

Chapter 2

Direct Current Circuits

2.1 Introduction

Nowadays, our lives are increasingly dependent upon the availability of devices that make extensive use of electric circuits. The knowledge of the electrical current flowing in each component of a circuit, and the corresponding voltage drop, is essential to obtain the desired function from the device exploiting it. When we obtain these values, we say that the circuit is solved.

Circuits where voltages and currents are not time dependent are called direct current circuits or, using an acronym, DC circuits. They are a useful starting point to confronting the task of constructing procedures useful to obtain circuit solutions.

This chapter deals with methods to obtain the solution of a DC circuit. First, we discuss in Sect. 2.2 the basic physical principles to be used and we recall the two Kirchhoff's laws. Then we introduce some basic concepts of graphs theory useful to analyze the topological properties of electrical circuits. Using these results, we move on in Sect. 2.3 to describe the implementation of the two principal approaches to circuit solution, the method of nodes and the method of meshes. As an example of the application of both methods, we give the detailed solution of the Wheatstone's bridge, a circuit used to obtain accurate measurement of a resistance. Next in Sect. 2.4, we turn our attention to circuits made only of components whose current is linearly dependent upon the applied voltage and we prove a number of theorems that simplify considerably the task of solving linear circuits. Finally, in Sect. 2.5, we discuss briefly the issue of power transfer among different parts of an electric circuit.

2.2 Kirchhoff's Laws

The tools adopted to solve electrical circuits are based on the two *Kirchhoff's laws*. The first, known as the *Kirchhoff's law of currents*, uses the concept of *node*, i.e., a point connecting three or more electrical components, and is based on the principle of

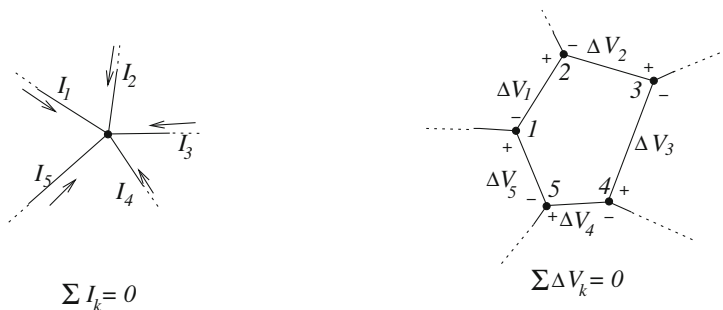


Fig. 2.1 Schematic illustration of the two Kirchhoff's laws: the diagram on the *left* refers to the law of the nodes, while diagram on the *right* illustrates the loops law. The two figures show, respectively, the positive direction of the current (indicated by the *arrows*) and the positive voltage (indicated by + and -)

conservation of electric charge. The second, known as the *Kirchhoff's law of voltages*, uses of the concept of *loop*, a closed path in a circuit, and is based on the consideration that the electrostatic field is conservative. The formulation of Kirchhoff's laws is the following:

1. **Kirchhoff's law of currents**, also known as the *law of the nodes* or the *first Kirchhoff's law*: the sum of all currents converging in a node is zero: $\sum_k I_k = 0$ (see Fig. 2.1).
2. **Kirchhoff's law of voltages**, also known as the *law of the loops* or the *second Kirchhoff's law*: the sum of the potential differences across components forming a closed loop is zero: $\sum_k \Delta V_k = 0$ (see Fig. 2.1).

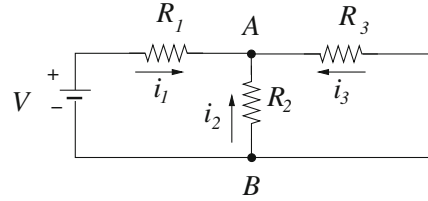
The two Kirchhoff's laws are based on the general principles of conservation of charge and energy, respectively, and are valid independently of the nature of the components in the circuit.

In the two Kirchhoff's laws both I_k and ΔV_k are algebraic quantities. By convention, the I_k are positive when directed toward the node. Likewise, the ΔV_k are positive when the voltage increases along the arbitrarily chosen direction of the loop.

2.2.1 Network Geometry

Consider a circuit with more than one loop, such as, for example, the circuit shown in Fig. 2.2. One can easily verify that the Kirchhoff's laws allow writing a number of equations that exceed the number of unknown currents required for the circuit solution. In the circuit, three closed loops are present and to each of them we can apply the law of voltages. Denoting with i_1 , i_2 , and i_3 the currents flowing, respectively, in resistors R_1 , R_2 , and R_3 (in the figure the positive directions of the currents are indicated by the arrows), we obtain

Fig. 2.2 Example of a circuit with two independent loops



$$\begin{cases} V = R_1 i_1 - R_2 i_2 \\ V = R_1 i_1 - R_3 i_3 \\ 0 = R_2 i_2 - R_3 i_3 \end{cases} \quad (2.1)$$

However, these three equations are not enough to obtain the values of the three unknown currents since they are linearly dependent. In fact, the third equation can be obtained by difference from the first two and it is easy to recognize that this is only a consequence of the geometrical properties of the network. We say, therefore, that the three loops in the circuit are not independent. To solve this circuit we must also take into account the Kirchhoff's law of currents applied to the two nodes in the circuit under test. In fact, it is easy to verify that for both nodes the same equation is obtained: $i_1 + i_2 + i_3 = 0$ which means that the two nodes are not independent. We leave to the reader the solution of the circuit with the first two Eq. (2.1) and the relation among the currents given above.

Our next task is to consider how to determine in a generic circuit the number of independent loops and the number of independent nodes. To answer this question, we need to exploit some simple and intuitive concepts borrowed from graph theory.

The graphs. In mathematics, a graph consists of a set of points called *vertices* or *nodes* of the graph, and of a set of *edges* or *branches* joining pairs of *nodes*. Graph theory is devoted to the study of the topological properties¹ of these structures. Consider a graph with n nodes and r branches such as, for example, the one shown in Fig. 2.3. By definition, a *tree* of the graph is a set of its branches connecting all its nodes without making a closed path. It is easy to verify that the number of branches in a tree must be $n - 1$; the set of branches that do not belong to the tree will form a so-called *co-tree* that consists of $r - (n - 1)$ branches.

Adding to the tree a branch of the corresponding *co-tree* will result in a closed loop. Since $r - n + 1$ is the number of branches of the *co-tree*, you can then get in this way $r - n + 1$ closed loops that are referred to as the *fundamental loops* of the graph. Obviously, the number of fundamental loops is independent from the tree initially selected.

Another operation defined on graphs is the *cut*, which consists of isolating some nodes of the graph with a surface (a line in the plane). Eliminating all branches intersected by the *cut*, the graph becomes divided into two parts *non-connected* between them. A cut that intersects only one branch of the tree (the other intersected

¹The topological properties of a geometrical entity are those invariant for elastic deformations.

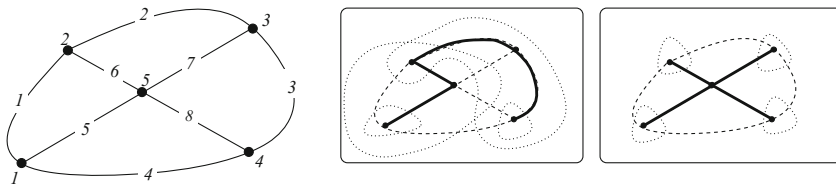


Fig. 2.3 Graph with five nodes and eight branches. In the two inserts, two trees of the graph (in *bold*) with their co-trees (*dashed branches*) are shown. The *dotted* closed curves represent the fundamental cuts intersecting only one branch belonging to the tree

branches belonging to the co-tree) is called *fundamental* or *independent*; the number of fundamental cuts is obviously equal to the number of branches in a tree, i.e., $n - 1$.

The parallel between graphs and electrical circuits is evident: in fact, nodes of the graphs can be identified with the nodes of the electrical circuits and the branches of the graph can be identified with the bipolar components (possibly in series) of electrical circuits. In particular, the number of fundamental loops in the graphs coincides with the number $m = r - n + 1$ of independent loops to which apply Kirchhoff's voltage law, and the number of independent cuts ($n - 1$) coincides with the number of independent nodes to which apply Kirchhoff's current law.

Considering again the circuit of Fig. 2.2, we note that the network represented there consists of three branches and two nodes. Therefore, the number of independent loops $r - (n - 1)$ is equal to two, while the number of independent nodes $n - 1$ is equal to one, as obtained previously from a direct analysis of equations resulting from Kirchhoff's laws.

2.3 Solution Methods for Electric Circuits

As previously mentioned, the solution of an electrical circuit consists in the determination of the voltage at each node and the current in each branch. The principles underlying the methods of solution are the two Kirchhoff's laws to be used together with relationships linking current and voltage in each component of the circuit. For resistive components, the *Ohm's law* provides these relations. In next paragraphs, we will work out two methods, the first said method *of the nodes* and the second said method *of the loops* that, using the two Kirchhoff's law, give the solution of a generic DC circuit.

2.3.1 The Method of Nodes

The method of nodes addresses the solution of a circuit by applying Kirchhoff's law of currents ($\sum I_k = 0$) to its independent nodes. Its name should be, more correctly,

the method of the voltage of the nodes. In fact, the application of this method starts with the arbitrary choice of a node of reference against which to measure the potential differences of all other nodes. The voltages of these $n - 1$ nodes are the unknowns of our problem, while the current in each branch can be expressed in terms of the node voltages and of the impedances of the electrical components in the branch. Application of Kirchhoff's law of currents to each independent node yields the $n - 1$ equations needed to obtain the circuit solution. Note that assigning a value of voltage to each node, with respect to the reference node, implies *the automatic verification of the Kirchhoff's law of voltages*; in fact, with this assignment, in a closed path in the circuit the sum of the voltage drops in each branch will be always identically zero. In other words, the value of the electric potential at a given point does not depend on the path we follow to reach it but only on the point itself.

The steps for solving electrical circuits with this method are as follows:

1. Among the n nodes of a circuit, choose a reference node against which to measure the voltages of the other $n - 1$ nodes.
2. Assign unknown voltage values ($V_1, V_2, \dots, V_k, \dots, V_{n-1}$) to these $n - 1$ nodes.
3. Evaluate the currents that converge in each node using the values of the voltages V_k and the impedances of the circuit elements through which they flow.
4. Write the $n - 1$ equations expressing Kirchhoff's current law.
5. Solve the system to obtain the $n - 1$ node voltages.
6. Compute the currents flowing in various circuit components using the known values of node voltages and the relationship between voltage and current for the circuit components connecting them. When the circuit consists only of resistors, this relation is the first Ohm's law.

As an application, we solve by this method the circuit of Fig. 2.2 that we already addressed in the previous section. In this circuit there are $n = 2$ nodes, and then $n - 1 = 1$ independent node. We choose the node B as the reference for the voltage of the other node (in this particular case only for the voltage in A) and denote by i_1, i_2 , and i_3 currents that, flowing in the resistors with the same index, converge in the node A . We denote by V_A the unknown voltage of node A , and we write the Kirchhoff's current law for this node:

$$\sum_{k=1}^3 i_k = i_1 + i_2 + i_3 = \frac{V - V_A}{R_1} + \frac{0 - V_A}{R_2} + \frac{0 - V_A}{R_3} = 0$$

To simplify this expression, we introduce the conductance $G_k = 1/R_k$ ($k = 1, 2, 3$) and the solution of the previous equation becomes

$$V_A = \frac{G_1}{G_1 + G_2 + G_3} V$$

With a simple algebraic manipulation, it can be shown that this expression of V_A coincides with the one that can be derived from the equations given in the previous Sect. 2.2.1.

2.3.2 The Method of Meshes

The method of the meshes is based on the Kirchhoff's law of voltages requiring that the sum of the voltage drops across components forming a closed path is zero. This method introduces a fictitious current, said the *mesh current*, for each of the independent loops in the circuit. Expressing the physical currents, those that actually flow in the various components, as the algebraic sum of the mesh currents that affect them, *the Kirchhoff's law of currents is everywhere automatically satisfied* because all currents are now sums of closed currents (the mesh currents).

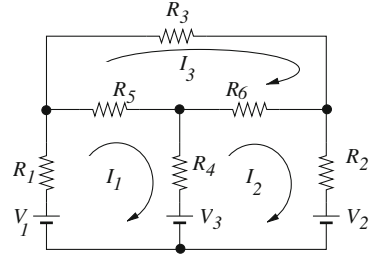
The mesh currents are now the unknowns of the problem and it is sufficient to use the Kirchhoff's law of the voltages applied to the independent meshes to obtain their value. The steps for solving electrical circuits with this method are as follows:

1. Count the number n of nodes and r of branches in the circuit and compute the number m of independent loops: $m = r - n + 1$. This is the number of equations needed to solve the circuit.
2. Select m between all possible independent loops² by arbitrarily choosing a tree on the graph of the circuit and proceeding as described in Sect. 2.2.1. The m loops selected in this way are those to which the Kirchhoff's law of voltages applies.
3. For each of the loops identified in the previous paragraph indicate a mesh current (unknown) with an arbitrary direction.
4. Compute the currents in the individual components by adding up the mesh currents flowing in them taking into account their mutual direction.
5. Apply the law of voltages to each loop and obtain the m equations of the form $\sum_k V_k = 0$ using the relationships between voltage and current for the components in the circuit. In the case that only resistors form the circuit, this relationship consists of the simple Ohm's law. Note that the e.m.f. of each generator in the mesh has to be taken positive if it generates a current flowing contrary to the positive direction chosen for the mesh and negative if it generates a current along that direction.
6. Solve the resulting system of equations to obtain the value of the m mesh currents. The currents flowing in single elements are obtained by adding up the mesh currents that flow in the element taking into account their mutual direction.
7. The voltages at the nodes of the circuit are obtained by selecting a node as reference and assigning an arbitrary voltage to it, typically the null value. The voltage of any other node is then obtained by applying sequentially the Ohm's law³ to the branches of a path that connects it to the reference node.

² The importance of choosing independent loops can be understood from the example in the following Sect. 2.3.3 on the solution of the Wheatstone's bridge shown in Fig. 2.5. If we had chosen for the solution of the circuit with the method of meshes, instead of the mesh supporting the current I_a , the one formed by the resistances R_1, R_2, R_3, R_4 , the determinant of the resulting system would have been zero, because the mesh R_1, R_2, R_3, R_4 is linearly dependent (it is the sum) of those supporting the currents I_b and I_c .

³ The voltage drop across a branch consisting of resistors and generators is simply equal to the sum of voltage drops across each individual element in the branch.

Fig. 2.4 Network with three independent loops



We now apply the method of the loops described above to the solution of the circuit of Fig. 2.4. The circuit has $n = 4$ nodes and $r = 6$ branches and then the number of independent loops is $m = r - n + 1 = 3$. We choose the three meshes⁴ and their positive direction as shown in the figure; we can write the following equations that follow from the application of the Kirchhoff's law of voltages to the loops with currents I_1 , I_2 , and I_3 , respectively:

$$\begin{cases} R_1 I_1 + R_5(I_1 - I_3) + R_4(I_1 - I_2) - V_1 + V_3 = 0 \\ R_4(I_2 - I_1) + R_6(I_2 - I_3) + R_2 I_2 - V_3 + V_2 = 0 \\ R_3 I_3 + R_6(I_3 - I_2) + R_5(I_3 - I_1) = 0 \end{cases}$$

Rearranging this system and isolating the unknowns I_1 , I_2 , and I_3 :

$$\begin{cases} (R_1 + R_4 + R_5)I_1 & -R_4 I_2 & -R_5 I_3 = V_1 - V_3 \\ -R_4 I_1 + (R_2 + R_4 + R_6)I_2 & & -R_5 I_3 = V_3 - V_2 \\ -R_5 I_1 & -R_6 I_2 + (R_3 + R_5 + R_6)I_3 = 0 \end{cases}$$

The solution of this system is obtained with a long calculation of determinants and minors that is left as an exercise to the interested reader.⁵ In case all the resistors are equal, $R_k = R$, one can easily get the following solution:

⁴The three meshes used for the solution are generated from T-shaped tree obtained by the branches that contain resistors R_5 , R_6 , and R_4 .

⁵As a hint to the solution we give the value of the determinant Δ of the system and the expression for the current I_1 :

$$\begin{aligned} \Delta &= R_1 R_2 R_3 + R_1 R_3 R_4 + R_2 R_3 R_4 + R_1 R_2 R_5 + R_2 R_3 R_5 + R_1 R_4 R_5 + R_2 R_4 R_5 + R_3 R_4 R_5 + \\ &\quad + R_1 R_2 R_6 + R_1 R_3 R_6 + R_1 R_4 R_6 + R_2 R_4 R_6 + R_3 R_4 R_6 + R_1 R_5 R_6 + R_2 R_5 R_6 + R_3 R_5 R_6 \\ I_1 &= \frac{1}{\Delta} \{ [R_4 R_5 + R_6(R_4 + R_5)](V_1 - V_2) + R_2(R_3 + R_5 + R_6)(V_1 - V_3) \\ &\quad + R_3(R_3 V_1 + R_6 V_1 - R_4 V_2 - R_6 V_3) \} \end{aligned}$$

$$\begin{cases} I_1 = \frac{2V_1 - V_2 - V_3}{4R} \\ I_2 = -\frac{2V_2 - V_1 - V_3}{4R} \\ I_3 = \frac{V_1 - V_2}{4R} \end{cases}$$

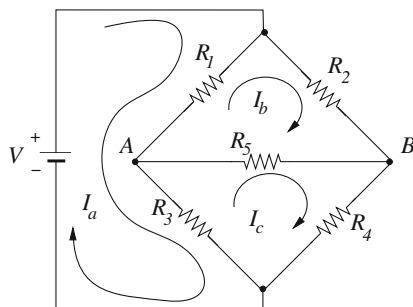
A Note on the Best Method Selection

In the preceding paragraphs, we illustrated two ways to deal with the challenge of solving electrical circuits: the method of nodes and that of meshes. The two methods are equivalent and the choice among them is largely subjective. A possible criterion for this choice, based on the economy of the calculations, is the comparison between the number of independent nodes $n - 1$ and the number of independent meshes $m = r - n + 1$; if $n - 1 < m$ it is convenient to use the method of nodes; otherwise it is more convenient to apply the meshes method. It can be shown that if $n < 4$ and the circuit has only voltage generators, the method of the nodes is the most convenient. See in this respect the Problem 2 at the end of this chapter.

2.3.3 Example of Circuit Solution: The Wheatstone's Bridge

A circuit particularly important for the accurate measurement of resistance values is the Wheatstone's bridge that is shown in Fig. 2.5. We first apply the method of the meshes for the solution of this circuit that has four nodes and six branches. The number of independent loops is therefore 3; the three loops selected for the solution are indicated in the figure by the currents I_a , I_b , and I_c , which circulate in them. The reader can easily verify that the three meshes have been identified from the tree that starts from node A and contains the three branches with resistors R_1 , R_5 , and R_3 .

Fig. 2.5 Solution of the Wheatstone's bridge by the method of the meshes. The circuit has three independent meshes for which we have defined the three mesh currents I_a , I_b , I_c



Applying Kirchhoff's law of voltages to these three meshes, we have

$$\begin{cases} R_1(I_a - I_b) + R_3(I_a - I_c) - V = 0 \\ R_2I_b + R_5(I_b - I_c) + R_1(I_b - I_a) = 0 \\ R_3(I_c - I_a) + R_5(I_c - I_b) + R_4I_c = 0 \end{cases}$$

(you need to pay close attention to how you add up the currents in the individual resistors). Rearranging the above equations, we have

$$\begin{cases} (R_1 + R_3)I_a - R_1I_b - R_3I_c = V \\ -R_1I_a + (R_1 + R_2 + R_5)I_b - R_5I_c = 0 \\ -R_3I_a - R_5I_b + (R_3 + R_4 + R_5)I_c = 0 \end{cases} \quad (2.2)$$

Denoting with Δ the determinant of the matrix of the coefficients of the currents,

$$\begin{aligned} \Delta &= \begin{vmatrix} R_1 + R_3 & -R_1 & -R_3 \\ -R_1 & R_1 + R_2 + R_5 & -R_5 \\ R_3 & -R_5 & R_3 + R_4 + R_5 \end{vmatrix} \\ &= R_3[R_4R_5 + R_2(R_4 + R_5)] + R_1[R_4(R_3 + R_5) + R_2(R_3 + R_4 + R_5)] \end{aligned}$$

we obtain the expression for the current I_b :

$$I_b = \frac{1}{\Delta} \begin{vmatrix} R_1 + R_3 & V & -R_3 \\ -R_1 & 0 & -R_5 \\ -R_3 & 0 & R_3 + R_4 + R_5 \end{vmatrix} = \frac{V}{\Delta} [R_5R_3 + R_1(R_3 + R_4 + R_5)]$$

Similarly, we get for I_c :

$$I_c = \frac{1}{\Delta} \begin{vmatrix} R_1 + R_3 & -R_1 & V \\ -R_1 & R_1 + R_2 + R_5 & 0 \\ -R_3 & -R_5 & 0 \end{vmatrix} = \frac{V}{\Delta} [R_1R_5 + R_3(R_1 + R_2 + R_5)]$$

Now we calculate the current I_5 passing in R_5 , taking as the positive direction that of I_b :

$$\begin{aligned} I_5 &= I_b - I_c = \frac{V}{\Delta} (R_3R_5 + R_1R_3 + R_1R_4 + R_1R_5 - R_1R_5 - R_1R_3 - R_2R_3 - R_3R_5) \\ I_5 &= \frac{V}{\Delta} (R_1R_4 - R_2R_3) \end{aligned} \quad (2.3)$$

The relation (2.3) characterizes the Wheatstone bridge. In particular, the Wheatstone bridge is *balanced* when it has $I_5 = 0$ which implies that between the four resistors R_1 , R_2 , R_3 , and R_4 , the following relation holds:

$$R_1R_4 = R_2R_3 \quad \text{or equivalently} \quad \frac{R_1}{R_2} = \frac{R_3}{R_4} \quad (2.4)$$

The value of an unknown resistance (R_1 for example) can be obtained by balancing the Wheatstone's bridge using for R_2 , R_3 , and R_4 precision resistors that can be varied in a known manner (resistance box).

To complete the solution of the circuit of the Wheatstone bridge we calculate the voltages of nodes A and B . Putting to zero the value of the potential of the negative terminal of the battery, we get for the node B

$$V_B = R_4 I_c = \frac{VR_4}{\Delta} [R_1 R_5 + R_3 (R_1 + R_2 + R_5)] \quad (2.5)$$

The voltage of node A can be obtained subtracting to the voltage in B (given the sign of I_5) the voltage drop on R_5 :

$$\begin{aligned} V_A = V_B - R_5 I_5 &= \frac{V}{\Delta} \{R_4 [R_1 R_5 + R_3 (R_1 + R_2 + R_5)] - R_5 (R_1 R_4 - R_2 R_3)\} \\ &= \frac{V}{\Delta} (R_1 R_3 R_4 + R_2 R_3 R_4 + R_3 R_4 R_5 + R_2 R_3 R_5) \end{aligned} \quad (2.6)$$

Of course when the bridge is balanced ($I_5 = 0$) we have $V_A = V_B$.

Having solved the circuit of the Wheatstone's bridge with the meshes method, we do it again with the method of the nodes. The nodes are in this case $n = 4$, and then we need to apply the first Kirchhoff's law to $n - 1 = 3$ nodes to obtain the equations for the solution of the circuit. We select the nodes A , B and the positive terminal of the generator but we note that the voltage of this last node is fixed by its connection to the battery. This observation reduces the unknowns, and therefore the equations we need, from three to two.⁶ Therefore, we choose the nodes A and B for which to write Kirchhoff's law of currents. We get

$$\begin{cases} -\frac{V_A}{R_3} + \frac{V - V_A}{R_1} + \frac{V_B - V_A}{R_5} = 0 \\ -\frac{V_B}{R_4} + \frac{V - V_B}{R_2} + \frac{V_A - V_B}{R_5} = 0 \end{cases}$$

To simplify the notation we introduce the conductance $G_k = 1/R_k$ and the previous system becomes

$$\begin{cases} (G_1 + G_3 + G_5)V_A - G_5 V_B = G_1 V \\ G_5 V_A - (G_2 + G_4 + G_5)V_B = -G_2 V \end{cases}$$

⁶ In general, it can be stated that if two nodes of a circuit are connected only by an ideal voltage generator, the number of equations necessary for its solution by the method of the nodes is reduced by one unit.

whose solution is

$$V_A = \frac{[G_2 G_5 + G_1(G_2 + G_4 + G_5)]}{(G_1 + G_3)(G_2 + G_4) + (G_1 + G_2 + G_3 + G_4)G_5} V$$

$$V_B = \frac{G_1(G_2 + G_5) + G_2(G_3 + G_5)}{(G_1 + G_3)(G_2 + G_4) + (G_1 + G_2 + G_3 + G_4)G_5} V$$

It is a simple algebraic exercise to verify that the formulas of V_A and V_B are the same as in Eqs. (2.6) and (2.5). The equilibrium condition of the bridge, i.e., the vanishing of the current that flows in R_5 ($V_A = V_B$), leads to the following condition on the conductance of the resistors in the circuit:

$$G_1 G_4 = G_2 G_3$$

which, as we have expected, is identical to relation (2.4) we found by solving the circuit with the method of the meshes.

2.3.4 Nonlinear Circuits

The methods for the solution of circuits, illustrated in the previous paragraphs, are based on the Kirchhoff's laws that, in turn, are based on the general principles of conservation of energy and electric charge and therefore can be applied, in principle, to circuits with nonlinear components. The difficulties encountered when trying to solve this kind of circuits are of a mathematical nature and derive from the nonlinear relationship between current and voltage in the nonlinear element, described by the so-called *characteristic*. This is the reason why in the examples of solution of DC circuits discussed above we considered only resistors that are linear components. However, Kirchhoff's laws apply to *all circuits*, even those with nonlinear components.

As an example of application of the Kirchhoff's laws to a nonlinear circuit, we analyze the circuit shown in Fig. 2.6, which makes use of a diode, a nonlinear component. For the diode, whose junction we assume of the type pn , the relationship between the voltage across it, v_D , and the current flowing through it, i_D , is $i_D = i_s(e^{v_D/v_0} - 1) \simeq i_s e^{v_D/v_0}$, where i_s and v_0 are constants (see Fig. 2.6). Applying the Kirchhoff's law for voltages to the unique mesh in the circuit and reversing the relationship between i_D and v_D we obtain

$$V_0 = Ri + v_D = Ri + v_0 \log \frac{i}{i_s} \quad (2.7)$$

Formula (2.7) allows calculating, in principle, the current i that flows in the circuit. However, this equation is *transcendent*, and does not have an explicit solution making it necessary to resort to numerical or geometrical methods to solve it. The plot in

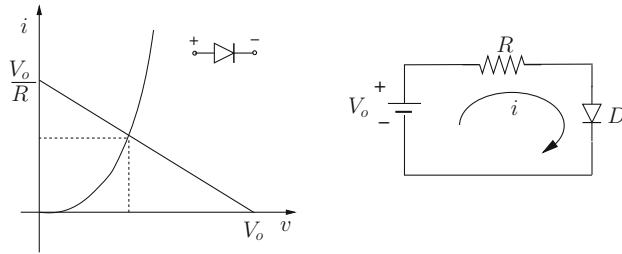


Fig. 2.6 Example of nonlinear circuit. On the *left* side of the figure the so-called “characteristic voltage–current” of a junction diode is represented while the *right* side shows the scheme of a simple circuit with a diode. The straight line in the plane $v = V_o - Ri$ is referred to as the load line and gives the current in the resistor for a given voltage drop across the diode. Since the current flowing in the two components must be the same, the intersection of the load line with the characteristic of the diode gives the solution of the circuit

Fig. 2.6 shows how it is possible to obtain graphically the solution required. In the diagram, the two curves are the exponential *characteristic* of the diode and the current flowing in the resistance R as a function of the voltage across the diode. The point where the two curves intersect determines the solution of the circuit: the current flowing in R is equal to that flowing in the diode as required by charge conservation.

2.4 Analysis of Linear Networks

Circuits whose components are *all* linear are referred to as *linear circuits*. In practice, in DC circuits only resistors are linear passive components. In fact, we recall here that the impedance operator of an ideal resistor is given by a simple multiplicative constant (the resistance), and it is therefore obviously linear. The linearity of a circuit has important consequences on its properties. The relationships implied by linearity facilitate the understanding of the behavior of the circuit and simplify considerably their solution. Below we give the detailed statements and the proofs of the most important theorems on linear circuits

2.4.1 Superposition Theorem

As it is known from elementary mathematical analysis, the solution of a system of n linear equations can always be expressed as a linear combination of the known terms. Therefore, in a linear circuit the currents in all branches and the voltage of each node can always be expressed as a linear superposition of the intensities of source generators (the “known terms” in the equations). In particular if there are only n voltage generators V_k , ($k = 1, \dots, n$) the current in the generic branch j can

always be expressed as

$$I_j = \sum_{k=1,n} A_k V_k \quad (2.8)$$

where A_k are constants determined only by the characteristics of the circuit components. Likewise, the voltage at a generic node can be obtained as a linear combination of the n V_k values. Similarly, if in the circuit only n current generators I_k are present, the voltage in the generic node j can be expressed as

$$V_j = \sum_{k=1,n} B_k I_k \quad (2.9)$$

where once again the B_k are constants determined only by the characteristics of the circuit components. A similar expression also gives the current in a generic branch. The generalization to the case where they are simultaneously present in both voltage and current generators is trivial. From this simple mathematical property, we derive the superposition theorem that can be stated in the two following forms.

For voltage generators. In any linear circuit with only n voltage generators $V_1, V_2, \dots, V_n$, the current flowing in a generic branch (or the voltage of a generic node) of the circuit is equal to the sum of the currents (or the voltages) produced by each voltage generator, taken individually, assuming that all the others have been short-circuited.

For current generators. In any linear circuit with only n current generators $I_1, I_2, \dots, I_n$, the current flowing in a generic branch (or the voltage of a generic node) of the circuit is equal to the sum of the currents (or of the voltages) produced by each current generator considered individually, assuming that all the others have been disconnected.

2.4.2 Thévenin's Theorem

The practical exploitation of electrical circuits is greatly simplified by the existence of equivalence theorems. They allow for an easy evaluation of circuits behavior when they are connected among themselves or to a single component, the load, whose parameters can depend on the specific application. Remarkably, using equivalence theorems we can avoid the tedious task of solving the circuit when the load parameters are changed.

Thévenin's theorem, in its formulation for DC circuits, states the equivalence between a generic linear circuit and a voltage generator with a resistor in series; the formulation of the theorem is the following:

Any linear circuit "seen" between two of its nodes, A and B, is equivalent to an ideal voltage generator V_{oc} whose tension is equal to the voltage measured between nodes A and B (open-circuit voltage), in series with the resistance "seen" between the two nodes which is called the equivalent resistance of Thévenin (see Fig. 2.7).

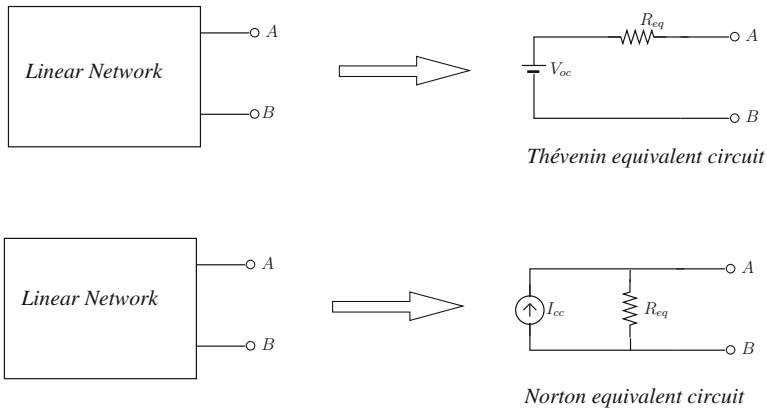


Fig. 2.7 Illustration of the theorems of Thévenin and Norton. On the *left*, we depict a generic linear network viewed from terminals A and B. On the *right*, the corresponding equivalent circuits according to Thévenin (*top*) and Norton (*bottom*)

The equivalent resistance R_{eq} can be obtained by analyzing the circuit between the two terminals A and B, combining resistors in series and in parallel according to their connections when all voltage generators present in the circuit have been *short-circuited* and all current generators have been *opened*. A method to obtain the value of R_{eq} when the layout of the circuit is not available will be given below.

The use of the equivalent circuit of Thévenin is particularly convenient when a resistance of the circuit, say R_L , can be considered as a variable load, i.e., a parameter that can vary from time to time, while all the other parameters of the circuit remain unchanged. This is elucidated in the following example.

Example. Consider the circuit of Fig. 2.8a whose solution can be obtained with the techniques already described (method of the nodes or of the meshes). Alternatively, you can deal with the solution of the circuit in question by calculating the equivalent of Thévenin considering R_L as the load of the circuit. Removing the load R_L , we apply Thévenin's theorem to the remaining circuit, see Fig. 2.8b. The resistance seen between A and B (Thévenin's equivalent resistance) amounts to $R_{eq} = R_1 \parallel R_2 = R_1 R_2 / (R_1 + R_2)$. With a simple calculation⁷ one gets the open-circuit voltage between terminals A and B as $V_{oc} = V_A - V_B = (R_1 V_2 + R_2 V_1) / (R_1 + R_2)$. The current flowing in R_L will then be equal to $V_{oc} / (R_{eq} + R_L)$.

Proof of Thévenin's Theorem.

According to the theorem of Thévenin, if V_{oc} is the open-circuit voltage between terminals A and B of a circuit and R_{eq} is the equivalent resistance seen between A and B, the current flowing in an additional load resistance r connected between A and B is

⁷The calculation of V_{oc} is immediate when using the superposition theorem.

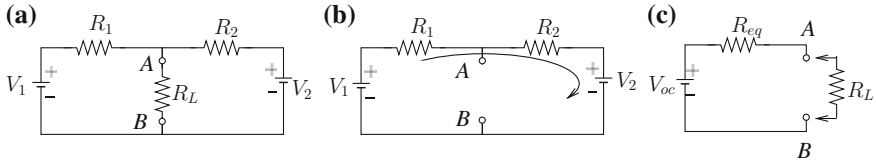


Fig. 2.8 Applying Thévenin's theorem: **a** original circuit; **b** circuit with the "load" R_L removed; **c** equivalent circuit of Thévenin with the load R_L connected

$$i = \frac{V_{oc}}{R_{eq} + r} \quad (2.10)$$

The proof of (2.10) is equivalent to the proof of Thévenin's theorem. Suppose to insert in series with the load r an additional generator whose e.m.f. is V_0 . Since the circuit is linear by assumption, denoting by $V_k, k = 1, \dots, n$ the generators belonging to the circuit, the current i' flowing in the resistance r , in the presence of the additional generator V_0 , is given by the linear combination:

$$i' = \sum_{k=0}^n A_k V_k \quad (2.11)$$

where A_k are appropriate constants with dimensions of a conductance. Let us now reduce to zero the e.m.f. of the generators pertaining to the circuit, i.e., those with $k \neq 0$. This procedure is equivalent to replacing each generator with a *short circuit*. Equation (2.11) becomes

$$i' = A_0 V_0$$

Under these conditions, the constant $1/A_0$ can be identified by construction with the sum of r and R_{eq} :

$$\frac{1}{A_0} = r + R_{eq}$$

We now bring back the circuit to its original condition by setting the V_k to their values. In addition, we set the e.m.f. of the generator inserted in the load branch to a value of $-V$ such that the current flowing in r is reduced to zero. From Eq. (2.11) we get

$$0 = -VA_0 + \sum_{k=1}^n A_k V_k$$

that gives

$$\sum_{k=1}^n A_k V_k = VA_0$$

In this circumstance the load branch added, the one including r , does not absorb current and therefore V must coincide with the open-circuit voltage between the points A and B (V_{oc}). Finally, setting $V_0 = 0$ we obtain the current in the load as

$$i = \sum_{k=0}^n A_k V_k = \sum_{k=1}^n A_k V_k = V_{A0} = \frac{V_{oc}}{r + R_0}$$

which proves Thévenin's theorem.

2.4.3 Norton's theorem

Norton's theorem for DC circuits establishes the equivalence between a generic linear circuit and a current generator with a resistor in parallel; the formulation of the theorem is the following:

Any linear circuit "seen" between two of its nodes, A and B , is equivalent to an ideal current generator whose output I_{sc} is equal to the short-circuit current between A and B in parallel with the resistance "seen" between the two nodes which is said (Norton's) equivalent resistance (see Fig. 2.7).

The proof of Norton's theorem is similar to that of Thévenin's theorem and is left to the reader as a useful exercise.

The same observation made for Thévenin's theorem about the impedance seen between two terminals, also applies to Norton's theorem. In fact, the equivalent resistances of Thévenin and Norton are identical and can be obtained in the same way.

Theorem of Open Circuit and Short Circuit

Thévenin's and Norton's theorems have the following remarkable corollary: *the impedance R_{eq} between two terminals of a linear circuit is equal to the ratio between the open circuit voltage V_{oc} and the short circuit current I_{sc} :*

$$R_{eq} = \frac{V_{oc}}{I_{sc}}$$

It should be remarked that the above expression, extremely useful for the evaluation of the *output impedance* of various circuits, should not be confused with Ohm's law, with which it only shares the mathematical expression.

Example. Consider the circuit of Fig. 2.9 where $V_s = 6.0$ V, $R_1 = 1.5$ k Ω , $R_2 = 2.2$ k Ω , $R_3 = 3.3$ k Ω with A and B nodes for which we need to calculate the Norton equivalent circuit.

We start with the calculation of the resistance R_{eq} "seen" between A and B . To get R_{eq} we must short-circuit the voltage generator appearing in the circuit. Therefore, R_{eq} is the parallel of R_1 , R_2 , and R_3 :

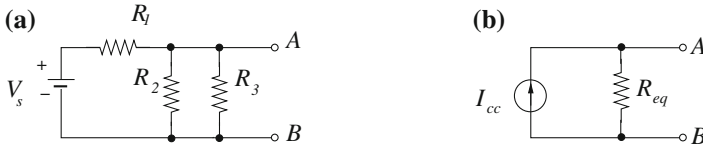


Fig. 2.9 Example of application of Norton's theorem

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \left(\frac{1}{1.5} + \frac{1}{2.2} + \frac{1}{3.3} \right) \times 10^{-3} \Omega^{-1} = 1.42 \times 10^{-3} \Omega^{-1}$$

that yields $R_{eq} = 437 \Omega$

To complete the Norton equivalent circuit we must determine the value of I_{sc} , the short-circuit current between A and B. We get very easily

$$I_{sc} = \frac{V_s}{R_1} = \frac{6.0}{1.5 \times 10^3} A = 4.0 \text{ mA}$$

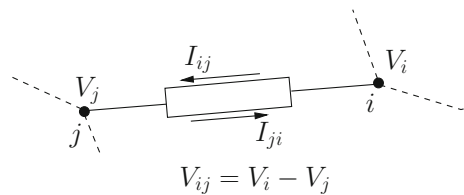
2.4.4 Reciprocity Theorem

The proof of this important theorem starts with the analysis of the power balance in a given circuit. In a generic branch connecting two nodes i and j of a network, the product of the voltage drop on the branch $V_{ij} = V_i - V_j$ times the current I_{ij} flowing through it (from node i to node j) yields the power P_{ij} that is generated, if negative, or dissipated, if positive, in the branch (see Fig. 2.10). Since energy is conserved, the total balance of power must be zero and the following equality must hold

$$\sum_{\text{branches}} V_{ij} I_{ij} = 0 \quad (2.12)$$

where the sum is extended to all the branches in the network. We now prove that Kirchhoff's laws imply the power balance. In order to proceed, we note that being $V_{ij} = V_i - V_j$, the first member of (2.12) becomes

Fig. 2.10 Notation and conventions for the power balance



$$\sum_{\text{branches}} V_{ij} I_{ij} = \sum_{\text{branches}} V_i I_{ij} - \sum_{\text{branches}} V_j I_{ij} = \sum_{\text{branches}} V_i I_{ij} + \sum_{\text{branches}} V_j I_{ji} \quad (2.13)$$

where we used the relation $I_{ji} = -I_{ij}$. Rearranging the terms, the previous expression becomes

$$\sum_i V_i \cdot \sum_j I_{ij} \quad (2.14)$$

where the first sum extends to all nodes and second only to the nodes connected to i and expresses just the sum of all the currents entering the node. Kirchhoff's law of the nodes states that the second sum is null for all i , and hence the equality (2.12) follows.

Suppose now to change the arrangement of the components on the network, without changing its topology, in order to build a new distribution of tensions V'_{ij} and currents I'_{ij} that obviously still respect Eq. (2.12). However, it is easy to see that, retracing the previous proof, we have also

$$\sum V'_{ij} \cdot I_{ij} = 0 \quad \text{e} \quad \sum V_{ij} \cdot I'_{ij} = 0 \quad (2.15)$$

In other words, the power balance remains in force even if the distribution of voltages and currents pertain to different distributions of components (*principle of conservation of virtual power*).

An important application of this result is the so-called reciprocity principle. To this aim, given a network having in the branch a the generator V_a we build a second network by removing the generator from the branch a and replacing it with a new generator V'_b now in the branch b different from a . Taking voltages of the first network and currents of the second we can write

$$V_a I'_a + \sum R_{ij} I_{ij} I'_{ij} = 0$$

where $R_{ij} I_{ij}$ are the voltage drops on the network resistances.

Taking instead currents of the first network and voltages of the second, we get

$$V'_b I_b + \sum R_{ij} I'_{ij} I_{ij} = 0$$

From these two relations we can easily deduce that

$$V_a I'_a = V'_b I_b$$

Now if we make $V_a = V'_b$ we get $I'_a = I_b$. This proves the following theorem:

Reciprocity Theorem

In a reciprocal network if a voltage generator V in the branch AA' produces a current I in the branch BB' , the same generator V in the branch BB' produces the same current I in the branch AA' . Similarly, in a reciprocal network if a current generator I in

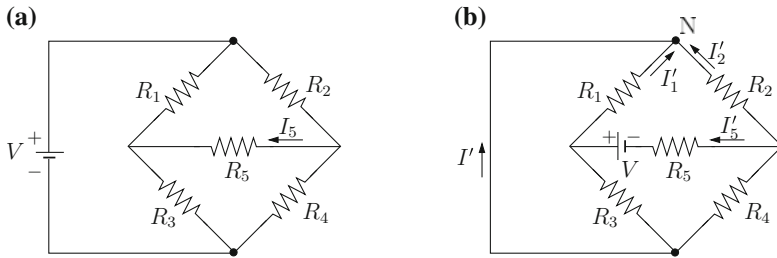


Fig. 2.11 Example of application of the reciprocity theorem

the branch AA' produces a voltage drop V between nodes B and B' , the same the generator I in the branch BB' produces the same voltage drop V between nodes A and A' .

It is easily shown that all linear networks are reciprocal while, if a network includes nonlinear elements or controlled generators, then in general the network is not reciprocal.

Example of Application of the Reciprocity Theorem

As an example, we apply the reciprocity theorem to the Wheatstone's bridge already studied in detail in Sect. 2.3.3 and shown again in Fig. 2.11a. The application of the reciprocity theorem allows calculating the current I_5 quickly and elegantly. Figure 2.11b shows the circuit in which the generator V has been removed from its original location (and replaced by a short circuit) and inserted into the branch where we need to calculate the current. The reciprocity theorem ensures that the current I_5 of Fig. 2.11a is equal to the current I' of Fig. 2.11b.

Consider this last figure and apply to the node N the second Kirchhoff's law to obtain $I' = -(I'_1 + I'_2)$. Both resistances R_1 and R_3 and resistances R_2 and R_4 are connected in parallel, and this leads us to

$$I'_1 = \frac{R_3}{R_1 + R_3} I'_5 \quad \text{e} \quad I'_2 = -\frac{R_4}{R_2 + R_4} I'_5$$

where we have used the properties of the current divider (see Problem 8 of Chap. 1) and I'_5 is the current flowing through R_5 . It is easy to get I'_5 as

$$I'_5 = \frac{V}{R_5 + R_1 \parallel R_3 + R_2 \parallel R_4}$$

Finally, we get the desired result:

$$I_5 = I' = \frac{V}{R_5 + R_1 \parallel R_3 + R_2 \parallel R_4} \left(\frac{R_4}{R_2 + R_4} - \frac{R_3}{R_1 + R_3} \right) \quad (2.16)$$

As a check of the reciprocity theorem, we leave to the reader the exercise to show that formula (2.16) is equal to expression (2.3), obtained for I_5 by the method of the meshes.

The reciprocity theorem has another interesting application to the solution of the circuit, known with the name of $R - 2R$ ladder, used in devices that convert digital signals into analog signals, see Problem 18 of Chap. 4.

2.5 Maximum Power Transfer Theorem

The simple circuit shown in Fig. 2.12 allows us to illustrate an important theorem, which is known as the theorem of *maximum power transfer*. In the diagram R_g represents the internal resistance of the voltage generator V_o and R the external load applied to it. Obviously, the current I that circulates in the circuit is given by $I = V_o/(R_g + R)$, and the voltage at the terminals of the load R is $V_R = RI$. The power P dissipated by the resistor R will be

$$P = V_R I = RI^2 = \frac{V_o^2 R}{(R_g + R)^2} \quad (2.17)$$

We now calculate the value of the load resistance R yielding the maximum power transfer from the generator to the load. By calculating the derivative of expression (2.17) with respect to R , we get

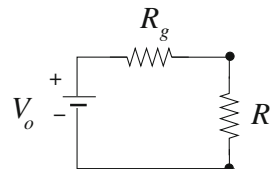
$$\frac{d}{dR} \frac{V_o^2 R}{(R_g + R)^2} = V_o^2 \frac{1}{(R_g + R)^2} \left(1 - \frac{2R}{R_g + R} \right)$$

that vanishes⁸ for $R = R_g$.

We can therefore state the following maximum power transfer theorem:

If a generator of internal resistance R_g is closed on a resistive load R , the power transferred from the generator to the load reaches a maximum if the value of the load resistance is equal to the internal resistance of the generator.

Fig. 2.12 Circuit for the proof of the theorem of maximum power transfer



⁸The second derivative of expression (2.17) is easily computed. Its value for $R = R_g$ is $-V_o^2/8R_g^3 < 0$, and confirms that in $R = R_g$ the expression (2.17) shows a maximum.

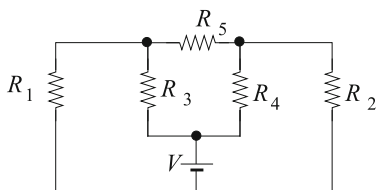
The previous discussion is related to the concept of impedance matching that will be dealt with in detail in Chap. 6.

Problems

Problem 1 Prove Norton's theorem following the layout of the proof of Thévenin's theorem illustrated in the text, adapting it where appropriate.

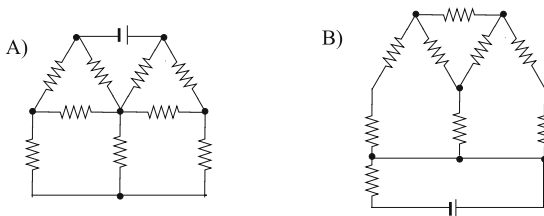
Problem 2 Recalling that at least three branches must converge in a node, show that a circuit with no more than three nodes is solved with the smallest number of equations using the method of the nodes when only voltage generators are present. Discuss why the same conclusion cannot be guaranteed in the presence of current generators.

Problem 3 What is the most effective method, i.e., the one using the smallest number of equations, to solve the circuit shown in the figure? [A. The method of the nodes.]



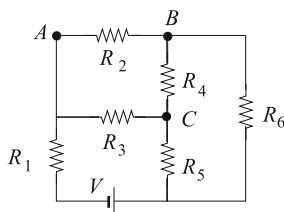
Problem 3

Problem 4 Which of the two circuits in the figure is solved more effectively with the method of the meshes? [A. Circuit B.]



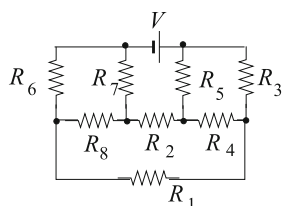
Problem 4

Problem 5 Solve the circuit shown in the figure with the most effective method and compute the voltage drop across R_5 , assuming that the values of the six resistances in the circuit are equal. [A. $V_C = V/4$, the two methods being equivalent.]



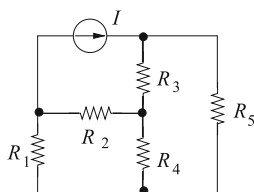
Problem 5

Problem 6 Solve the circuit shown in the figure with the most effective method and compute the voltage drop across R_1 , assuming that the values of the seven resistances in the circuit are equal. [A. $\Delta V_{R_1} = V/3$, with the mesh method.]



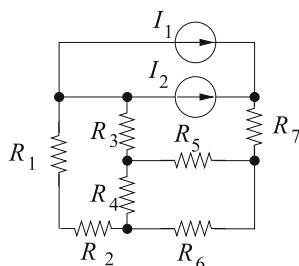
Problem 6

Problem 7 Solve the circuit shown in the figure with the most effective method and compute the current flowing in R_1 and R_5 , assuming that the values of the four resistances in the circuit are equal. [A. $I_{R_1} = I/2$, $I_{R_5} = I/2$ with the mesh method.]



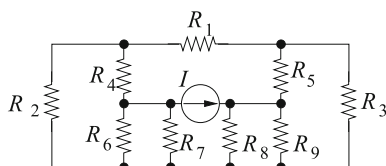
Problem 7

Problem 8 Solve the circuit shown in the figure with the most effective method and compute the current flowing in R_1 and R_5 , assuming that the values of the seven resistances in the circuit are equal. [A. $I_{R_1} = 4(I_1 + I_2)/11$, $I_{R_5} = 6(I_1 + I_2)/11$ with the mesh method.]



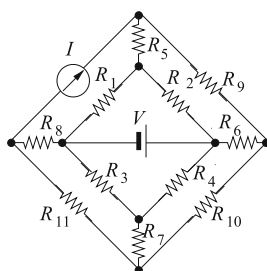
Problem 8

Problem 9 Solve the circuit shown in the figure with the most effective method and compute the current flowing in R_1 , R_2 , and R_3 , assuming that the values of the nine resistances in the circuit are equal. [A. $I_{R_1} = 2I/11$, $I_{R_2} = I/11$, $I_{R_3} = I/11$ with the mesh method.]



Problem 9

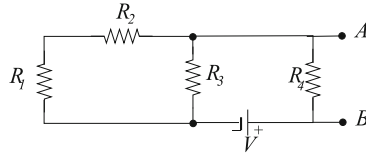
Problem 10 Solve the circuit shown in the figure with the most effective method and compute the current flowing in R_4 , R_9 , and R_{10} , assuming that the values of the nine resistances in the circuit are equal. [A. $I_{R_4} = \frac{9I}{21} - \frac{5V}{21R}$, $I_{R_9} = \frac{11I}{21} + \frac{5V}{63R}$, $I_{R_{10}} = \frac{I}{3} - \frac{2V}{9R}$ with the mesh method.]



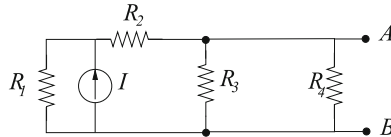
Problem 10

Problem 11 Compute the Norton equivalent with reference to the nodes A and B for the circuit in the figure. Use the following values for its parameters: $V = 5.0 \text{ V}$, $R_1 = R_2 = 2.0 \Omega$, $R_3 = R_4 = 4.0 \Omega$. [A. $I_{eq} = -2.5 \text{ A}$, $R_{eq} = 1.33 \Omega$.]

Problem 12 Compute the Norton equivalent with reference to the nodes A and B for the circuit in the figure. Use the following values for its parameters: $I = 10 \text{ A}$, $R_1 = R_2 = 2.0 \Omega$, $R_3 = R_4 = 4.0 \Omega$. [A. $I_{eq} = 5 \text{ A}$; $R_{eq} = 1.33 \Omega$.]

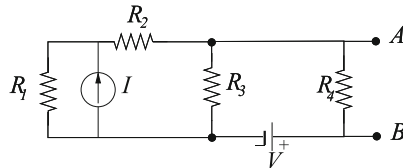


Problem 11



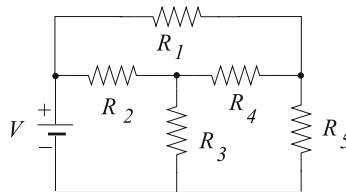
Problem 12

Problem 13 Compute the Norton equivalent with reference to the nodes A and B for the circuit in the figure. Use the following values for its parameters: $I = 10\text{ A}$, $V = 5.0\text{ V}$, $R_1 = R_2 = 2.0\ \Omega$, $R_3 = R_4 = 4.0\ \Omega$. [A. $I_{eq} = 2.5\text{ A}$, $R_{eq} = 1.33\ \Omega$.]



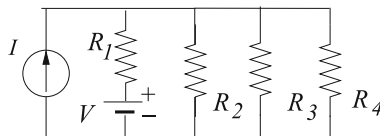
Problem 13

Problem 14 Make use of Thévenin's theorem to compute the current flowing in the resistance R_5 for the circuit in the figure with the following values for its parameters: $V = 10\text{ V}$, $R_1 = R_3 = R_5 = 200\ \Omega$, $R_2 = 600\ \Omega$, $R_4 = 400\ \Omega$. [A. 23 mA .]



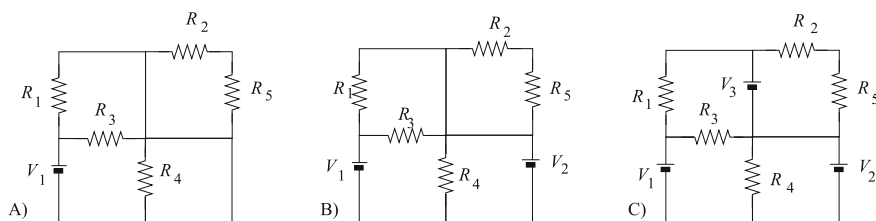
Problem 14

Problem 15 Make use of Thévenin's theorem to compute the power dissipated by the resistance R_4 for the circuit in the figure. Use the following values for its parameters: $V = 2.0\text{ V}$, $I = 4\text{ mA}$, $R_1 = 2.0\text{ k}\Omega$, $R_2 = 1.0\text{ k}\Omega$, $R_3 = 5.0\text{ k}\Omega$, $R_4 = 1.0\text{ k}\Omega$. [A. 3.4 mW .]



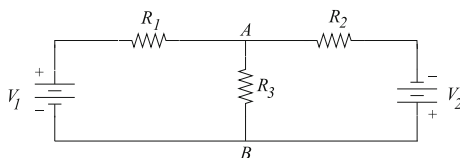
Problem 15

Problem 16 Solve the circuit indicated by (A) in the figure by an appropriate method obtaining the currents in its independent meshes and compute the current delivered by the generator V_1 . Then modify the circuit as shown in (B) by inserting a voltage generator V_2 . Use the theorems of reciprocity and superposition to evaluate the change in the output current from V_1 . Compute the value of V_2 needed to reduce this current to zero. Modify again the circuit by inserting a generator V_3 as shown in (C) and compute the new value of the current delivered by V_1 . [A. $I_1 = V_1(R_1 + R_3)/R_1R_3$; $I'_1 = (V_1 - V_2)(R_1 + R_3)/R_1R_3$; $V_1 = V_2$; $I''_1 = -V_3/R_1$.]



Problem 16

Problem 17 Compute the Thévenin equivalent with reference to the nodes A and B for the circuit in the figure and evaluate the voltage drop across R_3 assuming $V_1 = 25\text{ V}$, $V_2 = -10\text{ V}$, $R_1 = 10\text{ k}\Omega$, $R_2 = 5\text{ k}\Omega$, $R_3 = 20\text{ k}\Omega$. [A. $V_A - V_B = 1.43\text{ V}$.]



Problem 17

Electrical Measurements in the Laboratory Practice

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