Quasimodes in Integrable Systems and Semi-Classical Limit

M. Baldo and F. Raciti

Abstract Quasimodes are long-living quantum states that are localized along classical orbits. They can be considered as resonances, whose wave functions display semi-classical features. In some integrable systems, they have been constructed mainly by the coherent state method, and their connection with the classical motion has been extensively studied, in particular as a tool to perform the semi-classical limit of a quantum system. In this work, we present a method to construct quasimodes in integrable systems. Although the method is based on elementary procedures, it is quite general. It is shown that the requirement of a long lifetime and strong localization implies that the quasimode must be localized around a closed classical orbit. At a fixed degree of localization, the lifetime of the quasimode can be made arbitrarily longer with respect to the classical period in the asymptotic limit of large quantum numbers. It turns out that the coherent state method is a particular case of this general scheme.

1 Introduction

The semi-classical limit has been one of the basic issues that attracted continuous interest since the foundation of quantum mechanics. The eigenvalues of integrable systems can be obtained from the Bohr–Sommerfeld semi-classical quantization method, which is valid in the large quantum number limit, i.e. large actions with respect to the Planck constant h. More difficult is to obtain the semi-classical limit of the corresponding wave functions. It is usually assumed that the wave functions of the eigenstates cover uniformly the whole available phase space, which is the Liouville torus determined by the values of the quantum numbers. As such, they have no resemblance with any classical behaviour of the system. This is of course

M. Baldo
INFN, Sezione di Catania, Via S. Sofia 64, I-95123 Catania, Italy
e-mail: baldo@ct.infn.it

F. Raciti (✉)
Department of Mathematics and Computer Science, University of Catania, V.le A.Doria, 6, 95125-Catania, Italy
e-mail: fraciti@dmi.unict.it

T.M. Rassias, P.M. Pardalos (eds.), *Essays in Mathematics and its Applications*,
DOI 10.1007/978-3-319-31338-2_2

due to the fact that the eigenstates are stationary states which cannot be connected with any classical trajectory, which requires both localization and time dependence. The alternative method for a semi-classical description is through the introduction of wave packets, following the celebrated theorem by Ehrenfest. In the case of a particle motion, the time-dependent localized wave packet follows the classical trajectories. Its size actually spreads with time, but for macroscopic objects, the spreading time becomes asymptotically so large that the particle behaves in a completely classical fashion. For a particle moving in a potential, the theorem can be applied as a valid method for performing the semi-classical limit if the typical distance over which the potential is changing turns out to be much larger than the wavelengths that characterize the wave packet, that is, as it is well known, in the short wavelength limit.

For non-completely macroscopic objects, this means that the wave packet cannot move inside a container with sharp boundaries, noticeably a billiard; otherwise, the wave packet would spread rapidly as it hits one of the boundaries. However, it has been found that also in this typical case, localization can occur because of the presence of caustics [1], and if there is an exact degeneracy among different sites of the billiard, an exact eigenstate can be obtained by a linear combination of the wave functions localized in each site. These wave functions are generically indicated as "quasimodes". They exist also in a generic system. They have a counterpart in classical sound phenomena, like the whispering modes in an auditorium. However, the term quasimode has been used to indicate generically to states that are "close" to stationary mode, i.e. to eigenstates of the system [2, 12, 20, 22, 26]. We will indicate by quasimodes a more specific class of states. For a generic system, we will consider as quasimodes states that have the wave function sharply localized around a classical periodic orbit and have a long lifetime, in some sense to be specified later. Many authors have proposed different methods to construct quasimodes, either on the basis of particular eigenstate superposition suggested by Ehrenfest theorem [21] or on geometrical optics methods [5, 18]. In the latter work, it has been shown that in a billiard of generic shape, two types of quasimodes can be indeed present, the "whispering gallery" and "bouncing ball" modes.

Several authors [11, 25] have developed general methods to construct long-living quasimodes that asymptotically for $\hbar \to 0$ "are close" to eigenstates, in a mathematically well-defined sense. All these methods build up the quasimode around a stable periodic orbit, corresponding to an elliptical fixed point. For a Riemann manifold [11], the quasimode is localized around a stable geodesic. The general case for a smooth dynamics has been treated in [25]. An extensive analysis of the mathematical basis of quasimodes can be found in [3].

It has to be stressed that quasimodes have not to be confused with the phenomenon of scars in chaotic systems [15]. In this case, one refers to an exact eigenstate whose wave function shows some enhancement around an unstable classical periodic orbit. This phenomenon is especially present in billiards. It has been studied extensively in the literature [6, 15–17, 23, 27, 28]. Partial explanations of scars have been presented by many authors. In [7], an approach based on the Gutzwiller trace formula [14] for chaotic systems has been used, where the density of states is determined, in the semi-classical limit, solely by the classical (unstable) periodic orbits. If around a classical periodic orbit one takes also the contribution from closed non-periodic orbits, the average wave function can display some enhancement in the vicinity of the periodic orbit.

However, it is possible to relate quasimodes and scars. A method [23, 27] that has been developed to show this link is based on the so-called Gaussian beams, where a Gaussian wave packet is launched along an unstable periodic orbit. The short-time dynamics of the wave packet provides the correct superposition of eigenstates, belonging to a band in the smoothed spectrum, that gives a wave function localized along the unstable periodic orbit. In this way, one gets a short-living quasimode. At the same time, the substantial overlap of the eigenstates, involved in the superposition, with the quasimode gives an explanation of the presence of scars. These results establish a link between quasimode and scars in chaotic systems. Since the periodic orbit is unstable, the quasimode so constructed has a short lifetime, determined at least by the Lyapunov exponents of the orbit. It seems clear that a long-living quasimode, sharply localized around an unstable classical orbit, with a lifetime arbitrarily larger than the classical period, cannot exist, even in the semi-classical limit. In particular, this method can have a limited application to billiards, since, as it is well known, a wave packet that hits a sharp edge spreads out rapidly. However, in [27], it was shown that following the evolution of the wave packet, one can obtain a wave function close to an eigenstate that in some cases displays scarred structure.

In integrable systems, the situation is substantially different. Periodic orbits can be grouped in families that can be obtained by a smooth variation of an orbital parameter. As an example, in the circular billiard, one can smoothly vary the orientation of the orbit. Furthermore, each periodic orbit corresponds to a parabolic fixed point in the Poincaré map, since neighbouring trajectories diverge linearly with time. The systems have symmetries that the eigenfunctions must respect. On the contrary, a given generic periodic orbit can have only discrete symmetries, different from the ones of the system, and therefore, a quasimode, localized around a periodic orbit, cannot approach any eigenstate, even asymptotically for $\hbar \to 0$, while this is possible in the case of an isolated stable periodic orbit (in a generic system, chaotic or not). Of course a suitable linear combination of such quasimodes can approach asymptotically an eigenstate, but this cannot be considered a semi-classical object. This does not prevent the quasimode to have a long lifetime, possibly arbitrarily larger than the classical period. At the same time, localization around the orbit can be in principle achieved by a proper superposition of almost degenerate eigenstates, like in the case of isolated unstable orbits in a chaotic system. To which extent localization and long lifetime can be reached in integrable systems is the subject of the present paper.

For rectangular billiards, a method to construct quasimodes was developed in [9], based on the introduction of coherent states, in analogy with the case of the harmonic oscillator.

In this paper, we present a general procedure to build quasimodes in integrable systems that, although based on elementary methods, allows for a systematic study of their properties. In particular, we will construct quasimodes that, in the proper asymptotic limit, are localized with arbitrary precision around a periodic orbit and at the same time have a lifetime that can be made arbitrarily larger than the period of the orbit. In this way, the quasimodes can be considered as resonances within the point spectrum of the system, as we are going to consider systems completely confined in a restricted region like billiards. Notice however that they are particular resonances, since the wave function is localized around a periodic orbit and they do not respect in general the symmetries of the system and of the eigenfunctions. In any case, because of their properties, they are surely semi-classical objects.

The quasimodes could be the basis for a different way of performing the semi-classical limit, but a systematic study of this possibility has not been fully developed.

Finally, one has to mention that there are exceptional cases where quasimodes are also eigenstates because of the asymptotically large and exact degeneracy present in the spectrum, like for the harmonic oscillator. This feature is clearly connected with the well-known fact that a wave packet does not spread indefinitely, but on the contrary, its size oscillates indefinitely while following the classical trajectory (a closed orbit) [24].

In Sect. 2, we present the method in its general form for two-dimensional systems and for integrable systems, either billiards or Hamiltonian ones. In Sect. 3, we present results for the quasimodes in billiards of different shapes, and we discuss their lifetimes. The connection with coherent states is also discussed. In Sect. 4, we consider Hamiltonian systems. Besides the special case of the harmonic oscillators, we analyse a generic two-dimensional system, and we discuss the quasimodes associated with trajectories that close after several revolutions. Section 5 is devoted to the conclusions and prospects.

2 The General Method

We describe the proposed method by recalling for completeness some elementary results for classical integrable systems. The treatment will be restricted, as in the rest of the paper, to two degrees of freedom. The extension to higher dimension looks possible but not obvious. In the semi-classical limit, we quantize the two-dimensional integrable system by the Bohr–Sommerfeld scheme, where each action integral along a topological distinct path in the invariant torus is imposed to be an integer multiple of the Planck constant h. Therefore, in two-dimensional integrable systems, the energy levels $E(n, l)$ are a function of two integers (quantum numbers) n, l. We are looking for linear combinations of almost degenerate levels that are localized as much as possible. Starting from a particular pair of quantum numbers n_0, l_0 and the corresponding energy $E_0 = E(n_0, l_0)$, we look for the set of quantum numbers that in linear approximation correspond to levels degenerate with E_0. Formally,

$$\begin{aligned} n &= n_0 + \delta n \\ l &= l_0 + \delta l \\ \delta E &= \tfrac{\partial E}{\partial n}\delta n + \tfrac{\partial E}{\partial l}\delta l = 0 \end{aligned} \tag{1}$$

Since n and l are integers, in order to fulfil the condition $\delta E = 0$, it is necessary that the variations δn and δl be in a constant fractional ratio

$$\frac{\delta n}{\delta l} = \frac{p}{q} = -\frac{\partial E}{\partial l} \Big/ \frac{\partial E}{\partial n} \tag{2}$$

where p and q are two integers that are prime with each other. The partial derivatives of the energy are the frequencies of the classical motion along each degree of freedom

$$\omega_n = \frac{\partial E}{\partial n} \quad ; \quad \omega_l = \frac{\partial E}{\partial l} \tag{3}$$

It follows that the classical orbit associated with (n_0, l_0) closes after $N = pq$ periods of the faster degrees of freedom. Notice however that the trajectory is closed only to order $1/n$, since the frequencies around the tori are discrete upon quantization. This shows the well-known result that sets of quasidegenerate levels, otherwise called "shells" [8], are associated with closed classical orbits. Notice that the condition (2) is also a constraint on the reference quantum numbers (n_0, l_0). Because the semi-classical limit corresponds to asymptotically large quantum numbers, this condition can be fulfilled to any degree of precision.

An estimate of the level of degeneracy can be obtained by calculating the second derivative of the energy along the direction defined by Eq. (2). Here we are of course treating the quantum numbers as continuous variables, which is justified in the semi-classical limit. Along this direction, one can take, e.g. the quantum number n as a linear function of the other quantum number l. As a consequence, also the energy is a function only of the quantum number l. Let us assume for simplicity that the energy appears explicitly only in the action integral J corresponding to the quantum number n, that is, the corresponding Bohr–Sommerfeld quantization ($\hbar = 1$)

$$J(E, l) = 2\pi n \tag{4}$$

is the equation that determines the semi-classical energy for a given pair (n, l) of quantum numbers (assuming for simplicity no exact quantal degeneracy). This will be the case in all explicit models considered in the rest of the paper. The more general case can be treated along the same lines. In Eq. (4), both E and n are considered functions of l. Taking the first derivative of Eq. (4), one gets

$$\frac{dJ}{dl} = \left(\frac{\partial J}{\partial E}\right)\frac{dE}{dl} + \frac{\partial J}{\partial l} = -2\pi\frac{\omega_l}{\omega_n} \tag{5}$$

which fixes the condition on the quantum numbers (n_0, l_0). At the reference point (n_0, l_0), the derivative of the energy is zero by construction. Taking into account this fact, the second derivation at (n_0, l_0) of the equation reads

$$\frac{d^2J}{dl^2} = \left(\frac{\partial J}{\partial E}\right)\frac{d^2E}{dl^2} + \frac{\partial^2 J}{\partial l^2} = 0 \tag{6}$$

where we have used the vanishing of the second derivative of n with respect to l due to linear dependence of n on l, according to Eq. (2). From this, one gets at (n_0, l_0)

$$\frac{d^2E}{dl^2} = -\frac{\partial^2 J}{\partial l^2} \Big/ \frac{\partial J}{\partial E} \tag{7}$$

The spread ΔE in energy within a range of values Δl of the quantum number l around the reference value l_0 can be estimated up to second order as

$$\Delta E = \left|\frac{1}{2}\frac{d^2E}{dl^2}\right| \Delta l^2 \tag{8}$$

It follows that a linear combination of eigenstates within this range will correspond to a state with a lifetime τ of order $1/\Delta E$, $(\hbar = 1)$. Notice that the energy derivative of J is associated with the characteristic time T of the corresponding closed classical orbit, in particular to its period, and therefore,

$$\frac{\tau}{T} = 1 \Big/ \left|\left(\frac{\partial^2 J}{\partial l^2}\Delta l^2\right)\right| \tag{9}$$

Within the same range Δl, one can construct a linear combination of eigenstates $\psi_{n,l}$ to obtain a wave function Ψ localized in coordinate space. To some extent, the type of linear combination is arbitrary. The standard choice is a Gaussian superposition

$$\Psi(\mathbf{r}) = \sum_l \exp[-(l - l_0)^2/\Delta l^2]\psi_l(\mathbf{r}) \tag{10}$$

where the summation is only over l because it is performed along the direction defined by Eq. (2) (i.e. n is a function of l) and $\mathbf{r}$ is the two-dimensional position vector (coordinate space). The localization will be in the coordinate (cyclic) variable ϕ, canonical conjugated to l. The localization $\Delta\phi$ will be then of the order of $1/\Delta l$. The main goal is now to see to what extent this localization put constraints to the lifetime τ of this state. It turns out that the second derivative of J with respect to l is asymptotically of the order of $1/l$, and therefore,

$$\frac{\tau}{T} \approx l/(\Delta l)^2 \tag{11}$$

which means that, at a fixed localization $\Delta\phi$, the ratio between the lifetime and the classical orbital period is asymptotically arbitrarily large. This result is valid for all the particular systems we are going to consider. The extension of these properties to a general system looks likely but not obvious.

It remains to demonstrate that the localization is around a definite classical orbit. This can be shown by introducing the standard semi-classical expression for the wave functions

$$\Psi_l(\mathbf{r}) \;\approx\; \exp(iS_l(r))\exp(il\phi) \tag{12}$$

where $S_l(r)$ is the reduced action, i.e. the wave function is the exponential of the total action. Expanding the action around $l = l_0$ and taking the stationary phase approximation of the superposition in l of Eq. (10) gives (apart from an irrelevant phase)

$$\Psi(\mathbf{r}) \;\approx\; \exp\left(-\frac{|\phi - \phi_S(\mathbf{r})|^2}{2\Delta\phi^2}\right) \tag{13}$$

where

$$\phi_S(r) \;=\; -\left(\frac{dS_l(r)}{dl}\right)_{l=l_0} \tag{14}$$

and $\phi = \phi_S(r)$ is indeed the equation of the trajectory. It has to be noticed that the superposition (10) is invariant under a shift of ϕ by a multiple of the quantity $\delta\phi$

$$\delta\phi \;=\; \frac{2\pi}{q} \tag{15}$$

because the summation over l is performed with a step q. This implies that the quasimode has a discrete symmetry of $\Delta\phi$. This means also that there are actually multiple points of stationary phase, regularly spaced in ϕ by $\delta\phi$. The approximate expression of Eq. (13) must be then summed up over these q stationary points, which gives the discrete symmetry. All that will be more clear in the explicit applications of the method, where the meaning of this shift $\delta\phi$ will be more evident.

The superposition of Eq. (10) can be performed also numerically, as we will do in the specific examples where the eigenfunctions are analytically known. Since in this case no approximation is used for calculating the quasimode wave function (10), the symmetry discussed above is automatically included. It has to be stressed that in this case there is some freedom in the choice of the eigenfunctions Ψ_{nl}, since they can be normalized but they can still be multiplied by a phase which eventually can be dependent on the quantum numbers (n, l). This can modify the superposition (10), since the phases will result in a different interference pattern. This is a quantum feature that cannot be eliminated even in the semi-classical limit. The choice of the phases can modify the region where the quasimode is localized. To get the proper choice, one can look at the approximate expression (12) and check if in the asymptotic limit this expression is indeed recovered. As a particular case, we can associate to each wave function a phase factor $\exp(il\phi_0)$. According to Eq. (13), this would just simply shift the variable ϕ by a fixed amount ϕ_0. Since ϕ_0 is arbitrary, one can see that one can associate to each classical orbit a family of orbits with similar characteristics but with a different geometry.

3 Quasimodes in Integrable Billiards

The billiards are the simplest systems where quasimodes can be constructed. At classical level, they cannot be described easily by means of a Hamiltonian, because of the discontinuity of the trajectory velocity at the point of bounce on the billiard boundary. However, between two bounces, the motion is free, and one can describe the trajectory as piecewise continuous. Moreover, one can still consider the Liouville–Arnold tori in phase space.

3.1 The Rectangular Billiard

Despite billiards not being Hamiltonian systems, they can be treated within the general method we have introduced. Let us start with the simplest billiard, the square billiard (SB). This case has been extensively studied in [9]. It has also been shown [10] that the quasimodes present wave functions characterized by a vortex structure. In that work, the quasimodes were constructed by coherent states, defined as in the case of the harmonic oscillator. We will treat briefly this case, following the scheme of Sect. 2, and show that one can construct quasimodes more generally, being the coherent states a particular class of them.

For the square billiard, the constants of motion are the (quasi)momenta k^x and k^y along the two side directions, and quantization gives

$$\begin{aligned} k^x{}_n &= n\tfrac{\pi}{R} \\ k^y{}_m &= m\tfrac{\pi}{R} \end{aligned} \tag{16}$$

where R is the side length of the square and n, m two integers, positive or negative. The motion is free, and therefore, the energy levels $E(n,m)$ are readily obtained

$$E(n,m) = n^2 + m^2 \tag{17}$$

where for simplicity we put $\pi/R = 1$. The condition (2) of stationary energy around the point (n_0, m_0) in this case is

$$\frac{\delta n}{\delta m} = -\frac{m_0}{n_0} = -\frac{p}{q} \tag{18}$$

Eq. (18) gives in a straightforward way the conditions on both the reference quantum numbers (n_0, m_0) and the direction $(\delta n, \delta m)$ along which the quantum numbers must move. The eigenfunctions are just the products of standing waves along x and y, which are sine (cosine) functions for even (odd) quantum numbers (taking the origin at the centre of the square). The two quantum numbers n and m are equivalent, and the superposition of Eq. (10) can be done in anyone of the two. For illustration, let us take $m_0 = n_0$ and a step of one unit for both quantum numbers in the superposition of Eq. (10). One can write

$$\Psi(x, y) = \sum_l \exp(i(m_0 + l)y + l\phi_0)\psi_l[(n_0 - l)x] \tag{19}$$

where for even values of $n_0 - l$, the eigenfunction ψ_l is a sine function, while for odd ones, it is a cosine function. The additional phase ϕ_0 has a given value. Here, the summation on l is extended in an interval between $l = -N_0$ and $l = N_0$, being N_0 large enough but much smaller than n_0 and m_0. Notice that we took a superposition with a constant factor rather than the Gaussian form of Eq. (10). The summation can be calculated exactly, since it involves geometrical series. The result is the superposition of four terms of the type

$$S_{\pm\pm}(x, y) = \sin[(N_0 + 1)(y \pm x \pm \phi_0)] / \sin(y \pm x \pm \phi_0) \tag{20}$$

where the choices of the signs are independent of each other. The wave function is therefore concentrated along the four straight lines $y \pm x \pm \phi_0 = 0$. For the choice $\phi_0 = \pi/2$, the numerical calculation gives the result depicted in Fig. 1. This has been obtained for $(n_0, m_0) = (400, 400)$ and $N_0 = 10$. The wave function is clearly concentrated along a classical trajectory. Notice that the considered superposition with a constant weight produces oscillations outside the region of maximum contribution, as it can be seen from Eq. (20). A similar procedure can be obtained with a Gaussian weight, as in Eq. (10). The approximate result is expected to be the superposition of the same four branches, but with smoother profiles. The numerical evaluation produces a plot practically indistinguishable from the one depicted in Fig. 1. This indicates that the type of weight for the superposition is not crucial, provided of course that the width of the superposition is similar. Other choices of ϕ_0 correspond to different classical trajectories. At classical level, this phase can be interpreted as fixing the time laps of the motions along the x and y directions. For $\phi_0 = \pi/3$, the result is reported in Fig. 2.

As ϕ_0 is varied, a family of classical trajectories is generated.

Other trajectories with a different topology are generated by different steps in the summation for the quantum numbers n an m. For $\delta n/\delta m = 1/2$ and $\phi_0 = \pi/2$ and $\phi_0 = \pi/3$, the result is reported in Fig. 3 and in Fig. 4, respectively.

The connection of the present method with the one in [9], based on the coherent states, can be obtained by considering the asymptotic form of the combinatorial factors that are used in the superposition. In fact, one has

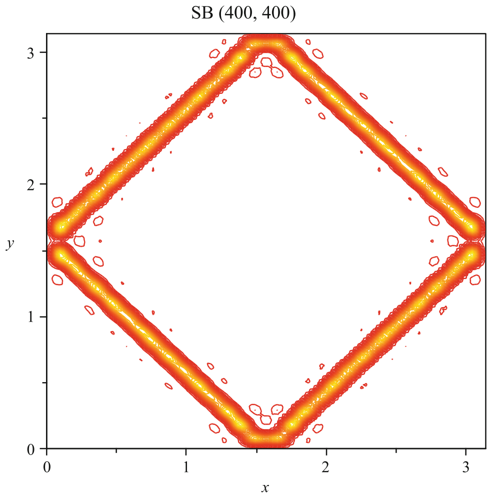

Fig. 1 Quasimode in the square billiard corresponding to the quantum numbers $(n_0, m_0) = (400, 400)$ and $p = q = 1$. The phase is $\phi_0 = \pi$ [see Eq. (19)]

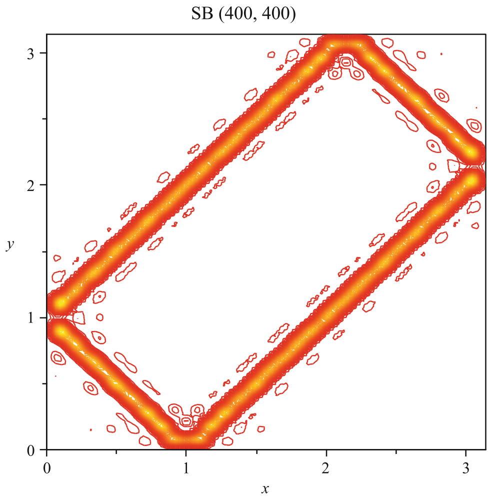

Fig. 2 The same as in Fig. 1, but for $\phi_0 = \pi/4$

$$\begin{pmatrix} N \\ K \end{pmatrix} \approx \exp[-(K - K_0)^2/K_0] \tag{21}$$

where we expanded the Stirling formula for the factorials around the value $K_0 = N/2$, where the combinatorial factor has its maximum. One can see that the coherent state representation corresponds to a Gaussian superposition with a particular choice for the width of the Gaussian.

Finally, for the energy spread of Eq. (8) and lifetime τ of Eq. (9), one gets

$$\Delta E = 1/n_0 \qquad ; \qquad \frac{\tau}{T} = n_0^3/E \approx n_0 \tag{22}$$

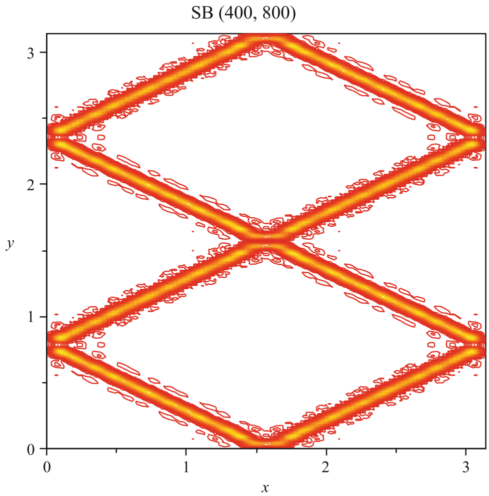

Fig. 3 Quasimode corresponding to $p = 1$ and $q = 2$ with the values of (n_0, m_0) reported in the title. The phase $\phi_0 = \pi$. See the text for details

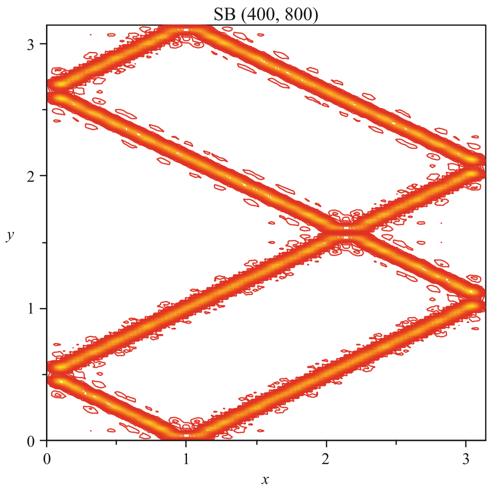

Fig. 4 The same as in Fig. 3, but with $\phi_0 = \pi/2$

which asymptotically for large quantum numbers have the anticipated trend. In the particular cases considered above, one has $\tau/T \approx 4$. With the same degree of localization, one can get an increasing value of τ/T as the quantum numbers increase.

3.2 The Circular Billiard

The next example we are considering is the circular billiard (SB), that was partially analysed in [4]. Here, we report a more extensive analysis, a set of results that illustrate the general method and a study of the localization around a classical trajectory. The semi-classical Bohr–Sommerfeld quantization of the circular billiard leads to a formula of the type of Eq. (4), which explicitly reads [18] ($\hbar = 2M = 1$, being M the mass of the particle)

$$\sqrt{k_{nl}^2 R^2 - l^2} - l\beta_0 = \left(n + \frac{1}{2}\right)\pi + \frac{\pi}{4} \tag{23}$$

where in this case l is the angular momentum, n is the quantum number associated with the radial motion, R is the radius of the billiard and k_{nl} is the momentum, which is then an implicit function of l and n, i.e. it is the eigenvalue. This formula can be obtained by the usual action integral along a closed path on the Liouville torus. Notice that this action integral is actually twice the LHS expression. The additional term $1/2$ at the RHS is introduced as a minimal quantum correction, and the additional term $\pi/4$ is needed because of the presence of reflections at a sharp boundary. These are standard corrections, but in any case, they can be safely neglected since we are working in the large quantum number limit. In this case, the motion is free, and the corresponding energy is just the kinetic energy

$$E_{nl} = k_{nl}^2 \tag{24}$$

In Eq. (23), the angle β_0 is related to the momentum by

$$\cos(\beta_0) = l/k_{nl}R \tag{25}$$

and therefore, β_0 is also an implicit function of the quantum numbers n, l. At the classical level, the angle $2\beta_0$ can be interpreted as the angle spanned by the vector $\mathbf{r}$ that fixes the position of the particle, between two bouncing on the billiard wall. If this angle is a rational fraction of 2π, i.e. $2\beta_0 = \frac{p}{q}2\pi$, then the particle orbit will close after pq hits on the wall (here we assume that p and q have no common factor). If instead it is an irrational fraction of 2π, the orbit will never close, and in the long time limit, the position of the hits will fill uniformly the circular boundary. The general Eq. (2) in this case reads

$$\frac{\beta_0}{\pi} = \cos^{-1}\left(\frac{l_0}{k_{nl}R}\right) = \frac{\delta n}{\delta l} = -\frac{p}{q} \tag{26}$$

and the corresponding classical orbit around which the quasimode is localized is indeed a closed orbit which closes after q bounces. The localization can be constructed according to the general prescription of Eq. (10). The action integral $S_l(r)$ in this case can be calculated analytically. After some manipulations, it reads

$$\begin{aligned} S_l(r) &= \int^r dr' \sqrt{E - \tfrac{l^2}{r'^2}} \\ &= \sqrt{k^2 r^2 - l^2} - l\beta(r) + C \end{aligned} \tag{27}$$

where C is a constant and

$$\cos(\beta(r)) = \frac{l}{kr} \tag{28}$$

Taking the lower limit of the integral as $r_0 = l/kR$ and the upper limit $r = R$, one gets the LHS of Eq. (23). In fact, this expression, multiplied by 2, is just the action integral over a closed loop along the proper Liouville torus. Following the general procedure, one can consider the superposition of Eq. (10), and one gets for the quasimode wave function (13) in the semi-classical limit

$$\Psi(r,\phi) \approx \exp\left(-\frac{[\phi - \beta(r)]^2}{2\Delta\phi^2}\right) \tag{29}$$

where (r,ϕ) are the cylindrical coordinates and $\Delta\phi = 1/\Delta_l$, being Δ_l the spread in l values considered along the direction specified by Eq. (26). The wave function is therefore concentrated along the curve $\phi = \beta(r)$, which is the straight line between two successive bounces. As discussed in Sect. 2, this expression should be summed up over the series of shifted phases, resulting in the total wave function of the quasimode Ψ_{tot}

$$\Psi_{tot}(r,\phi) = \sum_{j=1}^{q} \Psi(r,\phi + j\Delta\phi) \tag{30}$$

where in this case the discrete symmetry is indeed a rotational symmetry in ordinary space. After q bounces, the trajectory closes, while p has the meaning of "winding number", i.e. the number of times the trajectory performs a complete rotation around the centre of the billiard before closing.

Notice that the spatial width can be estimated as $\Delta s \approx R\Delta\phi \approx R/\Delta_l$. Here, we have implicitly assumed that the superposition is centred around $\phi = 0$. Shifting ϕ by a certain angle ϕ_0 would rotate the wave function by ϕ_0.

The energy spread ΔE can be calculated according to Eq. (8). One finds for ΔE and the corresponding lifetime τ of the quasimode

$$\frac{\Delta E}{E} \approx \left(\frac{\Delta_l}{k_{n_0 l_0} R}\right)^2 \qquad ; \qquad \frac{\tau}{T} \approx \left(\frac{\Delta s}{R}\right)^2 l_0 \tag{31}$$

One can see that the lifetime is asymptotically large in the semi-classical limit with respect to the period of the corresponding classical orbit at a fixed value of the spatial width.

Also in this system, the eigenfunctions are analytically known, and the eigenvalue equation (23) can be also obtained by considering their asymptotic expression for large quantum numbers. The eigenfunctions are given by the cylindrical Bessel function J_l

$$\Psi_{nl} = \exp(il\phi) J_l(k_{nl} r) \tag{32}$$

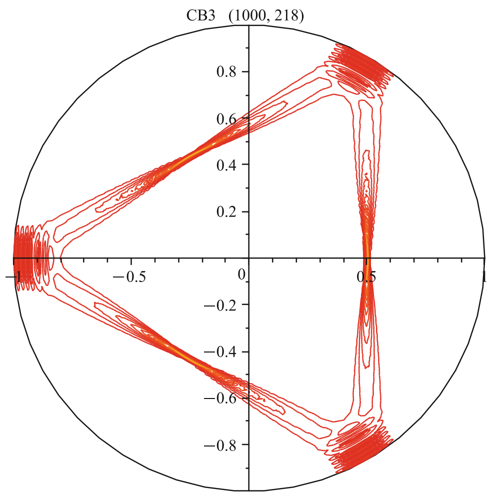

Fig. 5 Quasimode in the circular billiard corresponding to $p = 1$ and $q = 3$ and the central values (l_0, n_0) reported in the title

The exact eigenvalues are given by the zeros of the Bessel function at the boundary

$$J_l(k_{nl}R) = 0 \tag{33}$$

If one uses the asymptotic expression [13] for the Bessel function, one recovers the Bohr–Sommerfeld quantization of Eq. (23). The superposition of Eq. (10) can be done numerically with the exact eigenfunctions and eigenvalues, while of course the constraint on (n_0, l_0) implicitly implied by Eq. (2) can be only approximately satisfied, in principle with arbitrary precision in the large quantum number limit.

Let us consider some applications. If we take $p = 1$ and $q = 3$, we get the trajectory of triangular shape depicted in Fig. 5 for a specific case. A preliminary study of this case was already considered in [4]. In general, for $p = 1$ one gets, as it can be easily checked, the trajectory along a polygon of q sides, as in the case of Fig. 6, corresponding to $q = 7$. For large q, these quasimodes merge into the "whispering modes", studied in [5] for a generic billiard. For $p > 1$, a trajectory of "star" shape is generated. For $(p, q) = (2, 7)$ and $(p, q) = (3, 7)$, one gets the trajectories of Fig. 7 and of Fig. 8, respectively. For sake of illustration, we report in Table 1 the values of the quantum numbers used in the superposition and the corresponding energies in the case of Fig. 7.

For the star cases, it seems that a large concentration of the wave function is along a polygon on the points of the self-intersections of the classical trajectories. However, this is an artefact of the plotting system. In fact, close to the centre, the Bessel functions display an extremely oscillating behaviour that any plotting system is not able to follow, but instead it samples randomly the sharp and very high peaks of the wave function. This feature is not connected with the self-intersections that occur in the wave function, as it can be checked by looking at the square billiard, analysed in the previous subsection, where self-intersections occur but they do not display any similar behaviour. However, the interference that must occur between

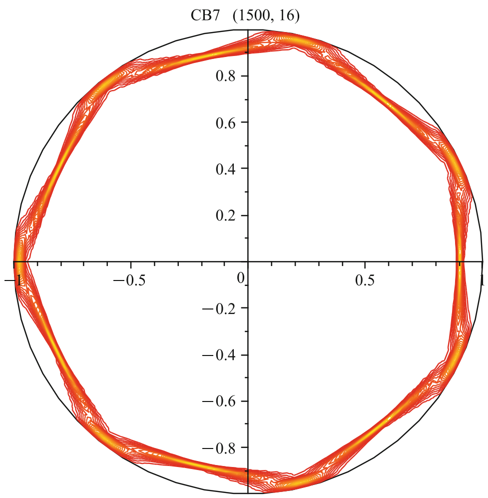

Fig. 6 Quasimode in the circular billiard corresponding to $p = 1$ and $q = 7$, with the central values (l_0, n_0) reported in the title. The quasimode is localized around the classical trajectory that is the polygon of seven sites inscribed in the circle. See the text

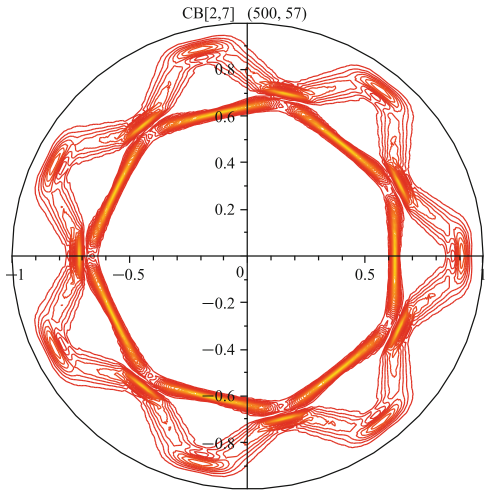

Fig. 7 Quasimode in the circular billiard corresponding to $p = 2$ and $q = 7$ and the central values (l_0, n_0) reported in the title. The quasimode is localized around the classical trajectory of "*star*" shape, in which the particle bounces seven times and performs two turns around the centre before closing

the two branches of the wave function that intersect seems to emphasize this effect. To illustrate the oscillations, we report in Fig. 9 the plot of the Bessel function corresponding to $l = 1000$ and $n = 69$. One can see the suppression of the wave function below the centrifugal barrier and the corresponding sharp rise at the barrier. In any case, it is important to realize that these wild oscillations will persist even in the extreme semi-classical limit (i.e. very large quantum numbers) and they are quantum phenomena that cannot be eliminated. This touches the well-known problem of the de-coherence that should occur in the classical limit, as suggested by several authors [29].

Fig. 8 Quasimode in the circular billiard corresponding to $p = 3$ and $q = 7$ and the central values (l_0, n_0) reported in the title. The quasimode is localized around the classical trajectory of "*star*" shape, in which the particle bounces seven times and performs three turns around the centre before closing

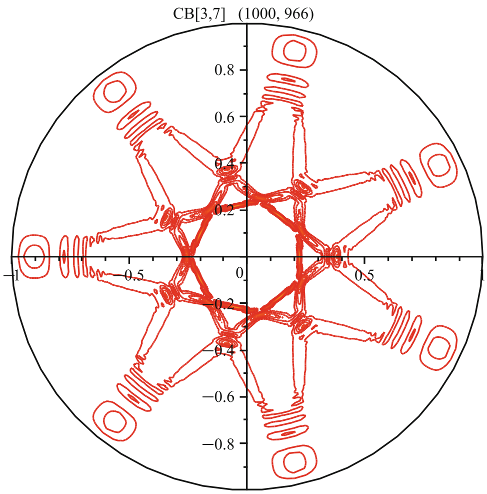

Table 1 Quantum numbers used for the quasimode of Fig. 7 and the corresponding energy (last column)

l	n	$E(n, l)$
430	77	797.5520632
437	75	798.3765400
444	73	799.1238211
451	71	799.7920827
458	69	800.3793629
465	67	800.8835484
472	65	801.3023598
479	63	801.6333358
486	61	801.8738136
493	59	802.0209079
500	**57**	802.0714854
507	55	802.0221372
514	53	801.8691447
521	51	801.6084410
528	49	801.2355660
535	47	800.7456120
542	45	800.1331610
549	43	799.3922075
556	41	798.5160678
563	39	797.4972677
570	37	796.3274064

In bold face are indicated the central quantum numbers (l_0, n_0) of the superposition (see text)

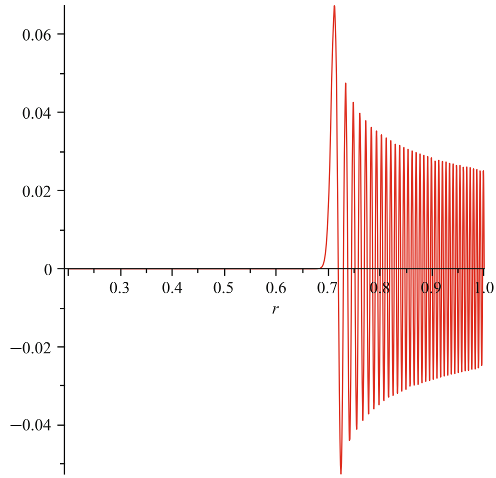

Fig. 9 Plot of the cylindrical Bessel function corresponding to the quantum numbers $(l_0, n_0) = (1000, 69)$. Notice the classically forbidden region and the rapid oscillation at the corresponding boundary

In all these examples, the lifetime τ of the quasimodes is in the range $\tau/T = 10$–15. It has to be noticed that the wave function is concentrated along a classical orbit only if the condition of quasidegeneracy of Eq. (1) is satisfied. In fact, localization can be easily obtained by a suitable superposition of eigenfunctions, but if that condition is not satisfied, the localization is not along a classical trajectory, even if the wave function has the same symmetry. An example is shown in Fig. 10, where the superposition is not taken along the direction fixed by Eq. (1), even if the considered eigenfunctions are chosen to be approximately degenerate, but not with the same degree of accuracy, and the steps are taken as for Fig. 7. In other words, the reference values n_0 and l_0 of the quantum numbers do not satisfy the condition implicit in Eq. (2). One can see that the localization is along some "curved star", which is not of course a classical trajectory, but with the symmetry fixed by the values of p and q.

4 Hamiltonian Systems

We now consider two-dimensional systems described by an integrable smooth Hamiltonian. We will concentrate on the motion of a particle in a central potential. In this case, the natural choices of the quantum numbers are the angular momentum and the one associated with the radial motion. In general, the classical trajectories are not closed for a generic potential. For particular values of the constant of motion, the trajectory can close after a certain number of turns around the centre of the potential. According to the general scheme, quasimodes are associated with such trajectories, on which they are concentrated. A special case is represented by the circular orbits, which needs a separate treatments since the radial quantum number vanishes. There are few central potentials that admit an exact quantum solution. The

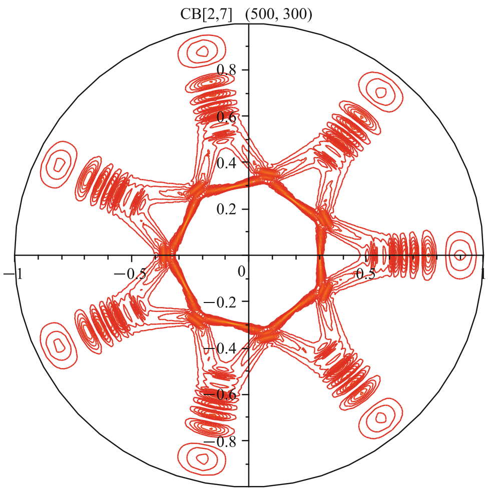

Fig. 10 Wave function localized around a non-classical path, which is not a quasimode. See the text for explanations

harmonic oscillator and the Kepler motion are the most known. However, these are very particular cases, since the trajectories are all closed and the spectrum displays very large exact degeneracy. We will consider briefly these cases that have been already considered in the literature in connection with quasimodes. We will then treat in an approximate way the general case and show the resemblance with the circular billiard.

4.1 The Harmonic Oscillator

It is well known that for the harmonic oscillator, the hypotheses of the Ehrenfest theorem are verified, i.e. a wave packet that moves inside the potential has its centre of mass moving indefinitely along a classical (closed) trajectory and its width does not spread but only oscillates with the frequency of the harmonic oscillator. In [10, 24], quasimodes were constructed using the coherent state representation, and it has been shown they concentrate indeed along a classical trajectory and furthermore the quasimode wave function has the largest strength in the positions where the classical motion is slower, e.g. at the point of maximum radial position ("periastron"), and actually, it is proportional to this time. As already noticed for the square billiard, the coherent state method is a particular case of the general method we have described. As we will show in the next subsection, in this case, the energy width of Eq. (8) vanishes, which is in line with the large degree of degeneracy in the spectrum. This shows the connection between the time-dependent treatment of Ehrenfest and the quasimodes.

We only illustrate in Fig. 11 a quasimode constructed with a uniform superposition of eigenfunctions, with a specific width, belonging to a given degenerate shell. The result is in line with [24].

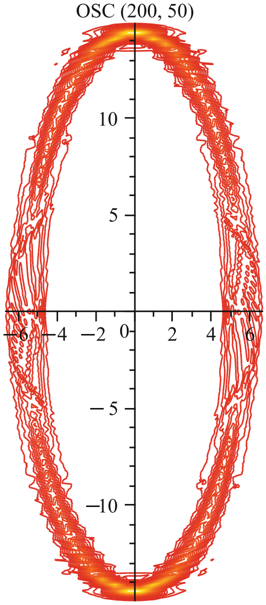

Fig. 11 Quasimode for the harmonic oscillator for the central values (l_0, n_0) reported in the title. The superposition is performed with exactly degenerate eigenfunctions, and therefore, it is actually an eigenstate of the harmonic oscillator

A similar treatment could be followed for the Kepler motion, for which no simple coherent states exist. It can be suggestive to guess a connection between quasimodes and the Rydberg states [19] in hydrogen-like atoms. The study of this case is left to future work.

4.2 *The General Potential Case*

Let us consider the bound motion in a generic central potential $V(r)$. For simplicity, we will assume that the potential is monotonically increasing with r. The action integral J_r for the radial motion can be written as

$$J_r(E, l) = 2 \int_{r_m}^{r_M} dr \sqrt{2m(E - V(r)) - l^2/r^2} \tag{34}$$

where r_m and r_M are the smaller and the larger radial coordinates where the square root vanishes. They correspond to the turning points of the trajectory. The derivative of J_r turns out to be the angle $\Delta\theta$ spanned by the radial vector as the particle moves from r_m to r_M and back

$$\Delta\theta = -\frac{\partial}{\partial l} J_r(E, l) = 2 \int_{r_m}^{r_M} \frac{dr}{r^2} \frac{l}{\sqrt{2m(E - V(r)) - l^2/r^2}} \tag{35}$$

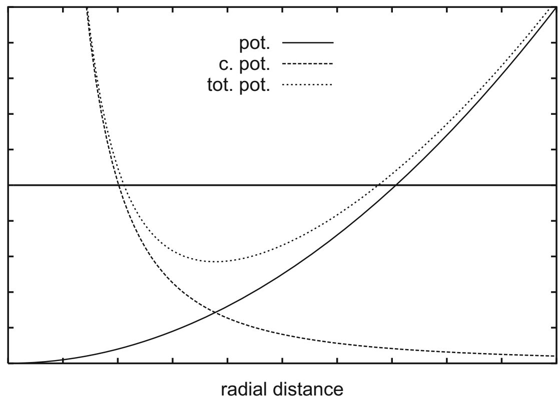

Fig. 12 Schematic representation of a generic central potential (pot) with a possible centrifugal potential (c.pot) and their sum (tot.pot). The *thick horizontal line* indicates a possible value of the total energy

For a generic potential, this angle depends both on the energy E and the angular momentum l. According to Eqs. (2) and (5), this angle must be a rational fraction of π, which is the condition on (n_0, l_0) and implies that the corresponding classical trajectory is closed. For the harmonic oscillator and the Kepler motion, this angle is however exactly π, independent of E and l. As a consequence, all the trajectories are closed (they close after two radial oscillations), and the second derivative of Eq. (7), which determines the energy spread, vanishes, as it was anticipated in the previous subsection.

In Fig. 12 is reported a schematic representation of the three terms appearing in the square root, the energy E, the potential $V(r)$ and the centrifugal potential l^2/r^2. Under our assumptions, the effective potential $U(r)$, the sum of $V(r)$ and the centrifugal potential, has a single minimum at a given radial distance r_0, and the energy must be larger than $U(r_0)$.

As shown in the appendix, after some manipulations of the integral, one finally gets

$$\Delta\theta \;=\; G(E, z_l) \qquad ; \qquad z_l \;=\; l/\sqrt{2mE} \tag{36}$$

where G is a smooth function, i.e. its derivatives with respect to the arguments are bounded. This term is identically π for the harmonic oscillator. The derivative of G with respect to l at a fixed energy E is therefore of the order at most of $1/\sqrt{E}$, which in the semi-classical limit becomes vanishing small. Furthermore, if the ratio $l/\sqrt{2mE}$ tends to a finite value, the derivative is equally of the order $1/l$. The result is valid for a generic potential, provided some reasonable conditions on the potential are fulfilled (see appendix). This finding is in line with the general statement about the lifetime of the quasimode with respect of the classical period, according to Eq. (9).

The motion in the potential can become close to the motion in a circular billiard if the potential resembles a container with a steep boundary. For the billiard, the ratio $l/\sqrt{2mE}$ is bound by the radius R of the billiard. The limiting values R and zero correspond to the condition on the angle $2\beta_0$ to approach asymptotically a value equal to zero (circular orbit) or to π (radial oscillation passing through the centre).

5 Conclusion

We have presented a general method to construct semi-classical quasimodes in integrable two-dimensional systems. These quantum states are localized around a given classical closed orbit, and at the same time, their lifetime τ can be made asymptotically large with respect to the classical period T of the orbit. At a fixed value of the localization width around the classical closed orbit, the ratio τ/T diverges for asymptotically large quantum numbers. This means that the system is considered first in the asymptotically large time limit, and then eventually in the limit usually indicated as the $\hbar \to 0$ limit. As it is well known, the two limits cannot be inverted in general, and therefore, the quasimodes have to be constructed with a definite procedure.

The method is general, and it can be applied both to billiards and to the Hamiltonian system. It includes as a particular case the one based on the generalized coherent states. Several examples have been illustrated for billiards, and, besides the harmonic oscillator, the treatment of a generic Hamiltonian system has been discussed. The construction of quasimodes suggests a method to perform the semi-classical limit in integrable systems, which however keeps some quantum features since these states are uniformly distributed along a classical periodic orbit. They can be considered as particular resonance states within the spectrum of the system. Their connection with the Ehrenfest semi-classical limit, based on (time-dependent) localized wave packets, has still to be clarified.

Appendix

In this appendix, we evaluate for a generic potential the dependence of the angle $\Delta\theta$ in Eq. (36) on the angular momentum l at a fixed energy E. In the expression of Eq. (35), it is convenient to introduce the new variable $y^2 = l^2/2mEr^2$. In the new variable, one gets

$$\Delta\theta = 2\int_{y_m}^{y_M} \frac{dy}{\sqrt{1 - [y^2 + V(r)/E]}} \tag{37}$$

where

$$r = r(y) = \frac{l}{\sqrt{2mE}\, y} \tag{38}$$

The limits of integration y_m and y_M are the values at which the square root vanishes. Under the assumption of a monotonically increasing potential at increasing r, there are only two values, corresponding to the extremes of the radial oscillations

$$\frac{1}{E} V(r(y_0)) + y_0^2 = 1 \tag{39}$$

where y_0 is either y_m or y_M. If the potential is smooth and the radial motion has a non-zero amplitude (i.e. the trajectory is not exactly circular), y_0 is a smooth function of l, which actually appears only in the combination $l/\sqrt{2mE}$. However, the integrand is singular at the integration limits, although the integral is of course converging. This does not allow to do any derivative with respect to l inside the integral to calculate the derivative of $\Delta\theta$. We have then to analyse the contribution to the integral from an interval close to the limits of integration. If the trajectory is not circular, at y_M, the function $R(y)$ inside the square root vanishes linearly, and the integrand can be written as

$$\frac{1}{\sqrt{1 - R(y)}} = \frac{1}{\sqrt{R'(y_M)(y_M - y)}} + S(y) \tag{40}$$

where R' is the derivative of R and the remainder $S(y)$ is a regular smooth function. The contribution to the integral from an interval $y_1 < y < y_M$, with y_1 some value close to y_M, can then be written as

$$\int_{y_1}^{y_M} \frac{dy}{\sqrt{R(y)}} = \frac{2}{\sqrt{R'(y_M)}}\sqrt{y_M - y_1} + \int_{y_1}^{y_M} dy\, S(y) \tag{41}$$

Explicitly, the derivative R' is

$$R'(y_M) = -\frac{1}{E}V'\left(\frac{l}{\sqrt{2mE}\, y_M}\right)\frac{l}{\sqrt{2mE}\, y_M^2} + 2y_M \tag{42}$$

where $V'(x) = dV/dx$. The same procedure can be followed for the lower limit y_m. At fixed E, the integral is therefore a smooth function of $l/\sqrt{2mE}$.

References

1. Arnold, V.I.: Modes and quasimodes. Funct. Anal. Appl. **6**(2), 94–101 (1972)
2. Barnett, S.M., Radmore, P.M.: Quantum theory of cavity quasimodes. Opt. Commun. **68**, 364–368 (1988)
3. Babic, V.M., Buldyrev, V.S.: Short Wave-Length Diffraction Theory. Springer, Berlin (1990)
4. Baldo, M., Raciti, F.: Building quasimodes in integrable billiards. Phys. Lett. A **223**, 417–420 (1996)
5. Balian, R., Bloch, R.: Eigenfrequency density oscillation. Ann. Phys. **69**, 76–170 (1971)
6. Berry, M.V.: Quantum scars of classical closed orbits in phase space. Proc. R. Soc. Lond. **423**, 219–231 (1989)
7. Bogomolny, E.B.: Smoothed wave functions of chaotic quantum systems. Physica D **31**, 169–189 (1988)
8. Bohr, A., Mottelson, B.R.: Nuclear Structure, vol. II. Benjamin, New York (1975)
9. Chen, Y.F., Huang, K.F., Lan, Y.P.: Localization of wave patterns on classical periodic orbits in a square billiard. Phys. Rev. E **66**, 046215–046221 (2002)

10. Chen, Y.F., Huang, K.F., Lan, Y.P.: Quantum manifestations of classical periodic orbits in a square billiard: formation of vortex lattices. Phys. Rev. E **66**, 066210–06621 (2002)
11. Colin de Verdiere, Y.: Quasimodes sur les variétées Riemanniennes. Invent. Math. **43**, 15–52 (1977)
12. Gamblin, D.: Construction de quasimodes de Rayleigh á grande durée. J. Funct. Anal. **236**, 201–243 (2006)
13. Gradshteyn I.S., Ryzhik, I.M.: Tables of Integrals, Series and Products, Formula 8.453. Academic, New York (1991)
14. Gutzwiller, M.C.: Periodic orbits and classical quantization conditions. J. Math. Phys. **12**, 343–358 (1971)
15. Heller, E.J.: Bound states eigenfunctions of classically chaotic Hamiltonian . Phys. Rev. Lett. **53**, 1515–1518 (1984)
16. Kaplan, L., Heller, E.J.: Linear and nonlinear theory of eigenfunction scars. Ann. Phys. **264**, 171–206 (1998)
17. Kaplan, L., Heller, E.J.: Measuring scars of periodic orbits. Phys. Rev. E **59**, 6609–6628 (1999)
18. Keller, J.B., Rubinow, S.I.: Asymptotic solution of eigenvalue problems. Ann. Phys. **9**, 24–75 (1960)
19. Mestayer, J.J., Wyker, B., Lancaster, J.C., Dunning, F.B., Reynhold, C.O., Yoshida S., Burgdörfer, J.: Realization of localized Bohr-like wave packets. Phys. Rev. Lett. **100**, 243004–243007 (2008)
20. Norris, A.N.: Rays,beams and quasimodes on thin shell structures. Wave Motion **21**, 127–147 (1995)
21. Paul, T., Uribe, A.: A construction of quasi-modes using coherent states. Ann. Inst. Henri Poincaré **59**, 357–381 (1993)
22. Perez, E.: Long time approximations for solutions of wave equations via standing waves from quasimodes. J. Math. Pures et Appl. **90**, 387–411 (2008)
23. Polavieja, G.G., Borondo F., Benito, R.M.: Scars in groups of eigenstates in a classically chaotic system. Phys. Rev. Lett. **73**, 1613–1616 (1994)
24. Pollet, J., Meplan, O., Gignoux, C.: Elliptic eigenstates for the quantum harmonic oscillator. J. Phys. A Math. Gen. **28**, 7287–7298 (1995)
25. Ralston, J.V.: On the construction of quasimodes associated with stable periodic orbits. Commun. Math. Phys. **51**, 219–242 (1976)
26. Stefanov, P: On the resonances of the Laplacian on waveguides. C.R. Acad. Sci. Paris **330**, 105–108 (2000)
27. Tomsovic, S., Heller, E.J.: Semiclassical construction of chaotic eigenstates. Phys. Rev.Lett. **70**, 1405–1408 (1993)
28. Wisniacki, D.A., Vergini, E., Benito, R.M., Borondo, F.: Scarring by homoclinic and heteroclinic orbits. Phys. Rev. Lett. **97**, 094101–094106 (2006)
29. Zurek, W.H.: Pointer basis of quantum apparatus: into what mixture does the wave packet collapse? Phys. Rev. D **24**, 1516–1525 (1981)

Springer
http://www.springer.com/978-3-319-31336-8

Essays in Mathematics and its Applications
In Honor of Vladimir Arnold
Rassias, T.M.; Pardalos, P. (Eds.)
2016, VIII, 663 p. 74 illus., 22 illus. in color., Hardcover
ISBN: 978-3-319-31336-8