Chapter 2
Preliminaries

`Abundant dulcibus vitiis.`

I immediately mention that always[1] $p \geq 2$ in these notes and often $p > n =$ the dimension of the space.[2] (At a first reading, one had better keep p large.) We aim at $p = \infty$.

We begin with a special case of Ascoli's theorem.

Theorem 2.1 (Ascoli) *Let the sequence of functions $f_k : \Omega \longrightarrow \mathbb{R}$ be*

equibounded:

$$\sup_{\Omega} |f_k(x)| \leq M < \infty \quad \textit{when} \quad k = 1, 2, \ldots,$$

equicontinuous:

$$|f_k(x) - f_k(y)| \leq C|x - y|^{\alpha} \quad \textit{when} \quad k = 1, 2, \ldots$$

Then there exists a continuous function f and a subsequence such that $f_{k_j} \longrightarrow f$ locally uniformly in Ω.

If the domain Ω is bounded, all functions can be extended continuously to the boundary, and the convergence is uniform in the closure $\overline{\Omega}$.

Proof We reproduce a well-known proof. First, we construct a subsequence which converges at the rational points. Let $q_1, q_2, q_3, \ldots$ be a numbering of the rational points in Ω. Since the sequence $f_1(q_1), f_2(q_1), f_3(q_1), \ldots$ is bounded by our assumption it has a convergent subsequence (Weierstrass' Theorem), say $f_{1j}(q_1)$, $j = 1, 2, 3, \ldots$ Consider the next point q_2. Now the sequence $f_{11}(q_2)$, $f_{12}(q_2)$,

[1]The p-harmonic operator is not pointwise defined for $p < 2$.

[2]All functions in the Sobolev space $W^{1,p}$ are continuous when $p > n$.

P. Lindqvist, *Notes on the Infinity Laplace Equation*,
SpringerBriefs in Mathematics, DOI 10.1007/978-3-319-31532-4_2

$f_{13}(q_2), \ldots$ is bounded and so we can extract a convergent subsequence, say $f_{21}(q_2), f_{22}(q_2), f_{23}(q_2), \ldots$ Continuing like this, extracting subsequences of subsequences, we have the scheme

$$
\begin{array}{lll}
f_{11}, f_{12}, f_{13}, f_{14} \ldots & \text{converges at} & q_1 \\
f_{21}, f_{22}, f_{23}, f_{24} \ldots & \text{converges at} & q_1, q_2 \\
f_{31}, f_{32}, f_{33}, f_{34} \ldots & \text{converges at} & q_1, q_2, q_3 \\
f_{41}, f_{42}, f_{43}, f_{44} \ldots & \text{converges at} & q_1, q_2, q_3, q_4 \\
\multicolumn{3}{l}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .}
\end{array}
$$

We see that the diagonal sequence $f_{11}, f_{22}, f_{33}, f_{44}, \ldots$ converges at every rational point. To simplify the notation, write $f_{k_j} = f_{jj}$.

We claim that the constructed diagonal sequence converges at each point in Ω, be it rational or not. To this end let $x \in \Omega$ be an arbitrary point and let q be a rational point very near it. Then

$$
\begin{aligned}
|f_{k_j}(x) - f_{k_i}(x)| &\le |f_{k_j}(x) - f_{k_j}(q)| + |f_{k_j}(q) - f_{k_i}(q)| + |f_{k_i}(q) - f_{k_i}(x)| \\
&\le 2C|x-q|^\alpha + |f_{k_j}(q) - f_{k_i}(q)|.
\end{aligned}
$$

Given $\varepsilon > 0$, we fix q so close to x that $2C|x-q|^\alpha < \frac{\varepsilon}{2}$, which is possible since the rational points are dense. By the convergence at the rational points, we infer that

$$
|f_{k_j}(x) - f_{k_i}(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
$$

when the indices i and j are large enough. By Cauchy's general convergence criterion, the sequence converges at the point x. We have established the existence of the pointwise limit function

$$
f(x) = \lim_{j\to\infty} f_{k_j}(x).
$$

Next, we show that the convergence is (locally) uniform. Suppose that $\overline{\Omega}$ is compact. Cover it by balls $B(x, r)$ with diameter $2r = \varepsilon^{\frac{1}{\alpha}}$. A finite number of these balls covers $\overline{\Omega}$:

$$
\overline{\Omega} \subset \bigcup_{m=1}^{N} B(x_m, r).
$$

Choose a rational point from each ball, say $q'_m \in B(x_m, r)$. Since only a finite number of these points are involved, we can fix an index N_ε such that

$$
\max_m |f_{k_j}(q'_m) - f_{k_i}(q'_m)| < \varepsilon \qquad \text{when} \quad i, j > N_\varepsilon.
$$

Let $x \in \overline{\Omega}$ be arbitrary. It must belong to some ball, say $B(x_m, r)$. Again we can write

$$\begin{aligned}
|f_{k_j}(x) - f_{k_i}(x)| &\le 2C|x - q'_m|^{\alpha} + |f_{k_j}(q'_m) - f_{k_i}(q'_m)| \\
&\le 2C(2r)^{\alpha} + |f_{k_j}(q'_m) - f_{k_i}(q'_m)| \\
&\le 2C\varepsilon + \varepsilon \qquad \text{when} \quad i, j > N_{\varepsilon}.
\end{aligned}$$

The index N_ε is independent of how the point x was chosen. This shows that the convergence is *uniform* in $\overline{\Omega}$. —If the domain Ω is unbounded, we notice that the above proof is valid for every fixed bounded subdomain, in which case the convergence is locally uniform. ☐

Next, we consider Lipschitz continuous functions. A function $f : \Omega \longrightarrow \mathbb{R}$ is Lipschitz continuous if for some constant L,

$$|f(x) - f(y)| \le L|x - y| \quad \text{when} \quad x, y \in \Omega.$$

Theorem 2.2 (Rademacher) *A Lipschitz continuous function f is totally differentiable a. e. in its domain: the expansion*

$$f(y) = f(x) + \langle \nabla f(x), y - x \rangle + o(|y - x|) \quad \textit{as} \quad y \to x$$

holds at almost every point $x \in \Omega$.

It is useful to know that convex functions are locally Lipschitz continuous. As we shall see in Chap. 7, a convex function has, indeed, even second derivatives a. e. in the way they should appear in Taylor's expansion.

Remark According to the above definition of Lipschitz continuity, for example the function

$$u(x, y) = \arctan\left(\frac{y}{x}\right)$$

is not Lipschitz continuous in the slit domain

$$\Omega = \left\{(x, y) | \, 1 < \sqrt{x^2 + y^2} < 2\right\} \setminus \{(x, 0) | -2 < x < -1\}.$$

(Here u varies between $-\pi$ and π, so that it is positive in the upper half-plane.) The reason is that the function has a jump of 2π across the slit:

$$u\left(-\tfrac{3}{2}, +\varepsilon\right) - u\left(-\tfrac{3}{2}, -\varepsilon\right) = 2\pi + o(\varepsilon).$$

We use the "straight" distance $|(-\frac{3}{2}, +\varepsilon) - (-\frac{3}{2}, -\varepsilon)| = 2\varepsilon$ and not the intrinsic metric $2\sqrt{\varepsilon^2 + \frac{9}{4}}$, which is the infimum of the lengths of all the curves in Ω joining the two points. Yet, in this example

$$\|\nabla u\|_{\infty,\Omega} = \frac{1}{2} < \infty.$$

In convex domains this kind of behaviour is out of the question.

Solvable Spaces We denote by $W^{1,p}(\Omega)$ the Sobolev space consisting of functions u that together with their first distributional derivatives

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right)$$

belong to the space $L^p(\Omega)$. Equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$$

it is a Banach space. In particular, the space $W^{1,\infty}(\Omega)$ consists of all Lipschitz continuous functions defined in Ω. The closure of $C_0^\infty(\Omega)$ with respect to the Sobolev norm is denoted by $W_0^{1,p}(\Omega)$. About Sobolev spaces we refer the reader to Chap. 7 of the book [GT].

Before proceeding, let us take advantage of the fact that only very large values of the exponent p are needed here. If $p > n =$ the number of coordinates in $\mathbb{R}^n$, the Sobolev space contains only continuous functions and the boundary values are taken in the classical sense. All domains[3] Ω are regular for the Dirichlet problem, when $\mathbf{p} > \mathbf{n}$.

Lemma 2.3 *Let $p > n$ and suppose that Ω is an arbitrary bounded domain in $\mathbb{R}^n$. If $v \in W_0^{1,p}(\Omega)$, then*

$$|v(x) - v(y)| \leq \frac{2pn}{p-n} |x-y|^{1-\frac{n}{p}} \|\nabla v\|_{L^p(\Omega)} \tag{2.1}$$

for a. e. $x, y \in \Omega$. One can redefine v in a set of measure zero and also extend it to the boundary so that $v \in C^{1-\frac{n}{p}}(\overline{\Omega})$ and $v|_{\partial\Omega} = 0$.

This is a variant of Morrey's inequality. It is important that the constant remains bounded for large p. If we do not require zero boundary values, the inequality still holds for many domains. For example, if Ω is a cube Q, the inequality holds for $v \in W^{1,p}(Q)$.

On the Constant Since the behaviour of the constant is decisive, as $p \to \infty$, I indicate how to obtain it for a smooth function $v \in C^1(Q) \cap W^{1,p}(Q)$. Integrating

$$\begin{aligned} v(x) - v(y) &= \int_0^1 \frac{d}{dx} v\big(x + t(y-x)\big)\, dt \\ &= \int_0^1 \big\langle y - x, \nabla v\big(x + t(y-x)\big)\big\rangle\, dt \end{aligned}$$

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[3]As always, a domain is an open connected set.

with respect to y over Q, we see that[4]

$$
\begin{aligned}
|v(x) - v_Q| &= \left| \fint_Q \int_0^1 \big\langle y - x, \nabla v\big(x + t(y-x)\big)\big\rangle \, dt \, dy \right| \\
&\leq \operatorname{diam}(Q) \fint_Q \int_0^1 \big|\nabla v\big(x + t(y-x)\big)\big| \, dt \, dy \\
&\leq \operatorname{diam}(Q) \int_0^1 \left(\fint_Q \big|\nabla v\big(x + t(y-x)\big)\big|^p \, dy \right)^{\frac{1}{p}} dt.
\end{aligned}
$$

The change of coordinates

$$
\xi = x + t(y - x), \quad d\xi = t^n dy
$$

in the inner integral yields

$$
\fint_Q \big|\nabla v\big(x + t(y-x)\big)\big|^p \, dy = \frac{1}{t^n} \fint_{Q_t} |\nabla v(\xi)|^p \, d\xi \leq \frac{1}{t^n} \fint_Q |\nabla v(\xi)|^p \, d\xi,
$$

since the intermediate domain of integration $Q_t \subset Q$. Therefore

$$
|v(x) - v_Q| \leq \frac{\operatorname{diam}(Q)}{|Q|^{\frac{1}{p}}} \int_0^1 t^{-\frac{n}{p}} \|\nabla v\|_{L^p(Q)} \, dt = \frac{1}{1 - \frac{n}{p}} \frac{\operatorname{diam}(Q)}{|Q|^{\frac{1}{p}}} \|\nabla v\|_{L^p(Q)}.
$$

(It was needed that $p > n$.) The triangle inequality yields

$$
|v(x) - v(y)| \leq |v(x) - v_Q| + |v(y) - v_Q| \leq \frac{2p}{p-n} \frac{\operatorname{diam}(Q)}{|Q|^{\frac{1}{p}}} \|\nabla v\|_{L^p(Q)}.
$$

To conclude, we can always choose an auxiliary cube $Q' \subset Q$ so that $|x - y| \leq \operatorname{diam}(Q')$. —In the general case, when v no longer has continuous first derivatives, one can use approximations with convolutions and conclude the proof with the aid of Ascoli's theorem.

I repeat that always[5] $\mathbf{p \geq 2}$ in these notes and often $\mathbf{p > n}$ = the dimension of the space.[6]

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[4]The notation

$$
f_Q = \fint_Q f \, dx = \frac{\int_Q f \, dx}{\int_Q dx}
$$

is used for the average of a function.

[5]The p-harmonic operator is not pointwise defined for $p < 2$.

[6]All functions in the Sobolev space $W^{1,p}$ are continuous when $p > n$.

The p-Laplace Equation for finite p We need some standard facts about the p-Laplace equation and its solutions. Let us consider the existence of a solution to the Dirichlet boundary value problem. Minimizing the variational integral

$$I(v) = \int_\Omega |\nabla v|^p \, dx \tag{2.2}$$

among all functions with the same given boundary values, we are led to the condition that the first variation must vanish:

$$\int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \eta \rangle \, dx = 0 \quad \text{when} \quad \eta \in C_0^\infty(\Omega), \tag{2.3}$$

where u is minimizing. Under suitable assumptions this is equivalent to

$$\int_\Omega \eta \, \nabla \cdot \big(|\nabla u|^{p-2} \nabla u \big) \, dx = 0,$$

Since this must hold for all test functions η we have

$$\Delta_p u \equiv \nabla \cdot \big(|\nabla u|^{p-2} \nabla u \big) = 0.$$

In other words, the p-Laplace Equation is the *Euler–Lagrange Equation* for the above variational integral. A more precise statement is:

Theorem 2.4 *Take $p > n$ and consider an arbitrary bounded domain Ω in $\mathbb{R}^n$. Suppose that $g \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is given. Then there exists a unique function $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ with boundary values g which minimizes the variational integral*

$$I(v) = \int_\Omega |\nabla v|^p \, dx$$

among all similar functions.

The minimizer is a weak solution to the p-Laplace Equation, i.e.

$$\int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \eta \rangle \, dx = 0 \quad \textit{when} \quad \eta \in C_0^\infty(\Omega).$$

On the other hand, a weak solution in $C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is always a minimizer (among functions with its own boundary values).

Proof The uniqueness of the minimizer follows easily from

$$\left| \frac{\nabla u_1 + \nabla u_2}{2} \right|^p < \frac{|\nabla u_1|^p + |\nabla u_2|^p}{2} \quad \text{when} \quad \nabla u_1 \neq \nabla u_2,$$

upon integration. To wit, if there were two minimizers u_1 and u_2, then $\frac{u_1+u_2}{2}$ would be admissible and

$$I(u_1) \leq I\Big(\frac{u_1+u_2}{2}\Big) < \frac{I(u_1)+I(u_2)}{2} = I(u_1),$$

unless $\nabla u_1 = \nabla u_2$ almost everywhere. To avoid the contradiction, we must have $u_1 = u_2$.

The Euler–Lagrange Equation can be derived from the minimizing property

$$I(u) \leq I(u + \varepsilon\eta).$$

(The function $v(x) = u(x) + \varepsilon\eta(x)$ is admissible.) We must have

$$\frac{d}{d\varepsilon} I(u + \varepsilon\eta) = 0 \quad \text{when} \quad \varepsilon = 0$$

by the infinitesimal calculus. This shows that the first variation vanishes, i.e., Eq. (2.3) holds.

To show that the minimizer exists, we use the Direct Method in the calculus of variations, due to Lebesgue, see the book [G]. Let

$$I_0 = \inf_v \int_\Omega |\nabla v|^p \, dx$$

where the infimum is taken over the class of admissible functions. Now $0 \leq I_0 \leq I(g) < \infty$. Consider a so-called minimizing sequence of admissible functions v_j:

$$\lim_{j\to\infty} I(v_j) = I_0.$$

We may assume that $I(v_j) < I_0 + 1$ for $j = 1, 2, \ldots$ We may also assume that

$$\min g \leq v_j(x) \leq \max g \quad \text{in} \quad \Omega,$$

since we may cut the functions at the constant heights $\min g$ and $\max g$. (The procedure decreases the integral!) We see that the Sobolev norms $\|v_j\|_{W^{1,p}}(\Omega)$ are uniformly bounded.[7] By weak compactness, there exists a function $u \in W^{1,p}(\Omega)$ and a subsequence such that

$$\nabla v_{j_k} \rightharpoonup \nabla u \quad \text{weakly in} \quad L^p(\Omega).$$

[7]The conventional way is not to cut the functions, but to use the Sobolev inequality

$$\|v_j - g\|_{L^p(\Omega)} \leq C\|\nabla(v_j - g)\|_{L^p(\Omega)}$$

to uniformly bound the norms

Since $p > n$ we know from Lemma 2.3 that $u \in C(\overline{\Omega})$ and we conclude that $u = g$ on $\partial\Omega$. We got the continuity for free! By the weak lower semicontinuity of convex integrals

$$I(u) \leq \liminf_{k\to\infty} I(v_{j_k}) \;=\; I_0.$$

Since u is admissible also $I(u) \geq I_0$. Therefore u is a minimizer and the existence is established.

It remains to show that the weak solutions of the Euler–Lagrange Equation are minimizers.[8] By integrating the inequality[9]

$$|\nabla(u+\eta)|^p \geq |\nabla u|^p + p\langle |\nabla u|^{p-2}\nabla u, \nabla\eta\rangle,$$

we obtain

$$\int_\Omega |\nabla(u+\eta)|^p\,dx \geq \int_\Omega |\nabla u|^p\,dx + 0 = \int_\Omega |\nabla u|^p\,dx.$$

Therefore u is a minimizer. □

Remark If the given boundary values g are merely continuous ($g \in C(\partial\Omega)$ but perhaps $g \notin W^{1,p}(\Omega)$), then there exists a unique p-harmonic function $u \in C(\overline{\Omega})$ with boundary values g. However it may so happen that $\int_\Omega |\nabla u|^p\,dx = \infty$. — Hadamard gave a counter example for the ordinary Dirichlet integral ($p = 2$).

(Footnote 7 continued)

$$\begin{aligned}\|v_j\|_{L^p(\Omega)} &\leq \|v_j - g\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} \leq C\|\nabla(v_j - g)\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}\\ &\leq C\big[\|\nabla v_j\|_{L^p(\Omega)} + \|\nabla g\|_{L^p(\Omega)}\big] + \|g\|_{L^p(\Omega)}\\ &\leq C\big[(I_0+1)^p + \|\nabla g\|_{L^p(\Omega)}\big] + \|g\|_{L^p(\Omega)} \equiv M < \infty,\end{aligned}$$

when $j = 1, 2, 3, \ldots$

[8] There are variational integrals for which this is not the case.

[9] Since $|w|^p$ is convex, the inequality

$$|b|^p \geq |a|^p + p\langle |a|^{p-2}a, b-a\rangle$$

holds for vectors.

Springer

http://www.springer.com/978-3-319-31531-7

Notes on the Infinity Laplace Equation
Lindqvist, P.
2016, IX, 68 p. 1 illus. in color., Softcover
ISBN: 978-3-319-31531-7