[\cdot] : \mathbb{D}_K^{1,\mathrm{ad}} \rightarrow \mathbb{D}_K^{1,\mathrm{Ber}}$ takes this valuation to the supremum norm on the disc $\mathbb{D}(a, r)$. One can also check that if instead we took $r \notin v(\overline{K})$ then the valuation would be equivalent to the Type III valuation corresponding to the disc $\mathbb{D}(a, r)$.

For $\# \in \{\mathrm{rig}, \mathrm{Ber}, \mathrm{ad}\}$ there are specialisation maps

$$\mathrm{sp} : \mathfrak{P}^{\#} \rightarrow \mathfrak{P} \simeq P$$

which are compatible with the inclusions $\mathfrak{P}^{\mathrm{rig}} \rightarrow \mathfrak{P}^{\mathrm{Ber}} \rightarrow \mathfrak{P}^{\mathrm{ad}}$, but *not* with the quotient map $[\cdot] : \mathfrak{P}^{\mathrm{ad}} \rightarrow \mathfrak{P}^{\mathrm{Ber}}$. The maps $\mathfrak{P}^{\mathrm{rig}} \rightarrow \mathfrak{P}$ and $\mathfrak{P}^{\mathrm{ad}} \rightarrow \mathfrak{P}$ are continuous (for the G -topology on $\mathfrak{P}^{\mathrm{rig}}$), but the map $\mathfrak{P}^{\mathrm{Ber}} \rightarrow \mathfrak{P}$ is anti-continuous, that is the inverse image of an open set is closed and vice versa. For every subset $S \subset A$ of a topological space A , we will let S° denote its interior and \overline{S} its closure.

Definition 2.6 For the closed subvariety $Y \subset P$ define the tubes

$$\begin{aligned}]Y[_{\mathfrak{P}}^{\text{rig}} &:= \text{sp}^{-1}(Y) \subset \mathfrak{P}^{\text{rig}}, \\]Y[_{\mathfrak{P}}^{\text{Ber}} &:= \text{sp}^{-1}(Y) \subset \mathfrak{P}^{\text{Ber}}, \\]Y[_{\mathfrak{P}}^{\text{ad}} &:= \text{sp}^{-1}(Y)^{\circ} \subset \mathfrak{P}^{\text{ad}}, \end{aligned}$$

note the fact that the adic tube is the interior of the ‘naïve’ tube $\text{sp}^{-1}(Y)$. Also note that this definition works for any closed subset of P , in particular we can also talk about the tubes $]Z[_{\mathfrak{P}}^{\#}$.

These tubes can be calculated locally as follows. Suppose that $\mathfrak{P} = \text{Spf}(A)$ is affine, and that $f_1, \dots, f_n \in A$ are functions such that $Y \subset P$ is the vanishing locus of the mod π -reductions \bar{f}_i . Then we have:

$$\begin{aligned}]Y[_{\mathfrak{P}}^{\text{rig}} &= \{x \in \mathfrak{P}^{\text{rig}} \mid v_x(f_i) < 1 \ \forall i\}, \\]Y[_{\mathfrak{P}}^{\text{Ber}} &= \{x \in \mathfrak{P}^{\text{Ber}} \mid v_x(f_i) < 1 \ \forall i\}, \\]Y[_{\mathfrak{P}}^{\text{ad}} &= \{x \in \mathfrak{P}^{\text{ad}} \mid v_{[x]}(f_i) < 1 \ \forall i\}, \end{aligned}$$

again note the difference in the description of the adic tube. Another way of stating the last of these formulae is that $]Y[_{\mathfrak{P}}^{\text{ad}}$ is the inverse image of $]Y[_{\mathfrak{P}}^{\text{Ber}}$ under the map $[\cdot] : \mathfrak{P}^{\text{ad}} \rightarrow \mathfrak{P}^{\text{Ber}}$.

Example 2.7 Let $Y = \text{Spec}(k)$, $\mathfrak{P} = \widehat{\mathbb{A}}_{\mathcal{V}}^1 = \text{Spf}(\mathcal{V}\langle T \rangle)$, with Y embedded in P via the zero section. For $\# \in \{\text{rig}, \text{Ber}\}$, we note that for rank 1 valuations the condition $v(T) < 1$ is equivalent to the condition $v(T) \leq r^{-1/n}$ for some n (recall that $r = |\pi|^{-1}$), and hence $]Y[_{\mathfrak{P}}^{\#}$ is covered by the closed discs

$$\left\{ x \in \mathbb{D}_K^{1,\#} \mid v_x(T) \leq r^{-1/n} \right\}$$

of radius < 1 . In other words, $]Y[_{\mathfrak{P}}^{\text{rig}}$ and $]Y[_{\mathfrak{P}}^{\text{Ber}}$ are the rigid and Berkovich open unit discs respectively.

Calculating $]Y[_{\mathfrak{P}}^{\text{ad}}$ requires a little bit more care. The key point is to show that if we take a possibly higher rank valuation v_x whose maximal generisation $v_{[x]}$ satisfies $v_{[x]}(T) < 1$, then there exists some n such that $v(T) \leq r^{-1/n}$ (see Lemma 2.9 below), hence again $]Y[_{\mathfrak{P}}^{\text{ad}}$ is the union of all closed discs of radius < 1 . Notice that this is *not* the same as the naïve ‘open unit disc’

$$\{x \in \mathbb{D}_K^1 \mid v_x(T) < 1\}$$

since for higher rank valuations v it is not true that $v(T) < 1 \Rightarrow v(T) \leq r^{-1/n}$ for some n . This latter space is the literal inverse image of $Y \subset P$ under the specialisation map, is a closed subspace of \mathbb{D}_K^1 and does *not* have the structure of an adic space

over K in general. Its interior is the ‘true’ adic open unit disc

$$\{x \in \mathbb{D}_K^1 \mid v_x(T) \leq r^{-1/n} \text{ for some } n\}.$$

Generalising this example, we have the following.

Lemma 2.8 *In the above situation we have the (admissible) affinoid coverings:*

$$]Y[_{\mathfrak{P}}^{\text{ad}} = \bigcup_{n \geq 1}]Y[_n^{\text{ad}},$$

where

$$]Y[_n^{\text{ad}} = \{x \in \mathfrak{P}^{\text{ad}} \mid v_x(\pi^{-1} f_i^n) \leq 1 \ \forall i\},$$

and

$$]Y[_{\mathfrak{P}}^{\text{rig}} = \bigcup_{n \geq 1}]Y[_n^{\text{rig}},$$

where

$$]Y[_n^{\text{rig}} = \{x \in \mathfrak{P}^{\text{rig}} \mid v_x(\pi^{-1} f_i^n) \leq 1 \ \forall i\}.$$

Proof Note that each $]Y[_n^{\text{rig}} = \text{Sp}(A_K \langle T \rangle / (\pi T - f^n)) = \{x \in \mathfrak{P}^{\text{rig}} \mid v_x(f) \leq r^{-1/n}\}$ is affinoid, and these form an admissible cover of $]Y[_{\mathfrak{P}}^{\text{rig}}$ by Proposition 1.1.9 of [3].

In the adic case, note that each $]Y[_n^{\text{ad}} = \text{Spa}(A_K \langle T \rangle / (\pi T - f^n))$ is an affinoid open subspace of \mathfrak{P}^{ad} , we must show that they form an open cover of $]Y[_{\mathfrak{P}}^{\text{ad}}$. Firstly, note that $v_x(\pi^{-1} f^n) \leq 1 \Rightarrow v_{[x]}(\pi^{-1} f^n) \leq 1 \Rightarrow v_{[x]}(f) \leq r^{-1/n} < 1$, the first implication following from Lemma 2.9 below and the second using the fact that $v_{[x]}$ is a valuation of rank 1, so can be viewed as a multiplicative map into $\mathbb{R}^{\geq 0}$. Hence $]Y[_n^{\text{ad}} \subset]Y[_{\mathfrak{P}}^{\text{ad}}$ for each n .

Now suppose that $x \in]Y[_{\mathfrak{P}}^{\text{ad}}$, that is $v_{[x]}(f_i) < 1$. Then there exists some n such that $v_{[x]}(\pi^{-1} f_i^n) < 1$. Hence again by Lemma 2.9 below we must have $v_x(\pi^{-1} f_i^n) \leq 1$ and hence $x \in]Y[_n^{\text{ad}}$ for some n . \square

Lemma 2.9 *Let $\mathcal{X} = \text{Spa}(B)$ be an affinoid adic space for some B topologically of finite type over K . Then for any point $x \in \mathcal{X}$ and any $f \in B$ we have*

$$\begin{aligned} v_x(f) \leq 1 &\Rightarrow v_{[x]}(f) \leq 1 \\ v_{[x]}(f) < 1 &\Rightarrow v_x(f) < 1. \end{aligned}$$

Proof Let $I \subset B$ denote the support of the valuation v_x corresponding to x , that is the ideal of elements with valuation 0. Let V_x be the valuation ring of the induced valuation $v : \text{Frac}(B/I) \rightarrow \{0\} \cup \Gamma$, and let $P_x \subset V_x$ denote the prime ideal of

elements whose valuation is < 1 . The radical $\mathfrak{p} := \sqrt{(\pi)} \subset P_x$ of (π) is a height one prime ideal of V_x to which we may associate a rank one valuation $v_{\mathfrak{p}} : \text{Frac}(B/I) \rightarrow \{0\} \cup \Gamma'$, which is the valuation associated to $[x]$ (see for Example II.3.3.(b) of [6]). Since $\mathfrak{p} \subset P_x$ it follows that $v(\lambda) \leq 1 \Rightarrow v_{\mathfrak{p}}(\lambda) \leq 1$ for all $\lambda \in B/I$, which proves the first claim. For the second, note that we may assume that $v_x(f) \neq 0$, that is $f \notin I$, and hence v_x and $v_{[x]}$ both extend uniquely to valuations on $B\langle f^{-1} \rangle$. To obtain the second claim we now just simply apply the first to f^{-1} . \square

In particular, $]Y[_{\mathfrak{P}}^{\text{ad}}$ is an adic space locally of finite type over $\text{Spa}(K)$. In II.B of [6], Fujiwara and Kato construct an equivalence

$$\mathcal{X} \mapsto \mathcal{X}_0$$

from the category of adic spaces locally of finite type over $\text{Spa}(K)$ to rigid analytic varieties over $\text{Sp}(K)$, which is such that $\text{Spa}(B)_0 = \text{Sp}(B)$ for any affinoid algebra B , and such that $(\mathfrak{P}^{\text{ad}})_0 = \mathfrak{P}^{\text{rig}}$ for any formal scheme \mathfrak{P} of the type considered above. The previous lemma allows us to deduce the same result for the tube $]Y[_{\mathfrak{P}}$.

Corollary 2.10 *There is an isomorphism*

$$(\mathcal{Y}[_{\mathfrak{P}}^{\text{ad}})_0 \cong \mathcal{Y}[_{\mathfrak{P}}^{\text{rig}}$$

as rigid analytic varieties over $\text{Sp}(K)$.

This should convince any doubtful reader that the definition above of $\mathcal{Y}[_{\mathfrak{P}}^{\text{ad}}$ is indeed the correct one. Note that in all cases $\# \in \{\text{rig}, \text{Ber}, \text{ad}\}$, the specialisation map gives rise to a map

$$\mathcal{Y}[_{\mathfrak{P}}^{\#} \rightarrow Y.$$

Definition 2.11 For the open subvariety $X \subset Y$ define the tubes

$$\begin{aligned} \mathcal{X}[_{\mathfrak{P}}^{\text{rig}} &:= \text{sp}^{-1}(X) \subset \mathcal{Y}[_{\mathfrak{P}}^{\text{rig}}, \\ \mathcal{X}[_{\mathfrak{P}}^{\text{Ber}} &:= \text{sp}^{-1}(X) \subset \mathcal{Y}[_{\mathfrak{P}}^{\text{Ber}}, \\ \mathcal{X}[_{\mathfrak{P}}^{\text{ad}} &:= \overline{\text{sp}^{-1}(X)} \subset \mathcal{Y}[_{\mathfrak{P}}^{\text{ad}}, \end{aligned}$$

again note the fact that the adic tube is the closure of the ‘naïve’ tube $\text{sp}^{-1}(X)$.

As before, these tubes can be calculated locally as follows. Suppose that $\mathfrak{P} = \text{Spf}(A)$ is affine, and suppose that $g_1, g_2, \dots, g_m \in A$ are functions such that $X = Y \cap (\cup_j D(\bar{g}_j))$, where \bar{g}_j the reduction of g_j . Then we have:

$$\begin{aligned} \mathcal{X}[_{\mathfrak{P}}^{\text{rig}} &= \left\{ x \in \mathcal{Y}[_{\mathfrak{P}}^{\text{rig}} \mid \exists j \text{ s.t. } v_x(g_j) \geq 1 \right\} \\ \mathcal{X}[_{\mathfrak{P}}^{\text{Ber}} &= \left\{ x \in \mathcal{Y}[_{\mathfrak{P}}^{\text{Ber}} \mid \exists j \text{ s.t. } v_x(g_j) \geq 1 \right\} \\ \mathcal{X}[_{\mathfrak{P}}^{\text{ad}} &= \left\{ x \in \mathcal{Y}[_{\mathfrak{P}}^{\text{ad}} \mid \exists j \text{ s.t. } v_{[x]}(g_j) \geq 1 \right\} \end{aligned}$$

and again note that $]X[_{\mathfrak{P}}^{\text{ad}}$ is the inverse image of $]X[_{\mathfrak{P}}^{\text{Ber}}$ under the map $[\cdot] : \mathfrak{P}^{\text{ad}} \rightarrow \mathfrak{P}^{\text{Ber}}$. For $\# \in \{\text{rig}, \text{Ber}, \text{ad}\}$ denote by

$$j :]X[_{\mathfrak{P}}^{\#} \rightarrow]Y[_{\mathfrak{P}}^{\#}$$

the canonical inclusion, note that for $\# = \text{Ber}, \text{ad}$ this is the inclusion of a *closed* subset, but for $\# = \text{rig}$ this is an *open* immersion. For $\# = \text{Ber}, \text{ad}$ and a sheaf \mathcal{F} on $]Y[_{\mathfrak{P}}^{\#}$ we define $j_X^{\dagger} \mathcal{F} := j_* j^{-1} \mathcal{F}$, however, the definition in the rigid case is slightly more involved.

Definition 2.12 A strict neighbourhood of $]X[_{\mathfrak{P}}^{\text{rig}}$ in $]Y[_{\mathfrak{P}}^{\text{rig}}$ is an open subset $V \subset]Y[_{\mathfrak{P}}^{\text{rig}}$ such that $]Y[_{\mathfrak{P}}^{\text{rig}} = V \cup]Z[_{\mathfrak{P}}^{\text{rig}}$ is an admissible open covering, where recall that Z is the complement of X in Y . For any such V , we let $j_V : V \rightarrow]Y[_{\mathfrak{P}}^{\text{rig}}$ be the canonical open immersion.

For a sheaf \mathcal{F} on $]Y[_{\mathfrak{P}}^{\text{rig}}$, define $j_X^{\dagger} \mathcal{F} = \text{colim}_V j_{V*} j_V^{-1} \mathcal{F}$, where the colimit is taken over all strict neighbourhoods V of $]X[_{\mathfrak{P}}^{\text{rig}}$ in $]Y[_{\mathfrak{P}}^{\text{rig}}$.

Definition 2.13 We say that the frame (X, Y, \mathfrak{P}) is proper if Y is proper over k . The rigid cohomology of a variety X/k is defined to be

$$H_{\text{rig}}^i(X/K) := H^i(]Y[_{\mathfrak{P}}^{\text{rig}}, j_X^{\dagger} \Omega_{]Y[_{\mathfrak{P}}^{\text{rig}}}^*)$$

whenever (X, Y, \mathfrak{P}) is a smooth and proper frame, this does not depend on the choice of such a frame.

We want to show that we can compute this instead as

$$H^i(]Y[_{\mathfrak{P}}^{\text{ad}}, j_X^{\dagger} \Omega_{]Y[_{\mathfrak{P}}^{\text{ad}}}^*) = H^i(]X[_{\mathfrak{P}}^{\text{ad}}, j^{-1} \Omega_{]Y[_{\mathfrak{P}}^{\text{ad}}}^*),$$

the equality following from the fact that j is a closed immersion, and therefore j_* is exact.

In order to do this we must first recall Berthelot's construction of a cofinal system of strict neighbourhoods from Sect. 1.2 of [3]. For \mathfrak{P} affine we have constructed affinoids $]Y]_n^{\text{rig}}$ and $]Y]_n^{\text{ad}}$, which depended on the choice of functions $f_i \in \mathcal{O}_{\mathfrak{P}}$ cutting out Y in the special fibre P .

Lemma 2.14 *For $n \gg 0$ the affinoids $]Y]_n^{\text{rig}}$ and $]Y]_n^{\text{ad}}$ are independent of the choice of the f_i . Hence they glue over an open affine covering of \mathfrak{P} .*

Proof For $]Y]_n^{\text{rig}}$ this is proved in 1.1.8 of [3], and since $(]Y]_n^{\text{ad}})_0 \cong]Y]_n^{\text{rig}}$ the claim for adic spaces then follows. \square

As noted in the proof of the lemma, we have $(]Y]_n^{\text{ad}})_0 \cong]Y]_n^{\text{rig}}$, and these are the same as the closed tubes $[Y]_{r^{-1/n}}$ of radius $r^{-1/n}$, constructed by Berthelot in [3]. We also have

$$]Y[_{\mathfrak{P}}^{\#} = \bigcup_{n \gg 0}]Y[_n^{\#}$$

and this is an admissible covering if $\# = \text{rig}$. Similarly, when \mathfrak{P} is affine and we have f_i, g_j as above, so that $Y = \cap_i Z(\bar{f}_i)$ and $X = Y \cap (\cup_j D(\bar{g}_j))$ we can define

$$\begin{aligned} U_m^{\text{rig}} &= \left\{ x \in]Y[_{\mathfrak{P}}^{\text{rig}} \mid \exists j \text{ s.t. } v_x(\pi^{-1} g_j^m) \geq 1 \right\}, \\ U_m^{\text{ad}} &= \left\{ x \in]Y[_{\mathfrak{P}}^{\text{rig}} \mid \exists j \text{ s.t. } v_x(\pi^{-1} g_j^m) \geq 1 \right\}, \end{aligned}$$

as well as

$$\begin{aligned} U_{m,j}^{\text{rig}} &= \left\{ x \in]Y[_{\mathfrak{P}}^{\text{rig}} \mid v_x(\pi^{-1} g_j^m) \geq 1 \right\}, \\ U_{m,j}^{\text{ad}} &= \left\{ x \in]Y[_{\mathfrak{P}}^{\text{rig}} \mid v_x(\pi^{-1} g_j^m) \geq 1 \right\}, \end{aligned}$$

so that $U_m^{\#} = \cup_j U_{m,j}^{\#}$, and this is an admissible open covering when $\# = \text{rig}$. As before, for $m \gg 0$ these are independent of the choice of the g_j and hence glue over an open affine covering of \mathfrak{P} . Finally we set

$$\begin{aligned} V_{n,m}^{\#} &=]Y[_n^{\#} \cap U_m^{\#} \\ V_{n,m,j}^{\#} &=]Y[_n^{\#} \cap U_{m,j}^{\#} \end{aligned}$$

so that $V_{n,m}^{\#} = \cup_j V_{n,m,j}^{\#}$ and this covering is admissible when $\# = \text{rig}$.

Lemma 2.15 *When \mathfrak{P} is affine, the $V_{n,m,j}^{\#}$ are affinoid, and $(V_{n,m,j}^{\text{ad}})_0 \cong V_{n,m,j}^{\text{rig}}$.*

Proof Suppose that $\mathfrak{P} \cong \text{Spf}(A)$ is affine, and choose f_i, g_j as above. Then

$$\begin{aligned} V_{n,m,j}^{\text{ad}} &\cong \text{Spa} \left(\frac{A_K \langle T_1, \dots, T_n, S \rangle}{(\pi T_i - f_i, \pi - g_j^m S)} \right) \\ V_{n,m,j}^{\text{rig}} &\cong \text{Sp} \left(\frac{A_K \langle T_1, \dots, T_n, S \rangle}{(\pi T_i - f_i, \pi - g_j^m S)} \right) \end{aligned}$$

and the lemma follows. □

Corollary 2.16 *For all frames (X, Y, \mathfrak{P}) , we have $(V_{n,m}^{\text{ad}})_0 \cong V_{n,m}^{\text{rig}}$.*

Now, for any increasing sequence of integers $\underline{m}(n) \rightarrow \infty$, we let

$$V_{\underline{m}}^{\#} = \bigcup_n V_{n,\underline{m}(n)}^{\#}.$$

The previous corollary tells us that $(V_{\underline{m}}^{\text{ad}})_0 \cong V_{\underline{m}}^{\text{rig}}$, and it is proved in 1.2.4 of [3] that the $V_{\underline{m}}^{\text{rig}}$ for varying \underline{m} form a cofinal system of strict neighbourhoods of $]X[_{\mathfrak{P}}^{\text{rig}}$ inside $]Y[_{\mathfrak{P}}^{\text{rig}}$. In order to show that the same is true in the adic world, we need the following lemma.

Lemma 2.17 *Let $\mathcal{X} = \text{Spa}(B)$ be an affinoid adic space of finite type over K . Let $V \subset \mathcal{X}$ be an open subset, and $g \in B$ such that*

$$V \supset \{x \in \mathcal{X} \mid v_{[x]}(g) \geq 1\}.$$

Then there exists some m such that

$$V \supset \{x \in \mathcal{X} \mid v_x(\pi^{-1}g^m) \geq 1\}.$$

Proof Let $T = \mathcal{X} \setminus V$ denote the complement of V , this is a quasi-compact topological space. We consider the continuous function

$$\|g\| : \mathcal{X} \rightarrow \mathbb{R}^{\geq 0}$$

defined as follows. For any point $x \in [\mathcal{X}]$ normalising the associated rank 1 valuation on B so that it restricts to the given norm on K defines a norm on the completed residue field $|\cdot| : \mathcal{H}(x) \rightarrow \mathbb{R}^{\geq 0}$ at x . Now if $x \in \mathcal{X}$ we set $\|g\|(x) = |g([x])|$ where $g([x])$ denotes the image of g in $\mathcal{H}([x])$ in the usual way.

This induces a continuous function

$$\|g\| : T \rightarrow \mathbb{R}^{\geq 0}$$

and by assumption $\|g\|(T) \subset [0, 1)$. But since T is quasi-compact, so must its image under $\|g\|$ be, and hence $\|g\|(T) \subset [0, \eta]$ for some $\eta < 1$. Hence

$$T \subset \{x \in \mathcal{X} \mid v_{[x]}(g) \leq \eta\}$$

and by Lemma 2.9, there exists some m such that

$$T \subset \{x \in \mathcal{X} \mid v_x(\pi^{-1}g^m) \leq 1\}.$$

The claim follows. □

Proposition 2.18 *As \underline{m} varies, the $V_{\underline{m}}^{\text{ad}}$ form a cofinal system of open neighbourhoods of $]X[_{\mathfrak{P}}^{\text{ad}}$ inside $]Y[_{\mathfrak{P}}^{\text{ad}}$.*

Proof Let $V \subset]Y[_{\mathfrak{P}}$ be an open subset containing $]X[_{\mathfrak{P}}$. It suffices to show that for all n there exists some m such that

$$[Y]_n \cap V \supset [Y]_n \cap U_m.$$

Since the $[Y]_n$ are quasi-compact and glue over an open affine covering of \mathfrak{P} , we may assume that \mathfrak{P} is affine, and hence the $[Y]_n$ are affinoid. Let $g_j \in \mathcal{O}_{\mathfrak{P}}$ be functions whose reductions \bar{g}_j satisfy $X = Y \cap (\cup_j D(\bar{g}_j))$, so that

$$\begin{aligned} [Y]_n \cap U_m &= \cup_j U_{m,j} \\ [Y]_n \cap U_{m,j} &= \{x \in [Y]_n \mid v_x(\pi^{-1}g_j^m) \geq 1\}. \end{aligned}$$

It thus suffices to show that for all j , there exists m such that $V \cap [Y]_n \supset [Y]_n \cap U_{m,j}$. But this is exactly the content of Lemma 2.17 above. \square

Before we prove the fundamental result of this section, Proposition 2.20, we need the following topological lemma.

Lemma 2.19 *Let $i : T \rightarrow V$ be the inclusion of a closed subspace T of a topological space V . Suppose that there exists a basis \mathcal{B} of open subsets of V such that for every $W \in \mathcal{B}$ and every open subset U' of V containing $T \cap W$, there exists an open neighbourhood U of T in V such that $U \cap W \subset U'$. Then for any sheaf \mathcal{F} on V there exists an isomorphism*

$$i_* i^{-1} \mathcal{F} \cong \operatorname{colim}_{U \supset T} j_{U*} j_U^{-1} \mathcal{F}$$

where the colimit runs over all open neighbourhoods U of T in V , and $j_U : U \rightarrow V$ denotes the corresponding inclusion.

Proof Note that by general nonsense, i^{-1} commutes with sheafification, we claim that the same is actually true for i_* . Indeed, for any presheaf \mathcal{G} there is a natural morphism

$$(i_* \mathcal{G})^a \rightarrow i_*(\mathcal{G}^a)$$

where $(-)^a$ denotes sheafification. To check that it is an isomorphism, we can check on stalks. For any point $x \notin T$, the stalks of both sides at x are 0, and for any point $x \in T$, the stalks of both sides at x is just the stalk \mathcal{G}_x .

It thus follows that $i_* i^{-1} \mathcal{F}$ is the sheafification of the presheaf

$$W \mapsto \operatorname{colim}_{U' \supset T \cap W} \Gamma(U', \mathcal{F}).$$

Since sheafification preserves colimits, it follows that $\operatorname{colim}_{U \supset T} j_{U*} j_U^{-1} \mathcal{F}$ is the sheafification of the presheaf

$$W \mapsto \operatorname{colim}_{U \supset T} \Gamma(U \cap W, \mathcal{F}).$$

thus there is a natural map

$$\operatorname{colim}_{U \supset T} j_{U*} j_U^{-1} \mathcal{F} \rightarrow i_* i^{-1} \mathcal{F}$$

which is induced by

$$\begin{aligned} \{U \supset T\} &\rightarrow \{U' \supset W \cap T\} \\ U &\mapsto U \cap W. \end{aligned}$$

The condition in the statement of the lemma is exactly that this is a cofinal map of directed sets for a basis for the topology of V . \square

Recall that we have an equivalence $(-)_0$ from the category of adic spaces locally of finite type over K to that of rigid analytic varieties over K . This functor is such that there is a canonical equivalence of topoi between \mathcal{X} with its usual topology and \mathcal{X}_0 with the G -topology. We will also use $(-)_0$ to denote the corresponding functor for sheaves, i.e. if \mathcal{F} is a sheaf on \mathcal{X} we will denote by \mathcal{F}_0 the associated sheaf on \mathcal{X}_0 .

Proposition 2.20 *Under the equivalence $(-)_0 : (Y[\mathfrak{p}^{\text{ad}}], \mathcal{O}_{Y[\mathfrak{p}^{\text{ad}}]}) \cong (Y[\mathfrak{p}^{\text{rig}}], \mathcal{O}_{Y[\mathfrak{p}^{\text{rig}}]})$ of ringed topoi induced by Corollary 2.10 and II.B.2(e) of [6], we have an isomorphism*

$$(j_X^\dagger \mathcal{F})_0 \cong j_X^\dagger (\mathcal{F}_0)$$

for any $\mathcal{O}_{Y[\mathfrak{p}^{\text{ad}}]}$ -module \mathcal{F} .

Proof First note that Proposition 2.18 also holds when we restrict to an affinoid subset of \mathfrak{p}^{ad} , and hence the conditions of Lemma 2.19 are met for the inclusion $]X[\mathfrak{p}^{\text{ad}}] \rightarrow]Y[\mathfrak{p}^{\text{ad}}]$, and we have

$$j_X^\dagger \mathcal{F} \cong \text{colim}_{\underline{m}} j_{m*} j_m^{-1} \mathcal{F}$$

where $j_{\underline{m}} : V_{\underline{m}}^{\text{ad}} \rightarrow]Y[\mathfrak{p}^{\text{ad}}]$ denotes the inclusion. The functor $(-)_0$ commutes with push-forward and pullback, and hence by 1.2.4 of [3], which proves an analogue of Proposition 2.18 in the rigid world, we have

$$\begin{aligned} (j_X^\dagger \mathcal{F})_0 &\cong (\text{colim}_{\underline{m}} j_{m*} j_m^{-1} \mathcal{F})_0 \\ &\cong \text{colim}_{\underline{m}} j_{m0*} j_{m0}^{-1} \mathcal{F}_0 \\ &\cong j_X^\dagger (\mathcal{F}_0) \end{aligned}$$

as required. \square

Corollary 2.21 *There is an isomorphism*

$$H^i(]Y[\mathfrak{p}^{\text{ad}}], j_X^\dagger \Omega_{]Y[\mathfrak{p}^{\text{ad}}]/K}^*) \cong H^i(]Y[\mathfrak{p}^{\text{rig}}], j_X^\dagger \Omega_{]Y[\mathfrak{p}^{\text{rig}}]/K}^*).$$

Proof This follows from the previous corollary together with the fact that there is an isomorphism $(\Omega_{]Y[\mathfrak{p}^{\text{ad}}]/K}^*)_0 \cong \Omega_{]Y[\mathfrak{p}^{\text{rig}}]/K}^*$. \square

2.2 Rigid Cohomology over Laurent Series Fields

Let $k, \mathcal{V}, K, \pi, r$ be as in the previous section. As discussed in the introduction, if we take our ground field to be $k((t))$, the Laurent series field over k , then rigid cohomology is a functor

$$X \mapsto H_{\text{rig}}^*(X/\mathcal{O}_K)$$

taking values in graded vector spaces over the Amice ring

$$\mathcal{O}_K = \left\{ \sum_i a_i t^i \in K[[t, t^{-1}]] \mid \sup_i |a_i| < \infty, a_i \rightarrow 0 \text{ as } i \rightarrow -\infty \right\}.$$

Again, as we said there, if we are to obtain a theory

$$X \mapsto H_{\text{rig}}^i(X/\mathcal{O}_K^\dagger)$$

taking values in the bounded Robba ring

$$\mathcal{O}_K^\dagger = \left\{ \sum_i a_i t^i \in \mathcal{O}_K \mid \exists \eta < 1 \text{ s.t. } |a_i| \eta^i \rightarrow 0 \text{ as } i \rightarrow -\infty \right\}$$

then we need to take into account overconvergence conditions along $t = 0$, or, in other words, we should compactify our varieties over $k[[t]]$ rather than over $k((t))$. This leads to the following definition.

Definition 2.22 A frame over $\mathcal{V}[[t]]$ is a triple (X, Y, \mathfrak{P}) where $X \rightarrow Y$ is an open immersion of a $k((t))$ -variety X into a separated, $k[[t]]$ -scheme Y of finite type, and $Y \rightarrow \mathfrak{P}$ is a closed immersion of Y into a separated, topologically finite type, π -adic formal $\mathcal{V}[[t]]$ -scheme. We say that a frame is proper if Y is proper over $k[[t]]$ and smooth if \mathfrak{P} is smooth over $\mathcal{V}[[t]]$ in a neighbourhood of X . A morphism of frames is simply a commutative diagram

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & \mathfrak{P}' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & \mathfrak{P} \end{array}$$

and is said to be proper if $Y' \rightarrow Y$ is, and smooth if $\mathfrak{P}' \rightarrow \mathfrak{P}$ is in a neighbourhood of X . We say that a $k((t))$ -variety X is embeddable if there exists a smooth and proper frame of the form (X, Y, \mathfrak{P}) .

Example 2.23 1. Two extremely important examples will be the frames

$$(\mathbb{A}_{k((t))}^1, \mathbb{P}_{k[[t]]}^1, \widehat{\mathbb{P}}_{\mathcal{V}[[t]]}^1)$$

and

$$(\mathrm{Spec}(k((t))), \mathrm{Spec}(k[[t]]), \widehat{\mathbb{A}}_{\mathcal{V}[[t]]}^n)$$

for $n \geq 0$.

2. If we let $\mathcal{O}_{\mathcal{E}_K}$ denote the ring of integers of \mathcal{E}_K , then this is a complete DVR with residue field $k((t))$ which is topologically of finite type and formally smooth over $\mathcal{V}[[t]]$. Hence we may consider a smooth frame $(X, \overline{X}, \mathfrak{X})$ over $\mathcal{O}_{\mathcal{E}_K}$ in the classical sense (see the previous section) as a particular example of a smooth frame over $\mathcal{V}[[t]]$.

Since we will now almost exclusively use adic (rather than rigid analytic or Berkovich spaces), we will henceforth (unless otherwise mentioned) use the word rigid space to mean an adic space locally of finite type over the bounded open unit disc $\mathbb{D}_K^b := \mathrm{Spa}(S_K, \mathcal{V}[[t]])$ in the sense of (1.1.2) and Definition 1.2.1 of [8]. Equivalently, using Theorem II.A.5.2. of [6] this is the same thing as a rigid space locally of finite type over $\mathrm{Spf}(\mathcal{V}[[t]])^{\mathrm{rig}}$ in the sense of Definitions II.2.2.18 and II.2.3.1 of [6]. We will usually identify these two kinds of rigid spaces, and use the same letter \mathcal{X} to denote both an adic space in the sense of Huber (locally of finite type over \mathbb{D}_K^b) and the associated rigid space in the sense of Fujiwara–Kato.

For any topologically finite type S_K -algebra B , we will denote by B^+ the integral closure of the image of $\mathcal{V}[[t]] \langle x_1, \dots, x_n \rangle$ in B under any presentation $S_K \langle x_1, \dots, x_n \rangle \rightarrow B$, this does not depend on the choice of presentation, and we will denote $\mathrm{Spa}(B, B^+) \cong \mathrm{Spf}(B^+)^{\mathrm{rig}}$ by $\mathrm{Spa}(B)$. If $B = A_K := A \otimes_{\mathcal{V}} K$ for some topologically finite type $\mathcal{V}[[t]]$ -algebra A , we will also write A^+ for the integral closure of A inside B , thus $A^+ = B^+$.

Remark 2.24 It is worth noting that since $\mathcal{V}[[t]]$ is Noetherian, we satisfy the hypothesis (1.1.1) of [8] as well as being in the ‘t.u. rigid Noetherian’ case of [6]. Note that by Proposition A.2.2 of [6] this implies that S_K is strongly Noetherian, i.e. the Tate algebras $S_K \langle x_1, \dots, x_n \rangle$ are all Noetherian.

If (X, Y, \mathfrak{P}) is a frame then we will let $\mathfrak{P}_K = \mathfrak{P}^{\mathrm{rig}}$ denote the generic fibre of \mathfrak{P} , this is a rigid space of finite type over $\mathbb{D}_K^b := \mathrm{Spa}(S_K, \mathcal{V}[[t]])$. We will also let P denote the special fibre of \mathfrak{P} , so that there is a homeomorphism $P \simeq \mathfrak{P}$. Then there is a specialisation map

$$\mathrm{sp} : \mathfrak{P}_K \rightarrow \mathfrak{P} \simeq P$$

as in Sect.II.3.1 of [6], which locally on \mathfrak{P} can be described as follows. If $\mathfrak{P} = \mathrm{Spf}(A)$, then points of \mathfrak{P}_K can be identified with certain valuations on A_K , and points of $\mathrm{Spf}(A)$ with open prime ideals of A . The specialisation map sends $v : A_K \rightarrow \{0\} \cup \Gamma$ to the prime ideal consisting of elements $a \in A$ such that $v(a) < 1$.

Let $[\mathfrak{P}_K] \subset \mathfrak{P}_K$ denote the subset of points whose corresponding valuation is of rank 1, by II.2.3.(c) and Proposition II.4.1.7 of [6] there is a map

$$[\cdot] : \mathfrak{P}_K \rightarrow [\mathfrak{P}_K]$$

which takes a point to its maximal generisation (see also Appendix B). The set $[\mathfrak{P}_K]$ is topologised via this quotient map, with respect to this topology it is Hausdorff, and $[\cdot]$ identifies $[\mathfrak{P}_K]$ with the maximal Hausdorff quotient of \mathfrak{P}_K (Proposition II.2.3.9 of [6]). With respect to this topology, the inclusion $[\mathfrak{P}_K] \rightarrow \mathfrak{P}_K$ is *not* continuous in general. This ‘maximal generisation’ map can be described locally exactly as in the proof of Lemma 2.9.

If $Z \subset P$ a closed subset, we define

$$]Z[_{\mathfrak{P}} = \text{sp}^{-1}(Z)^{\circ}.$$

to be interior of the inverse image of Z by the specialisation map. Exactly as in the previous section, if $\text{sp}_{[\cdot]} : [\mathfrak{P}_K] \rightarrow P$ denotes the induced specialisation map on the subset of rank 1 points, then we have $]Z[_{\mathfrak{P}} = [\cdot]^{-1}(\text{sp}_{[\cdot]}^{-1}(Z))$, and if locally we have $f_i \in \mathcal{O}_{\mathfrak{P}}$ whose reductions \bar{f}_i define Z inside P , then

$$]Z[_{\mathfrak{P}} = \{x \in \mathfrak{P}_K \mid v_{[x]}(f_i) < 1 \ \forall i\}$$

(Proposition II.4.2.11 of [6]). Specialisation induces a continuous map $\text{sp}_Z :]Z[_{\mathfrak{P}} \rightarrow Z$. If $U \subset Z$ is open, then we set

$$]U[_{\mathfrak{P}} := \overline{\text{sp}_Z^{-1}(U)}.$$

Again, we have $]U[_{\mathfrak{P}} = [\cdot]^{-1}(\text{sp}_{[\cdot]}^{-1}(U))$ which shows that $]U[_{\mathfrak{P}}$ only depends on U and \mathfrak{P} (and not on Z) and if locally we have $g_j \in \mathcal{O}_{\mathfrak{P}}$ such that $U = Z \cap (\cup_j D(g_j))$, then

$$]U[_{\mathfrak{P}} = \{x \in]Z[_{\mathfrak{P}} \mid \exists j \text{ s.t. } v_{[x]}(g_j) \geq 1\}.$$

Remark 2.25 1. We will often refer to $\text{sp}_Z^{-1}(U)$ as the interior tube of U , and denote it by $]U[_{\mathfrak{P}}^{\circ}$. We do not know if it is literally the interior of $]U[_{\mathfrak{P}}$. This should not cause too much confusion.

2. Since $\mathcal{O}_{\mathcal{E}_K} = \mathcal{V}[\![t]\!][t^{-1}]$, if U is actually a scheme over $k((t))$ then this interior tube is a rigid space locally of finite type over \mathcal{E}_K . Moreover if g_j are as above, then it can be described as

$$\{x \in]Z[_{\mathfrak{P}} \mid \exists j \text{ s.t. } v_x(g_j) \geq 1\}.$$

In particular, if $\mathfrak{P}_{\mathcal{E}_K}$ denotes the base change of \mathfrak{P} to $\mathcal{O}_{\mathcal{E}_K}$, then for $U/k((t))$, the interior tube of U in \mathfrak{P} and the interior tube of U in $\mathfrak{P}_{\mathcal{E}_K}$ (defined in a similar manner) are equal as rigid spaces over \mathcal{E}_K . Note that the underlying rigid analytic space (in the sense of Tate) is just Berthelot’s tube $]U[_{\mathfrak{P}}$.

3. Since $]U[_{\mathfrak{P}} = [\cdot]^{-1}(\mathrm{sp}_{[\cdot]}^{-1}(U))$ for any locally closed subscheme $U \subset P$, we can see that the formation of tubes behaves well with regard to unions and intersections of subschemes of P . For example, if $U = U_1 \cup U_2$ is a union of closed subschemes U_i , then $]U[_{\mathfrak{P}} =]U_1[_{\mathfrak{P}} \cup]U_2[_{\mathfrak{P}}$, and if U, V are subschemes of P such that $U \cap V = \emptyset$, then $]U[_{\mathfrak{P}} \cap]V[_{\mathfrak{P}} = \emptyset$. Note that neither of these is immediately obvious from the definitions. Another fact that follows along the same lines that we will need later on is that if we have a Cartesian diagram

$$\begin{array}{ccc} U' & \longrightarrow & \mathfrak{P}' \\ \downarrow & & \downarrow u \\ U & \longrightarrow & \mathfrak{P} \end{array}$$

with horizontal arrows locally closed immersions, then $u_K^{-1}(]U[_{\mathfrak{P}}) =]U'[_{\mathfrak{P}'}$.

If (X, Y, \mathfrak{P}) is a frame, we let $j :]X[_{\mathfrak{P}} \rightarrow]Y[_{\mathfrak{P}}$ denote the inclusion. As in the previous section, for a sheaf \mathcal{F} on $]Y[_{\mathfrak{P}}$ we define $j_X^\dagger \mathcal{F} := j_* j^{-1} \mathcal{F}$.

Definition 2.26 We define the rigid cohomology of the frame (X, Y, \mathfrak{P}) to be

$$H_{\mathrm{rig}}^i((X, Y, \mathfrak{P})/\mathcal{O}_K^\dagger) := H^i(]Y[_{\mathfrak{P}}, j_X^\dagger \Omega_{]Y[_{\mathfrak{P}}/S_K}^*) = H^i(]X[_{\mathfrak{P}}, j^{-1} \Omega_{]Y[_{\mathfrak{P}}/S_K}^*),$$

the equality following from exactness of j_* .

We will see shortly that these are indeed vector spaces over \mathcal{O}_K^\dagger , thus justifying the notation. Beforehand, however, we will first discuss how the cofinal systems of neighbourhoods we saw in the previous section can be constructed entirely similarly in the context of frames over $\mathcal{V}[[t]]$.

So suppose that we have a frame (X, Y, \mathfrak{P}) , with \mathfrak{P} affine, and let $f_i, g_j \in \mathcal{O}_{\mathfrak{P}}$ be functions such that, letting $\bar{f}_i, \bar{g}_j \in \mathcal{O}_P$ denote their mod- π reductions, we have

$$\begin{aligned} Y &= \cap_i V(\bar{f}_i) \subset P, \\ X &= Y \cap (\cup_j D(\bar{g}_j)). \end{aligned}$$

Define

$$\begin{aligned} [Y]_n &= \{x \in \mathfrak{P}_K \mid v_x(\pi^{-1} f_i^n) \leq 1 \ \forall i\}, \\ U_{m,j} &= \{x \in]Y[_{\mathfrak{P}} \mid v_x(\pi^{-1} g_j^m) \geq 1\}, \\ U_m &= \cup_j U_{m,j}, \\ V_{n,m,j} &= [Y]_n \cap U_{m,j}, \\ V_{n,m} &= [Y]_n \cap U_m, \end{aligned}$$

as in the previous section. Exactly as before, for $n, m \gg 0$, these do not depend on the choice of f_i, g_j , and hence glue over an open affine cover of \mathfrak{P} . Moreover,

we have $]Y[_{\mathfrak{P}} = \cup_n]Y_n[_$. We will also need a slightly different version of the U_m which better reflects the fact that for proper, non-empty frames we always have a non-trivial open immersion $X \rightarrow Y$. With this in mind, we choose g'_j such that $X = Y \cap D(t) \cap \left(\cup_j D(g'_j) \right)$, and define

$$\begin{aligned} U'_{m,j} &= \{x \in]Y[_{\mathfrak{P}} \mid v_x(\pi^{-1}g_j'^m) \geq 1, v_x(\pi^{-1}t^m) \geq 1\}, \\ U'_m &= \cup_j U_{m,j}, \\ V'_{n,m,j} &=]Y[_n \cap U_{m,j}, \\ V'_{n,m} &=]Y[_n \cap U_m, \end{aligned}$$

again these do not depend on the choice of the g'_j and hence glue over an open affine covering of \mathfrak{P} . Finally, for any increasing sequence of integers $\underline{m}(n) \rightarrow \infty$, we set

$$\begin{aligned} V_{\underline{m}} &= \bigcup_n V_{n,\underline{m}(n)}, \\ V'_{\underline{m}} &= \bigcup_n V'_{n,\underline{m}(n)}. \end{aligned}$$

Proposition 2.27 1. For all $n \geq 0$, both $V_{n,m}$ and $V'_{n,m}$ form a cofinal system of neighbourhoods of $]Y[_n \cap]X[_{\mathfrak{P}}$ in $]Y[_n$.
 2. As \underline{m} varies, both $V_{\underline{m}}$ and $V'_{\underline{m}}$ form a cofinal system of neighbourhoods of $]X[_{\mathfrak{P}}$ in $]Y[_{\mathfrak{P}}$.

Proof Exactly the same argument that proves Lemma 2.17 and Proposition 2.18 works here. \square

We can now prove that our notation for the rigid cohomology of a frame over (X, Y, \mathfrak{P}) is justified.

Lemma 2.28 The cohomology groups $H_{\text{rig}}^i((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$ are vector spaces over \mathcal{E}_K^\dagger .

Proof There is a morphism of frames

$$(X, Y, \mathfrak{P}) \rightarrow (\text{Spec}(k((t))), \text{Spec}(k[[t]]), \text{Spf}(\mathcal{V}[[t]]))$$

which induces a morphism of ringed spaces

$$(]Y[_{\mathfrak{P}}, j_X^\dagger \mathcal{O}_{]Y[_{\mathfrak{P}}}) \rightarrow (\mathbb{D}_K^b, j_{\text{Spec}(k((t)))}^\dagger \mathcal{O}_{\mathbb{D}_K^b})$$

where we recall that $\mathbb{D}_K^b = \mathrm{Spa}(S_K, \mathcal{V}[\![t]\!])$. By functoriality, the cohomology

$$H^i(\mathrm{I}Y[\mathfrak{p}, j_X^\dagger \Omega_{Y[\mathfrak{p}]/S_K}^*])$$

is naturally a module over $\Gamma(\mathbb{D}_K^b, j_{\mathrm{Spec}(k((t)))}^\dagger \mathcal{O}_{\mathbb{D}_K^b})$, and it therefore suffices to observe that by Proposition 2.27 and Lemma 2.19 we have

$$\Gamma(\mathbb{D}_K^b, j_{\mathrm{Spec}(k((t)))}^\dagger \mathcal{O}_{\mathbb{D}_K^b}) = \mathrm{colim}_m \frac{S_K \langle T \rangle}{(\pi - t^m T)} = \mathcal{O}_K^\dagger.$$

□

2.3 Sundry Properties of Rigid Spaces and Morphisms Between Them

In this section we will collect together a few technical results we shall need about rigid spaces and morphisms between them, and as such it can be safely skimmed and the results referred back to as necessary. There are certain properties of morphisms of rigid spaces, that we will need to use, such as finite, proper, etc., which are defined both by Huber in [8] and by Fujiwara and Kato in [6]. Since results proved both in [8] and in [6] will be useful for us, it will be necessary to know that the two definitions coincide. Thus part of this section is devoted to proving these equivalences. We will also need a result concerning the étale locus of a morphism of rigid spaces. First, however, we will prove a few results about the support of coherent sheaves on rigid spaces, and about the interaction of closed analytic subspaces with the kinds of open subspaces considered in the previous section. Unless otherwise mentioned, all rigid spaces will be assumed to be locally of finite type over \mathbb{D}_K^b .

Definition 2.29 Let \mathcal{X} be a rigid space. Then a closed analytic subspace of \mathcal{X} is a subspace defined by a coherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$. This is again a rigid space, with structure sheaf given by $\mathcal{O}_{\mathcal{X}}/\mathcal{I}$.

Remark 2.30 Note that by Proposition II.7.3.5 of [6] a closed analytic subspace of \mathcal{X} is exactly the image of a closed immersion of rigid spaces in the sense of Definition II.7.3.7 of *loc. cit.*

Proposition 2.31 Let \mathcal{F} be a coherent sheaf on a rigid space \mathcal{X} . Then the support $\mathrm{supp}(\mathcal{F})$ of \mathcal{F} is a closed analytic subspace of \mathcal{X} .

Proof The question is local on \mathcal{X} , which we may thus assume to be affine $\mathcal{X} \cong \mathrm{Spa}(B)$ for some topologically finite type S_K -algebra B . Then \mathcal{F} is the $\mathcal{O}_{\mathcal{X}}$ -module associated to some finite B -module M . If $\mathcal{F}^{\mathrm{alg}}$ is the coherent sheaf on $X = \mathrm{Spec}(A)$ associated to M , and $\varphi : \mathcal{X} \rightarrow X$ the canonical morphism of ringed spaces, then

$\mathcal{F} \cong \varphi^* \mathcal{F}^{\text{alg}}$ (see for example Sect. II.6.6 of [6]). Hence

$$\text{supp}(\mathcal{F}) \subset \varphi^{-1}(\text{supp}(\mathcal{F}^{\text{alg}}))$$

and $\text{supp}(\mathcal{F}^{\text{alg}})$ is the closed subset $V(I)$ of $X := \text{Spec}(B)$ defined by the ideal $I = \text{Ann}(M) \subset A$. By Proposition II.7.3.16 of [6], its inverse image under φ coincides with the closed analytic subspace of $\text{Spa}(B)$ corresponding to I .

It is therefore enough to show that in fact we have

$$\text{supp}(\mathcal{F}) = \varphi^{-1}(\text{supp}(\mathcal{F}^{\text{alg}})),$$

or in other words, that for any $x \in \text{Spa}(B)$ we have $\mathcal{F}_{\varphi(x)}^{\text{alg}} \neq 0 \Rightarrow \mathcal{F}_x \neq 0$. Since \mathcal{F} and \mathcal{F}^{alg} are coherent, we may use Nakayama's lemma to replace the stalks $\mathcal{F}_{\varphi(x)}^{\text{alg}}$ and \mathcal{F}_x by the fibres $\mathcal{F}_{\varphi(x)}^{\text{alg}}/\mathfrak{m}_{\varphi(x)}$ and $\mathcal{F}_x/\mathfrak{m}_x$. If we let $K_{\varphi(x)}$ and K_x denote the residue fields at $\varphi(x)$ and x respectively, then we have

$$\mathcal{F}_x/\mathfrak{m}_x \cong \left(\mathcal{F}_{\varphi(x)}^{\text{alg}}/\mathfrak{m}_{\varphi(x)} \right) \otimes_{K_{\varphi(x)}} K_x$$

and the claim follows. \square

Proposition 2.32 *Let \mathcal{X} be a quasi-compact rigid space, and $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Let $T \subset \mathcal{X}$ be a closed analytic subspace, and suppose that*

$$T \cap \{x \in \mathcal{X} \mid v_x(f) \geq 1\} = \emptyset.$$

Then there exists some m such that

$$T \cap \{x \in \mathcal{X} \mid v_x(\pi^{-1}f^m) \geq 1\} = \emptyset.$$

Proof Since \mathcal{X} is quasi compact, the question is local, so we may assume that \mathcal{X} , and hence T , is affinoid, say $T \cong \text{Spa}(B)$. Let $g \in B$ be the pullback of f , we are required to show that

$$v_x(g) < 1 \ \forall x \in T \Rightarrow \exists m \text{ s.t. } v_x(\pi^{-1}g^m) < 1 \ \forall x \in T.$$

By Lemma 2.9 (or rather, its analogue for rigid varieties over S_K , the proof carries over verbatim) we may restrict to height one points $x \in [T]$. But now this can be rephrased as

$$v_x(g) < 1 \ \forall x \in [T] \Rightarrow \exists m \text{ s.t. } v_x(g) < r^{-1/m} \ \forall x \in [T],$$

so if we let $\|\cdot\|_{\text{sup}} = \sup_{v \in \mathcal{M}(B)} v(\cdot)$ denote the spectral semi-norm on B , then it suffices to show that

$$v(g) < 1 \quad \forall v \in \mathcal{M}(B) \Rightarrow \|g\|_{\sup} < 1.$$

This then follows from compactness of $\mathcal{M}(B)$. \square

Remark 2.33 This proposition is closely related to the fact that the underlying set of a closed analytic subspace of \mathcal{X} is an overconvergent closed subset of \mathcal{X} , i.e. the inverse image of a closed subset of $[\mathcal{X}]$.

Now we turn to proving the equivalences we require between the definitions of separatedness, properness and finiteness given by Huber and Fujiwara/Kato.

Definition 2.34 A morphism $\mathcal{X} \rightarrow \mathcal{Y}$ of rigid spaces is separated if the diagonal morphism

$$\Delta : \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

is a closed immersion. Note that this is the definition given in both [6, 8].

Definition 2.35 From now on ‘rigid variety over S_K ’ will mean ‘rigid space separated and locally of finite type over \mathbb{D}_K^b ’.

Note that by Corollary II.7.5.12 (3) of [6], any morphism between rigid varieties is separated. Any morphism is also locally of finite type in the sense of Definition II.2.3.1 of [6] by Proposition II.2.3.2 of [6], and in the sense of Definition 1.2.1 of [8] by Lemma 3.5 (iv) of [7]. A morphism of rigid varieties over S_K is of finite type (in the sense of either [6] or [8]) if and only if it is quasi-compact. Indeed, this follows by Proposition II.7.1.5 (1) of [6] and is the definition of finite type in [8].

Definition 2.36 A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of finite type between rigid varieties over S_K is said to be:

1. proper in the sense of Huber if for any morphism $\mathcal{Z} \rightarrow \mathcal{Y}$ of adic spaces (i.e. not necessarily locally of finite type over \mathbb{D}_K^b) the map

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{Z}$$

is closed.

2. proper in the sense of Fujiwara–Kato if for any morphism $\mathcal{Z} \rightarrow \mathcal{Y}$ of Fujiwara–Kato rigid spaces (i.e. not necessarily locally of finite type of $\mathrm{Spf}(\mathcal{V}[[t]])^{\mathrm{rig}}$) the map

$$\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \rightarrow \mathcal{Z}$$

is closed.

Note that a priori the two definitions are not the same, since the category of objects we are base changing by could be different. However, by Corollary II.7.5.16 of [6] it suffices to check the universal closedness for Fujiwara–Kato properness for $\mathcal{Z} = \mathbb{D}_{\mathcal{Y}}^n$ for $n \geq 1$, and hence Huber properness implies Fujiwara–Kato properness.

To show the converse, we first recall some notation. For an adic formal scheme \mathfrak{X} of finite ideal type, not necessarily of finite type over $\mathcal{V}[[t]]$, we let $\mathfrak{X}^{\text{rig}}$ denote the associated coherent rigid space in the sense of Sect. II.2.1 of [6]. For a rigid variety \mathcal{X} over S_K and $x \in \mathcal{X}$ we denote the stalk of the integral structure sheaf $\mathcal{O}_{\mathcal{X},x}^+$ of \mathcal{X} at x by A_x , the stalk of the structure sheaf $\mathcal{O}_{\mathcal{X},x}$ by $B_x = A_x[\pi^{-1}]$, the maximal ideal of B_x by \mathfrak{m}_x , and the residue field B_x/\mathfrak{m}_x by K_x . Corollary II.3.2.8 of [6] tells us that $\mathfrak{m}_x \subset A_x$, and that $V_x := A_x/\mathfrak{m}_x$ is a valuation ring inside K_x . Let k_x denote the residue field of V_x .

Following Sect. 1.1 of [8], we define an affinoid field to be a pair (A^\flat, A^+) where A^+ is a valuation ring with quotient field A^\flat , and the valuation topology on A^\flat is induced by a valuation of rank 1. Following Sect. II.3.3 of [6], we define a rigid point of a rigid variety \mathcal{X} over S_K to be a morphism of rigid spaces $\text{Spf}(V)^{\text{rig}} \rightarrow \mathcal{X}$ (not necessarily of finite type over \mathbb{D}_K^b) where V is an a -adically complete valuation ring for some $a \in \mathfrak{m}_V \setminus \{0\}$.

Lemma 2.37 *If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is Fujiwara–Kato proper, then it is Huber proper.*

Proof This more or less follows from the respective valuative criteria, with a little care taken to ensure that rigid points of \mathcal{X} correspond to morphisms $\text{Spa}(A^\flat, A^+) \rightarrow \mathcal{X}$ where (A^\flat, A^+) is an affinoid field.

So let us assume that f is Fujiwara–Kato proper; to show that it is Huber proper, we will apply Huber’s valuative criterion for properness, i.e. Lemma 1.3.10 of [8]. What we have to demonstrate is that given a diagram of adic spaces

$$\begin{array}{ccc} \text{Spa}(A^\flat, A^+) & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \text{Spa}(B^\flat, B^+) & \longrightarrow & \mathcal{Y} \end{array}$$

such that $\widehat{A^\flat} = \widehat{B^\flat}$, there is a unique morphism $\text{Spa}(B^\flat, B^+) \rightarrow \mathcal{X}$ making the diagram commute. Actually, since $\mathcal{X} \rightarrow \mathcal{Y}$ is separated, by the valuative criterion for separatedness (i.e. Lemma 1.3.10 of [8]) it suffices to show that there exists such a morphism. Note that since $\text{Spa}(A^\flat, A^+)$ only depends on the completion $(\widehat{A^\flat}, \widehat{A^+})$ we may assume that $A^\flat = B^\flat$ is complete, and we have valuation rings $B^+ \subset A^+ \subset A^\flat$.

Then by 1.1.8 of [8], the morphism $\text{Spa}(A^\flat, A^+) \rightarrow \mathcal{X}$ corresponds to a pair (x, φ) where $x \in \mathcal{X}$ and $\varphi : K_x \rightarrow A^\flat$ is a continuous homomorphism such that $V_x = \varphi^{-1}(A^+)$. Hence this extends uniquely to a morphism $\widehat{V}_x \rightarrow A^+$ of complete valuation rings (where \widehat{V}_x is the π -adic completion of V_x) and hence a rigid point $\text{Spf}(A^+)^{\text{rig}} \rightarrow \mathcal{X}$, similarly the morphism $\text{Spa}(A^\flat, B^+) \rightarrow \mathcal{Y}$ corresponds to a rigid point of \mathcal{Y} , and there is a commutative diagram

$$\begin{array}{ccc}
 \mathrm{Spf} (A^+)^{\mathrm{rig}} & \longrightarrow & \mathcal{X} \\
 \downarrow & \nearrow \text{---} & \downarrow \\
 \mathrm{Spf} (B^+)^{\mathrm{rig}} & \longrightarrow & \mathcal{Y}.
 \end{array}$$

Since the morphism $B^+ \rightarrow A^+$ is a localisation at a prime ideal of B^+ , the morphism

$$\mathrm{Spf} (A^+)^{\mathrm{rig}} \rightarrow \mathrm{Spf} (B^+)^{\mathrm{rig}}$$

is a generisation in the sense of II.7.5(c) of [6], and hence applying Fujiwara–Kato’s valuative criterion of properness (Theorem II.7.5.17 of [6]) there exists a unique morphism $\mathrm{Spf} (B^+)^{\mathrm{rig}} \rightarrow \mathcal{X}$ making the diagram commute. Again, the rigid point $\mathrm{Spf} (B^+)^{\mathrm{rig}} \rightarrow \mathcal{X}$ corresponds to a point $x \in \mathcal{X}$ and a continuous homomorphism $\widehat{V}_x \rightarrow B^+$ of complete valuation rings, and hence a continuous homomorphism $V_x \rightarrow B^+$. This extends uniquely to a continuous homomorphism $K_x \rightarrow A^\flat$ and hence a morphism $\mathrm{Spa}(A^\flat, B^+) \rightarrow \mathcal{X}$ as required. \square

Henceforth we will simply refer to a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of rigid varieties over S_K being proper.

Definition 2.38 A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of rigid analytic varieties over S_K is said to be:

1. finite in the sense of Huber if locally on \mathcal{Y} it is of the form

$$\mathrm{Spa}(B) \rightarrow \mathrm{Spa}(A)$$

for some finite morphism $A \rightarrow B$ of topologically finite type S_K -algebras A, B ;

2. finite in the sense of Fujiwara–Kato if locally on \mathcal{Y} it arises as the generic fibre of a finite morphism $\mathfrak{X} \rightarrow \mathfrak{Y}$ between formal schemes of finite type over $\mathcal{V}[[t]]$.

It is clear that Fujiwara–Kato finiteness implies Huber finiteness.

Lemma 2.39 *If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is Huber finite, then it is Fujiwara–Kato finite.*

Proof We may suppose that f is associated to a finite morphism $A \rightarrow B$ of topologically finite type S_K -algebras. Let $A^+ \rightarrow B^+$ be the associated morphism of $+$ -parts, note that B^+ is the integral closure of A^+ in B . Choose $b_1, \dots, b_n \in B^+$ which generate B as an A -module, and hence topologically as an A -algebra. Since each b_i is integral over A^+ , the map

$$A^+ \rightarrow A^+[b_1, \dots, b_n]$$

is a finite formal model for $A \rightarrow B$. \square

Henceforth we will simply refer to a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of rigid varieties over S_K being finite. Having proved the required equivalence between the different definitions of properness and finiteness, we move on to the second main result of this section, concerning the openness of the étale locus of a morphism of rigid varieties over S_K .

Definition 2.40 Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of rigid spaces over S_K , denote by $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ the closed immersion defined by the diagonal, and $\mathcal{I} = \ker(\mathcal{O}_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \rightarrow \Delta_* \mathcal{O}_{\mathcal{X}})$ the kernel of the multiplication map. Then we define the module of differentials

$$\Omega^1_{\mathcal{X}/\mathcal{Y}} = \Delta^*(\mathcal{I}) = \mathcal{I}/\mathcal{I}^2,$$

by 1.6 of [8] this is a coherent $\mathcal{O}_{\mathcal{X}}$ -module.

While the following definitions are not those given in Sect. 1 of [8], Propositions 1.6.8 and 1.7.5 of *loc. cit.* show that they are equivalent.

Definition 2.41 A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of rigid varieties over S_K is said to be:

1. unramified if $\Omega^1_{\mathcal{X}/\mathcal{Y}} = 0$;
2. flat if for each $x \in \mathcal{X}$, the local ring $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$;
3. étale if it is flat and unramified.

It follows immediately from Proposition 2.31 that the locus where a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ is not unramified is a closed analytic subspace of \mathcal{X} . The following result says that the same is true for étaleness.

Proposition 2.42 Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of rigid varieties over S_K . Then f is étale away from a closed analytic subspace of \mathcal{X} .

Proof We may assume that f is unramified, and the question is local on both \mathcal{X} and \mathcal{Y} , which we may thus assume to be affinoid. Hence by Proposition 1.6.8 of [8] we may factor $\mathcal{X} \rightarrow \mathcal{Y}$ as a closed immersion $g : \mathcal{X} \rightarrow \mathcal{Z}$ followed by an étale map $h : \mathcal{Z} \rightarrow \mathcal{Y}$. Thus it suffices to show that the closed immersion $g : \mathcal{X} \rightarrow \mathcal{Z}$ is an isomorphism away from a closed analytic subspace of \mathcal{Z} . But now this just follows from the fact that g is an isomorphism away from the support of the coherent sheaf

$$\ker(\mathcal{O}_{\mathcal{Z}} \rightarrow g_* \mathcal{O}_{\mathcal{X}})$$

together with Proposition 2.31. □

Remark 2.43 Note that this closed subspace will in general be the whole of \mathcal{X} , and this result will only be useful when we already know that f is generically étale, i.e. étale on some open subset of \mathcal{X} .

2.4 Independence of the Frame

In this section we prove that for a smooth and proper frame (X, Y, \mathfrak{P}) , the rigid cohomology

$$H_{\text{rig}}^i((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$$

only depends on X , and gives rise to a functor

$$X \mapsto H_{\text{rig}}^i(X/\mathcal{E}_K^\dagger)$$

on the category of embeddable varieties (we will extend this to non-embeddable varieties in Chap. 4.) We will follow closely Berthlot's original proof of independence for rigid cohomology in [2, 3], the key results being the Strong Fibration Theorem and the overconvergent Poincaré Lemma below.

Remark 2.44 We will often refer to a claim concerning frames, or a property of frames (X, Y, \mathfrak{P}) as being local on X or \mathfrak{P} . By being local on \mathfrak{P} we mean that it suffices to check it on every frame $(X \cap \mathfrak{P}_i, Y \cap \mathfrak{P}_i, \mathfrak{P}_i)$ associated to an open cover $\{\mathfrak{P}_i\}$ of \mathfrak{P} , and by being local on X we mean that it suffices to check it on every frame (X_i, Y, \mathfrak{P}) associated to an open cover $\{X_i\}$ of X .

Proposition 2.45 (Strong Fibration Theorem) *Suppose that*

$$\begin{array}{ccccc} & & Y' & \xrightarrow{i'} & \mathfrak{P}' \\ & \nearrow j' & \downarrow v & & \downarrow u \\ X & \xrightarrow{j} & Y & \xrightarrow{i} & \mathfrak{P} \end{array}$$

is a diagram of frames over $\mathcal{V}[[t]]$, such that v is proper, and u is étale in a neighbourhood of X . Then u_K induces an isomorphism between a cofinal system of neighbourhoods of $]X[_{\mathfrak{P}}$ in $]Y[_{\mathfrak{P}}$ and a cofinal system of neighbourhoods of $]X[_{\mathfrak{P}'}$ in $]Y'[_{\mathfrak{P}'}$.

Proof We follow closely the proof of Théorème 1.3.5 of [3]. We may replace Y' by the closure of X in Y' , and hence assume that $u^{-1}(X) \cap Y' = X$. Using the standard neighbourhoods V_m constructed in the previous section, it therefore suffices to prove that for all $n \gg 0$ there exists some $d \geq n$, such that

$$[Y']_d \cap u_K^{-1}([Y]_n \cap U_m) \rightarrow [Y]_n \cap U_m$$

is an isomorphism for $m \gg 0$. The question is local on \mathfrak{P} , which we may thus assume to be affine, isomorphic to $\text{Spf}(A)$. After base changing to $[Y]_n$ we may assume that $[Y]_n = \mathfrak{P}_K$ and that the closed immersion $Y \rightarrow P$ of Y into the reduction of \mathfrak{P} is nilpotent. The question is also local on X , which we may thus assume to be of the

form $D(tg) \cap Y$ for some $g \in A$. Thus we have

$$U_m = \{x \in \mathfrak{P}_K \mid v_x(p^{-1}(tg)^m) \geq 1\}$$

and we wish to show that there exists some $d \geq n$ such that

$$[Y']_d \cap u_K^{-1}(U_m) \xrightarrow{\sim} U_m$$

for $m \gg 0$. Let P and P' denote the reductions of \mathfrak{P} and \mathfrak{P}' respectively, and let $U'_m = u_K^{-1}(U_m)$. Write $P'_Y = Y \times_P P'$ and $P'_X = X \times_P P'$, note that P'_Y has the same underlying space as P' . Since $P'_X \rightarrow X$ admits a section around which it is étale, it follows that X is open and closed in P'_X , and since P'_X is open in P'_Y , X must be open in P'_Y .

Let $D = P'_Y \setminus X$ be the closed complement, since $X \subset Y'$ it follows that P'_Y is the union of its two closed subschemes Y' and D , and P'_X is the union of its two components X and $P'_X \setminus X = D \cap u^{-1}(X)$. Thus we have

$$\begin{aligned} \mathfrak{P}'_K &=]Y'[_{\mathfrak{P}'} \cup]D[_{\mathfrak{P}'} \\]P'_X[_{\mathfrak{P}'} &= u_K^{-1}(]X[_{\mathfrak{P}}) =]X[_{\mathfrak{P}'} \cup (]D[_{\mathfrak{P}'} \cap u_K^{-1}(]X[_{\mathfrak{P}})) \end{aligned}$$

the first being an open covering and the second being a decomposition into components. Arguing by quasi-compactness, we can see that there must be some d, l such that

$$\mathfrak{P}'_K = [Y']_d \cup [D]_l.$$

Since $u^{-1}(X) \cap Y' = X$, it follows that

$$]Y'[_{\mathfrak{P}'} \cap]D[_{\mathfrak{P}'} \cap u_K^{-1}(]X[_{\mathfrak{P}}) =]X[_{\mathfrak{P}'} \cap]D[_{\mathfrak{P}'} =]X \cap D[_{\mathfrak{P}'} = \emptyset,$$

so *a fortiori* $[Y']_d \cap [D]_l \cap u_K^{-1}(]X[_{\mathfrak{P}}) = \emptyset$. By the maximum principle applied on the separated quotient of $[Y']_d \cap [D]_l$ together with Lemma 2.9, we can see that we must in fact have

$$[Y']_d \cap [D]_l \cap U'_m = \emptyset$$

for $m \gg 0$. Thus $U'_m = ([Y']_d \cap U'_m) \cup ([D]_l \cap U'_m)$ is a decomposition of U'_m into components.

Define $T_K^m = [Y']_d \cap U'_m$, this is an open and closed subset of U'_m , and hence a quasi-compact rigid space over S_K . By Remark 2.25(2) and the weak fibration theorem (Proposition 1.3.1 of [3]), $T_K^m \rightarrow U_m$ induces an isomorphism between the interior tubes $]X[_{\mathfrak{P}'}^{\circ}$ and $]X[_{\mathfrak{P}}^{\circ}$. By Proposition 2.42, the locus where $T_K^m \rightarrow U_m$ is not étale is a closed analytic subset of T_K^m , and the fact that it is étale on the interior tubes together with Proposition 2.32 implies that $T_K^m \rightarrow U_m$ is étale for all $m \gg 0$. Since u_K is proper, we must also have that $T_K^m \rightarrow U_m$ is proper, thus by Proposition 1.5.5 of [8] it is finite.

Proposition II.7.2.4 of [6] now shows that (again, for $m \gg 0$) $T_K^m \rightarrow U_m$ is the morphism associated to a coherent \mathcal{O}_{U_m} -algebra \mathcal{A} , say, and by the ‘classical’ weak fibration theorem we know that this morphism is an isomorphism on the interior tube $]X[_{\mathfrak{P}}$. Hence by applying Proposition 2.31 to the kernel and cokernel of $\mathcal{O}_{U_m} \rightarrow \mathcal{A}$, and then using Proposition 2.32 we can see that it is an isomorphism on U_m for $m \gg 0$. This completes the proof. \square

To be able to use this, we will need to know that, locally, a smooth morphism of frames $(X, Y, \mathfrak{P}') \rightarrow (X, Y, \mathfrak{P})$ factors into an étale morphism of frames followed by a projection $(X, Y, \widehat{\mathbb{A}}_{\mathfrak{P}}^d) \rightarrow (X, Y, \mathfrak{P})$, where Y is embedded in $\widehat{\mathbb{A}}_{\mathfrak{P}}^d$ via the zero section. This is the content of the following lemma.

Lemma 2.46 *Let*

$$\begin{array}{ccccc} & & Y' & \xrightarrow{i'} & \mathfrak{P}' \\ & \nearrow j' & \downarrow v & & \downarrow u \\ X & \xrightarrow{j} & Y & \xrightarrow{i} & \mathfrak{P} \end{array}$$

be a morphism of frames over $\mathcal{V}[[t]]$, such that u is smooth in a neighbourhood of X . Let $\mathcal{S}' \subset \mathcal{O}_{\mathfrak{P}'}$ denote the ideal of Y' in \mathfrak{P}' , and I' that of Y' inside $P_Y' = \mathfrak{P}' \times_{\mathfrak{P}} Y$. Suppose that there are sections $t_1, \dots, t_d \in \Gamma(\mathfrak{P}', \mathcal{S}')$ inducing a basis $\bar{t}_1, \dots, \bar{t}_d$ of the conormal sheaf I'/I'^2 in a neighbourhood of X . Then the morphism $\varphi : \mathfrak{P}' \rightarrow \widehat{\mathbb{A}}_{\mathfrak{P}}^d$ defined by t_1, \dots, t_d maps Y' into Y and is étale in a neighbourhood of X .

Proof The proof is identical to that of Théorème 1.3.7 in [3]. \square

The other fundamental result that we require is a suitable version of the Poincaré Lemma, however, we will first need a result in homological algebra.

Lemma 2.47 *Let $\mathcal{K} \rightarrow \mathcal{L}$ be a morphism of bounded below complexes of sheaves on a topological space T , and assume that the induced map*

$$\mathbf{R}\Gamma(U, \mathcal{K}) \rightarrow \mathbf{R}\Gamma(U, \mathcal{L})$$

is a quasi-isomorphism for all open subsets U of T . Then $\mathcal{K} \rightarrow \mathcal{L}$ is a quasi-isomorphism.

Proof By considering a mapping cone, it suffices to show that if $\mathbf{R}\Gamma(U, \mathcal{K}) \simeq 0$ for all U then $\mathcal{K} \simeq 0$ (note that viewed from a higher categorical perspective, this is essentially a triviality). Since \mathcal{K} is bounded below, if it is non-zero we may apply a suitable shift to assume that $\mathcal{H}^j(\mathcal{K}) = 0$ for $j < 0$ and $\mathcal{H}^0(\mathcal{K}) \neq 0$. Hence

$$\mathbf{R}^0\Gamma(U, \mathcal{K}) \cong \Gamma(U, \mathcal{H}^0(\mathcal{K})).$$

and the hypothesis that $\mathbf{R}\Gamma(U, \mathcal{K}) \simeq 0$ for all U implies that $\Gamma(U, \mathcal{H}^0(\mathcal{K})) = 0$ for all U . Therefore $\mathcal{H}^0(\mathcal{K}) = 0$, and we obtain a contradiction. Hence we must have $\mathcal{K} = 0$. \square

Proposition 2.48 (Poincaré Lemma) Let (X, Y, \mathfrak{P}) be a frame over $\mathcal{V}[[t]]$, and let

$$u : (X, Y, \widehat{\mathbb{A}}_{\mathfrak{P}}^d) \rightarrow (X, Y, \mathfrak{P})$$

be the natural morphism of frames. Then the induced morphism

$$j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}}} \rightarrow \mathbf{R}u_{K*} j_X^\dagger \Omega_{|Y|_{\widehat{\mathbb{A}}_{\mathfrak{P}}^d}/|Y|_{\mathfrak{P}}}^*$$

is a quasi-isomorphism.

Proof The question is local on \mathfrak{P} , which we may thus assume to be affine, $\mathfrak{P} \cong \mathrm{Spf}(A)$. Let $f_i \in A$ be functions whose reductions \bar{f}_i define the ideal of Y inside P and choose $g_j \in A$ such that $X = (D(t) \cap (\bigcup_j D(\bar{g}_j))) \cap Y$.

First suppose that $d = 1$, and let $\mathbb{D}_{S_K}^1$ denote the unit disk $(\widehat{\mathbb{A}}_{\mathcal{V}[[t]]}^1)_K = \mathrm{Spa}(S_K \langle z \rangle)$ over S_K , with co-ordinate z . If we let $[Y]_n = \{x \in \mathfrak{P}_K \mid v_x(\pi^{-1} f_i^n) \leq 1 \ \forall i\}$, then $|Y|_{\mathfrak{P}} = \bigcup_n [Y]_n$, and we may base change to a formal model of $[Y]_n$ and hence assume that $|Y|_{\mathfrak{P}} = \mathfrak{P}_K$. Now define

$$\begin{aligned} [Y]_n' &= \{x \in \mathfrak{P}_K \times_{S_K} \mathbb{D}_{S_K}^1 \mid v_x(\pi^{-1} z^n) \leq 1\} \\ U_m' &= \left\{x \in |Y|_{\widehat{\mathbb{A}}_{\mathfrak{P}}^1} \mid v_x(\pi^{-1} t^m) \geq 1, \exists j \text{ s.t. } v_x(\pi^{-1} g_j^m) \geq 1\right\} \\ U_m &= \{x \in \mathfrak{P}_K \mid v_x(\pi^{-1} t^m) \geq 1, \exists j \text{ s.t. } v_x(\pi^{-1} g_j^m) \geq 1\} \\ U_{m,j}' &= \left\{x \in |Y|_{\widehat{\mathbb{A}}_{\mathfrak{P}}^1} \mid v_x(\pi^{-1} t^m) \geq 1, v_x(\pi^{-1} g_j^m) \geq 1\right\} \\ U_{m,j} &= \{x \in \mathfrak{P}_K \mid v_x(\pi^{-1} t^m) \geq 1, v_x(\pi^{-1} g_j^m) \geq 1\} \end{aligned}$$

so that, by Proposition 2.27 and the preceding discussion, $|Y|_{\widehat{\mathbb{A}}_{\mathfrak{P}}^1} = \bigcup_n [Y]_n'$, and $[Y]_n' \cap U_m'$ (resp. U_m) is a cofinal system of neighbourhoods of $|X|_{\widehat{\mathbb{A}}_{\mathfrak{P}}^1} \cap [Y]_n'$ in $[Y]_n'$ (resp. $|X|_{\mathfrak{P}}$ in \mathfrak{P}_K). Since both $j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}}}$ and $\mathbf{R}u_{K*} j_X^\dagger \Omega_{|Y|_{\widehat{\mathbb{A}}_{\mathfrak{P}}^d}/|Y|_{\mathfrak{P}}}^*$ are supported on X , to check they are quasi-isomorphic we may do so on any open neighbourhood of $|X|_{\mathfrak{P}}$, and moreover on any open cover of such a neighbourhood. In particular, it therefore suffices to prove that they are quasi-isomorphic after restricting to each $U_{m_0,j}$ for some fixed m_0 , hence we may assume that there is only one g_j , or in other words that there exists some g such that

$$\begin{aligned} U_m' &= \left\{x \in |Y|_{\widehat{\mathbb{A}}_{\mathfrak{P}}^1} \mid v_x(\pi^{-1} t^m) \geq 1, v_x(\pi^{-1} g^m) \geq 1\right\} \\ U_m &= \{x \in \mathfrak{P}_K \mid v_x(\pi^{-1} t^m) \geq 1, v_x(\pi^{-1} g^m) \geq 1\}. \end{aligned}$$

Hence each $[Y]_n' \cap U_m'$ (resp. U_m) is affinoid, and if we let $B_{n,m}'$ (resp. B_m) denote the affinoid algebra over S_K corresponding to $[Y]_n' \cap U_m'$ (resp. U_m), then we have

$$B'_{n,m} \cong B_m \langle r^{1/n} z \rangle := \frac{B_m \langle z, T \rangle}{(\pi T - z^n)}.$$

We next claim that the result holds for global sections on \mathfrak{P}_K , that is

$$\operatorname{colim}_m B_m \rightarrow \mathbf{R}\Gamma(\mathfrak{P}_K, \mathbf{R}u_{K*} j_X^\dagger \Omega_{Y[\widehat{\mathbb{A}}_K^1/\mathfrak{P}_K]}^*)$$

is an isomorphism. Indeed, letting $j_{m,n} : [Y]_n' \cap U_m' \rightarrow [Y]_n'$ denote the natural inclusion we have a sequence of quasi-isomorphisms

$$\begin{aligned} \mathbf{R}\Gamma(\mathfrak{P}_K, \mathbf{R}u_{K*} j_X^\dagger \Omega_{Y[\widehat{\mathbb{A}}_K^1/\mathfrak{P}_K]}^*) &\cong \mathbf{R}\Gamma([Y]_{\widehat{\mathbb{A}}_K^1}, j_X^\dagger \Omega_{Y[\widehat{\mathbb{A}}_K^1]/[Y]_{\mathfrak{P}_K}}^*) \\ &\cong \mathbf{R}\lim_n \mathbf{R}\Gamma([Y]_n', j_X^\dagger \Omega_{Y[\widehat{\mathbb{A}}_K^1]/[Y]_{\mathfrak{P}_K}}^*|_{[Y]_n'}) \\ &\cong \mathbf{R}\lim_n \mathbf{R}\Gamma([Y]_n', \operatorname{colim}_m j_{m,n*} \Omega_{[Y]_n' \cap U_m' / U_m'}^*) \\ &\cong \mathbf{R}\lim_n \operatorname{colim}_m \mathbf{R}\Gamma([Y]_n', \mathbf{R}j_{m,n*} \Omega_{[Y]_n' \cap U_m' / U_m'}^*) \\ &\cong \mathbf{R}\lim_n \operatorname{colim}_m \mathbf{R}\Gamma([Y]_n' \cap U_m', \Omega_{[Y]_n' \cap U_m' / U_m'}^*) \\ &\cong \mathbf{R}\lim_n \operatorname{colim}_m \Gamma([Y]_n' \cap U_m', \Omega_{[Y]_n' \cap U_m' / U_m'}^*) \\ &\cong \mathbf{R}\lim_n \operatorname{colim}_m (B_m \langle r^{1/n} z \rangle \rightarrow B_m \langle r^{1/n} z \rangle dz), \end{aligned}$$

where:

- the first, second and fifth isomorphisms are simply compositions of derived functors;
- the third isomorphism follows from the characterisation of j_X^\dagger given by Lemma 2.19;
- the fourth isomorphism follows from quasi-compactness of $[Y]_n'$;
- the sixth isomorphism follows from the fact that $[Y]_n' \cap U_m'$ is affinoid;
- the last isomorphism follows from the above description of $B'_{n,m}$.

Thus what we want to prove is that

$$\operatorname{colim}_m B_m \rightarrow \mathbf{R}\lim_n \operatorname{colim}_m (B_m \langle r^{1/n} z \rangle \rightarrow B_m \langle r^{1/n} z \rangle dz)$$

is a quasi-isomorphism. Write

$$\begin{aligned} H_n^i &= \operatorname{colim}_m H^i (B_m \langle r^{1/n} z \rangle \rightarrow B_m \langle r^{1/n} z \rangle dz) \\ H^j &= H^j (\mathbf{R}\lim_n \operatorname{colim}_m (B_m \langle r^{1/n} z \rangle \rightarrow B_m \langle r^{1/n} z \rangle dz)) \end{aligned}$$

so that we have

$$\begin{aligned} H^0 &\cong \lim_n H_n^0 \\ H^2 &\cong \lim_n^1 H_n^1, \end{aligned}$$

an exact sequence

$$0 \rightarrow \lim_n H_n^1 \rightarrow H^1 \rightarrow \lim_n^1 H_n^0 \rightarrow 0$$

and $H^j = 0$ for $j \neq 0, 1, 2$. Since $H_n^0 = \text{colim}_m B_m$ for all n , we can easily see that $\lim_n^1 H_n^0 = 0$.

We next claim that the transition maps $H_n^1 \rightarrow H_{n-1}^1$ are all zero (this is essentially the main content of the proposition). Concretely, this is stating that any differential form $\omega \in \text{colim}_m B_m \langle r^{1/n} z \rangle dz$ is closed after passing to $\text{colim}_m B_m \langle r^{1/(n-1)} z \rangle dz$. This can be checked for fixed m , where it follows from a simple calculation.

Hence we have $H^0 = \text{colim}_m B_m$, $H^1 = H^2 = 0$ and hence

$$\text{colim}_m B_m \rightarrow \mathbf{R}\Gamma(\mathfrak{P}_K, \mathbf{R}u_{K*} j_X^\dagger \Omega_{]Y[_{\widehat{\mathbb{A}}^1_{\mathfrak{P}}}/]Y[_{\mathfrak{P}}})$$

is a quasi-isomorphism as claimed. Since a similar calculation holds when we replace \mathfrak{P} by any open affinoid subset, the claim in relative dimension 1 follows from Lemma 2.47 above.

In the general case we consider the tower

$$(X, Y, \widehat{\mathbb{A}}_{\mathfrak{P}}^d) \xrightarrow{u^{(d)}} (X, Y, \widehat{\mathbb{A}}_{\mathfrak{P}}^{d-1}) \xrightarrow{u^{(d-1)}} \dots \xrightarrow{u^{(1)}} (X, Y, \mathfrak{P})$$

and we know that at each stage,

$$j_X^\dagger \mathcal{O}_{]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^{k-1}}} \rightarrow \mathbf{R}u_{K*}^{(k)} j_X^\dagger \Omega_{]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^k}/]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^{k-1}}}$$

is a quasi-isomorphism. We want to deduce that in fact

$$j_X^\dagger \Omega_{]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^{k-1}}/]Y[_{\mathfrak{P}}} \rightarrow \mathbf{R}u_{K*}^{(k)} j_X^\dagger \Omega_{]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^k}/]Y[_{\mathfrak{P}}}$$

is a quasi-isomorphism. To do so, we consider the Gauss–Manin filtration F^\bullet on

$$j_X^\dagger \Omega_{]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^k}/]Y[_{\mathfrak{P}}}$$

arising from the composition $]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^k} \rightarrow]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^{k-1}} \rightarrow]Y[_{\mathfrak{P}}$. This is defined by

$$F^i(j_X^\dagger \Omega_{]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^k}/]Y[_{\mathfrak{P}}}^l) := \text{im}(j_X^\dagger \Omega_{]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^k}/]Y[_{\mathfrak{P}}}^{l-i} \otimes (u_K^{(k)})^* j_X^\dagger \Omega_{]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^{k-1}}/]Y[_{\mathfrak{P}}}^i \rightarrow j_X^\dagger \Omega_{]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^k}/]Y[_{\mathfrak{P}}}^l).$$

Since the terms in the exact sequence

$$0 \rightarrow (u_K^{(k)})^* j_X^\dagger \Omega_{]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^{k-1}}/]Y[_{\mathfrak{P}}}^1 \rightarrow j_X^\dagger \Omega_{]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^k}/]Y[_{\mathfrak{P}}}^1 \rightarrow j_X^\dagger \Omega_{]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^k}/]Y[_{\widehat{\mathbb{A}}_{\mathfrak{P}}^{k-1}}}^1 \rightarrow 0$$

are locally free, we can deduce that

$$\mathrm{Gr}_F^i(j_X^\dagger \Omega_{|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^k / |Y[\mathfrak{P}}]}^*) \cong (u_K^{(k)})^* j_X^\dagger \Omega_{|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^{k-1} / |Y[\mathfrak{P}}]}^i \otimes j_X^\dagger \Omega_{|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^k / |Y[\hat{\mathbb{A}}_{\mathfrak{P}}^{k-1}]}^{*-i}$$

and hence

$$\begin{aligned} \mathbf{R}u_{K*}^{(k)} \mathrm{Gr}_F^i(j_X^\dagger \Omega_{|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^k / |Y[\mathfrak{P}}]}^*) &\cong j_X^\dagger \Omega_{|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^{k-1} / |Y[\mathfrak{P}}]}^i \otimes \mathbf{R}u_{K*}^{(k)}(j_X^\dagger \Omega_{|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^k / |Y[\hat{\mathbb{A}}_{\mathfrak{P}}^{k-1}]}^{*-i}) \\ &\cong j_X^\dagger \Omega_{|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^{k-1} / |Y[\mathfrak{P}}]}^i[-i]. \end{aligned}$$

Thus the spectral sequence associated to the Gauss–Manin filtration degenerates and gives

$$\mathbf{R}u_{K*}^{(k)}(j_X^\dagger \Omega_{|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^k / |Y[\mathfrak{P}}]}^*) \cong j_X^\dagger \Omega_{|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^{k-1} / |Y[\mathfrak{P}}]}^*.$$

Repeatedly applying this we see that

$$\mathbf{R}u_{K*}(j_X^\dagger \Omega_{|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^d / |Y[\mathfrak{P}}]}^*) \cong j_X^\dagger \mathcal{O}_{|Y[\mathfrak{P}]}$$

as required. \square

Corollary 2.49 *With hypotheses as in Lemma 2.48 the induced morphism*

$$j_X^\dagger \Omega_{|Y[\mathfrak{P}] / S_K}^* \rightarrow \mathbf{R}u_{K*} j_X^\dagger \Omega_{|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^d / S_K]}^*$$

is a quasi-isomorphism.

Proof To prove this we simply use the Gauss–Manin filtration on $j_X^\dagger \Omega_{|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^d / S_K]}^*$ arising from the composition $|Y[\hat{\mathbb{A}}_{\mathfrak{P}}^d] \rightarrow |Y[\mathfrak{P}] \rightarrow \mathbb{D}_K^b$ in exactly the same way as above. (Note that local freeness of $j_X^\dagger \Omega_{|Y[\mathfrak{P}] / S_K}^*$ follows from combining smoothness of \mathfrak{P} over $\mathcal{V}[\![t]\!]$ in a neighbourhood of X with Propositions 2.31 and 2.32.) \square

Finally, we will need to know a certain degree of locality on X . If $Z = Y \setminus X$ and E is any sheaf on $|Y[\mathfrak{P}]$ then we define $\underline{\Gamma}_Z^\dagger E$ by the exact sequence

$$0 \rightarrow \underline{\Gamma}_Z^\dagger E \rightarrow E \rightarrow j_X^\dagger E \rightarrow 0.$$

Note that j_X^\dagger and $\underline{\Gamma}_Z^\dagger$ are exact, and we have $j_X^\dagger j_{X'}^\dagger E \cong j_{X \cap X'}^\dagger E$ and $\underline{\Gamma}_Z^\dagger \underline{\Gamma}_{Z'}^\dagger E \cong \underline{\Gamma}_{Z \cap Z'}^\dagger E$.

Lemma 2.50 *Let (X, Y, \mathfrak{P}) be a $\mathcal{V}[\![t]\!]$ -frame, and $X = \cup_{j=1}^n X_j$ a finite open cover of X . Then for any sheaf E on $|Y[\mathfrak{P}]$ there is an exact sequence of sheaves*

$$0 \rightarrow j_X^\dagger E \rightarrow \prod_j j_{X_j}^\dagger E \rightarrow \prod_{j_0 < i_{j_1}} j_{X_{j_0} \cap X_{j_1}}^\dagger E \rightarrow \dots \rightarrow j_{\cap_j X_j}^\dagger E \rightarrow 0$$

on $|Y[\mathfrak{P}]$.

Proof We follow the proof of Proposition 2.1.8 of [3], and proceed by induction on the size of the covering n . The case $n = 1$ is obvious, so assume that $n \geq 2$ and let $X' = \cup_{j=2}^n X_j$. The induction hypothesis implies that

$$0 \rightarrow j_{X'}^\dagger E \rightarrow \prod_{1 < j} j_{X_j}^\dagger E \rightarrow \prod_{1 < j_0 < i_j} j_{X_{j_0} \cap X_{j_1}}^\dagger E \rightarrow \cdots \rightarrow j_{\cap_{j=2}^n X_j}^\dagger E \rightarrow 0$$

is exact. Let $Z' = Y \setminus X'$ so that the complex

$$K^\bullet := (0 \rightarrow E \rightarrow \prod_{1 < j} j_{X_j}^\dagger E \rightarrow \prod_{1 < j_0 < i_j} j_{X_{j_0} \cap X_{j_1}}^\dagger E \rightarrow \cdots \rightarrow j_{\cap_{j=2}^n X_j}^\dagger E \rightarrow 0)$$

is a resolution of $\Gamma_{Z'}^\dagger E$. Letting $Z_1 = Y \setminus X_1$ we get an exact sequence of complexes

$$0 \rightarrow \Gamma_{Z_1}^\dagger K^\bullet \rightarrow K^\bullet \rightarrow j_{X_1}^\dagger K^\bullet \rightarrow 0.$$

We can identify $j_{X_1}^\dagger K^\bullet$ with the complex

$$0 \rightarrow j_{X_1}^\dagger E \rightarrow \prod_{1 < j} j_{X_1 \cap X_j}^\dagger E \rightarrow \cdots \rightarrow \cdots j_{\cap_j X_j}^\dagger E \rightarrow 0$$

and hence we can identify the total complex of the double complex associated to $K^\bullet \rightarrow j_{X_1}^\dagger K^\bullet$ with

$$0 \rightarrow E \rightarrow \prod_j j_{X_j}^\dagger E \rightarrow \prod_{j_0 < i_j} j_{X_{j_0} \cap X_{j_1}}^\dagger E \rightarrow \cdots \rightarrow j_{\cap_j X_j}^\dagger E \rightarrow 0.$$

This is quasi-isomorphic to $\Gamma_{Z_1}^\dagger K^\bullet$, which by exactness of $\Gamma_{Z_1}^\dagger$ is in turn quasi-isomorphic to $\Gamma_{Z_1}^\dagger \Gamma_{Z'}^\dagger E \cong \Gamma_Z^\dagger E$. Thus we have a quasi-isomorphism

$$\Gamma_Z^\dagger E \cong \left(0 \rightarrow E \rightarrow \prod_j j_{X_j}^\dagger E \rightarrow \prod_{j_0 < i_j} j_{X_{j_0} \cap X_{j_1}}^\dagger E \rightarrow \cdots \rightarrow j_{\cap_j X_j}^\dagger E \rightarrow 0 \right)$$

and hence an exact sequence

$$0 \rightarrow j_X^\dagger E \rightarrow \prod_j j_{X_j}^\dagger E \rightarrow \prod_{j_0 < i_j} j_{X_{j_0} \cap X_{j_1}}^\dagger E \rightarrow \cdots \rightarrow j_{\cap_j X_j}^\dagger E \rightarrow 0$$

as claimed. \square

We can now put this all together to prove that, up to isomorphism, the rigid cohomology of (X, Y, \mathfrak{A}) only depends on X .

Theorem 2.51 *Let*

$$\begin{array}{ccccc}
 & & Y' & \xrightarrow{i'} & \mathfrak{P}' \\
 & \nearrow j' & \downarrow v & & \downarrow u \\
 X & \xrightarrow{j} & Y & \xrightarrow{i} & \mathfrak{P}
 \end{array}$$

be a morphism of frames over $\mathcal{V}[\![t]\!]$, such that v is proper, and u is smooth in a neighbourhood of X . Then the natural maps

$$H_{\text{rig}}^i((X, Y, \mathfrak{P})/\mathcal{O}_K^\dagger) \rightarrow H_{\text{rig}}^i((X, Y', \mathfrak{P}')/\mathcal{O}_K^\dagger)$$

are isomorphisms for all $i \geq 0$.

Proof We closely follows the proof of Theorem 6.5.2 in [9]. It suffices to prove that the natural morphism

$$j_X^\dagger \Omega_{Y[\mathfrak{P}]/S_K}^* \rightarrow \mathbf{R}u_{K*} j_X^\dagger \Omega_{Y'[\mathfrak{P}']/S_K}^*$$

is a quasi-isomorphism, this question is clearly local on \mathfrak{P} , and is local on X by Lemma 2.50. Also note that we may at any point replace Y or Y' by closed subschemes containing X .

First assume that $v = \text{id}$ is the identity, so we actually have a diagram

$$\begin{array}{ccccc}
 & & & & \mathfrak{P}' \\
 & & \nearrow i' & & \downarrow u \\
 X & \xrightarrow{j} & Y & \xrightarrow{i} & \mathfrak{P}
 \end{array}$$

with u smooth around X . In this case the question is also local on \mathfrak{P}' , and hence we may assume that the conclusions of Lemma 2.46 hold. Hence we may factor u as

$$(X, Y, \mathfrak{P}') \xrightarrow{u''} (X, Y, \widehat{\mathbb{A}}_{\mathfrak{P}}^d) \xrightarrow{u'} (X, Y, \mathfrak{P})$$

where v is étale in a neighbourhood of X . Hence by Proposition 2.45 we have

$$j_X^\dagger \Omega_{Y[\widehat{\mathbb{A}}_{\mathfrak{P}}^d]/S_K}^* \cong \mathbf{R}u_{K*}' j_X^\dagger \Omega_{Y[\mathfrak{P}']/S_K}^*,$$

by Proposition 2.48 we have

$$j_X^\dagger \Omega_{Y[\mathfrak{P}]/S_K}^* \cong \mathbf{R}u_{K*}' j_X^\dagger \Omega_{Y[\widehat{\mathbb{A}}_{\mathfrak{P}}^d]/S_K}^*$$

and combining these two then gives the result.

Next assume that v is projective, then exactly as in Lemma 6.5.1 of [9] by localising on \mathfrak{P} and X , and replacing Y' by some closed subscheme containing X we may assume that we have a morphism of frames

$$\begin{array}{ccccc} & & Y' & \xrightarrow{i''} & \mathfrak{P}'' \\ & \nearrow j' & \downarrow v & & \downarrow u' \\ X & \xrightarrow{j} & Y & \xrightarrow{i} & \mathfrak{P} \end{array}$$

with u' étale around X . We consider the diagram of frames

$$\begin{array}{ccc} (X, Y', \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{P}'') & \longrightarrow & (X, Y', \mathfrak{P}') \\ \downarrow & & \downarrow \\ (X, Y', \mathfrak{P}'') & \longrightarrow & (X, Y, \mathfrak{P}) \end{array}$$

and by the case $v = \text{id}$ already proven, we deduce that

$$\mathbf{R}u'_{K*} j_X^\dagger \Omega_{Y'[\mathfrak{P}'']/S_K}^* \cong \mathbf{R}u_{K*} j_X^\dagger \Omega_{Y[\mathfrak{P}]/S_K}^*$$

since both are isomorphic to $\mathbf{R}u''_{K*} j_X^\dagger \Omega_{Y'[\mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{P}'']/S_K}^*$, where $u'' : \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{P}'' \rightarrow \mathfrak{P}$ is the canonical map. Again using Proposition 2.45 we deduce that

$$j_X^\dagger \Omega_{Y[\mathfrak{P}]/S_K}^* \cong \mathbf{R}u'_{K*} j_X^\dagger \Omega_{Y'[\mathfrak{P}'']/S_K}^* \cong \mathbf{R}u_{K*} j_X^\dagger \Omega_{Y'[\mathfrak{P}']/S_K}^*$$

as required.

Finally we consider the general case. Thanks to Chow's Lemma (see 7.5.13 and 7.5.14 of [10]) we may blow-up \mathfrak{P}' along a closed subscheme of Y' outside X , and obtain a diagram

$$\begin{array}{ccccc} & & Y'' & \longrightarrow & \mathfrak{P}'' \\ & \nearrow & \downarrow v' & & \downarrow u' \\ X & \longrightarrow & Y' & \longrightarrow & \mathfrak{P}' \\ & \searrow & \downarrow v & & \downarrow u \\ & & Y & \longrightarrow & \mathfrak{P} \end{array}$$

where $v \circ v'$ is projective. Since the closed subscheme we are blowing up is contained in $V(\pi) \subset \mathfrak{P}'$, the induced map u'_K is an isomorphism on generic fibres, as well as between tubes. Hence we get

$$j_X^\dagger \Omega_{Y'[\mathfrak{P}']/S_K}^* \cong \mathbf{R}u'_{K*} j_X^\dagger \Omega_{Y''[\mathfrak{P}'']/S_K}^*.$$

The projective case already proven then implies that

$$j_X^\dagger \Omega_{|Y[_{\mathfrak{P}}]/S_K} \cong \mathbf{R}(u \circ u')_{K*} j_X^\dagger \Omega_{|Y''[_{\mathfrak{P}''}]/S_K}^* \cong \mathbf{R}u_{K*} j_X^\dagger \Omega_{|Y[_{\mathfrak{P}'}]/S_K}^*$$

as required. \square

Of course, by considering the fibre product of two frames this implies that the rigid cohomology groups of any two smooth and proper frames of the form (X, Y, \mathfrak{P}) and (X, Y', \mathfrak{P}') are isomorphic. Exactly as in the discussion following Corollaire 1.5 of [4], we then get a functor

$$X \mapsto H_{\text{rig}}^i(X/\mathcal{E}_K^\dagger)$$

from the category of embeddable $k((t))$ -varieties to \mathcal{E}_K^\dagger -vector spaces. We can thus summarise the results of this section as follows.

Theorem 2.52 *There are functors*

$$X \mapsto H_{\text{rig}}^i(X/\mathcal{E}_K^\dagger)$$

from the category of embeddable varieties over $k((t))$ to vector spaces over \mathcal{E}_K^\dagger , which can be calculated as $H^i(|Y[_{\mathfrak{P}}], j_X^\dagger \Omega_{|Y[_{\mathfrak{P}}]/S_K}^)$ for any smooth and proper frame (X, Y, \mathfrak{P}) . Moreover, the functoriality morphism*

$$f^* : H_{\text{rig}}^i(X/\mathcal{E}_K^\dagger) \rightarrow H_{\text{rig}}^i(X'/\mathcal{E}_K^\dagger)$$

associated to a morphism $f : X' \rightarrow X$ of embeddable varieties can be calculated as that induced by a morphism of smooth and proper frames $(X', Y', \mathfrak{P}') \rightarrow (X, Y, \mathfrak{P})$.

We will extend this to include coefficients in the next section, and to non-embeddable varieties in Chap. 4.

Remark 2.53 Actually, we get slightly more, since the proof shows that we can define the rigid cohomology $H_{\text{rig}}^i((X, Y)/\mathcal{E}_K^\dagger)$ of any embeddable pair (X, Y) consisting of an open immersion of a $k((t))$ -variety into a flat, finite type $k[[t]]$ -scheme. We do so by choosing a closed immersion $Y \rightarrow \mathfrak{P}$ into a finite type formal $\mathcal{V}[[t]]$ -scheme, smooth over $\mathcal{V}[[t]]$ around X .

One interesting special case of this is when Y is a compactification of X as a $k((t))$ -variety. In this case, we choose an embedding of Y into a finite type formal $\mathcal{O}_{\mathcal{E}_K}$ -scheme, smooth around X , then such a formal scheme is also of finite type over $\mathcal{V}[[t]]$, and smooth over $\mathcal{V}[[t]]$ around X . Hence $H_{\text{rig}}^i((X, Y)/\mathcal{E}_K^\dagger)$ is just the usual rigid cohomology $H_{\text{rig}}^i(X/\mathcal{E}_K)$. More generally, if Y is actually a $k((t))$ -variety, then $H_{\text{rig}}^i((X, Y)/\mathcal{E}_K^\dagger)$ is just the usual partially overconvergent rigid cohomology $H_{\text{rig}}^i((X, Y)/\mathcal{E}_K)$.

Another interesting special case is when Y is taken to be a model for X over $k[[t]]$, in this case $H_{\text{rig}}^i((X, Y)/\mathcal{E}_K^\dagger)$ is a version of convergent cohomology taking values in \mathcal{E}_K^\dagger rather than \mathcal{E}_K . However, since this will not be finite dimensional in general, we see no reason to believe that this should be an \mathcal{E}_K^\dagger -structure on the usual convergent cohomology $H_{\text{conv}}^i(X/\mathcal{E}_K) := H_{\text{rig}}^i((X, X)/\mathcal{E}_K)$.

2.5 Relative Coefficients and Frobenius Structures

In this section we introduce the coefficients of the cohomology theory $X \mapsto H_{\text{rig}}^i(X/\mathcal{E}_K^\dagger)$, namely overconvergent isocrystals (relative to \mathcal{E}_K^\dagger). We follow closely the definition of overconvergent isocrystals given in Chap. 7 of [9], which is the inspiration for most of the definitions and results here. The definitions we give will transparently not depend on any choice of a smooth and proper frame containing X , however, the key results will be a characterisation in terms of modules with overconvergent connection on a given frame, as well a characterisation of the pullback functor induced by a morphism of varieties in terms of a morphism of frames. We also define cohomology groups with values in an overconvergent isocrystal, which a priori *does* depend on a choice of frame, however, the results of the previous section (or rather, their proofs) will easily imply its independence from such choices. We then discuss Frobenius structures on isocrystals, and introduce the fundamental category of coefficients, the category $F\text{-Isoc}^\dagger(X/\mathcal{E}_K^\dagger)$ of overconvergent F -isocrystals on a $k((t))$ -variety X , and give a characterisation in terms of modules with overconvergent connection on a frame, together with a Frobenius structure. Nothing in this section should contain any surprises for those familiar with the theory of rigid cohomology, however, given the novel setting, we thought it best to proceed as slowly and thoroughly as we considered reasonable.

The categories of coefficients that we will consider in this section are *relative* coefficients, that is their differential structure is \mathcal{E}_K^\dagger -linear. This is the set-up most closely linked to classical rigid cohomology, and is also that in which it is perhaps most natural to state and prove the version of the p -adic monodromy theorem we will need in order to show finite dimensionality of \mathcal{E}_K^\dagger -valued rigid cohomology for smooth varieties. The reason for introducing such coefficients is two-fold: firstly, it is forced upon us by our strategy for the proof of finite dimensionality for smooth varieties in Chap. 3, since we will construct generic pushforwards in dimension 1, thus introducing twisted coefficients. Most importantly though, as alluded to in the introduction (although not explored in this book), it is these categories of coefficients that will provide an important intermediary between purely local objects on some variety X over $k((t))$ and more global ones living on a model for X .

For our eventual purposes of studying questions such as the weight monodromy conjecture and independence of ℓ , however, these coefficients will not be enough. In Chap. 5, we will introduce and study categories of ‘absolute’ coefficients, i.e. those

for which the connection is relative to K , rather than \mathcal{E}_K^\dagger . These objects will then come with a natural connection on their cohomology groups, the Gauss–Manin connection, and these groups will therefore become (φ, ∇) -modules over \mathcal{E}_K^\dagger . For now, though, we will start with the definition of the categories of ‘relative’ coefficients that we will be interested in, that is overconvergent isocrystals on varieties over $k((t))$ and frames over $\mathcal{V}[[t]]$.

Definition 2.54 1. Let $X/k((t))$ be a variety. An X -frame over $\mathcal{V}[[t]]$ is a frame (U, W, Ω) over $\mathcal{V}[[t]]$ together with a $k((t))$ -morphism $U \rightarrow X$. A morphism of X -frames is a morphism of frames commuting with the given morphism to X .

2. An overconvergent isocrystal on X/\mathcal{E}_K^\dagger is a collection of coherent $j_U^\dagger \mathcal{O}_{|W|_\Omega}$ -modules, \mathcal{E}_Ω , one for each X -frame (U, W, Ω) , together with isomorphisms $\psi_u : u^* \mathcal{E}_\Omega \rightarrow \mathcal{E}_{\Omega'}$ for every morphism of X -frames $u : (U', W', \Omega') \rightarrow (U, W, \Omega)$, which satisfy the following cocycle condition: for any commutative triangle

$$\begin{array}{ccc} (U'', W'', \Omega'') & \xrightarrow{u'} & (U', W', \Omega') \\ & \searrow u'' \quad \swarrow u & \\ & (U, W, \Omega) & \end{array}$$

of X -frames the induced diagram

$$\begin{array}{ccc} u''^* \mathcal{E}_\Omega = u'^* u^* \mathcal{E}_\Omega & \xrightarrow{u'^* \psi_u} & u'^* \mathcal{E}_{\Omega'} \\ & \searrow \psi_{u''} \quad \swarrow \psi_{u'} & \\ & \mathcal{E}_{\Omega''} & \end{array}$$

commutes. The category of such objects is denoted $\text{Isoc}^\dagger(X/\mathcal{E}_K^\dagger)$.

3. Let (X, Y, \mathfrak{P}) be a $\mathcal{V}[[t]]$ -frame. An overconvergent isocrystal on $(X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger$ is a collection of coherent $j_U^\dagger \mathcal{O}_{|W|_\Omega}$ -modules, \mathcal{E}_Ω , one for each frame (U, W, Ω) over (X, Y, \mathfrak{P}) , together with isomorphisms $u^* \mathcal{E}_\Omega \rightarrow \mathcal{E}_{\Omega'}$ for every morphism of frames $u : (U', W', \Omega') \rightarrow (U, W, \Omega)$ such that the diagram

$$\begin{array}{ccc} (U', W') & \longrightarrow & (U, W) \\ & \searrow & \downarrow \\ & & (X, Y) \end{array}$$

commutes, which satisfy a similar cocycle condition. The category of such objects is denoted $\text{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$.

These are abelian tensor categories, which admit internal hom objects, and are such that the natural realisation functors $\mathcal{E} \mapsto \mathcal{E}_\Omega$ commute with tensor products.

There is an obvious functor

$$\mathrm{Isoc}^\dagger(X/\mathcal{O}_K^\dagger) \rightarrow \mathrm{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{O}_K^\dagger)$$

induced by the forgetful functor from frames over (X, Y, \mathfrak{P}) to frames over X . It is straightforward to verify that the category $\mathrm{Isoc}^\dagger(X/\mathcal{O}_K^\dagger)$ is local for the Zariski topology on X , and that $\mathrm{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{O}_K^\dagger)$ is local for the Zariski topology on \mathfrak{P} . Zariski locality of $\mathrm{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{O}_K^\dagger)$ with respect to X follows from the lemma below.

Lemma 2.55 *Let (X, Y, \mathfrak{P}) be a $\mathcal{V}[[t]]$ -frame. Then restriction followed by push-forward induces an equivalence of categories*

$$\mathrm{colim}_V \mathrm{Coh}(\mathcal{O}_V) \rightarrow \mathrm{Coh}(j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}}})$$

where the colimit runs over all open neighbourhoods V of $|X|_{\mathfrak{P}}$ inside $|Y|_{\mathfrak{P}}$.

Proof Entirely similar to the proof that we will give later on for modules with connection, Lemma 2.66. \square

If $u : (X, Y', \mathfrak{P}') \rightarrow (X, Y, \mathfrak{P})$ is a morphism inducing an isomorphism between cofinal systems of neighbourhoods of $|X|_{\mathfrak{P}}$ in $|Y|_{\mathfrak{P}}$ and $|X|_{\mathfrak{P}'}$ in $|Y'|_{\mathfrak{P}'}$, then the pullback functor

$$u^* : \mathrm{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{O}_K^\dagger) \rightarrow \mathrm{Isoc}^\dagger((X, Y', \mathfrak{P}')/\mathcal{O}_K^\dagger)$$

is an equivalence of categories. The first step in interpreting overconvergent isocrystals on an embeddable variety is the following.

Proposition 2.56 *Let $u : (X, Y', \mathfrak{P}') \rightarrow (X, Y, \mathfrak{P})$ be a smooth and proper morphism of frames over $\mathcal{V}[[t]]$. Then the pullback functor*

$$u^* : \mathrm{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{O}_K^\dagger) \rightarrow \mathrm{Isoc}^\dagger((X, Y', \mathfrak{P}')/\mathcal{O}_K^\dagger)$$

is an equivalence of categories.

Proof The proof, as the proof of the corollary below, goes exactly as in Chap. 7 of [9], and is very similar to the proof of Theorem 2.51 above. The question is local on \mathfrak{P} and X , and we may also at any point replace either Y or Y' by a closed subscheme containing X . Again, we divide the proof into three stages.

First assume that the induced map $Y' \rightarrow Y$ is the identity, so that we actually have a diagram

$$\begin{array}{ccccc} & & & \mathfrak{P}' & \\ & & i' \nearrow & \downarrow u & \\ X & \xrightarrow{j} & Y & \xrightarrow{i} & \mathfrak{P} \end{array}$$

with u smooth around X . Here the question is also local on \mathfrak{P}' , and hence we may assume that we can factor u as

$$(X, Y', \mathfrak{P}') \xrightarrow{v} (X, Y, \widehat{\mathbb{A}}_{\mathfrak{P}}^d) \xrightarrow{w} (X, Y, \mathfrak{P})$$

where v is proper and étale. Since w admits a section inducing the identity on X , it is formal that it induces an equivalence

$$\mathrm{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \xrightarrow{\sim} \mathrm{Isoc}^\dagger((X, Y, \widehat{\mathbb{A}}_{\mathfrak{P}}^d)/\mathcal{E}_K^\dagger)$$

and the fact that v induces an equivalence

$$\mathrm{Isoc}^\dagger((X, Y, \widehat{\mathbb{A}}_{\mathfrak{P}}^d)/\mathcal{E}_K^\dagger) \rightarrow \mathrm{Isoc}^\dagger((X, Y', \mathfrak{P}')/\mathcal{E}_K^\dagger)$$

follows from the strong fibration theorem, i.e. Proposition 2.45.

Next we assume that $Y' \rightarrow Y$ is projective, hence by localising on X and \mathfrak{P} and replacing Y' if necessary we may assume that we have a morphism of frames

$$\begin{array}{ccccc} & & Y' & \xrightarrow{i''} & \mathfrak{P}'' \\ & \nearrow j' & \downarrow v & & \downarrow u' \\ X & \xrightarrow{j} & Y & \xrightarrow{i} & \mathfrak{P} \end{array}$$

with u' étale around X (as in the proof of Theorem 2.51). We consider the diagram of frames

$$\begin{array}{ccc} (X, Y', \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{P}'') & \longrightarrow & (X, Y', \mathfrak{P}') \\ \downarrow & & \downarrow \\ (X, Y', \mathfrak{P}'') & \longrightarrow & (X, Y, \mathfrak{P}) \end{array}$$

and by the case $v = \mathrm{id}$ already proven, together with the strong fibration theorem, we deduce that

$$\begin{aligned} \mathrm{Isoc}^\dagger((X, Y', \mathfrak{P}')/\mathcal{E}_K^\dagger) &\cong \mathrm{Isoc}^\dagger((X, Y', \mathfrak{P}' \times_{\mathfrak{P}} \mathfrak{P}'')/\mathcal{E}_K^\dagger) \\ &\cong \mathrm{Isoc}^\dagger((X, Y', \mathfrak{P}'')/\mathcal{E}_K^\dagger) \cong \mathrm{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger). \end{aligned}$$

Finally we consider the general case. As in the proof of Theorem 2.51 we may blow up \mathfrak{P}' along a closed subscheme of Y' outside of X to obtain a frame (X, Y'', \mathfrak{P}'') such that Y'' is projective over Y . Now, since the blow-up induces an isomorphism between a cofinal system of neighbourhoods of $]X[_{\mathfrak{P}'} \text{ in }]Y'[_{\mathfrak{P}'}$ and $]X[_{\mathfrak{P}''} \text{ in }]Y''[_{\mathfrak{P}''}$, we therefore have

$$\mathrm{Isoc}^\dagger((X, Y', \mathfrak{P}')/\mathcal{E}_K^\dagger) \cong \mathrm{Isoc}^\dagger((X, Y'', \mathfrak{P}'')/\mathcal{E}_K^\dagger)$$

But by the projective case already proven, we have

$$\mathrm{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \cong \mathrm{Isoc}^\dagger((X, Y'', \mathfrak{P}'')/\mathcal{E}_K^\dagger)$$

and the proof is complete. \square

Corollary 2.57 *Let (X, Y, \mathfrak{P}) be a smooth and proper frame over $\mathcal{V}[\![t]\!]$. Then the forgetful functor*

$$\mathrm{Isoc}^\dagger(X/\mathcal{E}_K^\dagger) \rightarrow \mathrm{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$$

is an equivalence of categories.

We can now interpret $\mathrm{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$ more concretely, first in terms of modules with stratifications, and then in terms of modules with connections. Let us first recall the definitions of stratifications and connections in rigid geometry, as well as the notion of an overconvergent stratification for frames.

For a frame (X, Y, \mathfrak{P}) and any $n \geq 1$, we let \mathfrak{P}^n denote the n -fold fibre product of \mathfrak{P} with itself over $\mathcal{V}[\![t]\!]$, the diagonal embedding allows us to consider the frames (X, Y, \mathfrak{P}^n) , and the natural inclusions and projections between the different \mathfrak{P}^n induce morphisms of frames

$$(X, Y, \mathfrak{P}^n) \rightarrow (X, Y, \mathfrak{P}^m)$$

which we will generally denote by the same letters, e.g. Δ for the diagonal morphism

$$(X, Y, \mathfrak{P}) \rightarrow (X, Y, \mathfrak{P}^2),$$

p_1, p_2 for the two morphisms of frames

$$(X, Y, \mathfrak{P}^2) \rightarrow (X, Y, \mathfrak{P})$$

and p_{12}, p_{23}, p_{13} for the projections from the triple product to the double product.

Also, for any smooth rigid variety \mathcal{X} over S_K , we let $\mathcal{X}^{(n)}$ denote the n th infinitesimal neighbourhood of \mathcal{X} inside $\mathcal{X} \times_{S_K} \mathcal{X}$, and $p_i^{(n)} : \mathcal{X}^{(n)} \rightarrow \mathcal{X}$ for $i = 1, 2$ the two natural projections.

Definition 2.58 Let (X, Y, \mathfrak{P}) be a smooth frame over $\mathcal{V}[\![t]\!]$.

1. An overconvergent stratification on a $j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}}}$ -module \mathcal{E} is an isomorphism

$$\epsilon : p_2^* \mathcal{E} \rightarrow p_1^* \mathcal{E}$$

of $j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}^2}}$ -modules, called the Taylor isomorphism, such that $\Delta^*(\epsilon) = \mathrm{id}$ and $p_{13}^*(\epsilon) = p_{12}^*(\epsilon) \circ p_{23}^*(\epsilon)$ on $|Y|_{\mathfrak{P}^3}$. The category of coherent $j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}}}$ -modules with overconvergent stratification is denoted $\mathrm{Strat}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$.

2. A stratification on an $\mathcal{O}_{|Y|_{\mathfrak{P}}}$ -module \mathcal{E} is a collection of compatible isomorphisms

$$p_2^{(n)*} \mathcal{E} \xrightarrow{\sim} p_1^{(n)*} \mathcal{E}$$

satisfying a cocycle condition similar to that for overconvergent stratifications. The category of coherent $j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}}}$ -modules with a stratifications as $\mathcal{O}_{|Y|_{\mathfrak{P}}}$ -modules is denoted $\text{Strat}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$.

3. An integrable connection on a $j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}}}$ -module \mathcal{E} is a map

$$\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_{|Y|_{\mathfrak{P}}/S_K}^1$$

which satisfies the Leibniz rule and is such that the induced map

$$\nabla^2 : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_{|Y|_{\mathfrak{P}}/S_K}^2$$

is zero. The category of coherent $j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}}}$ -modules with an integrable connection is denoted $\text{MIC}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$

Thus in the usual way there is an equivalence of categories

$$\text{Strat}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \rightarrow \text{MIC}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$$

and pulling back via the natural morphism

$$|Y|_{\mathfrak{P}}^{(n)} \rightarrow |Y|_{\mathfrak{P}^2}$$

gives a functor

$$\text{Strat}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \rightarrow \text{Strat}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger).$$

Modules with overconvergent stratifications are related to overconvergent isocrystals through the following construction. Given a smooth and proper frame (X, Y, \mathfrak{P}) and $\mathcal{F} \in \text{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$, we have isomorphisms

$$p_2^* \mathcal{F}_{\mathfrak{P}} \rightarrow \mathcal{F}_{\mathfrak{P}^2} \leftarrow p_1^* \mathcal{F}_{\mathfrak{P}}$$

and hence an isomorphism $p_2^* \mathcal{F}_{\mathfrak{P}} \rightarrow p_1^* \mathcal{F}_{\mathfrak{P}}$ which satisfies the cocycle condition on \mathfrak{P}^3 . This induces a functor

$$\text{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \rightarrow \text{Strat}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$$

given by $\mathcal{F} \mapsto \mathcal{E} := \mathcal{F}_{\mathfrak{P}}$. This functor is easily checked to be an equivalence. Thus we obtain a series of functors

$$\begin{aligned} \text{Isoc}^\dagger(X/\mathcal{E}_K^\dagger) &\rightarrow \text{Isoc}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \rightarrow \text{Strat}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \\ &\rightarrow \text{Strat}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \rightarrow \text{MIC}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \end{aligned}$$

where everything except $\text{Strat}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \rightarrow \text{Strat}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$ is an equivalence. We will shortly show that in fact it is fully faithful, however, before we do so we will need the following lemma.

Lemma 2.59 *Let $]X[_{\mathfrak{P}}^\circ$ denote the interior tube of X as in Remark 2.25, this is a rigid space locally of finite type over \mathcal{E}_K . Then the restriction functor*

$$\text{Coh}(j_X^\dagger \mathcal{O}_{]Y[_{\mathfrak{P}}}) \rightarrow \text{Coh}(\mathcal{O}_{]X[_{\mathfrak{P}}^\circ})$$

is faithful.

Proof The statement is local on $]Y[_{\mathfrak{P}}$, so by Lemma 2.55 we may replace $]Y[_{\mathfrak{P}}$ by $[Y]_n$. Suppose that $f : \mathcal{E} \rightarrow \mathcal{F}$ is a morphism of coherent $j_X^\dagger \mathcal{O}_{[Y]_n}$ -modules which restrict to zero on $]X[_{\mathfrak{P}}^\circ \cap [Y]_n$, by Lemma 2.55 we may assume that there is a neighbourhood V of $]X[_{\mathfrak{P}}^\circ \cap [Y]_n$ such that f arises from a morphism $f_V : E_V \rightarrow F_V$ of coherent \mathcal{O}_V -modules. The support of $\text{im}(f_V)$ is contained in a closed analytic subspace of V , and does not meet $]X[_{\mathfrak{P}}^\circ \cap [Y]_n$. Hence by Proposition 2.32 there must exist a neighbourhood V' of $]X[_{\mathfrak{P}}^\circ \cap [Y]_n$ contained in V such that the support of $\text{im}(f_V)$ does not meet V' . Hence f_V is zero on V' and thus f is zero. \square

Proposition 2.60 *The functor*

$$\text{Strat}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \rightarrow \text{Strat}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger).$$

is fully faithful, and commutes with tensor products and internal hom.

Proof Since both categories admit a faithful functor to the category of coherent $j_X^\dagger \mathcal{O}_{]Y[_{\mathfrak{P}}}$ -modules, it suffices to prove that the functor is full. That is, we must show that if a morphism $\mathcal{E} \rightarrow \mathcal{F}$ between modules with overconvergent stratification commutes with the finite level Taylor isomorphisms, then it commutes with the full overconvergent Taylor isomorphism. By looking at the difference between the two natural maps $p_2^* \mathcal{E} \rightarrow p_1^* \mathcal{F}$ arising from the diagram

$$\begin{array}{ccc} p_2^* \mathcal{E} & \longrightarrow & p_1^* \mathcal{E} \\ \downarrow & & \downarrow \\ p_2^* \mathcal{F} & \longrightarrow & p_1^* \mathcal{F} \end{array}$$

(as well as the maps $p_2^{(n)*} \mathcal{E} \rightarrow p_1^{(n)*} \mathcal{F}$ for all n), it suffices to show that if $\psi : p_2^* \mathcal{E} \rightarrow p_1^* \mathcal{F}$ is such that $\psi|_{]Y[_{\mathfrak{P}}^{(n)}} = 0$ for all n , then $\psi = 0$. Note that this is a question purely about coherent $j_X^\dagger \mathcal{O}_{]X[_{\mathfrak{P}}}$ -modules.

Let \mathfrak{P}' denote $\mathfrak{P} \otimes_{\mathcal{V}[[t]]} \mathcal{O}_{\mathcal{E}_K}$ and Y' denote $Y \otimes_{k[[t]]} k((t))$. Then Lemma 2.59 above implies that the restriction functor from coherent $j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}}}$ -modules to coherent $j_X^\dagger \mathcal{O}_{|Y'|_{\mathfrak{P}'}}$ -modules is faithful. Similarly the restriction functor from $j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}^2}}$ -modules to $j_X^\dagger \mathcal{O}_{|Y'|_{\mathfrak{P}^2}}$ -modules is faithful, so it suffices to prove the corresponding statement for the tubes over \mathcal{E}_K , that is if we have a morphism $\psi : p_2^* \mathcal{E} \rightarrow p_1^* \mathcal{F}$ between coherent $j_X^\dagger \mathcal{O}_{|Y'|_{\mathfrak{P}^2}}$ -modules such that $\psi|_{|Y'|_{\mathfrak{P}^2}}^{(n)} = 0$ for all n , then $\psi = 0$. But this is just a translation into the language of adic spaces of Lemma 7.2.7 of [9].

The statement about tensor product and internal hom is straightforward. \square

Definition 2.61 We say that an integrable connection is overconvergent if it is in the essential image of this functor. The category of modules with overconvergent connections is denoted

$$\mathrm{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \subset \mathrm{MIC}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger).$$

In other words, a connection is overconvergent if the Taylor series converges in a neighbourhood of $|X|_{\mathfrak{P}^2}$. The following theorem summarises the results of this section so far.

Theorem 2.62 *Let X be a $k((t))$ -variety, and (X, Y, \mathfrak{P}) a smooth and proper frame containing X . Then the realisation functor $\mathcal{E} \mapsto \mathcal{E}_{\mathfrak{P}}$ induces an equivalence of categories*

$$\mathrm{Isoc}^\dagger(X/\mathcal{E}_K^\dagger) \rightarrow \mathrm{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$$

from overconvergent isocrystals on X/\mathcal{E}_K^\dagger to coherent $j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}}}$ -modules with an overconvergent, integrable connection.

Of course, this equivalence is natural in the sense that if $u : (X', Y', \mathfrak{P}') \rightarrow (X, Y, \mathfrak{P})$ is a morphism of smooth and proper frames over $\mathcal{V}[[t]]$, and $f : X' \rightarrow X$ the induced morphism of $k((t))$ -varieties, then the diagram

$$\begin{array}{ccc} \mathrm{Isoc}^\dagger(X/\mathcal{E}_K^\dagger) & \xrightarrow{f^*} & \mathrm{Isoc}^\dagger(X'/\mathcal{E}_K^\dagger) \\ \downarrow & & \downarrow \\ \mathrm{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) & \xrightarrow{u^*} & \mathrm{MIC}^\dagger((X', Y', \mathfrak{P}')/\mathcal{E}_K^\dagger) \end{array}$$

commutes up to natural isomorphism. The following particular case of this naturality will be useful.

Corollary 2.63 *Let $u : (X, Y', \mathfrak{P}') \rightarrow (X, Y, \mathfrak{P})$ be a morphism of frames inducing the identity on X , and let $\mathcal{F} \in \mathrm{Isoc}^\dagger(X/\mathcal{E}_K^\dagger)$ be an overconvergent isocrystal, with realisations $\mathcal{E} \in \mathrm{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$ and $\mathcal{E}' \in \mathrm{MIC}^\dagger((X, Y', \mathfrak{P}')/\mathcal{E}_K^\dagger)$. Then $\mathcal{E}' \cong u^* \mathcal{E}$ as modules with connection.*

We now define cohomology groups with coefficients in an overconvergent isocrystal. So suppose that X is an embeddable variety and $\mathcal{E} \in \text{Isoc}^\dagger(X/\mathcal{E}_K^\dagger)$ is an overconvergent isocrystal. For every smooth and proper frame (X, Y, \mathfrak{P}) over $\mathcal{V}[[t]]$ we can realise \mathcal{E} on (X, Y, \mathfrak{P}) to give a $j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}}}$ -module $\mathcal{E}_{\mathfrak{P}}$ with an overconvergent integrable connection. Then we define the cohomology groups

$$H_{\text{rig}}^i((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger, \mathcal{E}) := H^i(|Y|_{\mathfrak{P}}, \mathcal{E} \otimes \Omega_{|Y|_{\mathfrak{P}}/S_K}^*),$$

as in the constant coefficient case these are vector spaces over \mathcal{E}_K^\dagger . The fact that this is compatible with the definition we gave in the constant case is a consequence of the following lemma.

Lemma 2.64 *Let (X, Y, \mathfrak{P}) be a frame. Then for any $\mathcal{O}_{|Y|_{\mathfrak{P}}}$ -modules \mathcal{E}, \mathcal{F} we have a natural isomorphism*

$$j_X^\dagger(\mathcal{E} \otimes_{\mathcal{O}_{|Y|_{\mathfrak{P}}}} \mathcal{F}) \cong j_X^\dagger \mathcal{E} \otimes_{\mathcal{O}_{|Y|_{\mathfrak{P}}}} \mathcal{F}$$

of $j_X^\dagger \mathcal{O}_{|Y|_{\mathfrak{P}}}$ -modules.

Proof First note that since $j_X^\dagger \mathcal{E}$ is supported on $|X|_{\mathfrak{P}}$, so is $j_X^\dagger \mathcal{E} \otimes_{\mathcal{O}_{|Y|_{\mathfrak{P}}}} \mathcal{F}$, and hence it suffices to prove the lemma after applying j^{-1} . But now we simply have

$$\begin{aligned} j^{-1}(j_X^\dagger(\mathcal{E} \otimes_{\mathcal{O}_{|Y|_{\mathfrak{P}}}} \mathcal{F})) &= j^{-1}(\mathcal{E} \otimes_{\mathcal{O}_{|Y|_{\mathfrak{P}}}} \mathcal{F}) \\ &= j^{-1} \mathcal{E} \otimes_{j^{-1} \mathcal{O}_{|Y|_{\mathfrak{P}}}} j^{-1} \mathcal{F} \\ &= j^{-1} j_* j^{-1} \mathcal{E} \otimes_{j^{-1} \mathcal{O}_{|Y|_{\mathfrak{P}}}} j^{-1} \mathcal{F} \\ &= j^{-1}(j_* j^{-1} \mathcal{E} \otimes_{\mathcal{O}_{|Y|_{\mathfrak{P}}}} \mathcal{F}) \\ &= j^{-1}(j_X^\dagger \mathcal{E} \otimes_{\mathcal{O}_{|Y|_{\mathfrak{P}}}} \mathcal{F}) \end{aligned}$$

as required. □

Note that this lemma also implies that rigid cohomology can be calculated as

$$H_{\text{rig}}^i((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger, \mathcal{E}) := H^i(|X|_{\mathfrak{P}}, j^{-1}(\mathcal{E} \otimes \Omega_{|Y|_{\mathfrak{P}}/S_K}^*)).$$

Theorem 2.65 *Up to natural isomorphism $H_{\text{rig}}^i((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger, \mathcal{E})$ only depends on X and \mathcal{E} and not on the choice of smooth and proper frame (X, Y, \mathfrak{P}) . Moreover, the pullback morphism*

$$u^{-1} : H_{\text{rig}}^i((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger, \mathcal{E}) \rightarrow H_{\text{rig}}^i((X', Y', \mathfrak{P}')/\mathcal{E}_K^\dagger, f^* \mathcal{E})$$

determined by any morphism of frames $u : (X', Y', \mathfrak{P}') \rightarrow (X, Y, \mathfrak{P})$ only depends on the morphism $f : X' \rightarrow X$ induced by u . Hence we get a functor

$$(X, \mathcal{E}) \rightarrow H_{\text{rig}}^i(X/\mathcal{E}_K^\dagger, \mathcal{E})$$

in the sense that for any pair of morphisms $f : X' \rightarrow X$ and $f^* \mathcal{E} \rightarrow \mathcal{E}'$ there is an induced morphism

$$H_{\text{rig}}^i(X/\mathcal{E}_K^\dagger, \mathcal{E}) \rightarrow H_{\text{rig}}^i(X'/\mathcal{E}_K^\dagger, \mathcal{E}').$$

Proof Identical to the case of constant coefficients $\mathcal{E} = \mathcal{O}_{X/\mathcal{E}_K^\dagger}^\dagger$ treated in the previous section (Theorem 2.51 and the following discussion). \square

We next show a characterisation of overconvergence of a connection, analogous to Theorem 4.3.9 of [9]. This characterisation will be local, so we will need to know that overconvergence itself is a suitably local property. Clearly $\text{MIC}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$ and $\text{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$ are local on \mathfrak{P} , we will need to know that they are also local on X .

We will let (X, Y, \mathfrak{P}) be a smooth frame, and for all open neighbourhoods V of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$ we will let $\text{MIC}(\mathcal{O}_V/S_K)$ denote the category of coherent \mathcal{O}_V -modules with integrable connection relative to S_K .

Lemma 2.66 *Restriction via $]X[_{\mathfrak{P}} \rightarrow V$ followed by push-forward along $]X[_{\mathfrak{P}} \rightarrow]Y[_{\mathfrak{P}}$ induces an equivalence of categories*

$$\text{colim}_V \text{MIC}(\mathcal{O}_V/S_K) \rightarrow \text{MIC}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$$

where the colimit runs over all open neighbourhoods V of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$.

Proof Let us first prove the corresponding result where we replace $]Y[_{\mathfrak{P}}$ by the quasi-compact tubes $[Y]_n$, so that the quasi-compact opens $V_{n,m} = [Y]_n \cap U_m$ form a cofinal system of neighbourhoods of $]X[_{\mathfrak{P}} \cap [Y]_n$ inside $[Y]_n$. By using internal hom for coherent modules with connection, full faithfulness boils down to showing that for any coherent module with integrable connection \mathcal{E} on some $V_{n,m}$,

$$\text{colim}_{m' \geq m} \Gamma(V_{n,m'}, \mathcal{E})^{\nabla=0} \xrightarrow{\sim} \Gamma([Y]_n, j_* j^{-1} \mathcal{E})^{\nabla=0}.$$

Since the natural morphism

$$\text{colim}_{m' \geq m} \Gamma(V_{n,m'}, \mathcal{E}) \rightarrow \Gamma([Y]_n, j_* j^{-1} \mathcal{E})$$

is horizontal, it certainly suffices to prove that this latter morphism is an isomorphism, which follows from Lemma 2.19 together with the fact that global sections commute with filtered direct limits on quasi-compact topological spaces. Note that this argument also shows that

$$\text{colim}_V \text{Coh}(\mathcal{O}_V) \rightarrow \text{Coh}(j_X^\dagger \mathcal{O}_{[Y]_n})$$

is fully faithful.

To show essential surjectivity, let \mathcal{E} be some coherent $j_X^\dagger \mathcal{O}_{[Y]_n}$ -module with connection, we first claim that \mathcal{E} itself comes from a coherent \mathcal{O}_V -module for some V . By the full faithfulness for coherent modules, this is local on a finite open covering of $[Y]_n$, hence we may assume that \mathcal{E} has a presentation. Again using the full faithfulness for coherent modules, this presentation must come from a presentation

$$\mathcal{O}_V^k \rightarrow \mathcal{O}_V^{k'} \rightarrow E_V \rightarrow 0$$

on some V . Now an entirely similar argument to above, using internal hom for abelian sheaves and Lemma 2.19 shows that the integrable connection on \mathcal{E} must come from some integrable connection on $E_V|_{V'}$ for some $V' \subset V$.

We now turn to the original case. So suppose that $\psi_V : E_V \rightarrow F_V$ is a morphism of coherent \mathcal{O}_V -modules with connection on some neighbourhood V of $]X[_{\mathfrak{P}}$, such that the induced morphism between coherent $j_X^\dagger \mathcal{O}_{[Y]_{\mathfrak{P}}}$ -modules is zero. Then for all n the induced morphism of $j_X^\dagger \mathcal{O}_{[Y]_n}$ -modules is zero, and hence there exists some m such that the restriction of φ_V to $V \cap V_{n,m}$ is zero. By taking the union over all n , there thus exists some sequence $\underline{m}(n) \rightarrow \infty$ such that the restriction of ψ_V to $V \cap V_{\underline{m}}$ is zero. Hence

$$\operatorname{colim}_V \operatorname{MIC}(\mathcal{O}_V/S_K) \rightarrow \operatorname{MIC}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$$

is faithful. Similarly, if E_V and F_V are coherent modules with connection on some neighbourhood V of $]X[_{\mathfrak{P}}$, and $\psi : \mathcal{E} \rightarrow \mathcal{F}$ is a horizontal morphism between the induced coherent $j_X^\dagger \mathcal{O}_{[Y]_{\mathfrak{P}}}$ -modules, then for all n we can find some $m = \underline{m}(n)$ such that $V_{n,m} \subset V$ and $\psi|_{[Y]_n}$ comes from a morphism $\psi_n : E_V|_{V_{n,m}} \rightarrow F_V|_{V_{n,m}}$. By increasing each $m(n)$ in turn, we can ensure that $\psi_n|_{[Y]_{n-1}}$ agrees with ψ_{n-1} . Hence taking the union over all n gives us a morphism $E_V|_{V_{\underline{m}}} \rightarrow F_V|_{V_{\underline{m}}}$ for some \underline{m} . Hence the functor is full.

To show essential surjectivity, suppose that we have some coherent $j_X^\dagger \mathcal{O}_{[Y]_{\mathfrak{P}}}$ -module with integrable connection \mathcal{E} . Then for all n we know that there exists some $m = m(n)$ such that $\mathcal{E}|_{[Y]_n}$ comes from some coherent module with connection E_n on $V_{n,m}$. Again, by possibly increasing each $\underline{m}(n)$ in turn, we can ensure that we have isomorphisms $E_n|_{V_{n-1, \underline{m}(n-1)}} \cong E_{n-1}$, and hence we can glue the E_n to give a coherent module with connection $E_{\underline{m}}$ on $V_{\underline{m}}$ inducing \mathcal{E} . \square

Corollary 2.67 *Both the category $\operatorname{MIC}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$ and the condition of being overconvergent are local on X .*

Proof Assume that we have an open cover $X = \cup_j X_j$, and compatible objects $\mathcal{E}_j \in \operatorname{MIC}((X_j, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$. By the previous lemma these extend to a compatible collection of coherent \mathcal{O}_{V_j} -modules with connection on some open neighbourhoods V_j of $]X_j[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$. Hence these glue to give a coherent module with connection on $V = \cup_j V_j$, which is a neighbourhood of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$. An entirely similar argument shows that morphisms glue as well. Thus $\operatorname{MIC}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$ is local on

X , and since $\text{Strat}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$ is also local on X , so is the overconvergence condition. \square

Thus we can test overconvergence locally, and we have the following more concrete criterion. Let (X, Y, \mathfrak{P}) be a smooth frame such that \mathfrak{P} is affine and $X = Y \cap D(\bar{g})$ for \bar{g} the reduction of some $g \in \mathcal{O}_{\mathfrak{P}}$. Assume further that $\Omega_{\mathfrak{P}/\mathcal{V}[\![t]\!]}^1$ has a basis dt_1, \dots, dt_n in a neighbourhood of X , for some functions $t_i \in \mathcal{O}_{\mathfrak{P}}$. Let $(\mathcal{E}, \nabla) \in \text{MIC}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$, and let $\partial_i : \mathcal{E} \rightarrow \mathcal{E}$ be the derivations corresponding to dt_i . For any multi-index $\underline{k} = (k_1, \dots, k_l)$ we set $\partial^{\underline{k}} = \partial_1^{k_1} \dots \partial_l^{k_l}$.

For any coherent sheaf \mathcal{E} on an affinoid rigid space $\text{Spa}(A)$, the space of global sections $M := \Gamma(\text{Spa}(A), \mathcal{E})$ is a finitely generated A -module. For any presentation of M , we therefore obtain a Banach norm on M arising from the product norm on A^n (after fixing a Banach norm on A), and as usual one can check that up to equivalence this does not depend on the choice of presentation.

Proposition 2.68 *Let V be an open neighbourhood of $]X[_{\mathfrak{P}}$ inside $]Y[_{\mathfrak{P}}$ such that dt_1, \dots, dt_l are a basis for Ω_{V/S_K}^1 and (\mathcal{E}, ∇) arises from a module with integrable connection (E_V, ∇_V) on V . Then (\mathcal{E}, ∇) is overconvergent if and only if for all n , there exists some m and some $d \geq n$ such that $]Y[_d \cap U_m \subset V$, and for every section $e \in \Gamma([Y]_d \cap U_m, E_V)$ we have*

$$\left\| \frac{\partial^{\underline{k}} e}{\underline{k}!} \right\| (r^{-\frac{|\underline{k}|}{n}}) \rightarrow 0$$

where $\|\cdot\|$ is some choice of Banach norm on $\Gamma([Y]_n \cap U_m, E_V)$ as above.

Proof Define $\tau : \mathfrak{P} \times_{\mathcal{V}[\![t]\!]} \mathfrak{P} \rightarrow \widehat{\mathbb{A}}_{\mathfrak{P}}^l$ by $\tau_i = p_1^*(t_i) - p_2^*(t_i)$. Let $V_{n,m} = [Y]_n \cap U_m$, be the standard neighbourhoods of $[Y]_n \cap]X[_{\mathfrak{P}}$ inside $[Y]_n$, and let $W_{n,m} \subset]Y[_{\mathfrak{P}^2}$ denote the similar standard neighbourhoods of $]X[_{\mathfrak{P}^2}$ inside $]Y[_{\mathfrak{P}^2}$. By Proposition 2.45 and its proof, there exists some $d \geq n$ such that for all $m \gg 0$, τ induces an isomorphism $W \cong V_{n,m} \times_{\mathbb{D}_K^l} \mathbb{D}_{S_K}^l(r^{-1/n})$ for some open $W_{n,m} \subset W \subset W_{d,m}$, where

$$\mathbb{D}_{S_K}^l(r^{-1/n}) = \text{Spa}(S_K \langle r^{1/n} X_1, \dots, r^{1/n} X_l \rangle)$$

is the polydisc of radius $r^{-1/n}$. Then we have two maps $p_1, p_2 : V_{n,m} \times \mathbb{D}_{S_K}^l(r^{-1/n}) \rightrightarrows V_{d,m}$, and possibly after increasing m we may assume that $V_{d,m} \subset V$.

If we let $M = \Gamma(V_{d,m}, E_V)$, $A = \Gamma(V_{d,m}, \mathcal{O}_{]Y[_{\mathfrak{P}}})$ and $B = \Gamma(V_{n,m}, \mathcal{O}_{]Y[_{\mathfrak{P}}})$ then the formal Taylor morphism

$$E_V|_{V_{d,m}} \rightarrow \varprojlim_n (p_2^{(n)*} E_V|_{V_{n,m} \times \mathbb{D}_{S_K}^l(r^{-1/n})})$$

can be identified with the map

$$M \rightarrow M \otimes_A B[[\tau]]$$

$$e \mapsto \sum_{\underline{k}} \frac{\partial^{\underline{k}} e}{\underline{k}!} \tau^{\underline{k}}$$

where $\tau^{\underline{k}} = \tau_1^{k_1} \dots \tau_l^{k_l}$. Then \mathcal{E} is overconvergent if and only if we can choose m so that this Taylor series actually converges on $V_{n,m} \times \mathbb{D}_{S_K}^l(r^{-1/n})$, or in other words, if we can choose m such that

$$\sum_{\underline{k}} \frac{\partial^{\underline{k}} e}{\underline{k}!} \tau^{\underline{k}} \in M \otimes_A B\langle r^{1/n} \tau \rangle$$

for all e . The proposition follows. \square

Corollary 2.69 *For any smooth frame (X, Y, \mathfrak{P}) the full subcategory*

$$\mathrm{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \subset \mathrm{MIC}((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$$

is stable under subobjects and quotients.

Proof This just follows from the fact that for any map $E_F \rightarrow F_V$ of coherent \mathcal{O}_V -modules, the map

$$\Gamma([Y]_n \cap U_m, F_V) \rightarrow \Gamma([Y]_n \cap U_m, E_V)$$

is a strict morphism of Banach $\Gamma([Y]_n \cap U_m, \mathcal{O}_V)$ -modules. \square

We will also need to know functoriality of coefficients and cohomology under certain extensions of \mathcal{E}_K^\dagger , in particular the following three cases.

1. The finite extension of \mathcal{E}_K^\dagger corresponding to a finite separable extension of $k((t))$.
2. The extension determined by some Frobenius lift $\sigma : \mathcal{E}_K^\dagger \rightarrow \mathcal{E}_K^\dagger$.
3. The extension $\mathcal{E}_K^\dagger \rightarrow \mathcal{E}_K$.

Firstly, denote by $F/k((t))$ be a finite separable extension of $k((t))$, by $A \subset F$ its ring of integers, and let l/k the induced extension of residue fields. Let L be the unramified extension of K lifting l/k , with extension \mathcal{W}/\mathcal{V} of rings of integers. Let $\mathcal{E}_K^{\dagger, F}$ and \mathcal{E}_K^F be the unramified extensions of \mathcal{E}_K^\dagger and \mathcal{E}_K respectively lifting $F/k((t))$, with rings of integers $\mathcal{O}_{\mathcal{E}_K^{\dagger, F}}$ and $\mathcal{O}_{\mathcal{E}_K^F}$ respectively. These are unique up to unique isomorphism, and can be described concretely as follows. Choose a uniformiser u for F , so that $F \cong l((u))$. Then we have

$$\mathcal{E}_K^F \cong \left\{ \sum_i a_i u^i \in L[[u, u^{-1}]] \mid \sup_i |a_i| < \infty, a_i \rightarrow 0 \text{ as } i \rightarrow -\infty \right\}$$

$$\mathcal{O}_K^{\dagger, F} \cong \left\{ \sum_i a_i u^i \in \mathcal{O}_K^F \mid \exists \eta < 1 \text{ s.t. } |a_i| \eta^i \rightarrow 0 \text{ as } i \rightarrow -\infty \right\}$$

$$\mathcal{O}_{\mathcal{O}_K^F} \cong \mathcal{O}_K^F \cap \mathcal{W}[[u, u^{-1}]], \quad \mathcal{O}_{\mathcal{O}_K^{\dagger, F}} \cong \mathcal{O}_K^{\dagger, F} \cap \mathcal{W}[[u, u^{-1}]].$$

Thus $\mathcal{W}[[u]] \subset \mathcal{O}_K^F$, and we let $S_K^F = \mathcal{W}[[u]] \otimes_{\mathcal{W}} L \subset \mathcal{O}_K^F$. Note that although the notation does not suggest so, the ring S_K^F depends on the choice of parameter u . Thus the rings

$$(\mathcal{W}[[u]], S_K^F) \subset (\mathcal{O}_{\mathcal{O}_K^{\dagger, F}}, \mathcal{O}_K^{\dagger, F}) \subset (\mathcal{O}_{\mathcal{O}_K^F}, \mathcal{O}_K^F)$$

are of exactly the same form as the rings

$$(\mathcal{V}[[t]], S_K) \subset (\mathcal{O}_{\mathcal{O}_K^{\dagger}}, \mathcal{O}_K^{\dagger}) \subset (\mathcal{O}_{\mathcal{O}_K}, \mathcal{O}_K)$$

but associated to the pair (L, u) rather than the pair (K, t) . The base extension

$$(k((t)), k[[t]], \mathcal{V}[[t]]) \rightarrow (F, A, \mathcal{W}[[u]])$$

then determines a base change functor

$$\mathrm{Isoc}^{\dagger}(X/\mathcal{O}_K^{\dagger}) \rightarrow \mathrm{Isoc}^{\dagger}(X_F/\mathcal{O}_K^{\dagger, F}),$$

which we will generally denote by $\mathcal{O} \mapsto \mathcal{O}_F$. *A priori*, this construction depends on the choice of parameter u . However, since $\mathcal{O}_{\mathcal{O}_K^{\dagger, F}} = \mathrm{colim}_m \mathcal{W}[[u]] \langle r^{-1/m} u^{-1} \rangle$ is independent of u , one can use the standard neighbourhoods $V'_{n, m}$ of Sect. 2.2 together with Lemma 2.55 to show that neither the category $\mathrm{Isoc}^{\dagger}(X_F/\mathcal{O}_K^{\dagger, F})$, nor the base extension functor

$$\mathrm{Isoc}^{\dagger}(X/\mathcal{O}_K^{\dagger}) \rightarrow \mathrm{Isoc}^{\dagger}(X_F/\mathcal{O}_K^{\dagger, F}),$$

nor the cohomology of such objects, as vector spaces over $\mathcal{O}_K^{\dagger, F}$, depend on the choice of u .

To discuss Frobenius structures and pullback, we fix a Frobenius σ_K on K , that is a field automorphism preserving \mathcal{V} and lifting the absolute q -power Frobenius on k .

Definition 2.70 A Frobenius on $\mathcal{V}[[t]]$ is a π -adically continuous endomorphism $\sigma_{\mathcal{V}[[t]]} : \mathcal{V}[[t]] \rightarrow \mathcal{V}[[t]]$ which is semi-linear over σ_K and lifts the absolute q -power Frobenius on $k[[t]]$.

Such a Frobenius extends uniquely to a continuous endomorphism of $\mathcal{O}_{\mathcal{O}_K}$, and hence \mathcal{O}_K , and this preserves the subrings $\mathcal{O}_{\mathcal{O}_K^{\dagger}}, \mathcal{O}_K^{\dagger}$ and S_K . We will henceforth assume that we have chosen a Frobenius on $\mathcal{V}[[t]]$, and we endow the rings $S_K, \mathcal{O}_{\mathcal{O}_K^{\dagger}}, \mathcal{O}_K^{\dagger}, \mathcal{O}_{\mathcal{O}_K}$ and \mathcal{O}_K with the induced Frobenii, all of which we will denote by the same letter σ . We will further assume that $\sigma(t) = ut^q$ for some $u \in (S_K)^{\times}$ congruent to 1

modulo π . This assumption on Frobenius structures will be crucial when we discuss logarithmic (φ, ∇) -modules in Chap. 5.

Exactly as above, if we let X' denote the base change of X by the q -power Frobenius on k , then we get a pullback functor

$$\sigma^* : \mathrm{Isoc}^\dagger(X/\mathcal{O}_K^\dagger) \rightarrow \mathrm{Isoc}^\dagger(X'/\mathcal{O}_K^\dagger)$$

which we can compose with pullback via relative Frobenius $X \rightarrow X'$, which is $k((t))$ -linear, to get a σ -linear Frobenius pullback functor

$$F^* : \mathrm{Isoc}^\dagger(X/\mathcal{O}_K^\dagger) \rightarrow \mathrm{Isoc}^\dagger(X/\mathcal{O}_K^\dagger).$$

Definition 2.71 An overconvergent F -isocrystal on X/\mathcal{O}_K^\dagger is an object $\mathcal{E} \in \mathrm{Isoc}^\dagger(X/\mathcal{O}_K^\dagger)$ together with an isomorphism $\varphi : F^*\mathcal{E} \rightarrow \mathcal{E}$. The category of overconvergent F -isocrystals on X/\mathcal{O}_K^\dagger is denoted $F\text{-Isoc}^\dagger(X/\mathcal{O}_K^\dagger)$.

Remark 2.72 Note that this definition depends on the choice of Frobenius σ on $\mathcal{V}[[t]]$.

It is not difficult to see that this construction is compatible with the previous construction associated to a finite separable extension $F/k((t))$. That is, if we have chosen a Frobenius on $\mathcal{W}[[u]]$ compatible with that on $\mathcal{V}[[t]]$, then this induces a Frobenius pullback on $\mathrm{Isoc}^\dagger(X_F/\mathcal{O}_K^{\dagger,F})$ and we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Isoc}^\dagger(X/\mathcal{O}_K^\dagger) & \longrightarrow & \mathrm{Isoc}^\dagger(X_F/\mathcal{O}_K^{\dagger,F}) \\ \downarrow F^* & & \downarrow F^* \\ \mathrm{Isoc}^\dagger(X/\mathcal{O}_K^\dagger) & \longrightarrow & \mathrm{Isoc}^\dagger(X_F/\mathcal{O}_K^{\dagger,F}) \end{array}$$

at least up to natural isomorphism. Thus there is an induced base extension functor

$$F\text{-Isoc}(X/\mathcal{O}_K^\dagger) \rightarrow F\text{-Isoc}^\dagger(X_F/\mathcal{O}_K^{\dagger,F})$$

which we will again denote by $\mathcal{E} \mapsto \mathcal{E}_F$. Again, this is compatible with pullback via morphisms of $k((t))$ varieties $U \rightarrow X$.

Finally, we consider the extension $\mathcal{O}_K^\dagger \rightarrow \mathcal{O}_K$. Let (X, Y, \mathfrak{P}) be a smooth and proper frame over $\mathcal{V}[[t]]$, and let $(X, Y_{k((t))}, \mathfrak{P}_{\mathcal{O}_{\mathcal{E}_K}})$ denote the base change of this frame to $\mathcal{O}_{\mathcal{E}_K}$, this is a smooth and proper frame over $\mathcal{O}_{\mathcal{E}_K}$ in the usual sense of Berthelot's rigid cohomology. Since $\mathcal{O}_{\mathcal{E}_K} \cong \mathcal{V}[[t]]\langle t^{-1} \rangle$, there is a natural open immersion of rigid spaces

$$]Y_{k((t))}[_{\mathfrak{P}_{\mathcal{O}_{\mathcal{E}_K}}} \rightarrow]Y[_{\mathfrak{P}}$$

over S_K such that $(j_X^\dagger \mathcal{O}_{Y[\mathfrak{P}]})|_{Y_{k((t))}[\mathfrak{P} \otimes_{\mathcal{E}_K}]} = j_X^\dagger \mathcal{O}_{Y_{k((t))}[\mathfrak{P} \otimes_{\mathcal{E}_K}]}$ (which follows, for example, by the concrete description of a cofinal system of neighbourhoods in both cases). This induces a functor

$$\mathrm{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \rightarrow \mathrm{MIC}^\dagger((X, Y_{k((t))}, \mathfrak{P} \otimes_{\mathcal{E}_K})/\mathcal{E}_K)$$

which is simply given by restriction. Here the latter category is the usual category of coherent modules with overconvergent connection as defined for example in Chap. 6 of [9]. Actually, the definition there is in terms of Tate's rigid spaces rather than Huber's adic spaces, but exactly the same methods as used in Sect. 2.1 will show that the two points of view are equivalent. The induced functor

$$\mathrm{Isoc}^\dagger(X/\mathcal{E}_K^\dagger) \rightarrow \mathrm{Isoc}^\dagger(X/\mathcal{E}_K)$$

is independent of the choice of frame (X, Y, \mathfrak{P}) and will be denoted $\mathcal{E} \mapsto \hat{\mathcal{E}}$ (the notation is meant to suggest a 'quasi-completion', that is π -adic completion in the horizontal variable t^{-1} but not the vertical variables). Again, this is easily seen to be compatible with all previous constructions of Frobenius base change, base change via a finite separable extension of $k((t))$ and pullback via a morphism of $k((t))$ -varieties.

All of these 'base changes' induce corresponding base change morphisms on overconvergent de Rham cohomology of frames, and hence on rigid cohomology. We therefore have canonical base change morphisms

$$\begin{aligned} H_{\mathrm{rig}}^i(X/\mathcal{E}_K^\dagger, \mathcal{E}) \otimes_{\mathcal{E}_K^\dagger} \mathcal{E}_K^{\dagger, F} &\rightarrow H_{\mathrm{rig}}^i(X_F/\mathcal{E}_K^{\dagger, F}, \mathcal{E}_F) \\ H_{\mathrm{rig}}^i(X/\mathcal{E}_K^\dagger, \mathcal{E}) \otimes_{\mathcal{E}_K^\dagger, \sigma} \mathcal{E}_K^\dagger &\rightarrow H_{\mathrm{rig}}^i(X/\mathcal{E}_K^\dagger, F^* \mathcal{E}) \\ H_{\mathrm{rig}}^i(X/\mathcal{E}_K^\dagger, \mathcal{E}) \otimes_{\mathcal{E}_K^\dagger, \sigma} \mathcal{E}_K &\rightarrow H_{\mathrm{rig}}^i(X/\mathcal{E}_K, \hat{\mathcal{E}}) \end{aligned}$$

which are all compatible, in the sense that we leave it to the reader to make precise. In particular, if $\mathcal{E} \in F\text{-Isoc}^\dagger(X/\mathcal{E}_K^\dagger)$ then we get a natural σ -linear morphism

$$H_{\mathrm{rig}}^i(X/\mathcal{E}_K^\dagger, \mathcal{E}) \rightarrow H_{\mathrm{rig}}^i(X/\mathcal{E}_K^\dagger, \mathcal{E})$$

which commutes with the extensions $\mathcal{E}_K^\dagger \rightarrow \mathcal{E}_K^{\dagger, F}$ and $\mathcal{E}_K^\dagger \rightarrow \mathcal{E}_K$.

We will end this section by noting a couple of easy corollaries of the naturality of Theorem 2.62, which give a concrete interpretation of Frobenius pullbacks and Frobenius structures, and will be useful in later chapters.

Definition 2.73 Let (X, Y, \mathfrak{P}) be a frame. A Frobenius on (X, Y, \mathfrak{P}) is a σ -linear endomorphism σ^* of \mathfrak{P} lifting the absolute q -power Frobenius on P .

Note that such a Frobenius induces a σ -linear pullback functor

$$\sigma^* : \mathrm{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \rightarrow \mathrm{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger),$$

more generally, if $u : (X', Y', \mathfrak{P}') \rightarrow (X, Y, \mathfrak{P})$ is a Frobenius semi-linear morphism of smooth and proper frames over $\mathcal{V}[[t]]$ then we get a pullback functor

$$u^* : \mathrm{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \rightarrow \mathrm{MIC}^\dagger((X', Y', \mathfrak{P}')/\mathcal{E}_K^\dagger)$$

which is σ -linear over \mathcal{E}_K^\dagger .

Definition 2.74 Let (X, Y, \mathfrak{P}) be a frame with Frobenius σ . Then a Frobenius structure on an object $\mathcal{E} \in \mathrm{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$ is an isomorphism $\varphi : \sigma^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ in $\mathrm{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$. The category of modules with an overconvergent integrable connection together with a Frobenius structure is denoted $\varphi\text{-MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger)$.

Proposition 2.75 1. Let $u : (X, Y', \mathfrak{P}') \rightarrow (X, Y, \mathfrak{P})$ be a Frobenius semi-linear morphism of smooth and proper frames over $\mathcal{V}[[t]]$, such that the induced morphism $X \rightarrow X$ is the absolute q -power Frobenius. Then the Frobenius pullback functor

$$F^* : \mathrm{Isoc}^\dagger(X/\mathcal{E}_K^\dagger) \rightarrow \mathrm{Isoc}^\dagger(X/\mathcal{E}_K^\dagger)$$

can be identified with the functor

$$u^* : \mathrm{MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger) \rightarrow \mathrm{MIC}^\dagger((X, Y', \mathfrak{P}')/\mathcal{E}_K^\dagger).$$

2. Let (X, Y, \mathfrak{P}) be a smooth and proper frame over $\mathcal{V}[[t]]$ with Frobenius φ . Then there is an equivalence of categories

$$F\text{-Isoc}^\dagger(X/\mathcal{E}_K^\dagger) \cong \varphi\text{-MIC}^\dagger((X, Y, \mathfrak{P})/\mathcal{E}_K^\dagger).$$

References

1. Berkovich, V.: Spectral theory and analytic geometry over non-Archimedean fields. In: Mathematical Survey and Monographs, vol. 33. American Mathematical Society, Providence (1990)
2. Berthelot, P.: Géométrie rigide et cohomologie des variétés algébriques de caractéristique p . Mem. Soc. Math. France **23**, 7–32 (1986)
3. Berthelot, P.: Cohomologie rigide et cohomologie rigide à supports propres, première partie. Preprint (1996)
4. Berthelot, P.: Finitude et pureté cohomologique en cohomologie rigide. Invent. Math. **128**(2), 329–377 (1997)
5. Frensel, J., van der Put, M.: Rigid analytic geometry and its applications. In: Progress in Mathematics, vol. 218. Birkhäuser (2004)
6. Fujiwara, K., Kato, F.: Foundations of rigid geometry I. preprint (2013). [arXiv:math/1308.4734v1](https://arxiv.org/abs/1308.4734v1), to appear in EMS Monographs in Mathematics
7. Huber, R.: A generalization of formal schemes and rigid analytic varieties. Math. Z. **217**(4), 513–551 (1994)

8. Huber, R.: Étale cohomology of rigid analytic varieties and adic spaces. Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig (1996)
9. Le Stum, B.: Rigid Cohomology, Cambridge Tracts in Mathematics, vol. 172. Cambridge University Press (2007)
10. Raynaud, M., Gruson, L.: Critères de platitude et de projectivité. Techniques de “platification” d’un module. *Invent. Math.* **13**, 1–89 (1971)

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