Chapter 2
Local Theory

1 Case of Division Algebras

Let F_u be a local non-Archimedean field, $G(F_u)$ the multiplicative group of a division algebra D_u central of rank n over F_u, and $G'(F_u) = \mathrm{GL}(n, F_u)$. There is an embedding of the set of conjugacy classes γ in $G(F_u)$ as the set of elliptic conjugacy classes γ' in $G'(F_u)$, defined by $p_\gamma = p_{\gamma'}$; here p_γ is the characteristic polynomial of γ and $p_{\gamma'}$ is that of γ'. In a fundamental but unpublished work [DK] (see also [DKV84]) of the late 1970s, Deligne and Kazhdan proved:

THEOREM 1.1. *There is a bijection from the set of equivalence classes of irreducible $G(F_u)$-modules π_u to the set of equivalence classes of irreducible square-integrable $G'(F_u)$-modules π'_u, defined by the character relation $\chi_{\pi'_u}(\gamma') = (-1)^{n-1}\chi_{\pi_u}(\gamma)$ for every regular γ in $G(F_u)$ with image γ' in $G'(F_u)$. Here $\chi_{\pi'_u}$ denotes the character* [HC78] of π'_u, and χ_{π_u} that of π_u.

By virtue of [Ka86.2], it suffices to prove this for F_u of characteristic zero. In fact, all of our arguments hold also in the positive characteristic case, except for the reference [Ka86.1] to the orthonormality relations for characters used in the proof of Proposition 1.9. These relations follow if we use the local integrability of characters in the positive characteristic case of [Le96].

A proof of Theorem 1.1 had been sketched in [JL70] for $n = 2$. The proof of [JL70], as well as that of [DK], relies on global techniques, principally the Selberg trace formula, and on local studies of transfer of orbital integrals between $G(F_u)$ and $G'(F_u)$. There are several proofs of this local transfer; the best is the one given later in this chapter, where the relations between germs of characters and orbital integrals (due to [Ho74, HC78, Ka86.1]) are exploited. The purpose of the present section is to prove Theorem 1.1 without transferring locally the orbital integrals (except in a trivial case) and *consequently deduce this transfer* (see Corollary 1.10 below) *by global means*. These means include, in addition to the trace formula, the Hecke L-function theory of [GJ72]. The observation that the transfer of orbital integrals

Y.Z. Flicker, *Arthur's Invariant Trace Formula and Comparison of Inner Forms*,
DOI 10.1007/978-3-319-31593-5_2

can be deduced from the lifting Theorem 1.1 was already made in the context of [FK87.2], §27.3. The deduction relies on results of [BDK86] or [F95], and [Ka86.1].

This section is based on [F90.2]. The point in our present proof is that the theory of L-functions is used to show the finiteness of the set of representations which appear in the trace formulae, under some conditions. This observation was made already in [DK] (see [DKV84], pp. 78–82), which was concerned not only with Theorem 1.1, but also it contained a discussion—in the context of $\mathrm{GL}(n)$—of some fundamental ideas later developed in [BDK86] and [Ka86.1]. Here we show that this observation suffices to complete the proof of Theorem 1.1 and Corollary 1.10. We then obtain a simple proof of the Deligne-Kazhdan theorem in the division algebra case. Theorem 1.1 is used as the first step in the inductive proof of the theorem in the simple algebra case, developed later in this chapter. The wish to provide such a simple proof (essentially due to [DK]) to this first induction step used later in this chapter, following [F87.1, III], and consequently to dissipate some misconceptions concerning the difficulty of this case were a main motivation for us to write [F90.2] and this section. Namely, all of our arguments can be found in [DK], and in particular the usage of the Hecke theory, but it was assumed in some expositions that the transfer of orbital integrals had to be proven first, and this led to a proof longer and more complicated than necessary.

Our proof is not elementary. It uses deep theorems in harmonic analysis (e.g., Kazhdan's orthonormality relations for characters). Consequently the reader should have a basic knowledge of representation theory to understand the proof. However, the reduction in this section of the correspondence to "standard theorems" is considerably shorter than other reductions. The rigidity observation of [F90.2]—that using the theory of L-functions one can show that there are only finitely many discrete spectrum automorphic representations whose components at almost all places are fixed—was extended in [Ba05] from the context of the multiplicative group of a division algebra to that of simple algebras. Thus the L-function theory is sufficiently developed for our purposes in the case of $\mathrm{GL}(n)$ and its inner forms: the multiplicative group of a simple algebra. In particular, it applies also in the analogous situation of base change for $\mathrm{GL}(n)$ (proven by Arthur-Clozel [AC89]), where the local theory of base-change lifting for $\mathrm{GL}(n)$ can be established using purely global means such as the trace formula and the L-functions theory of [JS81]; transfer of orbital integrals is obtained as a corollary (see Corollary 1.10, following [F90.2]). It will be interesting to develop this Hecke theory for other groups, for example, to satisfy the needs of the metaplectic correspondence (see [FK87.2]), symmetric-square lifting from $\mathrm{SL}(2)$ to $\mathrm{PGL}(3)$, or base change from $\mathrm{U}(3, E/F)$ to $\mathrm{GL}(3, E)$ (see [F06]).

In the global proof one takes a number field F, totally imaginary for simplicity, which has a place u such that the completion of F at u is the local field F_u in Theorem 1.1. Fix a finite place $u' \neq u$ of F. Let D be a division algebra central of rank n over F, whose invariant $\mathrm{inv}_u D$ at u is equal to the invariant $\mathrm{inv}\, D_u$ of D_u (equivalently $D_u \simeq D \otimes_F F_u$) and such that $\mathrm{inv}_v D = 0$ for all $v \neq u, u'$. Then $D \otimes_F F_v$ is the matrix algebra $M(n, F_v)$ for every $v \neq u, u'$. Put $G_v = (D \otimes_F F_v)^\times$ and $G'_v = \mathrm{GL}(n, F_v)$ for every place v of F. Note that the multiplicative group G of D is an inner form of $G' = \mathrm{GL}(n)$. Note that the center Z of G is isomorphic to that of G' and to the multiplicative group. To simplify the notation, we deal here only with representations and functions which transform trivially under the center. Put $\overline{G}$ for G/Z.

Choose an F-rational invariant differential form of maximal degree on G. It defines Haar measures $\mathrm{d}g_v$ on G_v and $\mathrm{d}g'_v$ on G'_v for all v and product measures $\mathrm{d}g = \otimes \mathrm{d}g_v$ on $G(\mathbb{A})$ and $\mathrm{d}g' = \otimes \mathrm{d}g'_v$ on $G'(\mathbb{A})$.

The trace formula is stated for a function $f = \otimes f_v$ in $C_c^\infty(\overline{G}(\mathbb{A}))$. It involves orbital integrals

$$\widetilde{\Phi}(\gamma,f) = \int_{\overline{G}(\mathbb{A})/G_\gamma(F)} f(g\gamma g^{-1})\,\mathrm{d}g = |G_\gamma(\mathbb{A})/Z(\mathbb{A})G_\gamma(F)| \int_{G(\mathbb{A})/G_\gamma(\mathbb{A})} f(g\gamma g^{-1})\,\mathrm{d}g,$$

(see 1.9 below for Φ without tilde) and traces

$$\operatorname{tr}\pi(f) = \prod_v \operatorname{tr}\pi_v(f_v), \qquad \text{where } \pi_v(f_v) = \int_{\overline{G}_v} f_v(g)\pi_v(g)\,\mathrm{d}g.$$

We take the component f_u to be supported on the set of γ in $G(F_u)$ such that γ^n is regular. Then for γ in $G(F)$ we have $\Phi(\gamma,f) \neq 0$ only when γ is (semisimple) regular, in which case the centralizer G_γ of γ in G is a torus.

THEOREM 1.2 (Trace Formula for G). *For any f as above, we have*

$$\sum_\gamma \widetilde{\Phi}(\gamma,f) = \sum_\pi m(\pi)\operatorname{tr}\pi(f). \tag{1.2.1}$$

The sum on the left ranges over the set of conjugacy classes γ in $\overline{G}(F)$ such that γ^n is regular. The sum on the right ranges over the set of equivalence classes of automorphic $G(\mathbb{A})$-modules π with trivial central character; $m(\pi)$ denotes the multiplicity of π in the space of automorphic forms.

The proof of this is elementary. It is Corollary 4.4 of this chapter.

The trace formula for $G'(\mathbb{A})$ will be stated for a function $f' = \otimes f'_v$ in $C_c^\infty(\overline{G}'(\mathbb{A}))$, with the following properties. Fix a finite place $u'' \neq u, u'$ of F. Let $f'_{u''}$ be a normalized coefficient of a (local) cuspidal $G'_{u''}$-module $\pi^0_{u''}$. Thus $\operatorname{tr}\pi_{u''}(f'_{u''}) = 0$ for any irreducible $\pi_{u''}$ inequivalent to $\pi^0_{u''}$, and $\operatorname{tr}\pi^0_{u''}(f'_{u''}) = 1$. We use the word cuspidal as in [BZ76].

Let $f'_{u'}$ be a pseudo-coefficient (its existence was proven in [Ka86.1]) of the Steinberg $G'_{u'}$-module $\mathrm{st}_{u'}$. Then $\operatorname{tr}\mathrm{st}_{u'}(f'_{u'}) = 1$, and $\operatorname{tr}\pi_{u'}(f'_{u'}) = 0$ for every irreducible tempered $G'_{u'}$-module $\pi_{u'}$ inequivalent to $\mathrm{st}_{u'}$. Moreover, the orbital integrals of $f'_{u'}$ vanish on the regular non-elliptic set, and

$$\widetilde{\Phi}(\gamma,f'_{u'}) = \int_{\overline{G}'_{u'}} f'_{u'}(g\gamma g^{-1})\,\mathrm{d}g$$

is equal to $\chi_{\mathrm{st}_{u'}}(\gamma) = (-1)^{n-1}$ on the regular elliptic set.

Finally let f'_u be a function supported on the set of γ in $G'(F_u)$ such that γ^n is regular.

Again by Corollary 4.4, we have

THEOREM 1.3 (Trace Formula for G'). *For any f' as above, we have*

$$\sum_{\gamma'} \Phi(\gamma', f') = \sum_{\pi'} \operatorname{tr} \pi'(f'). \tag{1.3.1}$$

The sum on the left ranges over the set of elliptic regular conjugacy classes γ' in $\overline{G}'(F)$ such that γ^n is regular. On the right the sum ranges over the set of cuspidal $G'(\mathbb{A})$-modules π' with trivial central character.

Note that the multiplicity of each such π' in the cuspidal spectrum for $G'(\mathbb{A})$ is one ([Shal74]).

The trace formula for G' will be used with a function $f' = \otimes f'_v$ whose components at u' and u'' are as described above. The component f'_u is taken to be supported on the set of γ in $G(F_u)$ with regular γ^n; moreover we assume that its orbital integrals vanish on the non-elliptic set of $G'(F_u)$. In this section we call such f'_u a *regular-discrete* function. The isomorphism $G_v \simeq G'_v$ for $v \neq u, u'$, can be and is used to transfer f'_v to a function f_v on G_v. Let $f_{u'}$ be a normalized matrix coefficient of the trivial $G_{u'}$-module $1_{u'}$. Then $\operatorname{tr} 1_{u'}(f_{u'}) = 1$, and $\operatorname{tr} \pi_{u'}(f_{u'}) = 0$ for any irreducible $\pi_{u'}$ inequivalent to $1_{u'}$. Moreover, $\widetilde{\Phi}(\gamma, f_{u'}) = 1$ for all γ in $G_{u'}$. Finally, take f_u to be a regular-discrete function on $G(F_u)$ (namely, f_u is supported on the set of γ in $G(F_u)$ such that γ^n is regular), with $\widetilde{\Phi}(\gamma, f_u) = \widetilde{\Phi}(\gamma', f'_u)$ for every γ regular in $G(F_u)$; γ' is the image of γ in $G'(F_u)$. We say in this case that f_u and f'_u have *matching orbital integrals*, and note that it is a well-known, relatively simple result of Harish-Chandra (see Proposition 2.9 below) that for every regular-discrete f'_u on $G'(F_u)$ there exists such f_u on $G(F_u)$, and for every regular-discrete f_u on $G(F_u)$ there exists such f'_u on $G'(F_u)$, with matching orbital integrals. The existence of matching functions in general is a more difficult problem, which we solve below on using Theorem 1.1; its solution is not required for the proof of Theorem 1.1.

PROPOSITION 1.4. *For the $f' = \otimes f'_v$ and $f = \otimes f_v$ related as above, we have*

$$\sum_{\pi'} \operatorname{tr} \pi'(f') = \sum_{\pi} m(\pi) \operatorname{tr} \pi(f). \tag{1.4.1}$$

The sums are those of (1.2.1) *and* (1.3.1).

PROOF. By the choice of f and f', the sums over γ and γ' in (1.2.1) and (1.3.1) range over isomorphic sets ($\gamma \leftrightarrow \gamma'$ iff $p_\gamma = p_{\gamma'}$), and $\widetilde{\Phi}(\gamma, f) = \widetilde{\Phi}(\gamma', f')$ for all $\gamma \leftrightarrow \gamma'$. Note that for regular γ, the centralizers G_γ of γ in G, and $G'_{\gamma'}$ of γ' in G', are isomorphic elliptic tori; this isomorphism is used to transfer measures between these groups. The proposition follows. □

Let π'_{0u} be a square-integrable $G'(F_u)$-module. By a standard construction result, see Proposition 16.2 below, there exists a cuspidal $G'(\mathbb{A})$-module π'_0 whose component at u is the chosen π'_{0u}, at u' it is the Steinberg $\mathrm{st}_{u'}$, and at u'' it is the cuspidal $\pi^0_{u''}$. Denote by $\mathbb{A}^u$ the ring of F-adèles without u-component. Denote by $\pi^u_0 = \bigotimes_{v \neq u} \pi^u_{0v}$ the $G(\mathbb{A}^u)$-module $1_{u'} \otimes (\bigotimes_{v \neq u,u'} \pi_{0v})$. Here we identify π'_{0v} with a G_v-module π_{0v} for $v \neq u, u'$, by $G_v \simeq G'_v$. The standard-type isolation argument (see Proposition 17.1) implies the

PROPOSITION 1.5. *For the given square-integrable π'_{0u}, there exist irreducible $G(F_u)$-modules π_u, such that for any matching regular-discrete f_u and f'_u, we have*

$$(-1)^{n-1} \operatorname{tr} \pi'_{0u}(f'_u) = \sum_{\pi_u} m(\pi_u \otimes \pi^u_0) \operatorname{tr} \pi_u(f_u). \tag{1.5.1}$$

REMARK 1.6. In the proof of (1.5.1), it is worthwhile to note that the choice of $f'_{u''}$ in (1.4.1) implies that the π' of (1.4.1) are all cuspidal. Hence each component of π' is non-degenerate. By [Ze80, (9.7b)], if $\operatorname{tr} \pi'_{u'}(f'_{u'}) \neq 0$, then $\pi'_{u'} \simeq \mathrm{st}_{u'}$ and so $\operatorname{tr} \mathrm{st}_{u'}(f'_{u'}) = (-1)^{n-1}$.

The usual argument, of [JL70, DK], or Section 18 below, to deduce Theorem 1.1 from (1.5.1), is based on evaluation of (1.5.1) at f_u which is a normalized coefficient of some π_u which occurs in (1.5.1) with $m(\pi_u \otimes \pi^u) \neq 0$. To do this, one has to show that there exist f'_u with orbital integrals matching those of f_u. We shall argue differently. Using the Hecke theory of [GJ72], we prove (following [DK]) that the sum in (1.5.1) is finite *uniformly* in f_u. In fact, since f_u is bi-invariant under some compact open subgroup K_u of the compact (modulo Z_u) group $G(F_u)$, there are only finitely many π_u with $\operatorname{tr} \pi_u(f_u) \neq 0$. However, the size of the finite set of such π_u increases as K_u decreases, and a priori the sum in (1.5.1) may be infinite (for a variable f_u). In order to use the orthonormality relations for characters (see the passage from Proposition 1.8 to Proposition 1.9 below), we need to know that the sum in (1.5.1) (and so in (1.8.1) below) is finite uniformly in, or independently of, f_u. Thus we prove

PROPOSITION 1.7. *The sum over π_u in* (1.5.1) *is finite* (*uniformly in f_u*).

PROOF. Let $\psi = \prod_v \psi_v$ be a nontrivial additive character of $\mathbb{A}/F$. Denote by $L(s, \pi_v)$ the L-function and by $\epsilon(s, \pi_v, \psi_v)$ the ϵ-factor, attached to π_v and ψ_v for every place v of F, in [GJ72, Theorem 3.3]. Consider $\pi = \pi_u \otimes \pi^u_0$ which occurs in (1.5.1) with $m(\pi) \neq 0$. Since π is automorphic, $\epsilon(s, \pi) = \prod_v \epsilon(s, \pi_v, \psi_v)$ is independent of ψ (see [GJ72, p. 149]), and $L(s, \pi) = \prod_v L(s, \pi_v)$ satisfies the functional equation $L(s, \pi) = \epsilon(s, \pi) L(1 - s, \pi^\vee)$; here $\pi^\vee$ signifies the contragredient of π. So, if π_u and π_{0u} contribute to (1.5.1) (namely, $m(\pi_u \otimes \pi^u_0) \neq 0$, $m(\pi_{0u} \otimes \pi^u_0) \neq 0$), we have

$$\frac{L(1-s, \pi_u^\vee)\epsilon(s, \pi_u, \psi_u)}{L(s, \pi_u)} = \frac{L(1-s, \pi_{0u}^\vee)\epsilon(s, \pi_{0u}, \psi_u)}{L(s, \pi_{0u})}. \tag{1.7.1}$$

Denote by K_u the multiplicative group of a maximal order M_u in the division algebra D_u underlying $G(F_u)$. Then K_u is open in $G(F_u)$, and $G(F_u)/Z_u$ is compact. Hence there are only finitely many irreducible $G(F_u)$-modules π_u with a trivial central character which are unramified (trivial on K_u). If π_u is not trivial on K_u, then $L(s,\pi_u) = 1 = L(s,\pi_u^\vee)$ by [GJ72, Prop. 4.4], identically in s. In this case (1.7.1) implies that $\epsilon(s,\pi_u,\psi_u)$ is independent of π_u (as long as $m(\pi_u \otimes \pi_0^u) \neq 0$). Denote by $c(\pi_u)$ the positive integer ("conductor") such that π_u is trivial on $1 + \boldsymbol{\pi}_u^{c(\pi_u)+1} M_u$ but not on $1 + \boldsymbol{\pi}_u^{c(\pi_u)} M_u$, where $\boldsymbol{\pi}_u$ is the local uniformizer in the ring R_u of integers in F_u. Choose ψ_u to be trivial on R_u, but not on $\boldsymbol{\pi}_u^{-1} R_u$. It is well known (see, e.g., [BF83, Theorem 3.2.11, p. 39]) that there exists a number α such that $\epsilon(s,\pi_u,\psi_u) = \alpha q_u^{-c(\pi_u)s}$; q_u is the cardinality of the residue field $R_u/(\boldsymbol{\pi}_u)$. Consequently $c = c(\pi_u)$ is independent of π_u. Since $G(F_u)/Z_u(1 + \boldsymbol{\pi}_u^c M_u)$ is finite, there are only finitely many irreducible $G(F_u)$-modules π_u with a trivial central character and a fixed conductor c. The proposition follows. □

Now that we know that the sum in (1.5.1) ranges over a finite set depending only on π'_{0u}, we can apply the Weyl integration formula

$$\int_{G/Z} f(g)dg = \int' \Phi(t,f)dt, \quad \text{where} \quad \int' \text{ signifies } \quad \sum_{\{T\}} [W(T)]^{-1} \int_{T/Z} \Delta(t)^2.$$

Here the sum ranges over a set of representatives for the conjugacy classes of tori T in G, $[W(T)]$ denotes the cardinality of the Weyl group of T, and Δ is a Jacobian. It implies

PROPOSITION 1.8. *For every regular γ in $G(F_u)$ and γ' in $G'(F_u)$ with $p_\gamma = p_{\gamma'}$, we have*

$$(-1)^{n-1}\ \chi_{\pi'_{0u}}(\gamma') = \sum_{\pi_u} m(\pi_u \otimes \pi_0^u)\ \chi_{\pi_u}(\gamma). \tag{1.8.1}$$

The sum is the same as in (1.5.1).

An immediate application of the orthonormality relations (due to [Ka86.1, Theorem K]), of characters of square-integrable representations, implies

PROPOSITION 1.9. *The sum in* (1.8.1) *consists of a single entry π_{0u} with $m(\pi_{0u} \otimes \pi_0^u) \neq 0$; moreover, $m(\pi_{0u} \otimes \pi_0^u) = 1$.*

This completes the proof of one half of Theorem 1.1, asserting that for each square-integrable π'_u, there exists a corresponding π_u. To prove the opposite direction, one starts with a $G(F_u)$-module π_u^0 and constructs a cuspidal $G(\mathbb{A})$-module π_0 whose component at u is π_u^0, at u'' it is the cuspidal $\pi_{u''}^0$, and it is $1_{u'}$ at u'. Then (1.5.1) is obtained and the proof proceeds as above.

Finally we use Theorem 1.1 to transfer orbital integrals. Since the following discussion is purely local, the index u is omitted. Recall that for a regular γ in G', the centralizer G'_γ is a torus, and we put

$$\Phi(\gamma,f') = \int_{G'/G'_\gamma} f'(g\gamma g^{-1})\,\mathrm{d}g.$$

Following [Ka86.1] we say that a function f' is *discrete* if $\Phi(\gamma,f') = 0$ for every regular non-elliptic γ in G'. The space of discrete f' is denoted by $A(G')$. Theorem 1.1 has the following:

COROLLARY 1.10. *For every f on G, there is f' in $A(G')$, and for every f' in $A(G')$, there is f on G, with $\Phi(\gamma,f) = \Phi(\gamma',f')$ for all regular γ in G and γ' in G' with $p_\gamma = p_{\gamma'}$.*

The proof consists of two parts.

LEMMA 1.11. *For every f there is f' in $A(G')$, and for every f' in $A(G')$, there is f, such that $(-1)^{n-1}\operatorname{tr}\pi(f) = \operatorname{tr}\pi'(f')$ for all π,π' corresponding as in Theorem 1.1.*

PROOF. Given f, define a form Φ on the free abelian group $R(G')$ generated by the equivalence classes of irreducible tempered G'-modules π' by $\Phi(\pi') = (-1)^{n-1}\operatorname{tr}\pi(f)$ if π' is square-integrable, and it corresponds to π, and by $\Phi(\pi') = 0$ if π' is irreducible, tempered but not square-integrable. It is clear that Φ is a good form in the terminology of [BDK86] or [F95], hence a trace form by the Theorem of [BDK86] or [F95]. Namely, there exists f' on G with $\Phi(\pi') = \operatorname{tr}\pi'(f')$ on $R(G')$. Since $\operatorname{tr}\pi'(f') = 0$ for every π' in $R_I(G')$ (in the notations of [Ka86.1]), we have that f' lies in $A(G')$. The proof of the opposite implication (given f' in $A(G')$, there is f on G) is analogous. □

LEMMA 1.12. *If f' in $A(G')$ and f on G satisfy $(-1)^{n-1}\operatorname{tr}\pi(f) = \operatorname{tr}\pi'(f')$ for all π,π' corresponding by Theorem 1.1, then $\Phi(\gamma,f) = \Phi(\gamma',f')$ for all regular γ,γ' with $p_\gamma = p_{\gamma'}$.*

PROOF. The Weyl integration formula for G implies that

$$\sum_{\{T\}}[W(T)]^{-1}\int_{T/Z}\Delta(t)^2[\Phi(t,f)-\Phi(t',f')]\chi_\pi(t)dt = 0$$

for every G-module π. Since G/Z is compact, the characters χ_π form an orthonormal basis with respect to the inner product

$$\langle\chi,\chi'\rangle = \sum_{\{T\}}[W(T)]^{-1}\int_{T/Z}\Delta(t)^2\chi(t)\overline{\chi}'(t)\,\mathrm{d}t.$$

The lemma follows, and so does the corollary. □

REMARK 1.13. Our Theorem and Corollary are the initial, special case of the correspondence of representations of $\mathrm{GL}(n)$ and its inner forms; see Theorem 13.8 below for the general statement for the multiplicative group of any simple, not only division, algebra. Our local Theorem has a global variant, relating cuspidal representations on $\mathrm{GL}(n, \mathbb{A})$ and $G(\mathbb{A})$, for any inner form G of $\mathrm{GL}(n)$; see Theorem 13.12 below, in the context of π' with two cuspidal components, Theorem 25.2 for general representations, and Theorem 26.18 in the context of π' with a single cuspidal component.

2 Orbital Integrals

2.1 Points

Let F be a global field of characteristic 0. Let $\mathbb{A}$ denote its ring of adèles. The completion of F at the place v will be denoted F_v. Let G be a reductive group over F, which is often identified with its group of $\overline{F}$-points, $G(\overline{F})$, with $\overline{F}$ a fixed algebraic closure of F. If E is an extension of F, put $G(E)$ for the group of E-points of G. We write $G(\mathbb{A})$ for the group of adèle points. These conventions apply also to subgroups of G defined over F.

2.2 Parabolic Subgroups

We usually let P denote a parabolic F-subgroup of G. We let N denote its unipotent radical. Fix a minimal parabolic F-subgroup P_0 with Levi decomposition $P_0 = M_0N_0$, with M_0 the Levi subgroup. Unless otherwise specified, we consider only *standard* P, the F-rational parabolics containing P_0. By a Levi subgroup of P, we mean the unique Levi that contains M_0.

2.3 Rank, Semisimple Regular, Elliptic Elements

In this subsection let F be a local or global field of characteristic 0. Let $\mathfrak{g}$ be the Lie algebra of G. For $x \in G(F)$, consider the polynomial $\det[(t + 1 - \mathrm{Ad}(x))|_{\mathfrak{g}}]$ in t. Let d be the degree of the first nonzero power of t in the polynomial, as x runs over G. It is called the *rank* of G. Let $D(x)$ be the coefficient of t^d. Then x is called *semisimple regular* if $D(x) \neq 0$. It is then semisimple, and its centralizer G_x in G is a torus. A semisimple x is called *elliptic* if the center of $G_x(F)/Z(F)$ (F local) or $G_x(\mathbb{A})/G_x(F)Z(\mathbb{A})$ (F global) is compact. If x is elliptic regular, then G_x is an elliptic torus of G. We define a general regular element in 2.9. But outside the present Section 2, we often use “regular” to mean “semisimple regular,” to simplify the terminology, in the hope that no confusion would occur.

2.4 Orbital Integrals

Fix a Haar measure $\mathrm{d}y$ on $G(F)/Z(F)$. Let G_x denote the centralizer of x in G. Let $x \in G(F)$ and fix a Haar measure d_x on $G_x(F)/Z(F)$. If $G_x(F)$ and $G_{x'}(F)$ are isomorphic, we take d_x and $\mathrm{d}_{x'}$ to be equal under the fixed isomorphism. Define the *orbital integral* of f at x by

$$\Phi^G(x,f) = \Phi(x,f) = \int_{G(F)/G_x(F)} f(yxy^{-1}) \frac{\mathrm{d}y}{\mathrm{d}_x}.$$

The orbital integral depends on the choice of measures. The orbit of a regular element is closed. Thus $\Phi(x,f)$ converges for all regular x and $f \in C_c^\infty(G(F))$. By Theorem 2.32 or [Ra72], $\Phi(f)$ converges for $f \in C_c^\infty(G(F))$ and any $x \in G(F)$. If x is regular, let G_x^{sp} be the split component of the center of the centralizer G_x, which is a torus. Define

$$\widetilde{\Phi}^G(x,f) = \widetilde{\Phi}(x,f) = \int_{G(F)/G_x^{\mathrm{sp}}(F)} f(yxy^{-1}) \frac{\mathrm{d}y}{\mathrm{d}_x}.$$

2.5 Jacobian

Let F be local. Recall that $x \in G(F)$ has a Jordan decomposition $x = su = us$ into semisimple and unipotent elements. Let $\mathfrak{g}_s$ be the Lie algebra of the centralizer G_s. Put

$$\Delta(x) = |\det(1 - \mathrm{Ad}(s))|_{\mathfrak{g}/\mathfrak{g}_s}|^{1/2}.$$

Put

$$I^G(x,f) = I(x,f) = \Delta(x)\Phi(x,f).$$

For regular x, set

$$\widetilde{I}^G(x,f) = \widetilde{I}(x,f) = \Delta(x)\widetilde{\Phi}(x,f).$$

Consider $G = \mathrm{GL}(n)$. Since $\Delta(x)$ is unchanged by field extension, assume $x = \mathrm{diag}(x_1, \ldots, x_n)$, with $x_i \neq x_j$. Then computing $\mathrm{Ad}(x)$ with the standard basis $\{e_{ij}\}_{i \neq j}$ gives $\mathrm{Ad}(x)$ as diagonal with entries $x_i x_j^{-1}$, and

$$\Delta(x) = \frac{\left|\prod_{i<j}(x_i - x_j)^2\right|^{1/2}}{|\det x|^{(n-1)/2}}.$$

2.6 Constant Term

Let $P = MN$ be an F-parabolic subgroup of G. Let $K \subset G(F)$ be a maximal compact subgroup such that $G(F) = KP(F) = P(F)K$. If $\mathfrak{n}$ is the Lie algebra of N, put $\delta_P(m) = |\det \mathrm{Ad}(m)|_{\mathfrak{n}}|$. For $f \in C_c^\infty(G(F))$, put

$$f_N(m) = \delta_P(m)^{1/2} \int_{N(F)} \int_K f(k^{-1}mnk)\,\mathrm{d}k\,\mathrm{d}n.$$

The function f_N depends on N, but its orbital integral at $m \in M(F)$, regular in G, depends only on M. Indeed, a standard computation, see [FK87.2, Section7], shows $I^G(m,f) = I^M(m,f_N)$, where I^M denotes the normalized orbital integral with respect to M. Note that f_N lies in $C_c^\infty(M)$.

Denote by $J^\infty(G(F))$ the space of $f \in C_c^\infty(G(F))$ such that $I(g,f) = 0$ for all regular $g \in G(F)$. Put $\overline{C}_c^\infty(G(F)) = C_c^\infty(G(F))/J^\infty(G(F))$. The image of f_N in $\overline{C}_c^\infty(M(F))$ will be denoted by f_M, since it depends on the Levi subgroup M, but not the unipotent radical N.

2.7 Germ Expansion

We recall the germ expansion of orbital integrals of f in $C_c^\infty(G(F))$. Let $\overline{O}(x)$ be the closure of the conjugacy class $O(x)$ of x in $G(F)$. It is the disjoint union of the conjugacy classes $O(su_i)$, $1 \le i \le r$, of elements su_i with semisimple part s, so that the following properties hold:

(1) $u_1 = e$.
(2) For each t, $1 \le t \le r$, the union $O_t = \bigcup_{i=1}^t O(su_i)$ is closed.
(3) $O(su_t)$ is open in O_t.

The (closed) set A_s of elements in $G(F)$ whose semisimple part is conjugate to s is of the form $\overline{O}(x)$ for some x, and there are f_i in $C_c^\infty(G(F))$ with $I(su_j,f_i) = \delta_{ij}$ and $f_i = 0$ on $O(su_j)$ for $j < i$. We have the following proposition, giving the germ expansion.

PROPOSITION 2.8. (1) *Given f in $C_c^\infty(G(F))$ and a semisimple s in $G(F)$, there exists a neighborhood V_f of s in $G(F)$ so that*

$$I(x,f) = \sum_i I(x,f_i)I(su_i,f)$$

for all regular x in V_f.

(2) *Conversely, given a function $I(x)$ on $G(F)$ such that for each semisimple s in $G(F)$, there is a neighborhood V of s in $G(F)$ with*

$$I(x) = \sum_i I(x, f_i) I(su_i)$$

for all regular $x \in V$, there exists f in $C_c^\infty(G(F))$ with $I(x) = I(x, f)$.

The remainder of this section concerns a proof due to Shalika [Shal72] and Harish-Chandra [HC70], extended by Vigneras [Vi82] to the metaplectic group. We give a standard approach in Proposition 2.18, and a variant, with considerable overlap, using the uniqueness of the invariant measure on the orbit as in the proof of Corollary 4.4 below, in Proposition 2.25. We often denote $G(F)$ by G, $T(F)$ by T, etc., to simplify the notation. We start with

PROPOSITION 2.9. *Let $f \in C_c^\infty(G)$. Let T be a maximal torus in G. The orbital integral $\Phi(t,f) = \int_{T\backslash G} f(g^{-1}tg)dg$ converges at all $t \in T^{\mathrm{reg}}$. As a function of t in T^{reg}, it is locally constant. There is a compact subset $T(f)$ of T such that $\Phi(t,f)$ is zero for $t \in T^{\mathrm{reg}} - T(f) \cap T^{\mathrm{reg}}$.*

Note that the algebraic connected component $Z_G(s)^0$ of the centralizer $Z_G(s)$ of a semisimple element s of a connected reductive algebraic group G is reductive ([SS70, p. 197]) and connected. If $g = su = us$ is the Jordan decomposition of $g \in G$ as a product of commuting semisimple s and unipotent u, then the centralizer $Z_G(g)$ is the centralizer $Z_{Z_G(s)}(u)$ of u in $Z_G(s)$.

Let X be a subset of the group G. A $g \in X \subset G$ is *regular in X* if its conjugacy class $O(g)$ in G has maximal possible dimension, thus $\dim O(g) \geq \dim O(x)$ for all $x \in X$. Put X^{reg} for the set of regular elements in X.

The *torus* of $g \in G$ is the center T of $Z_G(g)^0$. A *torus of* G is a torus of a semisimple element of G. A *torus-unipotent pair*, or just a *pair*, is a pair (T, u) consisting of a torus T in G and a unipotent u which commutes with each element of T.

To prove Proposition 2.9, we use the following result of [HC70, p. 52], stated there for u being the identity e of G and T a maximal torus.

PROPOSITION 2.10. *Let s be a semisimple element of G. Let T be a torus containing s. Let u be a unipotent element which commutes with every element of T. Then there is a neighborhood V of s in T such that for any compact subset K' of G, there is a compact C in the homogeneous space $M\backslash G$, $M = Z_G(s)^0$ $(= (Z_G(s)^0)(F))$ such that if $g^{-1}Vug \cap K'$ is nonempty, then $Mg \in M\backslash G$ belongs to C.*

This implies the convergence of the orbital integral of f at a semisimple s. Further, if K' is open and compact, $s \in T^{\mathrm{reg}}$, and the unipotent u commutes with each element of the torus T, then there is a neighborhood V of s in T^{reg} such that $g^{-1}Vug \subset K'$ if $g^{-1}sug \in K'$. This shows that the orbital integral is locally constant on the regular part of Tu.

Proof. We may assume K' contains the semisimple parts of its elements. Indeed, the semisimple part of $g \in G$ is a polynomial in g, obtained from the eigenvalues by the Chinese remainder theorem, so that the map $g = su = us \to s$ is open and closed. Then $g^{-1}Vug \cap K' \neq \emptyset$ implies $g^{-1}Vg \cap K' \neq \emptyset$. So we may assume that $u = e$. Moreover, we may take T to be maximal. Indeed, if T_1 is a subtorus of a maximal T, and we found V for T as in the proposition, take $V_1 = T_1 \cap V$. Then $g^{-1}V_1 g \cap K' \neq \emptyset$ implies $g^{-1}Vg \cap K' \neq \emptyset$, so there is C in $M\backslash G$ as required.

We may assume T and G are split. Indeed, denote by E a finite field extension of F over which G and T split. Then G is closed in $G(E)$, $T = T(E) \cap G$, and $M = M(E) \cap G$, since $M = G_s^0$ is defined over F. Hence $M\backslash G \hookrightarrow M(E)\backslash G(E)$. The proposition follows for (G, T) provided the map is topological, in the sense that: (a) $M\backslash G$ is closed in $M(E)\backslash G(E)$ and (b) $M\backslash G \hookrightarrow M(E)\backslash G(E)$ is a homeomorphism onto its image. To see these, let $\{Mg_j; j \geq 1\}$ be a sequence in $M\backslash G$ which converges in $M(E)\backslash G(E)$ to $M(E)g$. We need to show that $Mg \in M\backslash G$ (namely, $M(E)g \cap G \neq \emptyset$ so that g can be chosen in G) and Mg_j converges to Mg in $M\backslash G$. Choose then representatives $g_j \in G$ and $g \in G(E)$. Then $g_j^{-1}sg_j \to g^{-1}sg$ as $j \to \infty$ in $G(E)$. Hence $g_j^{-1}sg_j$ remains in a compact in $G(E)$. This implies that Mg_j are all in a compact in $M\backslash G$, implying (a) and (b).

Suppose then that T splits over F. Let Σ be the set of roots of T in G. Let Σ^+ be a choice of positive roots. Then there is a maximal compact subgroup K and a unipotent subgroup N such that $G = TNK$, the map $T \times N \times K \to G$, $(t, n, k) \mapsto tnk$, is continuous, $B = TN$ is a Borel subgroup (G is split), and Σ^+ is the set of roots of T in N. It suffices to show there is an open neighborhood V of s in T such that for any compact $K' \subset G$, there is a compact $C \subset M\backslash G$ with: if $n^{-1}Vn \cap K' \neq \emptyset$ for $n \in N$, then $\overline{n} \in C$, where $\overline{n} = Mn$ is the image of n in $M\backslash G$.

Write Σ^+ as a disjoint union of $\Sigma^s = \{\alpha \in \Sigma^+\,;\, \alpha(s) = 1\}$ and Σ_s. For each $\alpha \in \Sigma^+$, there is a unique subgroup N_α of N and an isomorphism $\theta_\alpha : \mathbb{G}_a \to N_\alpha$, where $\mathbb{G}_a$ is the additive group, such that $\theta_\alpha(\alpha(t)a) = \mathrm{Int}(t)\theta_\alpha(a)$ for all $a \in \mathbb{G}_a(F) = F$, $t \in T$. Recall that $M = Z_G(s)^0$. Put $N^s = M \cap N$. Then $N^s = \prod_{\alpha \in \Sigma^s} N_\alpha$ in any order. Label the roots $\alpha_1, \ldots, \alpha_r$ of Σ^+ so that $i < j$ if $\alpha_i < \alpha_j$ using the basis of Σ^+. The map

$$\beta : N_{\alpha_1} \times \cdots \times N_{\alpha_r} \to N, \quad (n_1, \ldots, n_r) \mapsto n_1 \cdots n_r,$$

is an isomorphism of varieties. The image N_k of $1 \times \cdots \times 1 \times N_{\alpha_k} \times \cdots \times N_{\alpha_r}$ is a subgroup with $N_k/N_{k+1} \xrightarrow{\sim} N_{\alpha_k}$. Also $[N, N_k] \subset N_{k+1}$. See [St74]. For each $\alpha \in \Sigma^+$, let pr_α be the projection of $N_{\alpha_1} \times \cdots \times N_{\alpha_r}$ onto N_α. Define $\varphi_\alpha(n) \in F$ by $\theta_\alpha(\varphi_\alpha(n)) = \mathrm{pr}_\alpha(\beta^{-1}(n))$, thus $\varphi_\alpha : N \to \mathbb{G}_a$.

We claim: $\varphi_{\alpha_j}(ab) = \varphi_{\alpha_j}(a) + \varphi_{\alpha_j}(b) + P_{\alpha_j}(a, b)$, where $P_{\alpha_j}(a, b)$ is a polynomial in $\varphi_{\alpha_k}(a)$ and $\varphi_{\alpha_k}(b)$ $(1 \leq k < j)$. Indeed, since $[N, N_{j+1}] \subset N_{j+2}$ and $N_j/N_{j+1} \to N_{\alpha_j}$, the restriction of φ_{α_j} to N_j is a homomorphism. Since N_j is a subgroup, for $b \in N_{j+1}$, we have $\varphi_{\alpha_j}(ab) = \varphi_{\alpha_j}(a)$. Since $\varphi_{\alpha_j}(ab)$ is a polynomial in the $\varphi_\alpha(a)$ and $\varphi_\alpha(b)$, the claim follows.

Let V be a compact open neighborhood of s in T on which none of the roots $\alpha \in \Sigma_s$ is identically 1. Write $N_s = \prod_{\alpha\in\Sigma_s} N_\alpha$ with $N = N^s N_s$, according to the order on the roots. If $n = n^s n_s$, then $n^{-1}tn = n_s^{-1}tn_1^s n_s$ with $n_1^s = t^{-1}(n^s)^{-1}tn^s \in N^s \subset M$. To prove the proposition, it suffices then to show that the φ_α are bounded on the set

$$S = \{n_s \in N_s\,;\, t^{-1}n_s^{-1}tn^s n_s \in K' \text{ for some } t \in V \text{ and } n^s \in N^s\}.$$

Since φ_{α_j} is 1 on N_k with $k > j$, and for $\alpha \in \Sigma^s$ we have $\varphi_\alpha = 1$ on N_s, we need consider only φ_α with $\alpha \in \Sigma_s$.

Suppose $n^s, n_s \in N_{\alpha_r} = N_r$. If $\alpha_r \notin \Sigma_s$, there is nothing to prove. If $\alpha_r \in \Sigma_s$, then $n^s = 1$ and $\varphi_{\alpha_r}(t^{-1}n_s^{-1}tn_s) = (1-\alpha_r(t^{-1}))\varphi_{\alpha_r}(n_s) + P_{\alpha_r}(t^{-1}n_st, n_s)$. Since $n_s \in N_{\alpha_r}$, we have by the claim that $P_{\alpha_r}(t^{-1}n_st, n_s)$ is 0. By assumption on t, we have $\alpha_r(t^{-1}) \neq 1$. Since K' is compact, $\varphi_{\alpha_r}(t^{-1}n_s^{-1}tn_s)$ is bounded for n_s with $t^{-1}n_s^{-1}tn_s \in K'$. This implies that $\varphi_{\alpha_r}(n_s)$ is bounded.

By induction, assume (1) the φ_α are bounded on the set of $n_s \in N_{k+1}$ such that there exists $n^s \in N_{k+1}$ with $n_s^{-1}tn^s n_s \in K'$, and (2) $n_s^{-1}tn^s n_s \in K'$ for some $n^s, n_s \in N_k$. If $\alpha_k \in \Sigma_s$, write $n_s = xy$ with $x \in N_{k+1}$ and $y \in N_{\alpha_k}$. Then $n_s \in N_{k+1}$ and $n_s^{-1}tn^s n_s = tn'_k n''_{k+1}$ where $n'_k = t^{-1}y^{-1}ty \in N_{\alpha_k}$ and $n''_{k+1} = y^{-1}t^{-1}x^{-1}tn^s xy \in N_{k+1}$. If $n_s^{-1}tn^s n_s \in K'$, then both n'_k and n''_{k+1} remain bounded. As before, this implies that y is bounded, and x is bounded by the induction assumption. The case of $\alpha_k \in \Sigma^s$ is similar, left for the reader. □

To show that the orbital integral is compactly supported on each torus, we prove

PROPOSITION 2.11. *Let K' be a compact subset of $G(F)$. Put $K'^G = \{g^{-1}xg\,;\, x \in K', g \in G\}$. Then the set $S = \{t \in T\,;\, t \in$ closure of $K'^G\}$ is relatively compact.*

PROOF. Let $\rho : G \to \mathrm{GL}(n)$ be a faithful F-rational representation of the linear algebraic group G. Write $\det(x + 1 - \rho(g)) = \sum_{j=0}^{\dim\rho} p_j(g)x^j$. The p_j are invariant polynomials. Hence they are bounded on the set S, since K' is compact. For $t \in T$, the image $\rho(t)$ is semisimple. Hence p_j is bounded on S implies that the eigenvalues of $\rho(t)$ are bounded on S. But ρ is faithful. Hence S is relatively compact. □

Denote by $O(x)$ the conjugacy class of $x \in G$. Write $d(O(x))$ for the dimension of $O(x)$ as a variety. Put $U_d = \bigcup\{O \in U\,;\, d(O) \le d\}$, $U_\infty = \bigcup\{O\,;\, O \in U\}$. Let U be the set of unipotent conjugacy classes in $G(F)$. Recall that to simplify the notation, we often write G for $G(F)$.

PROPOSITION 2.12. *For all $d \ge 0$ the set U_d is closed. For all $O \in U$, the union $O \bigcup U_{d(O)-1}$ is closed and O is open in $U_{d(O)}$ and in the closure $\overline{O}$ of O. The identity e of G lies in $\overline{O}$. The G-invariant measure $d\mu_O$ on O is unique up to a nonzero real multiple.*

This is well known. For $f \in C_c^\infty(G(F))$ and any $x \in O$, let $\Phi(x,f)$ or $\Phi(O,f)$ be $\int_O f\, \mathrm{d}\mu_O$. The integral converges by Theorem 2.32 of Deligne, or by Rao [Ra72]. Hence $f \mapsto \Phi(O,f)$ is an invariant distribution on G. When $O = \{e\}$ write $\Phi(e)$ for $\Phi(O)$. Put $\Phi(e,f) = f(e)$.

A distribution D on G (a complex valued linear form on $C_c^\infty(G(F))$) is said to vanish on an open set V if $D(f) = 0$ for all test functions f with support in V. If D vanishes on a family $\{V_\alpha\}$ of open sets, it vanishes on $\cup V_\alpha$, since $\mathrm{supp}(f)$ is compact. The *support* of the distribution D is the complement of the largest open set on which D vanishes. A distribution D on a subset X of G is called *invariant* if $D(f^g) = D(f)$ for all $g \in G(F)$ and $f \in C_c^\infty(X)$. For any subset X of G, write $I(X)$ for the set of invariant distributions on G supported in X.

PROPOSITION 2.13. *A basis for $I(U_\infty)$ is given by $\{\Phi(O)\,;\, O \in U\}$.*

PROOF. Suppose $D \in I(U_\infty)$ has $\mathrm{supp}(D) \subset U_d$. List the $O \in U$ of dimension d as $O_1, \dots, O_m$. Each O_j is open in U_d, and $D|_{O_j} = c_j\Phi(O_j)$ for some $c_j \in \mathbb{C}$, since $I(O_j)$ is spanned by $\Phi(O_j)$. Hence $\mathrm{supp}(D - \sum_{j=1}^m c_j\Phi(O_j)) \subset U_{d-1}$, and the proposition follows by induction. □

Let Ω be an open closed subset of $\mathfrak{g} = \mathrm{Lie}\, G$ containing 0 which is invariant under $\mathrm{Ad}(G)$, such that $\exp : \Omega \to G$ is defined, the image $\omega = \exp(\Omega)$ is open and closed and $\exp(\mathrm{Ad}(g)X) = (\mathrm{Int}(g))(\exp X)$ for all $g \in G$ and $X \in \Omega$, and $\mathcal{O}_F\Omega \subset \Omega$ where $\mathcal{O}_F$ is the ring of integers of F. For example, Ω can be the image under $\mathrm{Ad}(G)$ of $\mathfrak{t}(\mathcal{O}_F) + \boldsymbol{\pi}\mathfrak{g}(\mathcal{O}_F)$, where $\mathfrak{t}$ is a maximally split torus in $\mathfrak{g}$, both defined over $\mathcal{O}_F$, and $\mathfrak{t}(\mathcal{O}_F)$ and $\mathfrak{g}(\mathcal{O}_F)$ indicate the $\mathcal{O}_F$ valued points, $\boldsymbol{\pi}$ a generator of the maximal ideal in $\mathcal{O}_F$.

We use standard notation: $\mathrm{Int}(g)x = gxg^{-1}$, $g, x \in G$, $X \in \mathfrak{g}$, $\mathrm{Ad}(g)$ the differential of $\mathrm{Int}(g)$. Fixing an embedding of G in some $\mathrm{GL}(n)$, we embed $\mathfrak{g}$ in $M(n, F)$, and so $\mathrm{Ad}(g)X = gXg^{-1}$.

PROPOSITION 2.14. *We have $U_\infty \subset \omega$.*

PROOF. For every $O \in U$, we have $e \in \overline{O}$. As $e \in \omega$ and ω is open, we have $O \cap \omega \neq \varnothing$. But ω is G-invariant, so $O \subset \omega$. □

Each $O \in U$ is of the form $O(u)$ with $u = \exp X$ in U_∞. By the Jacobson-Morozov theorem, there is a nilpotent $Y \in \mathfrak{g}$ and a semisimple $H \in \mathfrak{g}$ such that X, Y, H satisfy the relations $[X, Y] = H$, $[H, X] = 2X$, $[H, Y] = -2Y$, and the Lie algebra $\mathfrak{s}$ generated by X, Y, H is isomorphic to $\mathfrak{sl}(2)$. Let S be the connected algebraic group obtained from $\mathfrak{s}$. Let T_H be the centralizer of H in S. There is a character $\chi : T_H \to F^\times$ with $\mathrm{Ad}(t)X = \chi(t)X$ for all $t \in T_H$. There is a central isogeny $\mathrm{SL}(2) \to S$. Hence $(F^\times)^2 \subset \chi(T_H)$, and for all $a \in F^\times$, the element a^2X is conjugate to X by some $t_a \in T$. Define, for each $a \in F^\times$, the automorphism α_a of O by $\alpha_a(\exp X) = \exp(a^2X)$, and $\alpha_a(\mathrm{Int}(g)\exp(X)) = (\mathrm{Int}(g))\alpha_a(\exp X)$ for all $g \in G$. Since α_a commutes with the action of G on O, we have $\mathrm{d}\mu_O \circ \alpha_a = \lambda(a)\mathrm{d}\mu_O$ for some $\lambda(a) \in \mathbb{R}_{>0}^\times$.

PROPOSITION 2.15.

(i) *For all $a \in F^\times$, we have $d\mu_O \circ \alpha_a = |a|^{d(O)}d\mu_O$.*
(ii) *For every $f \in C_c^\infty(\omega)$, put $f_a(\exp Z) = f(\exp(a^2Z))$ for all $Z \in \Omega$. Then $\Phi(O,f_a) = |a|^{-d(O)}\Phi(O,f)$.*

PROOF. The two claims being equivalent, we prove (ii). Denote by G_u the centralizer $\{g \in G\,;\, gug^{-1} = u\}$ of u in G and by $\mathfrak{g}_X$ the centralizer $\{Z \in \mathfrak{g}\,;\, \mathrm{ad}(Z)X = 0\}$ of X in $\mathfrak{g} = \mathrm{Lie}\, G$. Then $\Phi(O,f) = \int_{G_u\backslash G} f(g^{-1}ug)\, \mathrm{d}\dot{g}$ for a quotient measure $\mathrm{d}\dot{g}$ on $G_u\backslash G$. Since X is an eigenvector of $\mathrm{Ad}(T_H)$, the torus T_H normalizes G_u and $\mathfrak{g}_X$. Then

$$\Phi(O,f_a) = \int_{G_u\backslash G} f_a(g^{-1}ug)\,\mathrm{d}\dot{g} = \int_{G_u\backslash G} f(g^{-1}\exp(a^2X)g)\,\mathrm{d}\dot{g}$$
$$= \int_{G_u\backslash G} f(g^{-1}t\exp(X)t^{-1}g)\,\mathrm{d}\dot{g} \qquad (2.15.1)$$

where $t \in T_H$ has $\chi(t) = a^2$, and this is $\int_{G_u\backslash G} f((t^{-1}gt)^{-1}\exp(X)(t^{-1}gt))\,\mathrm{d}\dot{g}$ as $\int_{G_u\backslash G} f(g^{-1}ug)\,\mathrm{d}\dot{g}$ is invariant under $g \mapsto gx$. Changing $g \mapsto \mathrm{Int}(t)g$, we get

$$\Phi(O,f_a) = \int_{G_u\backslash G} f(g^{-1}ug)|\det(\mathrm{Ad}(t)|_{\mathfrak{g}_X\backslash\mathfrak{g}})|\,\mathrm{d}\dot{g}.$$

To prove the proposition, we need to show that $|\det(\mathrm{Ad}(t)|_{\mathfrak{g}_X\backslash\mathfrak{g}})| = |a|^{-d(O)}$. But $\mathfrak{g}$ is unimodular, so $|\det(\mathrm{Ad}(t)|_{\mathfrak{g}})| = 1$. It remains to establish

Claim: We have $|\det(\mathrm{Ad}(t)|_{\mathfrak{g}_X})| = |\chi(t)|^{d(O)/2}$.

To prove this claim, let ρ denote the restriction of the adjoint representation of G to the subgroup S. The irreducible constituents of ρ are denoted by $\mathfrak{g}_1, \ldots, \mathfrak{g}_r$, with dimensions $\dim \mathfrak{g}_j = d_j + 1$, so that $\mathfrak{g} = \oplus_{j=1}^r \mathfrak{g}_j$ and $\sum_{j=1}^r (d_j + 1) = \dim \mathfrak{g}$. Since $\mathfrak{s} \simeq \mathfrak{sl}(2)$, the representation of S on $\mathfrak{g}_j$ is completely determined by the dimension of $\mathfrak{g}_j$. Let X_j be the highest weight vector for T in $\mathfrak{g}_j$ for X. Let χ_j be the character $T_H \to F^\times$ defined by $\rho(t)X_j = \chi_j(t)X_j$ for $t \in T_H$. Then $\chi_j^2 = \chi^{d_j}$. Up to scalar multiples, X_j is the unique vector in $\mathfrak{g}_j$ with $\mathrm{ad}(X)X_j = 0$. Namely, $\dim(\mathfrak{g}_X \cap \mathfrak{g}_j) = 1$, $\{X_1, \ldots, X_r\}$ is a basis for $\mathfrak{g}_X$, and $\sum_{j=1}^r d_j = \dim \mathfrak{g} - \dim \mathfrak{g}_X = d(O)$. The action of T_H on $\mathfrak{g}_X$ with respect to the basis $\{X_1, \ldots, X_r\}$ is diagonal: $\mathrm{Ad}(t)X_j = \chi_j(t)X_j$. Then

$$|\det(\mathrm{Ad}(t)|\mathfrak{g}_X)| = \left|\prod_{j=1}^r \chi_j(t)\right| = \left|\prod_{j=1}^r \chi(t)^{d_j}\right|^{1/2} = |\chi(t)|^{d(O)/2}.$$

The claim, and the proposition, follow. □

Recall some results of Harish-Chandra ([HC70]). In topology, a continuous map $f : M \to N$ is called *submersive* if f is surjection, and for any $T \subset N$, we have that T is open or closed if and only if $f^{-1}(T)$ is so. In other words, N has the quotient topology relative to the surjection $M \to N$. In differential geometry submersive means moreover that the differential of f is surjective everywhere. Recall: $u = \exp(X)$ is unipotent and $\{X, Y, H\}$ is an $\mathfrak{sl}(2)$-triple; we put $G_Y = \{g \in G\,;\, \mathrm{Ad}(g)Y = Y\}$.

PROPOSITION 2.16. (i) *The map* $\psi : G \times G_Y \to G$, $(g, y) \mapsto g^{-1}(yu)g$ *is everywhere submersive.*

(ii) *There is an open subset* $V(O)$ *in* G_Y *containing* e *such that* ψ *maps* $G \times V(O)$ *to an invariant open set* Ω_O *in* G *containing the unipotent orbit* O*, such that* $\Omega_O \cap U_{d(O)} = O$.

(iii) *Let* $\pi : M \to N$ *be a smooth submersive map of manifolds. Let* ω_M *and* ω_N *be nowhere zero differential forms of maximal degree on* M *and* N*. Then there is a unique surjection* $C_c^\infty(M) \to C_c^\infty(N)$, $\alpha \mapsto f_\alpha$, *with* $\int_M (F \circ \pi)\alpha\omega_M = \int_N F f_\alpha \omega_N$ *for all locally integrable* F *on* N*. Further,* $\operatorname{supp} f_\alpha \subset \pi(\operatorname{supp} \alpha)$*. If* D *is a distribution on* N*, then the map* $\alpha \mapsto D(f_\alpha)$ *defines a distribution* D_M *on* M; D_M *determines* D *uniquely.*

(iv) *Let* D *be an invariant distribution on* G *supported in* $\psi(G \times V(O))$*. Then there is a unique distribution* D_O *on* $V(O)$ *with* $D(f_\alpha) = D_O(\beta_\alpha)$ *for all* α *in* $C_c^\infty(G \times V(O))$*, where* $\beta_\alpha(u) = \int_G \alpha(g, u)\, dg$.

For (ii) note that ψ maps $G \times \{e\}$ onto O. As O is open in $U_{d(O)}$, there is an open neighborhood $V(O)$ of e in G_Y such that $\psi(G \times V(O)) \cap U_{d(O)} = O$.

2.17 Orbital Integrals

Let G be a connected reductive F-group and T an F-torus. Write $\mathfrak{t}$ for the Lie algebra of $T(F)$, t_Ω for $\mathfrak{t} \cap \Omega$, and $\mathfrak{t}^{\mathrm{reg}}$ for the set of regular elements in $\mathfrak{t}$. Then $\exp : \mathfrak{t}_\Omega \to T(F)$ is defined, and $\mathcal{O}_F \cdot \mathfrak{t}_\Omega \subset \mathfrak{t}_\Omega$. Put T_Ω for $\exp(\mathfrak{t}_\Omega)$. It is an open neighborhood of e in $T(F)$. Denote the center of $G(F)$ by $Z(G)$.

Fix s in T. Write G_s^0 for the connected centralizer of s in G. It is a connected reductive F-group, with $\dim G_s^0 < \dim G$ if $s \notin Z(G)$. Write M for $G_s^0(F)$, T for $T(F)$, G for $G(F)$, and $Z(M)$ for the center of M. We view T as a torus in M. Write Σ_M for the set of roots of T in M and T_M^{reg} for $\{t \in T;\, \alpha(t) \neq 1 \text{ for all } \alpha \in \Sigma_M\}$. Then $T^{\mathrm{reg}} \subset T_M^{\mathrm{reg}}$. The latter is the set of elements of T which are regular as elements of M. Let $\mathfrak{m}$ be the Lie algebra of M, and $\mathfrak{t}_M^{\mathrm{reg}} = \{X \in \mathfrak{t};\, X \text{ is regular as an element of } \mathfrak{m}\}$. Write U^M for the set of unipotent conjugacy classes in G which intersect M. Put $U_\infty^M = U_\infty \cap M$. An $O \in U$ lies in U^M when $O = O(u)$ for some $u \in U_\infty^M$. If $O \subset U_\infty^M$ put $O_M = O \cap M$. The connected centralizer G_{su}^0 of su in the F-group G lies in G_s^0 for all $u \in U_\infty^M$, by the uniqueness of the Jordan decomposition. Write M_{su} for the group of F-points of the connected centralizer G_{su}^0 of su in the F-group G. Write $\Phi(sO, f) = \int_{M_{su}\backslash G} f(g^{-1}sug)\, d\dot{g}$ if $O = O(u)$, $u \in U_\infty^M$, for a quotient measure $d\dot{g}$ on $M_{su}\backslash G$. The integral converges by Theorem 2.32 or [Ra72]. So it defines an invariant distribution on G.

PROPOSITION 2.18. *For* s, M, T *as above, for each* $O \in U^M$*, there exist locally constant functions* $\Gamma(O, s, T; t)$ *(called germs) in* $t \in sT_\Omega \cap T_M^{\mathrm{reg}}$*, satisfying*

(i) $\Gamma(O, s, T; zt) = \Gamma(O, s, T; t)$ *for any* $z \in Z(M)$ *such that* t *and* zt *lie in* $zT_\Omega \cap T_M^{\mathrm{reg}}$;

(ii) $\Gamma(O, s, T; s\exp(a^2H)) = |a|^{-d(O_M)}\Gamma(O, s, T; s\exp(H))$, *for all* $a \in \mathcal{O}_F$, $H \in \mathfrak{t}_M^{\mathrm{reg}}$;

(iii) *for each* $f \in C_c^\infty(G)$, *there is an open neighborhood* $V(f)$ *of* s *in* $sT_\Omega \subset T$ *with*

$$\Phi(t,f) = \sum_{O \in U^M} \Gamma(O, s, T; t)\Phi(sO,f)$$

for all $t \in V(f) \cap T^{\mathrm{reg}}$.

The equation of (iii) is called the germ expansion of the orbital integral.

Note that if $s \in T^{\mathrm{reg}}$, then $M = T$ and the proposition asserts that $\Phi(t,f)$ is constant in an open neighborhood of s in T^{reg}. This is part of Proposition 2.9. The main case is where $s \in Z(G)$, and then $M = G$. The general case follows by a reduction argument.

Recall that $O = O(u)$, $u = \exp X$ unipotent, $\{X, Y, H\}$ is an $\mathfrak{sl}(2)$ triple, G_Y and $\mathfrak{g}_Y$ are the centralizers of Y in G and $\mathfrak{g}$, and $\psi : G \times G_Y \to G$, $\psi(g, y) = g^{-1}yug$ is a submersion.

PROPOSITION 2.19. *There exists an open set* $V(O) \subset G_Y$ *containing the unit* e *of* G *and satisfying Proposition* 2.16(ii)*, and also*: *if* $zu \in O(u)$ *for* $z \in V(O)$*, then* $z = e$.

Thus $V(O)$ contains representatives for the conjugacy classes which pass near u but only one representative for the class of u.

PROOF. Since $\varphi : G_u\backslash G \to O(u)$, $g \mapsto g^{-1}ug$, is an injective analytic isomorphism, for any open neighborhood V of u in G, there is an open neighborhood N of e in $G_u\backslash G$ with $\varphi(N) = O(u) \cap V$. The tangent space to $O(u)$ at u is $[X, \mathfrak{g}]$. The tangent space to $G_Y u$ at u is $\mathfrak{g}_Y = \{Z \in \mathfrak{g}\,;\, [Z, Y] = 0\}$.

Claim: $[X, \mathfrak{g}] \cap G_Y = \{0\}$. Indeed, $[X, \mathfrak{g}] =$ image of $\mathrm{ad}(X)$, for ad as in the proof of the claim in the proof of Proposition 2.15. Also $\mathfrak{g}_Y$ is the kernel of $\mathrm{ad}(Y)$, which is the space spanned by the lowest weight vectors in the irreducible constituents $\mathfrak{g}_j$ of ad. Then $[X, \mathfrak{g}] \cap \mathfrak{g}_Y = \mathrm{Im}(\mathrm{ad}(X)) \cap \ker(\mathrm{ad}(Y)) = \{0\}$ from the well-known structure of representations of $\mathfrak{sl}(2)$. Similar arguments show that $\mathfrak{g} = [X, \mathfrak{g}] \oplus \mathfrak{g}_Y$. Hence $O(u)$ and $G_Y u$ are transverse at u. So we can choose the open neighborhoods V of u in G and N of e in $G_u\backslash G$ to have $\varphi(N) = O(u) \cap V$ and $\varphi(N) \cap G_Y u = \{u\}$. Choose a neighborhood $V(O)$ of e in G_Y with $V(O)u \subset V$. Then $O(u) \cap V(O)u \subset \varphi(N) \cap G_Y u = \{u\}$, as required. □

2.20 *Start of Proof of Proposition 2.18*

Denote by ρ_0 a faithful finite dimensional F-rational representation of G, namely, an embedding of G in some $\mathrm{GL}(n)$. (The symbol ρ was already used in Proposition 2.15.) Put $\det(x + 1 - \rho_0(g)) = \sum_{j=0}^{\dim \rho_0} p_j(g)x^j$. For all $\epsilon > 0$ the set

$G(\epsilon) = \{g \in G;\ |p_j(g)| < \epsilon$ for all $j\,(0 \leq j < \dim \rho_0)\}$ is an invariant open neighborhood of U_∞, and $\bigcap_{\epsilon>0} G(\epsilon) = U_\infty$. In particular the sets $T(\epsilon) = G(\epsilon) \cap T$ make a basis for the set of open neighborhoods of e in T.

We now prove proposition 2.18 when $s = e$. We use induction on $\#U$. For the initial step, suppose $f \in C_c^\infty(G)$ is supported on $G - U_\infty$. Choose $\epsilon > 0$ so that $\mathrm{supp}(f) \subset G - G(\epsilon)$. Then $\Phi(t,f) = 0$ for all $t \in T(\epsilon) \cap T^{\mathrm{reg}}$.

For the induction step note that $\#U$ is finite as $\mathrm{char}(F) = 0$. List the orbits in U as $O_1, \ldots, O_m$ such that $d(O_j) \geq d(O_{j+1})$. Assume there are functions $\Gamma(O_j, e, T; t)$ in $T^{\mathrm{reg}} \cap T_\Omega$ for $j\ (0 \leq j < k)$ such that

$$\text{for every } f \text{ in } C_c^\infty(G) \text{ supported in } G - \bigcup_{j=k}^{m} O_j, \text{ there is } \epsilon = \epsilon(f) > 0 \text{ with} \tag{IND}$$

$$\Phi(t,f) = \sum_{j=1}^{k-1} \Gamma(O_j, e, T; t)\Phi(O_j,f) \quad \text{for} \quad t \in T(\epsilon) \cap T^{\mathrm{reg}}.$$

By the initial step, this holds for $k = 1$. To show (iii) of Proposition 2.18, we need to show there is a function $\Gamma(O_k, e, T; t)$ in $t \in T^{\mathrm{reg}} \cap T_\Omega$ for which (IND) holds with k replaced by $k + 1$. Assume, as we may, that $\mathrm{supp}(f) \subset \Omega(O_k) = \psi(G \times V(O_k))$, with $V(O_k)$ as in Proposition 2.18, ψ as in Proposition 2.16(i). Suppose $f = f_\alpha$ with $\alpha \in C_c^\infty(G \times V(O_k))$ as in Proposition 2.16(iii). Put

$$D_{O_k,t}(\beta_\alpha) = D_t(f_\alpha) = \Phi(t,f_\alpha) - \sum_{j=1}^{k-1} \Gamma(O_j, e, T; t)\Phi(O_j,f_\alpha), \quad t \in T^{\mathrm{reg}}.$$

Claim: If $\beta_\alpha(e) = 0$, then there is $\epsilon > 0$ with $D_t(f_\alpha) = 0$ for all $t \in T^{\mathrm{reg}} \cap T(\epsilon)$.

To verify this, put $\alpha_0 = \gamma \times \beta_\alpha \in C_c^\infty(G \times V(O_k))$, where $\gamma \in C_c^\infty(G)$ satisfies $\int_G \gamma(g)\,dg = 1$. Then $\beta_{\alpha_0} = \beta_\alpha$, hence $D_t(f_{\alpha_0}) = D_t(f)$. Also $\mathrm{supp}(f_{\alpha_0}) \subset \psi(\mathrm{supp}\,\alpha_0)$. Since $\Omega(O_k) \cap U_\infty \subset \bigcup_{j=1}^{k} O_j$, the claim will follow from the induction assumption if we show that $\psi(\mathrm{supp}(\alpha_0)) \cap O_k$ is empty. Since β_α is locally constant and $\beta_\alpha(e) = 0$, we have $e \notin \mathrm{supp}(\beta_\alpha)$. By the choice of $V(O_k)$ and Proposition 2.18, we have $\psi(\mathrm{supp}(\alpha_0)) \cap O_k = \varnothing$, so the claim follows.

2.21 *Induction Step*

We proceed to prove the induction step. The distribution $f_\alpha \mapsto \beta_\alpha(e)$ is invariant and supported in O_k. Hence there is $c \in \mathbb{C}$ with $\beta_\alpha(e) = c\Phi(O_k,f_\alpha)$. It is clear that $c \neq 0$ as we do not have $\int_G \alpha(g,e)\,dg = 0$ for all $\alpha \in C_c^\infty(G \times V(O_k))$. Choose $\beta \in C_c^\infty(V(O_k))$ with $\beta(e) = c$. Then $\beta_\alpha - \beta_\alpha(e)\beta$ satisfies the assumption of the claim. Hence there is $\epsilon > 0$ with

$$D_{O_k,t}(\beta_\alpha - \beta_\alpha(e)\beta) = 0 \quad \text{for all } t \in T^{\mathrm{reg}} \cap T(\epsilon).$$

Hence for all $t \in T(\epsilon) \cap T^{\mathrm{reg}}$, we have

$$\Phi(t,f_\alpha) = \sum_{j=1}^{k-1} \Gamma(O_j, e, T; t)\Phi(O_j,f_\alpha) + cD_{O_k,t}(\beta)\Phi(O_k,f_\alpha).$$

Write $\Gamma(O_k, e, T; t)$ for $cD_{O_k,t}(\beta)$ and $t \in T_\Omega \cap T^{\mathrm{reg}} \cap T(\epsilon)$. Extend it to all of $T_\Omega \cap T^{\mathrm{reg}}$ by $\Gamma(O_k, e, T; \exp(a^2H)) = |a|^{-d(O_k)}\Gamma(O_k, e, T; \exp(H))$ for all $H \in \mathfrak{t}_\Omega \cap \mathfrak{t}^{\mathrm{reg}}$, and $a \in \mathcal{O}_F$. By Proposition 2.15, $\Gamma(O_k, e, T; t)$ satisfies this identity on $T_\Omega \cap T^{\mathrm{reg}} \cap T(\epsilon)$, so $\Gamma(O_k, e, T; t)$ is well defined. The induction step follows.

Note that $\Gamma(O, e, T; t)$ is locally constant in t since so if $\Phi(t,f)$.

Now that we dealt with $s = e$, consider $s \in Z(G)$. Define $\Gamma(O, s, T; st)$ to be $\Gamma(O, e, T; t)$ and replace $f(g)$ by $f_s(g) = f(sg)$. Proposition 2.18 for $s \in Z(G)$ then follows from the case of $s = e$.

2.22 Reduction Step

Next suppose that $s \in T$ and $\dim M < \dim G$. We shall reduce Proposition 2.18 for s to a germ expansion on M. Let V be a neighborhood of s in T satisfying the conclusion of Proposition 2.10. Let $K' = \mathrm{supp}(f)$. Choose C as in that proposition. Then $g^{-1}Vg \cap K'$ is empty if the image of g is not in $C \subset M\backslash G$. The function

$$M\backslash G \ni Mg \mapsto \int_{T\backslash M} f(g^{-1}m^{-1}tmg)\,\mathrm{d}\dot{m}, \quad t \in V \cap T^{\mathrm{reg}},$$

vanishes outside C. Choose $\alpha \in C_c^\infty(G)$ such that $\overline{\alpha}(g) = \int_M \alpha(mg)\,\mathrm{d}g$ is 1 if $Mg \in C$ and 0 otherwise. Put $F(m) = \int_G \alpha(g)f(g^{-1}mg)\,\mathrm{d}g$. Then $F \in C_c^\infty(M)$ as in Proposition 2.16. For $t \in V \cap T^{\mathrm{reg}}$, we have

$$\begin{aligned}\int_{T\backslash G} f(g^{-1}tg)\,\mathrm{d}\dot{g} &= \int_{M\backslash G}\int_{T\backslash M} \overline{\alpha}(g)f(g^{-1}m^{-1}tmg)\,\mathrm{d}\dot{g}\,\mathrm{d}\dot{m} \\ &= \int_G\int_{T\backslash M} \alpha(g)f(g^{-1}m^{-1}tmg)\,\mathrm{d}g\,\mathrm{d}\dot{m} \\ &= \int_{T\backslash M} F(m^{-1}tm)\,\mathrm{d}\dot{m} = \Phi^M(t, F).\end{aligned}$$

Here $\Phi^M(t,f)$ is the orbital integral on M. A similar computation shows that $\Phi(sO,f)$, defined before Proposition 2.18, is equal to $\Phi(sO_M, F)$ for all $O \in U_M$(= the set of $O \in U$ which intersect M), where $O_M = O \cap M$, since $M_{su} \subset M$. Now s lies in the center of M. Hence the germ expansion holds for $\Phi^M(t,f)$ in an open neighborhood of s which intersects T_M^{reg}. This proves Proposition 2.18 with $\Gamma(O, s, T; t) = \Gamma(O_M, s, T; t)$. □

2.23 Extension of Proposition 2.8

We continue with a slight extension of Proposition 2.8, using an alternative approach, based on the uniqueness of the Haar measure.

Let T be a maximal torus of G containing the semisimple part s of $g \in G$. Let Σ be a root system of T in G. Let Σ_s be the subset of roots which do not take the value 1 at s. Recall that

$$\Delta(g) = \left| \prod_{\alpha \in \Sigma_s} (1 - \alpha(s)) \right|^{1/2} = |\det(1 - \mathrm{Ad}(s))|_{\mathfrak{g}/\mathfrak{g}_s}|^{1/2}$$

and

$$I(g,f) = \Delta(g) \int_{Z_G(g)\backslash G} f(x^{-1}gx)dx.$$

PROPOSITION 2.24.

(i) *Fix* $f \in C_c^\infty(G)$. *For each pair* (T,u), *the normalized orbital integral* $g \mapsto I(g,f)$ *is locally constant on* $(Tu)^{\mathrm{reg}}$ *and compactly supported on* Tu. *For each* $s \in T$, *there is a neighborhood* V_f *of* s *in* T, *and functions* $\Gamma(Tu, su_i)$ *on* $V_f u$ *whose germs are independent of* f, *where the* u_i *are as in* 2.7, *such that for* $t \in V_f \cap T^{\mathrm{reg}}$, *we have*

$$I(tu,f) = \sum_{i=1}^{r} I(su_i,f)\Gamma(Tu, su_i; tu).$$

(ii) *Conversely, a conjugacy invariant function on* G *satisfying* (i) *with the germs* $\Gamma(Tu, su_i)$ *is an orbital integral.*

There is an obvious extension of this to the space $C_c^\infty(G,\omega)$ of locally constant complex valued functions which transform under the center Z of G by a character ω of Z. We just need to add to "compactly supported" the term "mod Z".

The following is used also in the proof of Corollary 4.4. It originates from [GK75], p. 100.

2.25 An Extension Result

If G acts on a space X, write $C_0(X)$ for the subspace of $C_c^\infty(X)$ spanned by $g \cdot f - f$, where $f \in C_c^\infty(X)$ and $g \in G$, and $(g \cdot f)(x) = f(g^{-1}x)$. We denote by A_s the set of $g \in G$ with semisimple part conjugate to s.

PROPOSITION. *Suppose the normalized orbital integral* $I(g,f)$ *of* $f \in C_c^\infty(G)$ *vanishes on* A_s. *Then there exists* $\phi \in C_0(G)$ *with* $f = \phi$ *on* A_s.

PROOF. This is done by induction, using the following elementary

LEMMA. *Let A be a locally compact totally disconnected topological space. Let B be a locally closed nonempty subspace of A. Denote by* $C_c^\infty(A, F)$ *the space of* $f \in C_c^\infty(A)$ (*the space of compactly supported locally constant functions on* A) *which vanish on* $F = \overline{B} - B$. *Then the restriction map* $C_c^\infty(A, F) \to C_c^\infty(B)$ *is surjective.*

We shall also use the uniqueness up to scalar of the G-invariant Haar measure on $O(g)$, [BZ76], which is equivalent to the statement that $C_0(O(g))$ has codimension 1 in $C_c^\infty(O(g))$. See proof of Corollary 4.4 for more details.

Since $I(g,f)$ is 0 on A_s, and $O(s)$ is closed in A_s, the restriction of f to $O(s)$ is in the subspace $C_0(O(s))$ of $C_c^\infty(O(s))$, by the uniqueness of the Haar measure on $O(s)$. Hence there are finitely many pairs (g, h) in $G \times C_c^\infty(O(s))$ with $f = \sum(g \cdot h - h)$ on $O(s)$. The lemma permits extending h to $\overline{h} \in C_c^\infty(G)$.

The function $f_1 = f - \sum(g \cdot \overline{h} - \overline{h})$ is zero on $O(s)$ and $I(g,f_1) = 0$ for all $g \in A_s$. Note that $O(s)$ is the boundary of $A_2 = O(s) \cup O(su_2)$. Hence the restriction of f_1 to $O(su_2)$ lies in $C_c^\infty(O(su_2))$. Using again $[C_c^\infty(O(g)) : C_0(O(g))] = 1$, we conclude that there exist finitely many pairs $(g, h) \in G \times C_c^\infty(O(su_2))$ such that $f_1 = \sum(g \cdot h - h)$ on $O(su_2)$. By the lemma we extend h to $\overline{h} \in C_c^\infty(G, O(s))$. Then $f_2 = f_1 - \sum(g \cdot \overline{h} - \overline{h})$ is zero on A_2 and $I(g,f_2) = 0$ for all $g \in A_s$. Continuing by induction we conclude that there is $\phi \in C_0(G)$ with $f = \phi$ on A_s. □

PROPOSITION 2.26. *Let* $G = \mathrm{GL}(n, F)$. *Suppose* $f \in C_c^\infty(G)$ *is zero on* A_s, *where s is semisimple in G. Then there exists an open, G-invariant compactly generated neighborhood* V_f *of s, depending on f, on which f is zero.*

PROOF. The characteristic polynomial defines a continuous map $P : G \to F^n$, such that the inverse image of $P(s)$, s semisimple, is the set A_s of g in G with semisimple part conjugate to s. The image $P(\mathrm{supp} f)$ of the support of f is compact. In the totally disconnected space F^n, there is an open neighborhood N of $P(s)$ which does not intersect $P(\mathrm{supp} f)$. The inverse image of the open N is open and it contains s, hence also an open compact neighborhood sK' of s. Now $P(sK') \subset N$. Hence $\bigcup_{g \in G} gsK'g^{-1}$ does not intersect $\mathrm{supp} f$. □

COROLLARY 2.27. *Let* $G = \mathrm{GL}(n, F)$ *and* $f \in C_c^\infty(G)$. *For every semisimple s in G, there is a neighborhood* V_f *of f such that if* $A_s = \bigcup_{i=1}^t O(su_i)$ *and* f_i *are as in* 2.7, *for all* $g \in V_f$, *we have*

$$I(g,f) = \sum_{i=1}^{t} I(g,f_i) I(su_i,f).$$

PROOF. The orbital integrals of the function $f - \sum_{i=1}^t I(su_i,f) f_i$ are zero on all $g \in A_s$. Hence this function is equal to a function $\phi \in C_0(G)$ on an open G-invariant compactly generated neighborhood of s, depending on f, by the last two propositions. □

The formula displayed in the corollary is called the germ expansion of the orbital integral of f at s. Note that we limited the discussion to $G = \mathrm{GL}(n)$ as we used the characteristic polynomial.

2.28 Extension to Reductive Groups

To extend this analysis to general reductive connected G from the case of $\mathrm{GL}(n)$, we apply a sequence of reductions. Embed G in $\mathrm{GL}(n, F)$.

Suppose $s = 1$. The characteristic polynomial defines a continuous map $G \to F^n$. The inverse image of $P(1)$ is the set of unipotent elements in G. We obtain a germ expansion of the orbital integrals in a neighborhood of $s = 1$: if $f \in C_c^\infty(G)$ and $A_1 = \bigcup_{i=1}^t O(u_i)$, and f_i are as in 2.7, there is a neighborhood V_f of $s = 1$ such that for $g \in V_f$ we have

$$I(g,f) = \sum_{i=1}^{t} I(g,f_i)I(u_i,f).$$

If s lies in the center Z of G, consider ${}^s f(g) = f(s^{-1}g)$ and note that $I(sg, {}^s f) = I(g,f)$. Hence in the neighborhood sV_f of s, we have $I(g,f) = \sum_{i=1}^t I(s^{-1}g,f_i)I(su_i,f) = \sum_{i=1}^t I(g, {}^s f_i)I(su_i,f)$. The functions ${}^s f_i$ have the properties specified at the end of 2.7.

To deal with a general semisimple $s \in G$, we replace in a neighborhood of s the integral on G by an orbital integral on the reductive connected group $Z_G(s)^0$, in which s is central.

We use Proposition 2.10 to construct a function $f_s \in C_c^\infty(Z_G(s)^0)$ such that the orbital integral of f_s on $Z_G(s)^0$ is equal to that of f on G in a neighborhood of su in Tu, as follows.

Choose C as in Proposition 2.10 for $K' = \mathrm{supp}(f)$. We may assume that C is compact and open, since $Z_G(s)^0\backslash G$ is locally compact. Choose $h \in C_c^\infty(G)$ such that the integral $\overline{h}(g) = \int_M h(xg)\,\mathrm{d}x$, $M = Z_G(s)^0$, equals the characteristic function of C in $M\backslash G$. Define f_s to be $f_s(g) = \int_G h(x)f(x^{-1}gx)\,\mathrm{d}x$. Then for $g \in Vu$ we have $\Phi^G(g,f) = \Phi^M(g,f_s)$ (see notation in Subsection 2.6). Indeed,

$$\begin{aligned}\Phi^G(g,f) &= \int_{Z_G(g)\backslash G} f(x^{-1}gx)\,\mathrm{d}x = \int_{M\backslash G}\int_{Z_G(g)\backslash M} f(x^{-1}y^{-1}gyx)\,\mathrm{d}y\,\overline{h}(x)\,\mathrm{d}x \\ &= \int_G \mathrm{d}x \int_{Z_G(g)\backslash M} h(x)f(x^{-1}y^{-1}gyx)\,\mathrm{d}y = \int_{Z_G(g)\backslash M} f_s(y^{-1}gy)\,y = \Phi^M(g,f_s).\end{aligned}$$

For $g \in Vu$ we then have $I^G(g,f) = \frac{\Delta_G(g)}{\Delta_M(g)} I^M(g,f_s)$.

Suppose $A_{s,M} = \bigcup O(su_i)$ is the decomposition into orbits in M of the set of elements with semisimple part conjugate in M to s. Let $f_{s,i} \in C_c^\infty(M)$ be the functions satisfying 2.7. Then for $g \in M$ near s, we have

$$I^M(g,f_s) = \sum_{i=1}^{t} I^M(g,f_{s,i}) I^M(su_i,f_s).$$

A unipotent element u has conjugates arbitrarily close to the identity. If u, u' are unipotents in M, then su and su' are conjugate in M if and only if they are conjugate in G. In G we have $A_s = \bigcup_{i=1}^{t} O(su_i)$ with the same u_i as for M. We may assume that $f_{s,i} = f_i \Delta_G(su_i)/\Delta_M(su_i)$, where $f_i \in C_c^\infty(G)$ are as in 2.7. For g in a neighborhood of su in Tu, we then have

$$I^G(g,f) = \sum_{i=1}^{t} I^G(g,f_i) I^G(su_i,f).$$

We obtain then the following analogue of Corollary 2.27 for a general connected reductive G, not only for $G = \mathrm{GL}(n)$.

PROPOSITION 2.29. *Let $f \in C_c^\infty(G)$. For any semisimple s in G, torus T containing s, and unipotent u commuting with each element of T, there is a neighborhood V_f of s in T such that $I(g,f) = \sum_{i=1}^{t} I(g,f_i) I(su_i,f)$ for all $g \in V_f u$. Here $A_s = \bigcup_{i=1}^{t} O(su_i)$ and the f_i are as in 2.7.*

2.30 Proof of Part (i) *of Proposition 2.24*

To complete the proof of part (i) of Proposition 2.24, we introduce notation. Let s be semisimple in G. Put M for the connected centralizer $Z_G(s)^0$. Let T denote the center of M. Let u be a unipotent in M. Consider the normalizer $N(Tu) = \{x \in G\,;\, xTux^{-1} = Tu\}$, centralizer $Z(Tu) = \{x \in G\,;\, xtux^{-1} = tu$ for all t in $T\}$, and Weyl group $W(Tu) = N(Tu)/Z(Tu)$ of Tu in G. Then $W(Tu)$ injects into the Weyl group $W(T) = N(T)/Z(T)$ of T in G. It is the group of inner automorphisms of G mapping Tu to itself. The Weyl group $W(T)$ acts continuously on the left on $T \times (Z(T)\backslash G)$ by $w(t,g) = (n_w t n_w^{-1}, n_w g)$, $t \in T$, $g \in G$, $w \in W(T)$, $w = Z(T)n_w$, $n_w \in N(T)$. Write $O(T^{\mathrm{reg}}u)$ for the orbit of $T^{\mathrm{reg}}u$ in G.

LEMMA. *The map $(t,g) \mapsto g^{-1}tug$ induces a canonical homeomorphism*

$$W(Tu)\backslash(T^{\mathrm{reg}} \times (Z(Tu)\backslash G)) \to O(T^{\mathrm{reg}}u).$$

PROOF. The map is bijective and a local homeomorphism. As $W(Tu)$ is finite, the map is a homeomorphism. $\square$

It follows from the lemma that (i) for any $f \in C_c^\infty(G)$, the restriction of $I(g,f)$ to $T^{\mathrm{reg}}u$ is locally constant, and (ii) for any function I in $C_c^\infty(T^{\mathrm{reg}})$ invariant under $W(Tu)$, there exists f in $C_c^\infty(O(T^{\mathrm{reg}}u))$ such that $I(t) = I(tu,f)$ for all $t \in T^{\mathrm{reg}}$. Part (i) of Proposition 2.24 then follows from these and Proposition 2.29.

2.31 *Proof of Part* (ii) *of Proposition* 2.24

It remains to prove part (ii) of Proposition 2.24. The set of conjugacy classes of Tu, where (T, u) is a pair as in 2.9 consisting of a torus T and a unipotent u which commutes with each element of T, is finite. We recall that it can be numbered as $(Tu)_j$, $1 \le j \le r$, such that $X_j = \bigcup_{i=1}^{j} O((Tu)_i)$ is closed in G, and $O((T^{\text{reg}}u)_j)$ is open in X_j, and $X_r = G$.

We proceed as follows. For a fixed T, arrange the Tu in increasing dimensions: $Tu_1 < Tu_2$ if $\dim O(Tu_1) < \dim O(Tu_2)$. We get a finite sequence $S(T)$. Next arrange the sequences with $S(T_1) < S(T_2)$ if $T_1 \subset T_2$. Then $S(Z)$ is the first term, where Z is the center of G. Further we have $X_j = X_{j-1} \cup O((T^{\text{reg}}u)_j)$ $(2 \le j \le r)$.

Now we prove (ii) of Proposition 2.24, using the lemmas in 2.25 and 2.30 and part (i) of 2.24, inductively. Let I be the function of 2.24(ii). Its restriction $I|_Z$ to Z lies in $C_c^\infty(Z)$. Namely, there is $f \in C_c^\infty(Z)$ with $I(z) = I(z,f) = f(z)$ for $z \in Z$. Extend f to $f_0 \in C_c^\infty(G)$. Then $I_1 = I - I(\cdot, f_0)$ satisfies the requirements of (ii) of the proposition and also $I_1|_Z = 0$. By induction we assume, for j $(1 \le j < r)$, that I_j satisfies the requirements of (ii), and also $I_j|_{X_j} = 0$. The restriction $I_j|_{O((T^{\text{reg}}u)_{j+1})}$ lies in $C_c^\infty(O((T^{\text{reg}}u)_{j+1}))$ and it is $W((Tu)_{j+1})$ invariant. Hence there is $f \in C_c^\infty(O((T^{\text{reg}}u)_{j+1}))$ with $I_j(x) = I(x,f)$ for $x \in (T^{\text{reg}}u)_{j+1}$. We extend f to $f_j \in C_c^\infty(G, X_j)$, thus f_j is zero on X_j, and consider $I_{j+1} = I_j - I(\cdot, f_j)$. This function satisfies the requirements of (i) of 2.24 and also $I_{j+1}|_{X_{j+1}} = 0$. At the last step of induction, we obtain our initial I as an orbital integral, so (ii) of the proposition follows. $\square$

We next study invariant measures on conjugacy classes, following a letter of P. Deligne.

THEOREM 2.32. *Let G be a reductive group over a local field F of characteristic zero. Fix $x \in G$ and $f \in C_c^\infty(G)$. Then the orbital integral $\int_{G/G_x} f(gxg^{-1})\, d\dot{g}$ is finite.*

Reduction to case where x is unipotent. Put $x = su = us$ where s is semisimple and u is unipotent. Denote supp(f) by K'. Then $gsug^{-1} \in K'$ implies $gsg^{-1} \in K'$ (we may assume K' contains the semisimple parts of its elements since $gsg^{-1} = p(gxg^{-1})$ for some polynomial independent of g). Proposition 2.10 then implies that g lies in a compact C in G/M, $M = G_s^0$. Hence it suffices to replace G by G_s^0, that is to deal with the case where $s = e$, and assume $x = u$ is unipotent. Since the orbit of s is closed, its orbital integral converges so we may assume $u \neq e$ and that the center $Z(G)$ of G consists of e. Moreover, we may assume that $f \ge 0$.

2.33 *Unipotent Elements*

We first discuss the structure afforded by the unipotent element u. Fix a faithful linear representation of G on a vector space V over F, that is an embedding $G \to \mathrm{GL}(V)$.

We claim that there exists a unique decreasing filtration W^j of V such that

(i) $(u-e)W^j \subset W^{j+2}$,
(ii) $(u-e)^k : W^{-k}/W^{-k+1} \xrightarrow{\sim} W^k/W^{k+1}$.

To see this, let $i : \mathrm{SL}(2) \to G$ be an embedding with $i(I+X) = u$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $I = d(1)$ where $d(a) = \mathrm{diag}(a, a^{-1})$. We also put $Y = {}^tX$ for the transpose of X.

Consider the eigenspaces $V^j = \{v \in V;\, i(d(a))v = a^j v,\, a \in F^\times\}$ of the diagonal subgroup in $\mathrm{SL}(2, F)$, thus the vector space decomposition $V = \oplus_j V^j$. Then the filtration $W = \{W^j = \oplus_{k \geq j} V^k\}$ has the required properties.

In the special case of the adjoint representation of G on $\mathfrak{g} = \mathrm{Lie}(G)$, we have

(i) $W^0(\mathfrak{g})$ is the Lie algebra of a parabolic subgroup P.
(ii) $W^1(\mathfrak{g})$ is the Lie algebra of the unipotent radical N of P.
(iii) W^0/W^1 is the Lie algebra of the centralizer H of $i(d(a))$.
(iv) If $u = \exp(U)$, then $(\mathrm{ad}(U))(W^1) = W^3$, as $\mathrm{ad}(U) : W^j/W^{j+1} \twoheadrightarrow W^{j+2}/W^{j+3}$ and $U \in W^2$.

The surjectivity follows on using $i(I+Y)$ to construct a partial inverse. Consequently the conjugacy class in N of a conjugate x of u in P depends only on $\log(x)$ mod W^3. Let N^j denote the subgroup of N with Lie algebra W^j $(j > 0)$. Let N^* denote the set of conjugates of u in P. Then $N^* \subset N^2$, and the map from the set N^+ of N-conjugacy classes in N^*, to N^2/N^3 (which is a commutative group), identifies N^+ with the orbit under H of the image $\overline{u}$ of u in N^2/N^3. This orbit is open. Thus we identify N^+ with an open subset of $N^2/N^3$and hence transport the measure $\mathrm{d}n$ to N^+.

If $x \in N^*$, then we have

(i) a duality between W^{-k}/W^{-k+1} and W^k/W^{k+1}, using the Killing form;
(ii) an isomorphism $(x-e)^k : W^{-k}/W^{-k+1} \to W^k/W^{k+1}$.

Hence we get a bilinear form B_x on W^k/W^{k+1}. The isomorphism (ii) and the form B_x depend only on the image of x in $N^+ \subset N^2/N^3$.

For any subset A of G, write A_u for $A \cap G_u$, the centralizer of u in A. Then $G_u \subset P = HN$, and $P_u = H_u N_u$. Moreover,

$$N_u^j/N_u^{j+1} \simeq \ker(\mathrm{ad}(U) : W^j/W^{j+1} \to W^{j+2}/W^{j+3}).$$

Put (note that $(u-e)^{j+1}$ can be replaced by U^{j+1})

$$F^{-j} = \ker((u-e)^{j+1} : W^{-j}/W^{-j+1} \to W^{j+2}/W^{j+3}).$$

Then

$$W^j/W^{j+1} \simeq \oplus_{k \geq 0} U^{j+k}(F^{-j-2k}),$$

where the direct sum is B_u-orthogonal.

The determinant of the action of H_u on F^{-j} is trivial, due to the existence of B_u.

Note that G_u is unimodular, and thus $\delta_P|_{G_u} = \delta_{G_u}$ is 1, where $\delta_S(x) = \det(\mathrm{ad}(x)|\,\mathrm{Lie}\ S)$.

2.34 Factorization of an Integral

Let $\mathrm{d}\overline{g}$ denote the invariant measure on G/G_u, $\mathrm{d}\overline{p}$ that on P/G_u, $\mathrm{d}\overline{h}$ that on H/H_u, and $\mathrm{d}\overline{n}$ that on N/N_u. Denote by $\mathrm{d}g_u$ the invariant measure on G_u. Then

$$\mathrm{d}g = \delta_P(p)\mathrm{d}k\mathrm{d}p = \delta_P(p)\mathrm{d}k\mathrm{d}\overline{p}\mathrm{d}g_u = \mathrm{d}\overline{g}\mathrm{d}g_u$$

and

$$\mathrm{d}\overline{g} = \delta_P(p)\mathrm{d}k\mathrm{d}\overline{p}.$$

To verify the theorem, it suffices to show that

$$\int_{P/G_u} f(pup^{-1})\delta_P(p)\,\mathrm{d}\overline{p} < \infty.$$

Note that $\delta_N(H_u) = \delta_{N_u}(H_u) = 1$. Hence for g on P/G_u, we have

$$\int_{P/G_u} g(p)\,\mathrm{d}\overline{p} = \int_{H/H_u} \mathrm{d}\overline{h} \int_{N/N_u} g(hn)\,\mathrm{d}\overline{n}.$$

Hence

$$\int_{P/G_u} f(pup^{-1})\delta_P(p)\,\mathrm{d}\overline{p} = \int_{H/H_u} \delta_N(h)\,\mathrm{d}\overline{h} \int_{N/N_u} f(hnun^{-1}h^{-1})\,\mathrm{d}\overline{n}.$$

For any function $g \geq 0$ on N, we have (for some $c > 0$ independent of g)

$$\int_{N/N_u} g(nun^{-1})\,\mathrm{d}\overline{n} \leq c \int_{N_3} g(un)\,\mathrm{d}n.$$

Hence

$$\begin{aligned}
&\int_{N/N_u} f(hnun^{-1}h^{-1})\,\mathrm{d}\overline{n} \leq c \int_{N_3} f(hunh^{-1})\,\mathrm{d}n \\
&= c \int_{N_3} f(huh^{-1} \cdot hnh^{-1})\,\mathrm{d}n = \delta_{N_3}(h)^{-1} \int_{N_3} f(huh^{-1} \cdot n)\,\mathrm{d}n.
\end{aligned}$$

Consequently

$$\int_{P/G_u} f(pup^{-1})\delta_P(p)\,\mathrm{d}\overline{p} \le c \int_{H/H_u} \delta_{N/N_3}(h)\,\mathrm{d}\overline{h} \int_{N_3} f(huh^{-1}\cdot n)\,\mathrm{d}n.$$

Since f is compactly supported, the problem is to evaluate, for $F \in C_c^\infty(N_2/N_3)$, the integral

$$\int_{H/H_u} \delta_{N/N_3}(h)F(huh^{-1})\,\mathrm{d}\overline{h}.$$

2.35 Final Evaluation

By the isomorphism $H/H_u \simeq N^+ \subset N_2/N_3$, the measure $\mathrm{d}\overline{h}$ transfers to the invariant measure $\delta_{N_2/N_3}(h)^{-1}\,\mathrm{d}n$ on N^+, and so

$$\int_{H/H_u} \delta_{N/N_3}(h)F(huh^{-1})\,\mathrm{d}\overline{h} = \int_{N_2/N_3} \delta^*(n)F(n)\,\mathrm{d}n$$

where $\delta^*(huh^{-1}) = \delta_{N/N_2}(h)$. Every $x \in N^+ \subset N_2/N_3$ defines an isomorphism $\widetilde{x}: W^{-1}/W^0 \xrightarrow{\sim} W^1/W^2$. For $x = huh^{-1}$ we have a commutative diagram

$$\begin{array}{ccc} W^{-1}/W^0 & \xrightarrow[\widetilde{u}]{\sim} & W^1/W^2 \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ W^{-1}/W^0 & \xrightarrow[\widetilde{x}]{\sim} & W^1/W^2. \end{array}$$

Then $h^{-1} : W^{-1}/W^0 \to W^{-1}/W^0$ is the transpose of $h : W^1/W^2 \to W^1/W^2$ for the Killing form. Hence $\delta_{N/N_2}(h) = c \cdot \det(\widetilde{x} : W^{-1}/W^0 \to W^1/W^2)^{1/2}$, and

$$\int_{N_2/N_3} \delta^*(n)F(n)\,\mathrm{d}n = \int_{N_2/N_3} \det(\widetilde{n})^{1/2} \cdot F(n)\,\mathrm{d}n < \infty.$$

2.36 Euler-Poincaré Functions

In the next subsections, we follow Kottwitz [Ko88] in computing the orbital integrals of the Euler-Poincaré function, which we now introduce.

Let F be a non-Archimedean local field. Let G be a connected reductive group over F. Assume that the connected component of 1 in the center of G is anisotropic over F. Denote by $\mathbb{B}$ the building of $G(F)$ and by $\mathbb{F}$ the set of facets of $\mathbb{B}$. For $\sigma \in \mathbb{F}$, write $G(F)_\sigma = \{g \in G(F)\,;\, g\sigma = \sigma\}$ for the stabilizer of σ in $G(F)$. This stabilizer is a compact open subgroup in $G(F)$. Any $g \in G(F)_\sigma$ permutes the vertices of σ.

Denote by $\mathrm{sgn}_\sigma(g)$ the sign of this permutation. We get a character $\mathrm{sgn}_\sigma : G(F)_\sigma \to \{\pm 1\}$. Extend sgn_σ to a function on $G(F)$ by 0 on $g \in G(F) - G(F)_\sigma$. Choose a Haar measure $\mathrm{d}g$ on $G(F)$. Let $\mathbb{S}$ be a set of representatives for the orbits of $G(F)$ on $\mathbb{F}$. Define the *Euler-Poincaré function* on $G(F)$ to be

$$f_{EP} = \sum_{\sigma \in \mathbb{S}} (-1)^{\dim \sigma} |G(F)_\sigma|^{-1} \mathrm{sgn}_\sigma.$$

Here $|G(F)_\sigma|$ indicates the volume of $G(F)_\sigma$. Then $f_{EP} \in C_c^\infty(G(F))$. If $\mathrm{d}g$ is multiplied by a scalar, then f_{EP} is divided by that scalar. Changing $\mathbb{S}$ changes f_{EP}, but not its orbital integrals.

2.37 Poincaré Measures

Let F be a local field of characteristic 0. Let G be a connected reductive group over F. Let μ be an invariant measure on $G(F)$. Let Γ be a discrete cocompact subgroup of $G(F)$. Write $\mu(\Gamma \backslash G(F))$ for the volume of $\Gamma \backslash G(F)$ with respect to the invariant measure on $\Gamma \backslash G(F)$ obtained from μ. Serre [Sr71] showed that there exists an invariant measure μ_G on $G(F)$ having the following property. For every discrete, torsion-free, cocompact subgroup Γ of $G(F)$, the Euler-Poincaré characteristic of $H^\bullet(\Gamma, \mathbb{Q})$ is equal to $\mu_G(\Gamma \backslash G(F))$. Borel and Harder [BH78] showed that $G(F)$ always has discrete, torsion-free, cocompact subgroups. Hence μ_G is uniquely determined by the property just stated. Serre names μ_G the *Euler-Poincaré measure* of $G(F)$. It can be negative or zero. It is nonzero if and only if G has an anisotropic maximal F-torus. In the p-adic case, this happens if and only if the connected component of the identity in the center of G is anisotropic [Kn65]. If $\mu_G \neq 0$, its sign is $(-1)^{q(G)}$, where $q(G)$ is the F-rank of the derived group G^{ss} of G in the p-adic case, and half the dimension of the symmetric space G^{ss}/K attached to G^{ss} in the real case. See [Sr71, Propositions 23, 28].

2.38 Orbital Integrals of EP-Functions

Let I denote the connected component G_γ^0 of the identity in the centralizer G_γ of γ in G. Write

$$O_\gamma(f) = \int_{I(F)\backslash G(F)} f(g^{-1}\gamma g)\, \mathrm{d}g/\mathrm{d}i$$

for the orbital integral of $f \in C_c^\infty(G(F))$ at $\gamma \in G(F)$.

THEOREM. *The orbital integral* $O_\gamma(f_{EP})$ *is* 0 *unless* γ *is elliptic semisimple in* $G(F)$. *In the latter case, if* di *is the Euler-Poincaré measure on* $I(F)$ (*note that* $di \neq 0$), *then* $O_\gamma(f_{EP}) = 1$. *Further, the Shalika germ for the identity element* e *in* $G(F)$ *is identically* 1 *on every elliptic maximal torus of* G.

PROOF. (1) Let γ be a semisimple element of $G(F)$. Let $\mathbb{F}(\gamma)$ be the set of facets $\tau \in \mathbb{F}$ fixed by γ; thus $\mathbb{F}(\gamma) = \{\tau \in \mathbb{F}\,;\, \gamma\tau = \tau\}$. Write $\mathbb{B}(\tau) = \{x \in \mathbb{B}\,;\, \gamma x = x\}$ for the set of points x in the building $\mathbb{B}$ fixed by γ. For any τ in $\mathbb{F}(\gamma)$, put $\tau(\gamma) = \tau \cap \mathbb{B}(\gamma)$ for the set of points $x \in \tau$ fixed by γ. Then $\tau(\gamma) \neq \varnothing$, and $\mathbb{B}(\gamma)$ is a polysimplicial complex whose set of facets is $\{\tau(\gamma)\,;\, \tau \in \mathbb{F}(\gamma)\}$. One verifies that $\mathrm{sgn}_\tau(\gamma) = (-1)^{\dim(\tau)-\dim(\tau(\gamma))}$. If f^0 denotes the characteristic function of the maximal compact subgroup K in $G(F)$, and the measure on $G(F)$ is normalized by $|K| = 1$, then

$$O_\gamma(f^0) = \sum_g |I(F)\backslash I(F)gK|; \quad g \in I(F)\backslash G(F)/K\,, \quad g^{-1}\gamma g \in K.$$

Put $x = gk \in \mathbb{B}$, and note that the summand is $|I(F)_x|^{-1}$, to get

$$O_\gamma(f^0) = \sum_x |I(F)_x|^{-1}, \quad x \in I(F)\backslash \mathbb{B}(\gamma).$$

To compute $O_\gamma(f_{EP}) = \sum_{\sigma\in\mathbb{S}}(-1)^{\dim\sigma}|G(F)_\sigma|^{-1}O_\gamma(\mathrm{sgn}_\sigma)$, we compute

$$O_\gamma(\mathrm{sgn}_\sigma) = \sum_g |I(F)\backslash I(F)gG(F)_\sigma| \cdot \mathrm{sgn}_\sigma(g^{-1}\gamma g), \quad g \in I(F)\backslash G(F)/G(F)_\sigma.$$

Put $\tau = g\sigma$. Then $g^{-1}\gamma g \in G(F)_\sigma$ means $g^{-1}\gamma g\sigma = \sigma$, namely $\gamma\tau = \tau$. Also, $\gamma \in gG(F)_\sigma g^{-1} = G(F)_\tau$, $\mathrm{sgn}_\sigma(g^{-1}\gamma g) = \mathrm{sgn}_\tau(\gamma)$, and $\dim\tau = \dim\sigma$. The volume factor in the summand is $|I(F)_\gamma|^{-1}|G(F)_\sigma|$, so we get

$$O_\gamma(\mathrm{sgn}_\sigma) = \sum_\tau |G(F)_\sigma||I(F)_\tau|^{-1}\mathrm{sgn}_\tau(\gamma), \quad \tau \in I(F)\backslash(I(F)\sigma) \cap \mathbb{F}(\gamma).$$

Hence

$$O_\gamma(f_{EP}) = \sum_{\sigma\in\mathbb{S}}(-1)^{\dim\sigma}\sum_\tau |I(F)_\tau|^{-1}\mathrm{sgn}_\tau(\gamma) = \sum_\rho |I(F)_\rho|^{-1}(-1)^{\dim\rho}$$

with $\rho = \tau(\gamma) \in I(F)\backslash\{\tau(\gamma) = \tau \cap \mathbb{B}(\gamma)\,;\, \tau \in \mathbb{F}(\gamma)\}$; thus ρ runs through a set of representatives for the $I(F)$-orbits in the set of facets of $\mathbb{B}(\gamma)$.

If $\mathbb{B}(\gamma)$ is empty, the orbital integrals $O_\gamma(f_{EP})$ is 0. This can happen only if γ is non-elliptic. If $\mathbb{B}(\gamma) \neq \varnothing$, then it is contractible [Ko80, Lemma 7.2], and it satisfies all the conditions of [Sr71, 3.3] relative to the group I. Note that Serre's condition (v) is a consequence of the semisimplicity of γ. Proposition 24 of [Sr71] asserts that $\left(\sum_\rho (-1)^{\dim \rho} |I(F)_\rho|^{-1}\right) \mathrm{d}i$ is the Euler-Poincaré measure on $I(F)$. If γ is non-elliptic, then the Euler-Poincaré measure on $I(F)$ is 0, which implies that our orbital integral is also 0. If γ is elliptic, we are taking $\mathrm{d}i$ to be the Euler-Poincaré measure on $I(F)$. This measure is nonzero. Hence $O_\gamma(f_{EP})$ is 1.

(2) At this point we proved the first and second sentences in the theorem for semisimple elements. We next show that the orbital integrals of f_{EP} are 0 for non-semisimple elements. Let γ be a semisimple element of $G(F)$. Let I be the connected component of the identity G_γ^0 in its centralizer G_γ in G. Recall the germ expansion for f_{EP} about γ:

$$O_t(f_{EP}) = \sum_{u \in U} \Gamma_u(t) O_{\gamma u}(f_{EP}).$$

It holds for all regular semisimple t in $G(F)$ that lie in $I(F)$ and are close enough to γ. For u in the set U of representatives for the conjugacy classes of unipotent elements in $I(F)$, we write Γ_u for the germ associated with u. We have $O_t(f_{EP}) = 1$ if t is regular elliptic, and $O_t(f_{EP}) = 0$ if t is regular and not elliptic. Theorem 15 and Corollary 2 of Lemma 20 of [HC78] then imply that there is a real number c with $O_t(f_{EP}) = c\Gamma_1(t)$. Hence $\sum_{u \in U} c_u \Gamma_u = 0$, where $c_u = O_{\gamma u}(f_{EP})$ if $u \neq 1$, and $c_1 = O_\gamma(f_{EP}) - c$. Then [HC78, Lemma 24] implies that $O_{\gamma u}(f_{EP}) = 0$ for all $u \neq 1$, and the first sentence of the theorem follows, as well as the third.

□

2.39 *Steinberg Representations*

The function f_{EP} gives a pseudo-coefficient of the Steinberg representation of $G(F)$. Let (π, V) be an irreducible admissible representation of $G(F)$. Denote by $H_e^i(G(F), V)$ the continuous cohomology groups of [BW80, X.5.1]. These are finite dimensional complex vector spaces, trivial except for finitely many values of i. In the p-adic case, we write $q(G)$ for the F-rank of the derived group G^{ss}, which is semisimple.

THEOREM. (a) *We have* $\operatorname{tr} \pi(f_{EP}) = \sum_i (-1)^i \dim H_e^i(G(F), V)$. *Moreover,* $(-1)^{q(G)} f_{EP}$ *is a pseudo-coefficient for the Steinberg representation of* $G(F)$.

(b) *Assume that G is simple and π is unitary. Then* $\operatorname{tr} \pi(f_{EP})$ *is* 0 *except in the following two cases. The trace of f_{EP} on the trivial representation of $G(F)$ is* 1. *The trace of f_{EP} on the Steinberg representation of $G(F)$ is* $(-1)^{q(G)}$.

PROOF. From the definition of f_{EP}, we have $\operatorname{tr} \pi(f_{EP}) = \sum_{\sigma \in \mathbb{S}} (-1)^{\dim \sigma} \dim V_\sigma$, where V_σ denotes the biggest subspace of V on which $G(F)_\sigma$ acts by the character sgn_σ. By [BW80, X.2.4, X.5.1], the continuous cohomology of V is the cohomology of a complex, denoted by $C^\bullet(Y; V)^G$ by Borel-Wallach, whose ith term is isomorphic to $\bigoplus_\sigma V_\sigma$. The sum is taken over the $\sigma \in \mathbb{S}$ with $\dim \sigma = i$. This gives the first statement of (a). The second statement of (a) follows from [BW80, XI.3.8], which asserts that the Steinberg representation of $G(F)$ is the only tempered representation of $G(F)$ which has nonzero continuous cohomology.

Statement (b) follows from [BW80, XI.3.9]. This says that when G is simple and τ is unitary, then $H^i_e(G(F), V)$ is zero unless $i = 0$ and V is the trivial representation, or $i = q(G)$ and V is the Steinberg representation. In both cases $H^i_e(G(F), V)$ is one-dimensional. □

3 Automorphic Forms

3.1 Test Functions

Let F be a global field. At each non-Archimedean place v, let $\mathcal{O}_v \subset F_v$ be the ring of integers. Fix a special maximal compact subgroup $K_v \subset G(F_v)$. At almost all v, we can and will take $K_v = G(\mathcal{O}_v)$ to be a hyperspecial maximal compact subgroup, see [Ti79].

Let $Z_{0\mathbb{A}}$ be a closed subgroup of $Z(\mathbb{A})$ such that $Z_{0\mathbb{A}}Z(F)$ is closed and $Z(\mathbb{A})/Z_{0\mathbb{A}}Z(F)$ is compact. Suppose that $Z_{0\mathbb{A}} = \prod_v Z_{0v}$, product over all places of F, with Z_{0v} a closed subgroup of $Z(F_v)$. Set $Z_0 = Z_{0\mathbb{A}} \cap G(F)$. Note that $Z_{0\mathbb{A}}$, Z_{0v}, and Z_0 are *not* assumed to be the points of an algebraic group. However, any algebraic subgroup of Z does satisfy these conditions.

Fix a unitary character ω of $Z_{0\mathbb{A}}/Z_0$ with local components ω_v, which are characters of Z_{0v}. Fix also a place v. Let $C_c^\infty(G(F_v), Z_0(F_v), \omega)$ be the space of complex valued functions on $G(F_v)$ which transform under $Z_0(F_v)$ by ω_v^{-1}, compactly supported modulo Z_{0v}, smooth if v is Archimedean and locally constant if v is non-Archimedean. To simplify the notation, we will denote this space by $C_c^\infty(G(F_v))$. We omit the place v from the notation for the rest of this section, so that F is a local field.

Fix a product measure $\mathrm{d}x = \lambda \otimes_v \mathrm{d}x_v$ on $G(\mathbb{A})/Z_{0\mathbb{A}}$ so that the product of volumes $|K_v/K_v \cap Z_{0v}|$ converges. Consider a function $f = \otimes_v f_v$ on $G(\mathbb{A})$, where $f_v \in C_c^\infty(G(F_v))$ for all v. Let f_v^0 denote the function, defined for each non-Archimedean place v and supported on $Z_{0v}K_v$ with value $|K_v/K_v \cap Z_{0v}|^{-1}$ on K_v. We require $f_v = f_v^0$ at almost all places. Let $C_c^\infty(G(\mathbb{A}))$ be the space of functions spanned by all such f.

3.2 Kernel

Let $L = L(G(F)\backslash G(\mathbb{A}))$ be the space of complex valued functions ϕ on $G(F)\backslash G(\mathbb{A})$ which transform under $Z_{0\mathbb{A}}$ by the unitary character ω and are slowly increasing, see [BJ79], on $G(F)Z_{0\mathbb{A}}\backslash G(\mathbb{A})$. Define also $L^2 = L^2(G(F)\backslash G(\mathbb{A}))$ to be the space of complex valued functions ϕ on $G(F)\backslash G(\mathbb{A})$ which transform under $Z_{0\mathbb{A}}$ by ω and are absolutely square-integrable on $G(F)Z_{0\mathbb{A}}\backslash G(\mathbb{A})$. Let r be the representation of $G(\mathbb{A})$ on $L(G(F)\backslash G(\mathbb{A}))$ or L^2 by right translation. Define the convolution operator $r(f\,\mathrm{d}x), f \in C_c^\infty(G(\mathbb{A}))$, on $L(G(F)\backslash G(\mathbb{A}))$ or L^2, by

$$(r(f\,\mathrm{d}x)\phi)(y) = \int_{G(\mathbb{A})/Z_{0\mathbb{A}}} f(x)\phi(yx)\,\mathrm{d}x.$$

This equals

$$\int_{G(\mathbb{A})/Z_{0\mathbb{A}}} f(y^{-1}x)\phi(x)\,\mathrm{d}x = \int_{G(F)\backslash G(\mathbb{A})/Z_{0\mathbb{A}}} \Big[\sum_{\gamma} f(y^{-1}\gamma x)\Big]\phi(x)\,\mathrm{d}x.$$

Hence it is an integral operator on $G(F)\backslash G(\mathbb{A})/Z_{0\mathbb{A}}$ with kernel

$$K(x,y) = K_f(x,y) = \sum_{\gamma\in G(F)/Z_0} f(x^{-1}\gamma y).$$

3.3 Admissible and Automorphic Representations

A complex representation of $G(F_v)$ is called *smooth* if the stabilizer $\mathrm{Stab}_{G(F_v)}(\xi) = \{g \in G(F_v);\ g\xi = \xi\}$ of any $\xi \in V$ is open in $G(F_v)$. It is called *admissible* if for all open $U \subset G(F_v)$ the subspace $V^U = \{\xi \in V;\ u\xi = \xi \text{ for all } u \in R\}$ of U-fixed vectors is finite dimensional. Theorem 3.25 of [BZ76] asserts that a smooth $G(F_v)$-module of finite length is admissible. Conversely, an admissible representation of $G(F_v)$ is smooth and of finite length.

An irreducible admissible $G(\mathbb{A})$-module (π, V) is the restricted direct product $\pi = \otimes_v \pi_v$ of irreducible $G(F_v)$-modules π_v on the spaces V_v, almost all of which are *unramified*. That is, almost all have a (unique up to scalar) nonzero K_v-fixed vector ξ_v^0. All π_v are admissible and unitarizable if π is unitarizable (a dense submodule is a unitary module). Thus $V = \cup_S \otimes_{v\in S} V_v \otimes \otimes_{v\notin S}\xi_v^0$, union over all finite sets S of valuations of F whose complement consists of v where π_v is unramified.

An irreducible unitary $G(\mathbb{A})$-module (π, V) is called *automorphic* if it is a subquotient of the $G(\mathbb{A})$-module $L(G(F)\backslash G(\mathbb{A}))$.

3.4 Convolution Operators

For an admissible representation (π, V), we define

$$\pi(f) = \pi(f\,\mathrm{d}x) = \int_{G(\mathbb{A})/Z_{0\mathbb{A}}} f(x)\pi(x)\,\mathrm{d}x$$

on V, with $\mathrm{d}x$ the measure above. The space V is spanned by vectors $\otimes_v \xi_v$ with ξ_v in the space of π_v and it is K_v-invariant for almost all v. For almost all v, the operator $\pi_v(f_v)$ is the projection on the space of K_v-fixed vectors, so has $\operatorname{tr}\pi_v(f_v) = 1$. Hence almost all factors in $\operatorname{tr}\pi(f) = \prod_v \operatorname{tr}\pi_v(f_v)$ are 1.

3.5 Cuspidal Spectrum

A function $\phi \in L(G(F)\backslash G(\mathbb{A}))$ is *cuspidal* if for any proper F-parabolic subgroup of G with unipotent radical N, the integral

$$\int_{N(F)\backslash N(\mathbb{A})} \phi(nx)\,\mathrm{d}n$$

is 0 for any $x \in G(\mathbb{A})$. Let r_0 be the restriction of r to the space $L_0 = L_0(G(F)\backslash G(\mathbb{A}))$ of cuspidal functions. A standard result, see, e.g., [AFOO], is that the $G(\mathbb{A})$-module $L_0(G(F)\backslash G(\mathbb{A}))$ is the direct sum of irreducible $G(\mathbb{A})$-modules π with finite multiplicities $m(\pi)$. The operator $r_0(f)$ is trace class, and

$$\operatorname{tr} r_0(f) = \sum_{\pi} m(\pi)\operatorname{tr}\pi(f). \tag{3.5.1}$$

The sum is over the equivalence classes of π in $L_0(G(F)\backslash G(\mathbb{A}))$. It converges absolutely and all π are unitary. Note that L_0 coincides with the analogous space of cusp forms in L^2.

3.6 Stable Conjugacy

The difference between the notions of conjugacy and stable conjugacy can be studied using Galois cohomology. See Serre [Sr94], Ribes [Ri99], Platonov-Rapinchuck [PR93], Tate [Ta66], Kottwitz [Ko86].

Elements x, x' of $G(F)$ are (resp. *stably*) *conjugate* if there exists y in $G(F)$ (resp. $G(\overline{F})$) such that $x' = \mathrm{Ad}(y)x = yxy^{-1}$. Here F can be local or global. Let $A(x/F)$ be the set of $y \in G(\overline{F})$ with $\mathrm{Ad}(y)x$ in $G(F)$. The conjugacy classes within the stable class of x are parametrized by $B(x/F) = G(F)\backslash A(x,F)/G_x$. The map

$$y \mapsto \{\tau \mapsto y_\tau = y^{-1}\tau(y)\,;\ \tau \in \mathrm{Gal}(\overline{F}/F)\}$$

is a bijection

$$B(x/F) \xrightarrow{\sim} \ker[H^1(F,G_x) \to H^1(F,G)].$$

Here $H^1(F,A) = H^1(\mathrm{Gal}(\overline{F}/F), A(\overline{F}))$. Thus, given x, any x' stably conjugate to x determines an element of $B(x,F)$, and x' is conjugate to x if and only if it determines the identity in $H^1(F,G_x)$. If F is global, define $B(x/\mathbb{A})$ (resp. $B(x/\mathbb{A}^w)$) to be the pointed direct sum of $B(x/F_v)$ for all v (resp. $v \neq w$).

4 Trace Formula

4.1 Projection to Cuspidal Spectrum

The notation will be as in Section 3. Let u be a place of F. The function $f_u \in C_c^\infty(G(F_u))$ is called *cuspidal* if for any proper F_u-parabolic of $G(F_u)$ with unipotent radical $N(F_u)$, we have

$$\int_{N(F_u)} f_u(xny)\,\mathrm{d}n = 0$$

for any x, y in $G(F_u)$.

PROPOSITION. *If f has a cuspidal component at u, then $r(f)$ vanishes on the $G(\mathbb{A})$-invariant complement of $L_0(G(F)\backslash G(\mathbb{A}))$ in $L(G(F)\backslash G(\mathbb{A}))$. In particular, we have* $\mathrm{tr}\, r(f) = \mathrm{tr}\, r_0(f)$ *for such f.*

PROOF. Set $PG = G(\mathbb{A})/Z_{0\mathbb{A}}$. Then

$$\int_{N(F)\backslash N(\mathbb{A})} (r(f)\phi)(nx)\,\mathrm{d}n = \int_{N(F)\backslash N(\mathbb{A})} \int_{PG} f(y)\phi(nxy)\,\mathrm{d}y\,\mathrm{d}n$$

$$= \int_{N(F)\backslash N(\mathbb{A})} \int_{N(F)\backslash PG} \sum_{\gamma\in N(F)} f(x^{-1}n^{-1}\gamma y)\phi(y)\,\mathrm{d}y\,\mathrm{d}n$$

$$= \int_{N(F)\backslash PG} \left(\int_{N(F)\backslash N(\mathbb{A})} \sum_{\gamma\in N(F)} f(x^{-1}n^{-1}\gamma y)\,\mathrm{d}n \right) \phi(y)\,\mathrm{d}y$$

$$= \int_{N(F)\backslash PG} \left(\int_{N(\mathbb{A})} f(x^{-1}ny)\,\mathrm{d}n \right) \phi(y)\,\mathrm{d}y = 0.$$

The change in order of integration is justified by absolute convergence, since f is compactly supported on PG and $N(F)\backslash N(\mathbb{A})$ is compact. Now $r(f)$ preserves $L_0(G(F)\backslash G(\mathbb{A}))$ and its complement. So if ϕ lies in the complement, then $r(f)\phi \in L_0(G(F)\backslash G(\mathbb{A}))$ implies $r(f)\phi$ is 0. Hence $\mathrm{Im}\, r(f) = \mathrm{Im}\, r_0(f)$, and $\mathrm{tr}\, r(f) = \mathrm{tr}\, r_0(f)$. $\square$

Let F be a global field of characteristic 0.

PROPOSITION 4.2. *Let $C = \prod_v C_v$ be a compact subset of $G(\mathbb{A})$ with $C_v = G(\mathcal{O}_v)$ for almost all v. Then there are only finitely many regular conjugacy classes in $G(\mathbb{A})$ with a representative in $G(F)$ which intersect C nontrivially.*

PROOF. Fix a faithful representation of G in $\mathrm{GL}(n)$ over F, for some n. Define a map $G(\mathbb{A}) \to \mathbb{A}^{n-1} \times \mathbb{A}^{\times}$ by sending x to the ordered set of coefficients of its characteristic polynomial. The image of C is compact; that of $G(F)$ is discrete. Thus there are only finitely many semisimple conjugacy classes in $\mathrm{GL}(n, \mathbb{A})$ with a representative in $G(F)$ which intersect C nontrivially. Two semisimple conjugacy classes in $G(F)$ which are conjugate in $\mathrm{GL}(n, \mathbb{A})$ are conjugate in $\mathrm{GL}(n, F)$. The theorem of [St74, p. 102] asserts that a conjugacy class of $\mathrm{GL}(n, \overline{F})$ intersects $G(\overline{F})$ in only finitely many conjugacy classes of $G(\overline{F})$. By definition, a $G(\overline{F})$-conjugacy class with a representative in $G(F)$ is a stable conjugacy class. If γ_G is a stable conjugacy class in $G(F)$, then there exists a finite set V of places of F such that γ_G intersects $G(\mathcal{O}_v)$ at most at one conjugacy class for v outside V. This γ_G is contained in a stable conjugacy class $\gamma_{\mathbb{A}}$, where $\gamma_{\mathbb{A}}$ is the product over all v of stable conjugacy classes γ_v in $G(F_v)$. Since γ_v consists of finitely many classes for all v, the class γ_G consists of only finitely many conjugacy classes in $G(F)$ which intersect C. □

4.3 Geometric Side

Suppose f is as in Proposition 4.1. Then $r(f)$ is a trace class operator, whose trace is the integral of the kernel over the diagonal. Assume in addition f vanishes on the conjugacy class in $G(\mathbb{A})$ of any $\gamma \in G(F)$ which is not elliptic regular. Note that if γ is elliptic regular, then $\left|G_\gamma(\mathbb{A})Z(\mathbb{A})/G_\gamma(F)Z_{0\mathbb{A}}\right|$ is compact and its volume is finite. But this fails for other elements γ. Put $\widetilde{G}_\gamma$ for the centralizer of γ in G/Z. Put $\{\gamma\}$ for the set of conjugacy classes of elliptic regular elements in $G(F)$. By the conditions on f, the sum below ranges only over $\{\gamma\}$. Thus

$$\begin{aligned}
\operatorname{tr} r(f) &= \int_{G(\mathbb{A})/Z_{0\mathbb{A}}G(F)} \sum_{\gamma \in G(F)/Z_0} f(x\gamma x^{-1})\,\mathrm{d}x \\
&= \sum_{\{\gamma\}} \int_{G(\mathbb{A})/Z_{0\mathbb{A}}\widetilde{G}_\gamma(F)} f(x\gamma x^{-1})\,\mathrm{d}x \qquad (4.3.1)\\
&= \sum_{\{\gamma\}} \left(\left|G_\gamma(\mathbb{A})/G_\gamma(F)Z_{0\mathbb{A}}\right| / [\widetilde{G}_\gamma(F) : G_\gamma(F)]\right) \Phi(\gamma, f).
\end{aligned}$$

Each integral in (4.3.1) is absolutely convergent. The sum is finite by Proposition 4.2. Note that the integral in the middle over $G(\mathbb{A})/Z_{0\mathbb{A}}\widetilde{G}_\gamma(F)$ can be written as $1/[\widetilde{G}_\gamma(F) : G_\gamma(F)]$ times the integral over $G(\mathbb{A})/Z_{0\mathbb{A}}G_\gamma(F) = G(\mathbb{A})/G_\gamma(\mathbb{A}) \times G_\gamma(\mathbb{A})/G_\gamma(F)Z_{0\mathbb{A}}$.

The following is our simple trace formula.

COROLLARY 4.4. *Suppose that u, u', and u'' are places of F with $u \neq u'$, with f_u a cuspidal function, the orbital integral of $f_{u'}$ vanishes on the regular non-elliptic set of $G(F_{u'})$, and $f_{u''}$ vanishes on the singular set. Then* (3.5.1) *equals* (4.3.1)*, and the sum in* (4.3.1) *is finite.*

PROOF. Proposition 4.2 implies that if $f(x\gamma x^{-1}) \neq 0$ for $x \in G(\mathbb{A})$, then γ lies in one of finitely many regular conjugacy classes. Suppose γ lies in such a regular non-elliptic class. Then the invariant distribution $\Phi(\gamma) : h \mapsto \Phi(\gamma, h)$ on $C_c^\infty(G(F_{u'}))$ vanishes at $f_{u'}$. Let $C_0^\infty(G(F_{u'}))$ be the span of functions $h - h^g$, with $h \in C_c^\infty(G(F_{u'}))$ and $g \in G(F_{u'})$. Denote by $C_0^\infty(G(F_{u'}))_\gamma$ and $C_c^\infty(G(F_{u'}))_\gamma$ the restrictions of these sets to the orbit of γ. The uniqueness of the $G(F_{u'})$-invariant measure on the orbit of γ means that any distribution on $C_c^\infty(G(F_{u'}))_\gamma / C_0^\infty(G(F_{u'}))_\gamma$ is a scalar multiple of $\Phi(\gamma)$. Thus $C_c^\infty(G(F_{u'}))_\gamma / C_0^\infty(G(F_{u'}))_\gamma$ is one-dimensional, and $C_0^\infty(G(F_{u'}))$ is the kernel of $\Phi(\gamma)$. Hence there are $h_i \in C_c^\infty(G(F_{u'}))$ and $g_i \in G(F_{u'})$ such that

$$f_{u'} = \sum_i (h_i - h_i^{g_i})$$

on the orbit of γ. We may choose h_i to be zero outside a small neighborhood of γ.

Replacing in f the component $f_{u'}$ by $f_{u'} - \sum_i (h_i - h_i^{g_i})$ will not change side (3.5.1) of the trace formula, since $\operatorname{tr} \pi(h^g) = \operatorname{tr} \pi(h)$. On the other hand, the function f now vanishes on the orbit of γ, but its values on other conjugacy classes with a rational representative do not change. Consequently we may assume that if $f(x\gamma x^{-1}) \neq 0$, then γ is elliptic regular, and the corollary follows from 4.3. □

REMARK. (1) The fact that $f_{u'}$ can be *any* function whose orbital integrals vanish on the regular non-elliptic set of $G(F_{u'})$, and it is not assumed that it is supported on the elliptic regular set, is fundamental for the primary applications of this chapter.

(2) Cuspidal functions are obtained as linear combinations of matrix coefficients of cuspidal representations of local groups.

We give next a different form of the corollary. It is used in the next section. Also it provides an example of a simple trace formula derived from the full (noninvariant) trace formula of Chapter 3 below. By the *F-rank* of G, we mean the dimension of the quotient of a maximal F-split torus in G by the F-split component of a maximal F-torus in Z.

COROLLARY 4.5. *Let $f = \otimes_v f_v$, $f_v \in C_c^\infty(G(F_v))$, be a function whose components at u_i, $0 \leq i \leq r$ with $r \geq \operatorname{rank} G$, are supported on the elliptic regular set of $G(F_{u_i})$, and f_{u_i} is zero on the x in $G(F_{u_i})$ for which there are g in $G(F_{u_i})$ and $z \neq 1$ in $Z(F_{u_i})$ with $gxg^{-1} = zx$. Then*

$$\sum_{\{\gamma\}} \left| G_\gamma(\mathbb{A}) Z(\mathbb{A}) / G_\gamma(F) Z_0(\mathbb{A}) \right| \Phi(\gamma, f) = \sum_\pi c_\pi \operatorname{tr} \pi(f). \tag{4.5.1}$$

The sum over $\{\gamma\}$ is finite. It ranges over the conjugacy classes of regular x in $G(F)$ which are elliptic at the u_i. The sum over π is absolutely convergent. It ranges over automorphic $G(\mathbb{A})$-modules. The c_π are complex numbers.

PROOF. The assumption at u_1 alone implies that the sum $\sum J_{\mathfrak{o}}(f)$ is equal to our sum over $\{\gamma\}$. It is finite by Proposition 4.2. The sum $\sum J_\chi(f)$ consists of integrals of logarithmic derivatives of intertwining operators acting on induced representations. As the degrees of the derivatives are at most rank G, our $r+1$ assumptions imply the vanishing of all integrals. There remains a discrete sum of irreducible representations π whose components at u_i are elliptic. The c_π are integral and positive for cuspidal π. □

5 Density

To illustrate the power and typical usage of the simple trace formula, we give here a first application, the following density theorem, due to Kazhdan [Ka86.1, Appendix]. We use various notions and facts to be used in later sections, but we prefer to give the application first, begging the reader to take on faith several standard facts, to motivate by means of this example the value of the simple trace formula. The density theorem was conjectured by Harish-Chandra, who was looking for a purely local proof. The usage of the trace formula permits playing local against global techniques, to learn more about both. We shall have two forms of the density theorem. In the first, all (irreducible, admissible) π appear. In the second, the assumption is made only for tempered π.

Let F be a local field of characteristic 0 and G a reductive group over F, as in Section 2.

PROPOSITION 5.1. *Let f be a function in $C_c^\infty(G(F))$ such that $\operatorname{tr}\pi(f) = 0$ for all admissible irreducible $G(F)$-modules π. Then $\Phi(x,f) = 0$ for all regular x in $G(F)$.*

REMARK 5.2. Consequently $J^\infty(G(F))$ (defined in Subsection 2.6 to be the space of $f \in C_c^\infty(G(F))$ such that $I(g,f)$ is 0 for all regular $g \in G(F)$) consists of all f in $C_c^\infty(G(F))$ such that $f_G(\pi) = \operatorname{tr}\pi(f)$ is 0 for every $G(F)$-module π.

PROOF. In the proof, we denote F, G, and f by F', G', and f'. Due to the integration formula $I(x,f') = I^M(x,f'_N)$ (Section 2), we may assume that there exists an elliptic regular element x_0 in $G'(F)$ with $\Phi(x_0,f') \neq 0$ and that its centralizer T' in G' is an elliptic torus over F' which splits over the Galois extension F'' over F'.

We prove the following lemma. It is needed to apply global methods to the local problem.

LEMMA 5.3. *Let F' be a local field, G' a reductive group over F', T' a (maximal) torus of G' over F', and F'' a Galois field extension of F' such that T' and G' split over F''. Then there exists a Galois extension E/F of global fields such that at a set of places w of F of cardinality at least two we have $F_w \simeq F'$, $E_w = E \otimes_F F_w \simeq F''$,*

$\mathrm{Gal}(E_w/F_w) \simeq \Gamma$, *where* $\Gamma = \mathrm{Gal}(E/F)$, *and a pair* (T, G) *consisting of a reductive group* G *and a torus* T *over* F *with* $G(F_w) \simeq G'(F_w)$, $T(F_w) \simeq T'(F_w)$ *for all* w *such that* $G(F)$ *is dense in* $G(F_w)$ *and* $T(F)$ *is dense in* $T(F_w)$.

PROOF. By Hensel's Lemma, it is clear that there exist E and F with the required properties. Once (T, G) is found, since the set of w has cardinality at least two, it follows from [CF68, p. 361] that $(T(F), G(F))$ is dense in $(T(F_w), G(F_w))$. Now, it is well known (see [Sr94, p. III-1]) that if K/k is a Galois field extension, A is a torus in an algebraic group H, both defined over k, then the set of K/k-forms of (A, H) is parametrized by the first cohomology group $H^1(\mathrm{Gal}(K/k), \mathrm{Aut}_K(A, H))$ of $\mathrm{Gal}(K/k)$ in the group $\mathrm{Aut}_K(A, H)$ of automorphisms of the pair (A, H). This group $\mathrm{Aut}_K(A, H)$ consists of automorphisms of H over K which map A to A. The group $A(K)$ of K-points injects as a normal subgroup of $\mathrm{Aut}_K(A, H)$. Denote the quotient by W_K.

Let (A, H) be a pair consisting of a reductive group H over F with $H(E_w) \simeq G'(E_w)$ and a torus A of H over F with $A(E_w) \simeq T'(E_w)$. We have

$$\begin{tikzcd} 0 \arrow[r] & A(E) \arrow[r] \arrow[d, hook] & \mathrm{Aut}_E(A,H) \arrow[r] \arrow[d, hook] & W_E \arrow[r] \arrow[d, "\wr"] & 0 \\ 0 \arrow[r] & A(E_w) \arrow[r] & \mathrm{Aut}_{E_w}(A,H) \arrow[r] & W_{E_w} \arrow[r] & 0 \end{tikzcd}$$

Since $A(E)$ is normal in $\mathrm{Aut}_E(A, H)$, by [Sr94, Prop. 38, p. I-6], we have the associated commutative diagram

$$\begin{tikzcd} W_E^\Gamma \arrow[r] \arrow[d, "\wr"] & H^1(\Gamma, A(E)) \arrow[r] \arrow[d, "\psi"] & H^1(\Gamma, \mathrm{Aut}_E(A,H)) \arrow[r, "p"] \arrow[d, "\phi"] & H^1(\Gamma, W_E) \arrow[d, "\wr"] \\ W_{E_w}^\Gamma \arrow[r] & H^1(\Gamma, A(E_w)) \arrow[r] & H^1(\Gamma, \mathrm{Aut}_{E_w}(A,H)) \arrow[r, "p_w"] & H^1(\Gamma, W_{E_w}). \end{tikzcd}$$

The Tate-Nakayama theory [Ta66] and [Ko86] implies that ψ is surjective.

The pair (T', G') is determined by an element α_w in $H^1(\Gamma, \mathrm{Aut}_{E_w}(A, H))$. To produce a pair (T, G) as required, we have to find an element α in $H^1(\Gamma, \mathrm{Aut}_E(A, H))$ whose image under ϕ is α_w. Put $\beta = p_w(\alpha_w)$. It can be regarded as an element of $H^1(\Gamma, W_E)$. As in [Sr94], denote by ${}_{\alpha_w}A$ the torus determined by the cocycle α_w. Since it depends only on β, we denote ${}_{\alpha_w}A$ by ${}_\beta A$. For each γ in $H^1(\Gamma, W_E)$, there exists an element $\Delta(\gamma)$ in $H^2(\Gamma, {}_\gamma A(E))$ (constructed in [Sr94, p. I-70]), such that γ lies in the image of p_w if and only if $\Delta_w(\gamma_w) = 0$ (see [Sr94, Prop. 4, p. I-70]). Also, for each γ_w in $H^1(\Gamma, W_E)$, there is $\Delta_w(\gamma_w)$ in $H^2(\Gamma, {}_{\gamma_w}A(E_w))$ such that γ_w lies in the image of p_w if and only if $\Delta_w(\gamma_w) = 0$. The Tate-Nakayama theory

[Ta66] implies that $H^2(\Gamma,_\beta A(E))$ and $H^2(\Gamma,_\beta A(E_w))$ are isomorphic as groups. By their construction (in [Sr94, p. I-70]), $\Delta = \Delta_w$. Since $\beta = p_w(\alpha_w)$, we have $\Delta_w(\beta) = 0$; hence $\Delta(\beta) = 0$, and β lies in the image of p. By [Sr94, Cor. 2, p. I-67], the inverse image by p_w of β is the quotient of $H^1(\Gamma,_\beta A(E_w))$ by Im $W^\Gamma_{E_w}$, and $p^{-1}(\beta)$ is $H^1(\Gamma,_\beta A(E))/\,\mathrm{Im}\; W^\Gamma_E$. The Tate-Nakayama theory [Ta66] implies that the map $H^1(\Gamma,_\beta A(E)) \to H^1(\Gamma,_\beta A(E_w))$ is surjective. Hence there is α in $H^1(\Gamma, \mathrm{Aut}_E(A,H))$ with $\phi(\alpha) = \alpha_w$. The pair (T,G) determined by α has the required properties, and the lemma follows. □

Let E/F be a global field extension and (T,G) a pair defined over F with the properties of the lemma. In these notations $T_w = T'$ is the centralizer $Z_{G(F_w)}(x_0)$ of x_0 in $G(F_w)$. We have $T = T(F)$ dense in $T(F_w)$ and similarly for $G(F)$ in $G(F_w)$. Hence the centralizer $Z_{G(F_w)}(T(F_w))$ of $T(F_w)$ in $G(F_w)$ is equal to the centralizer $Z_{G(F_w)}(T(F))$ of $T(F)$ in $G(F_w)$ and contains the centralizer $Z_{G(F)}(T(F))$ of $T(F)$ in $G(F)$ as a dense subset. Choose x in $Z_{G(F)}(T(F))$ sufficiently near x_0 so that $T(F)$ is $Z_{G(F)}(x)$ and $\Phi(x,f_w) \neq 0$. Here we denote our local function f' by f_w.

As in [Ka86.1, Appendix], we use an argument from Galois cohomology. Note that T is a maximal torus (in the usual sense) in G. The set of $G(F_v)$-conjugacy classes in $G(F_v)$ which are contained in the $G(\overline{F}_v)$-conjugacy class of γ is in bijective correspondence (see Subsection 3.6) with a subset of

$$H^1(F_v, T) = H^1(\mathrm{Gal}(\overline{F}_v/F_v), T(\overline{F}_v)).$$

A similar assertion holds for $G(F)$-conjugacy classes. Let E/F be a finite Galois extension which is unramified outside a sufficiently large set $V = \{v_1, v_2, \ldots, v_k\}$ of places of F and over which T splits. Then $H^1(F_v,T)$ equals $H^1(\mathrm{Gal}(E_w/F_v), T(E_w))$, and Tate-Nakayama theory (see [Ta66] and [Ko86]) provides an isomorphism between this group and

$$\{\lambda^\vee \in X_*(T)\,;\, \mathrm{Norm}_{E_w/F_v}(\lambda^\vee) = 0\}/\{\lambda^\vee - \sigma\lambda^\vee\,;\, \lambda^\vee \in X_*(T), \sigma \in \mathrm{Gal}(E_w/F_v)\}, \tag{5.3.1}$$

and an isomorphism between

$$H^1(\mathrm{Gal}(E/F), T(\mathbb{A}_E)/T(E))$$

and

$$\{\lambda^\vee \in X_*(T)\,;\, \mathrm{Norm}_{E/F}(\lambda^\vee) = 0\}/\{\lambda^\vee - \sigma\lambda^\vee\,;\, \lambda^\vee \in X_*(T), \sigma \in \mathrm{Gal}(E/F)\}. \tag{5.3.2}$$

Here w stands for a fixed valuation on E which lies above a given v. Moreover, there is an exact sequence

$$H^1(\mathrm{Gal}(E/F), T(E)) \to \oplus_v H^1(\mathrm{Gal}(E_w/F_v), T(E_w)) \to H^1(\mathrm{Gal}(E/F), T(\mathbb{A}_E)/T(E)).$$

The first map is compatible with the embedding of $G(F)$-conjugacy classes into $\prod_v G(F_v)$, and the second arrow is given by the natural map

$$\oplus_v \lambda_v \mapsto \sum_v \lambda_v$$

from the direct sum of modules (5.3.1) into (5.3.2). Now, consider the conjugacy class of x. Any $x' \in G(F)$ with $\Phi(x',f) \neq 0$ maps to an element $\oplus_v \lambda_v$ such that $\sum_v \lambda_v = 0$. If v is one of the valuations $v_2, \ldots, v_k$, then x' is $G(F_v)$-conjugate to x, so that $\lambda_v = 0$. If v lies outside V, then x' is $G(F_v)$-conjugate to an element in K_v^G. Since (G,T) is unramified at v, we again have $\lambda_v = 0$ [Ko86, Proposition 7.1]. It follows that $\lambda_{v_1} = 0$. In other words, x' is $G(F_{v_1})$-conjugate to γ.

Let us rephrase this argument. The Tate-Nakayama theory [Ta66] and [Ko86] implies that the natural homomorphism from $H^1(F,T)$ to $H^1(\mathbb{A}^w,T)$ is an isomorphism, where $T = Z_G(x)$ is a torus and $H^1(\mathbb{A}^w,T)$ is the pointed direct sum of the groups $H^1(F_v,T)$ over all places $v \neq w$. If x' is an element of $G(F)$ which is stably conjugate to x in $G(F_v)$ for some place v, namely, x and x' are conjugate on $G(\overline{F}_v)$, then they are conjugate in $G(F')$ where F' is a finite extension of F, and hence in $G(\overline{F})$. Consequently they are stably conjugate in $G(F)$. If x' is an element of $G(F)$ which is conjugate to x in $G(F_v)$ for all $v \neq w$, then it determines the identity element in $H^1(\mathbb{A}^w,T)$, hence in $H^1(F,T)$, and hence it is conjugate to x.

Let V be a finite set of place of F where T is elliptic, of cardinality larger than the rank of G, not including the place w of the proposition. At each v in V, choose f_v in $C_c^\infty(G(F_v))$ which is supported on the elliptic regular set of $G(F_v)$ and with $\Phi(x,f_v) \neq 0$. Choose f in $C_c^\infty(G(\mathbb{A}))$ whose components at v in V are those chosen above and whose component at w is the function of the proposition. As noted in Proposition 4.2 there are only finitely many conjugacy classes in $G(\mathbb{A})$ with representative x' in $G(F)$, necessarily elliptic regular, with $\Phi(x',f) \neq 0$. We can replace finitely many of the components f_v (for $v \neq w$) of f by their product with the characteristic function of a small open and closed neighborhood of the orbit of x in $G(F_v)$ to assure that if $\Phi(x',f) \neq 0$ for x' in $G(F)$, then x' is conjugate to x in $G(F_v)$ for all $v \neq w$. Consequently, if $\Phi(y,f) \neq 0$ for y in $G(F)$, then it is conjugate to x.

We can now apply the trace formula identity (4.5.1) of Corollary 4.5 since f is chosen to satisfy the hypotheses of this corollary. The assumption of the proposition implies that the right side of (4.5.1) is equal to 0, since $\operatorname{tr}\pi(f) = 0$ for all π, while the left side of (4.5.1) is a nonzero scalar multiple of $\Phi(x,f)$. Since $\Phi(x,f_v) \neq 0$ for all $v \neq w$ by the choice of f_v, we conclude $\Phi(x,f_w) = 0$, as required. $\square$

REMARK 5.4. Kazhdan proved the Proposition in [Ka86.1, Appendix] using only the simple form of the trace formula given in Corollary 4.4. To be able to use it, he produced a cuspidal function f_u with $\Phi(x,f_u) \neq 0$ for the given x in G_u, so that $\Phi(x',f_u) = 0$ for any x' in G_u which is stably conjugate but not conjugate to x in G_u. This construction holds in the non-twisted case that we consider here, as well as in the case of base change, at a place which splits. However, in other twisted cases, it is more difficult to construct cuspidal functions. For example, there are no twisted-invariant cuspidal irreducible representations of $G_u = \mathrm{PGL}(3, F_u)$ if F_u has odd residual characteristic and the twisting is given by $x \mapsto J\,{}^t x^{-1} J$, $J = \text{antidiagonal}(1, -1, 1)$.

DEFINITION 5.5. *An irreducible admissible representation π of a local group G on a complex space V is called* unitarizable *if V makes a dense subspace in a unitary representation V'. Then V' is a Hilbert space, with inner product $\langle .,. \rangle$. A* matrix coefficient *of such π is a function on G of the form $f(g) = \langle \pi(g)\xi, \xi' \rangle$, where ξ, ξ' are nonzero vectors in V. Then π is called* square-integrable *if $|f|^2$ is integrable on G/Z. Since π is irreducible, by Schur's Lemma* ([BZ76]), *it has a central character, ω_π, which is unitary since π is unitarizable. For example, cuspidal and Steinberg representations are square-integrable. A π is called* tempered *if it is a constituent, necessarily a subrepresentation, of a representation $i_P(\sigma) = \mathrm{ind}_P^G(\delta_P^{1/2}\sigma)$, normalizedly induced (using $\delta_P^{1/2}$, see Subsection* 6.7 *below) from a square-integrable representation σ of the Levi factor M of a parabolic subgroup $P = MN$ of G. In particular, square-integrable representations are tempered.*

PROPOSITION 5.6. *For f in $C_c^\infty(G(F))$ with $\mathrm{tr}\,\pi(f) = 0$ for all tempered $G(F)$-modules π, we have $\Phi(x, f) = 0$ for all regular x in $G(F)$.*

REMARK 5.7. (i) Theorem 10 of [HC78] asserts that $I(f)$ is uniquely determined by its values on the regular set. Using this, the propositions imply that $\Phi(x, f) = 0$ for *all* x in $G(F)$.

(ii) The propositions are proven for a local field F of characteristic 0. They also hold for a local field F of positive characteristic by virtue of Theorem A of [Ka86.1]. Moreover, [Ka86.1, Theorem B] implies that if $\mathrm{tr}\,\pi(f) = 0$ for all $G(F)$-modules π, then f lies in the linear span of the commutators $[f_1, h_1] = f_1 * h_1 - h_1 * f_1$, where f_1 and h_1 lie in $C_c^\infty(G(F))$.

PROOF. As in Section 2, a minimal parabolic subgroup $P_0 = M_0 N_0$ is fixed. Here $P = MN$ denotes a parabolic subgroup containing P_0 such that M contains M_0. Let $A = A_M$ be the split component in the center of M. Let $\mathfrak{a} = \mathfrak{a}_P = \mathrm{Hom}(X(M)_\mathbb{Q}, \mathbb{R}) = \mathrm{Hom}(X(A)_\mathbb{Q}, \mathbb{R})$ where $X(M)_\mathbb{Q}$, $X(A)_\mathbb{Q}$ are the groups of rational characters of M, A defined over $\mathbb{Q}$ and $H : M \to \mathfrak{a}$ is the homomorphism defined by $\langle H(m), \chi \rangle = \log|\chi(m)|$ for all χ in $X(M)_\mathbb{Q}$. If τ is an irreducible $M(F)$-module with central character ω_τ, define λ_τ in $\mathfrak{a}^* = X(M)_\mathbb{Q} \otimes \mathbb{R}$ by $\langle \lambda_\tau, H(m) \rangle = \log|\omega_\tau(m)|$. This τ is called *positive* if $\langle \lambda_\tau, \alpha \rangle$ is positive for every root α of A in N. It is called *essentially tempered* if $\tau \otimes \chi$ is tempered for some χ in $X(M)_\mathbb{Q}$. The classification theorem of [BW80, XI, (2.11)] (or [La88, Si80]) used negative instead of positive, and quotient instead of subrepresentations. An equivalent formulation, established by Zelevinsky [Ze80] for $\mathrm{GL}(n)$, asserts then the following.

(i) If τ is essentially tempered and positive, then the unitarily induced $G(F)$-module $i_P(\tau)$ has a unique irreducible submodule $J_P(\tau)$.
(ii) Any irreducible $G(F)$-module is so obtained.
(iii) $J_P(\tau)$ is equivalent to $J_{P'}(\tau')$ if and only if $P = P'$ and τ is equivalent to τ'.

A $G(F)$-module is called *standard* if it is equivalent to $i_P(\tau)$ with a positive $M(F)$-module τ. By virtue of the relation $\operatorname{tr}(i_P(\tau))(f) = \operatorname{tr}\tau(f_N)$ (which follows from a standard computation of a character of an induced representations), the fact that $\operatorname{tr}\tau(f_N) = 0$ if and only if $\operatorname{tr}(\tau\otimes\chi)(f_N) = 0$ for any χ in $X(M)_{\mathbb{Q}}$, and the relation $I(m,f) = I^M(m,f_N)$ of orbital integrals for m in $M(F)$ regular in G, the proposition follows at once from Proposition 5.1 and the following lemma.

Let $R_{\mathbb{Z}}(G(F))$ be the Grothendieck (free abelian) group generated by $\operatorname{Irr}(G(F))$, the set of equivalence classes of (admissible) irreducible $G(F)$-modules. Put also $R(G(F)) = R_{\mathbb{Z}}(G(F)) \otimes \mathbb{C}$.

LEMMA 5.8. *The set of standard $G(F)$-modules is a basis of $R(G(F))$ over $\mathbb{Z}$.*

PROOF. Given an irreducible $G(F)$-module π, it is equivalent to $J_P(\tau)$ for some pair (P,τ). If π' is a submodule of $i_P(\tau)$ inequivalent to π, and $\pi' = J_{P'}(\tau')$, then $\lambda_{\tau'} < \lambda_\tau$ for the order $<$ on $\mathfrak{a}^*$ by [BW80, XI, (2.13)]. By [BZ77], π' and π have the same cuspidal datum (L,ε), consisting of a Levi subgroup L and an irreducible $L(F)$-module ε whose modules $\varepsilon_U = r_{L',U}\varepsilon$ of coinvariants (see Section 7) with respect to any parabolic subgroup $L'U$ of L are zero in the Grothendieck group $R(L')$ of L'. Hence π' lies in a fixed finite set, and by induction on λ_π, we may assume that each such π' is a linear combination over $\mathbb{Z}$ of standard $G(F)$-modules. Consequently $\pi = J_P(\tau) = i_P(\tau) - \sum\pi'$ also lies in the span of the standard $G(F)$-modules.

It remains to show that standard modules are linearly independent. Fixing a cuspidal datum (M,τ), it is shown above that all irreducible $G(F)$-modules attached to (M,τ) are linear combinations of standard $G(F)$-modules attached to (M,τ), and we obtain a (finite, square) unipotent matrix. Since irreducible $G(F)$-modules are linearly independent over $\mathbb{C}$, the standard $G(F)$-modules are linearly independent over $\mathbb{C}$, and the lemma follows. □

The proposition now follows. □

6 Characters

6.1 Invariant Distributions

Let F be a non-Archimedean local field of characteristic 0. We now recall some of the fundamental results of Howe [Ho74], Harish-Chandra [HC78], and Kazhdan [Ka86.1] about Fourier transforms of invariant distributions, characters, and orbital integrals. We will not reproduce the proofs. Some of the proofs, in particular those of [Ka86.1], rely on results from the next two sections. However, their statements are independent of those results. We prefer to record these fundamental statements first, as they clarify the relationship between characters and orbital integrals, and delay the study of induction and restriction.

Fix an F-valued symmetric non-degenerate $G(F)$-invariant bilinear form B on the Lie algebra $\mathfrak{g}$ of $G(F)$, a nontrivial additive character ψ of F and a Haar measure $\mathrm{d}X$ on $\mathfrak{g}$. Let $C_c^\infty(\mathfrak{g})$ denote the space of locally constant compactly supported functions on $\mathfrak{g}$. The Fourier transform $\phi \mapsto \widehat{\phi}$, where

$$\widehat{\phi}(X) = \int_{\mathfrak{g}} \psi(B(X,Y))\phi(Y)\,\mathrm{d}X,$$

is a linear bijection of $C_c^\infty(\mathfrak{g})$ onto itself. A *distribution* T on $\mathfrak{g}$ is a linear complex-valued function on $C_c^\infty(\mathfrak{g})$. Its Fourier transform $\widehat{T}$ is defined by $\widehat{T}(\phi) = -T(\widehat{\phi})$. For x in $G(F)$, put $\phi^x(X) = \phi(\mathrm{Ad}(x)X)$ and ${}^xT(\phi) = T(\phi^x)$. A distribution T is *invariant* if ${}^xT = T$ for all x in $G(F)$.

A distribution T *vanishes* on an open set U if it attains the value 0 at any $\phi \in C_c^\infty(\mathfrak{g})$ which is supported on U. If T vanishes on a family of open subsets U_α, it vanishes on $\cup_\alpha U_\alpha$, since ϕ is compactly supported. The *support* of T is the complement of the largest open set on which T vanishes. Given a set ω in $\mathfrak{g}$, let $J(\omega)$ be the space of all invariant distributions on $\mathfrak{g}$ which are supported on the closure of $\mathrm{Ad}(G)\omega$.

PROPOSITION 6.2. *If ω is compact and T lies in $J(\omega)$, then there exists a locally integrable function F on $\mathfrak{g}$ with*

$$T(\phi) = \int_{\mathfrak{g}} F(X)\phi(X)\,dX$$

for all ϕ in $C_c^\infty(\mathfrak{g})$.

PROOF. See [HC78, Theorem 3]. □

6.3 Measures on Nilpotent Orbits

Let G_X be the centralizer of X in G. Let $\mathrm{d}x$ be the unique (up to scalar) $G(F)$-invariant measure on the homogeneous space $G(F)/G_X(F)$. By Theorem 2.32 of Deligne and Rao [Ra72], the integral

$$\mu_{\mathcal{O}}(\phi) = \int_{G(F)/G_X(F)} \phi(\mathrm{Ad}(x)X)\,\mathrm{d}x$$

is well defined for $\phi \in C_c^\infty(\mathfrak{g})$. This is clear for regular X, whose orbit is closed, but not for unipotent orbits. It depends only on $\mathrm{d}x$ and the orbit $\mathcal{O} = \mathrm{Ad}(G)X$ of X. The Fourier transform $\widehat{\mu}_{\mathcal{O}}$ of the measure $\mu_{\mathcal{O}}$ is a function by Proposition 6.2.

Let $\mathfrak{N}$ be the set of nilpotent elements in $\mathfrak{g}$. It is a union of finitely many ("nilpotent") $G(F)$-orbits. Let ω be a compact subset of $\mathfrak{g}$. The local behavior of the Fourier transform of T in $J(\omega)$ is described by the following:

PROPOSITION 6.4. *There exists a $G(F)$-domain D (an open closed $G(F)$-invariant subset) of $\mathfrak{g}$ which contains 0 and a "nilpotent" distribution μ (a linear combination with complex coefficients, depending on T, of the unique up to a scalar multiple invariant measures $\mu_{\mathcal{O}}$ supported on $\mathcal{O}$, where $\mathcal{O}$ are the nilpotent orbits) so that $\widehat{T} = \widehat{\mu}$ on D.*

PROOF. See [HC78, Theorem 4]. □

6.5 Local Integrability of Characters

Fix a Haar measure $\mathrm{d}x$ on $G(F)$. For a smooth $G(F)$-module (π, V), we defined the endomorphism $\pi(f)$ of V by

$$\pi(f) = \pi(f\,\mathrm{d}x) = \int_{G(F)} f(x)\pi(x)\,\mathrm{d}x.$$

Clearly $\pi(f)$ depends linearly on $\mathrm{d}x$. If π is admissible, then $\pi(f)$ has finite rank. In this case, we denote the trace by $\operatorname{tr}\pi(f)$. It is easy to see that if π is admissible and irreducible, then there exists a complex-valued conjugacy invariant locally constant function χ on the *regular* set of $G(F)$ such that

$$\operatorname{tr}\pi(f) = \operatorname{tr}\pi(f\,\mathrm{d}x) = \int_{G(F)} f(x)\chi(x)\,\mathrm{d}x$$

for every f in $C_c^\infty(G(F))$ which is supported on the *regular* set of $G(F)$. The function χ is called the *character* of π. Note that $\operatorname{tr}\pi(f)$ depends on $\mathrm{d}x$, but $\chi(x)$ is independent of $\mathrm{d}x$.

PROPOSITION. *The character χ of an irreducible admissible $G(F)$-module π is a locally integrable function on $G(F)$. In particular,* $\operatorname{tr}\pi(f) = \int f(x)\chi(x)\,dx$ *for* every $f \in C_c^\infty(G(F))$.

PROOF. See [HC78, Theorem 1]. □

PROPOSITION 6.6. *Suppose that $\sigma \in G(F)$ is a semisimple element. Let $G_\sigma(F)$ and $\mathfrak{g}_\sigma$ be the centralizers of σ in $G(F)$ and $\mathfrak{g}$. Then there exists a neighborhood V of 0 in $\mathfrak{g}_\sigma$ and a $G_\sigma(F)$-invariant "nilpotent" distribution μ on $\mathfrak{g}_\sigma$, so that $\chi(\sigma \exp X) = \widehat{\mu}(X)$ for all X in V.*

PROOF. See [HC78, Theorem 5]. □

The above results by Harish-Chandra [HC78] are based on the technique developed by Howe [Ho74] in the case of $\mathrm{GL}(n)$. Kazhdan [Ka86.1] showed that the above local behavior in fact characterizes the characters and orbital integrals, at least on the elliptic set. This characterization extends to the entire (not necessarily elliptic) set $G(F)$ by Proposition 8.2 below.

6.7 Induction

To describe Kazhdan's theory, let S be the space of conjugation invariant functions s on G such that, for every semisimple σ in $G(F)$, there is a neighborhood V of 0 in the Lie algebra $\mathfrak{g}_\sigma$ of the centralizer $G_\sigma(F)$ in $G(F)$ and a $G_\sigma(F)$-invariant distribution μ on $\mathfrak{g}_\sigma$ supported on the nilpotent set of $\mathfrak{g}_\sigma$, so that $s(\sigma \exp X) = \widehat{\mu}(X)$ for all regular X in V. Let $\eta(X)$ be the coefficient of the smallest possible power of t in the polynomial $\det(t - \mathrm{ad}(X))$, $X \in \mathfrak{g}$. Then η is a nonzero polynomial function on $\mathfrak{g}$, and X is called *regular* if $\eta(X) \neq 0$.

Let $S_e(G(F))$ be the space of functions on the elliptic subset of $G(F)$ obtained by restriction of the functions in S.

Let $\Pi(G(F))$ be the set of equivalence classes of admissible irreducible $G(F)$-modules. We let $R_{\mathbb{Z}}(G(F))$ be the integral Grothendieck group of $G(F)$, that is of the category of smooth $G(F)$-modules, namely, the free abelian group generated by $\Pi(G(F))$. It is the group of virtual semisimplifications of the admissible representations of $G(F)$. Put $R(G(F)) = R_{\mathbb{Z}}(G(F)) \otimes_{\mathbb{Z}} \mathbb{C}$ for the Grothendieck group of $G(F)$.

Let M be a Levi subgroup of a parabolic subgroup $P = MN$ with unipotent radical N. Denote by $i_M^G\tau$ the $G(F)$-module normalizedly induced (i.e., twisted by $\delta_P^{1/2}$) from the smooth $M(F)$-module τ, which is trivially extended on $N(F)$ to $P(F)$. If V_τ denotes the space of τ, then $i_M^G\tau$ acts on

$$V = \{\phi : G(F) \to V_\tau;\ \phi(mngu) = \delta_P^{1/2}(m)\tau(m)\phi(g),\ u \in K_\phi,\\ m \in M(F),\ n \in N(F),\ g \in G(F)\}$$

—here K_ϕ is an open compact subgroup of $G(F)$ depending on ϕ—by right shifts: $((i_M^G\tau)(g))\phi(h) = \phi(hg)$. We write $\mathrm{ind}_M^G\,\tau$ for the unnormalized induced representation (thus with δ_P omitted).

Then i_M^G extends to functor from the category of smooth $M(F)$-modules to the category of smooth $G(F)$-modules. Its restriction is a homomorphism from $R(M(F))$ to $R(G(F))$. We denote by $R_I(G(F))$ the space in $R(G(F))$ of the images of i_M^G over all $M \neq G$. Put $\overline{R}(G(F))$ for the quotient $R(G(F))/R_I(G(F))$. Denote by χ the character of a member π in $R(G(F))$. It is a finite linear combination with complex coefficients of characters of irreducible $G(F)$-modules.

PROPOSITION 6.8. *The map $\overline{R}(G(F)) \to S_e(G(F))$, $\pi \mapsto \chi = \chi_\pi$, is an isomorphism.*

PROOF. See [Ka86.1, Theorem D]. □

In particular, any function on the elliptic set of $G(F)$ whose local behavior is given by the defining property of S is the restriction to the elliptic set of a character of a virtual $G(F)$-module.

6.9 Orbital Integrals on the Elliptic Set

Theorem C of [Ka86.1] gives another characterization of $S_e(G(F))$. Let $A_c^\infty(G(F))$ be the space of f in $C_c^\infty(G(F))$ whose orbital integrals vanish on the regular non-elliptic set. As in Subsection 2.6, let $J_c^\infty(G(F))$ be the space of functions in $C_c^\infty(G(F))$ whose orbital integrals vanish on the regular set of $G(F)$. Let $\overline{A}_c^\infty(G(F))$ be the quotient $A_c^\infty(G(F))/J_c^\infty(G(F))$. We can now state Theorem C of [Ka86.1].

PROPOSITION. *The map $\overline{A}_c^\infty(G(F)) \to S_e(G(F)), f \mapsto \widetilde{\Phi}(f)$, is an isomorphism.*

See Subsection 2.4 for the definition of $\widetilde{\Phi}(x,f)$.

The isomorphism defined by Propositions 6.8 and 6.9 yields the following:

COROLLARY 6.10. *The space $\overline{A}_c^\infty(G(F))$ and $\overline{R}(G(F))$ are isomorphic.*

PROPOSITION 6.11. *Let G be the multiplicative group of a simple algebra. Then $S_e(G(F))$ consists of the locally constant functions on the elliptic set of $G(F)$.*

PROOF. In this case, $G = M(m,D)^\times$, where D is a division algebra of rank d central over F. The group G is an inner form of the split group $G' = \mathrm{GL}(n)$, $n = md$. A stable conjugacy class in $G(F)$ consists of a single (rational) conjugacy class. A semisimple conjugacy class γ in $G(F)$ is determined by its characteristic polynomial p_γ (which has coefficients in F). A unipotent conjugacy class determines a conjugacy class of Levi subgroups, namely, a partition $\alpha = (m_i)$ of m. Here the m_i are positive integers and $\sum m_i = m$, $m_i \geq m_{i+1}$. There is a natural injection of the set of conjugacy classes in $G(F)$ into the set of conjugacy classes in $G'(F)$, denoted by $\gamma \mapsto \gamma'$ and defined by $p_{\gamma'} = p_\gamma$ and $(m_i) \mapsto (dm_i)$. Similarly there is an injection of the nilpotent classes in the Lie algebra $M(m,D)$ of G into the set of such classes in $M(n,F)$. The nilpotent orbit $\mathcal{O}$ in $M(m,D)$ determines the partition α of m, and the corresponding standard (upper triangular) parabolic subgroup of G is denoted by P_α. Put $\mathcal{O} = \mathcal{O}_\alpha$ and $\widehat{\mu}_\alpha$ for $\widehat{\mu}_{\mathcal{O}}$. Let θ_α be the character of the $G(F)$-module unitarily induced from the trivial $P_\alpha(F)$-module. Lemma 5 of [Ho74] asserts that there is a small neighborhood V of zero in $M(m,D)$ such that the Fourier transform $\widehat{\mu}_\alpha$ at X is equal to $\theta_\alpha(\exp X)$ for all X in V. The result is stated there only for $\mathrm{GL}(n)$, but the proof applies to any G as here. This is zero on the set of elliptic regular $\exp X$ if α is not the trivial partition (m) of m. Moreover, the character $\theta_{(m)}$ is identically one. Since the centralizer of any elliptic element in G is of the form $\mathrm{GL}(m',D')$, where D' is a central simple algebra over a field extension F' of F, the proposition follows. ☐

Combining this result (for $G(F)$ and $G'(F)$) with Proposition 6.9, we obtain the following:

COROLLARY 6.12. *For every f in $A_c^\infty(G(F))$, there exists f' in $A_c^\infty(G'(F))$, and for every f' in $A_c^\infty(G'(F))$, there exists f in $A_c^\infty(G(F))$, such that $\widetilde{\Phi}(\gamma,f) = \widetilde{\Phi}(\gamma',f')$ for all regular γ in $G(F)$ and γ' in $G'(F)$ with $p_\gamma = p_{\gamma'}$.*

This proves the assumptions 12.1 and 12.2 below in the special case of our $G(F)$ and $G'(F)$.

7 Coinvariants

7.1 Module of Coinvariants

Let F be a local non-Archimedean field. Let G be as in 2.1. Let (π, V) be an admissible $G(F)$-module. Then it is of finite length. Let $P = MN$ be an F-parabolic subgroup with a Levi subgroup M and unipotent radical N. Then the quotient of V by the subspace generated by $\{\pi(n)\xi - \xi\,;\, \xi \in V, n \in N(F)\}$ is an $M(F)$-module π'_N, since $M(F)$ normalizes $N(F)$. Denote by π''_N the image of π'_N in the Grothendieck group $R(M(F))$. The (normalized) *$M(F)$-module π_N of $N(F)$-coinvariant of π* is defined to be $\delta_P^{-1/2}\pi'_N$. It is shown in [BZ76] that if π is admissible, then π_N is admissible. Hence it is of finite length. The construction extends to a functor $r_M^G : \pi \mapsto \pi_N$, from the category $K(G(F))$ of smooth $G(F)$-modules to the category $K(M(F))$ of smooth $M(F)$-modules. This functor is exact. Let $i_M(\tau)$ be the $G(F)$-module $i_P^G(\tau) = \mathrm{ind}_P^G(\delta_P^{1/2}\tau)$ induced from the $P(F) = M(F)N(F)$-module $\delta_P^{1/2}\tau \otimes 1$. This construction defines a functor $i_M^G : \tau \mapsto i_M(\tau)$, from $K(M(F))$ to $K(G(F))$. It is exact. Frobenius reciprocity, see [BZ76], asserts that $\mathrm{Hom}_G(\pi, I_M(\tau)) = \mathrm{Hom}_M(\pi_N, \tau)$ for all irreducible $M(F)$-modules τ and $G(F)$-modules π. Hence $\pi_N \neq 0$ implies that there is a nonzero morphism of G-modules from π to $I_M(\pi_N)$.

7.2 Reduction of Characters to Levi Subgroups

We will now introduce notation needed to state a theorem of Deligne-Casselman [De76, Cas77]. Let A be a maximal F-split torus in G, B a minimal parabolic subgroup of G containing A, Δ the set of roots of A in B. Fix a lattice L in $A(F)$ so that $|\alpha(\lambda)| = 1$ if and only if $\alpha(\lambda) = 1$ for all λ in L and α in Δ, and so that $A(F)/L$ is compact. Put L^- for the set of λ in L with $|\alpha(\lambda)| \leq 1$ for all α in Δ. For any semisimple t in $G(F)$, there exists a positive integer m and $y \in G(F)$ so that $yt^my^{-1}\lambda s$, where λ lies in L^- and s is a compact element in $G(F)$ (the closure of the group generated by s is compact). Let P_λ be the standard (containing B) parabolic subgroup of G whose Levi component M_λ is the centralizer G_λ in $G(F)$, and put $P_t = M_tN_t$ for $y^{-1}P_\lambda y$.

The definition of the parabolic P_t is the same as in [Cas77]. To recall the definition of [Cas77], put $A_\theta = \cap_{\alpha\in\theta}\ker\alpha$ for any subset θ of Δ. Denote by A^- the set of x in A with $|\alpha(x)| \leq 1$ for all α in Δ. Given a semisimple t in $G(F)$ with $yt^my^{-1} = as$ for a in $A^-(F)$ and a compact element s, let θ be the set of α in Δ with $|\alpha(a)| = 1$. Denote by M_θ the centralizer in G of the torus A_θ. Then $M_\theta = M_\lambda$.

We can now state the following theorem of Casselman and Deligne.

PROPOSITION 7.3. *Let π be an admissible irreducible representation of $G(F)$. Let t be a regular element of $G(F)$ so that $P = P_t$ is a parabolic subgroup. Then $\chi_\pi(t) = \chi_{\pi'_N}(t)$ $(= \chi_{\pi''_N}(t))$. Since $\Delta(t) = \Delta_M(t)\delta_P(t)^{-1/2}$, we have $(\Delta\chi_\pi)(t) = (\Delta_M\chi_{\pi_N})(t)$ for such t.*

Here χ_π denotes the character of π. Deligne discovered the result where π is cuspidal [De76]. The extension to a general π is in [Cas77].

7.4 Reduction of Traces to Levi Subgroups

We now recall Lemma 5.1 of [Cas77]. Suppose t is in A^-. The associated λ in L then lies in L^-. We put $P = P_t = MN$. Let C be an open compact congruence subgroup of $G(F)$ with the properties of K_i in [Cas77, Lemma 2.1], and in particular

$$C = (C \cap \overline{N}(F))(C \cap M(F))(C \cap N(F)),$$

where $\overline{N}$ is the unipotent radical of the parabolic $P_{t^{-1}} = \overline{P}$ opposite to P. Let f_t be the function in $C_c^\infty(G(F))$ supported on $Z(F)CtC$ which attains the value $|Z(F)CtC/Z(F)|^{-1}$ on CtC. Let f_t^M be the function on $M(F)$ which is supported on $t(C \cap M(F))Z(F)$, transforms under $Z(F)$ by ω_π^{-1} (where ω_π is the central character of π), and attains the value $\delta_P^{1/2}(t)/|C \cap M(F)|$ on $t(C \cap M(F))$.

The following is Lemma 5.1 of [Cas77].

PROPOSITION 7.5. *We have* $\operatorname{tr}\pi(f_t) = \operatorname{tr}\pi_N(f_t^M)$ *for any $G(F)$-module π.*

7.6 Multiplicities

This proposition will be used below as follows. The τ be an irreducible constituent of the $M(F)$-module π_N. Denote its central character by ω_τ and its character by χ_τ. We are interested in the function f_t^M on $M(F)$ since

$$\operatorname{tr}\tau(f_t^M) = \int \chi_\tau(tx) f_t^M(tx)\,\mathrm{d}x = \omega_\tau(t)\delta_P^{1/2}(t)\operatorname{tr}\tau(1_C^M)$$

for t in the center of $M(F)$, where 1_C^M is the function in $C_c^\infty(M(F))$, transforming under $Z(F)$ by ω_π^{-1} and equal to $1/|C \cap M(F)|$ on $C \cap M(F)$. Then $\operatorname{tr}\tau(1_C^M)$ is the (non-negative integral) multiplicity of the trivial representation of $C \cap M(F)$ in τ. On the other hand, f_t is a C-bi-invariant function, where C is independent of t. Thus $\operatorname{tr}\pi(f_t) \neq 0$ only for π which have a nonzero C-invariant vector.

Let $W(M(F), G(F)) = N(M(F), G(F))/M(F)$ be the quotient by $M(F)$ of the normalizer $N(M(F), G(F))$ of $M(F)$ in $G(F)$, where $P(F) = M(F)N(F)$ is a parabolic subgroup of $G(F)$.

PROPOSITION 7.7. *Let x be a regular element in $G(F)$. Then the orbital integral $I(x,f_t)$ vanishes unless x is conjugate to an element of $M(F)$. For x in $M(F)$, we have*

$$I(x,f_t) = \sum_{w \in W(M(F),G(F))} I^M(wxw^{-1}, f_t^M).$$

REMARK. The proof of this proposition relies on Corollary 8.13. It is given here since the functions f_t and f_t^M do not appear in Section 8, and it is clear that the work of Section 8 does not depend on Proposition 7.7.

PROOF. Corollary 8.13 implies that given f_t^M, there exists a function f on $G(F)$ such that $I(x,f) = 0$ unless x is conjugate in $G(F)$ to an element of $M(F)$, and when x lies in $M(F)$, then

$$I(x,f) = \sum_{w \in W(M(F),G(F))} I^M(wxw^{-1}, f_t^M).$$

The Weyl integration formula and Proposition 7.3 imply that $\operatorname{tr} \pi(f) = \operatorname{tr} \pi_N(f_t^M)$ for every admissible $G(F)$-module π, since the parabolic subgroup P_x associated with any element x in the support of $I(x,f)$ is $P_t = P$. On the other hand, Proposition 7.5 implies that $\operatorname{tr} \pi(f_t) = \operatorname{tr} \pi_N(f_t^M)$, hence $\operatorname{tr} \pi(f_t) = \operatorname{tr} \pi(f)$, for every admissible $G(F)$-module π. But then Proposition 5.1 implies that $I(x,f) = I(x,f_t)$ for every regular x in $G(F)$, and the proposition follows. □

8 Trace Functions

8.1 Orbital Integrals of Elliptic Elements

Let F be a local non-Archimedean field, G and $C_c^\infty(G(F))$ as before. We continue to let $J_c^\infty(G(F))$ be the space of f in $C_c^\infty(G(F))$ whose orbital integrals vanish at each regular element in $G(F)$ and $A_c^\infty(G(F))$ the space of f in $C_c^\infty(G(F))$ whose orbital integrals vanish on every regular non-elliptic element in $G(F)$, $\overline{C}_c^\infty(G(F)) = C_c^\infty(G(F))/J_c^\infty(G(F))$ and $\overline{A}_c^\infty(G(F)) = A_c^\infty(G(F))/J_c^\infty(G(F))$.

Our final aim in this section is to prove the following:

PROPOSITION 8.2. *Let M be a Levi subgroup of G and f^M an element of $C_c^\infty(M(F))$ with the following property. For every m, m' in $M(F)$ which are regular in $G(F)$ and conjugate to each other by $G(F)$, we have*

$$I^M(m,f^M) = I^M(m',f^M). \tag{8.2.1}$$

Then there exists f in $C_c^\infty(G(F))$ with $f_M = f^M$ and $f_L = 0$ for every Levi subgroup L of G which does not contain a conjugate of M. See 2.6 for f_M.

This proposition, which concerns "lifting" of orbital integrals from a Levi subgroup of $G(F)$ to $G(F)$ itself, is proven below using representation theoretic techniques, in the spirit of Corollary 6.10. We will use the trace Paley-Wiener theorem of [BDK86] and [F95] and the geometric lemma of [BZ77, (2.12)], which we now proceed to state.

8.3 *Trace Paley-Wiener Theorem*

As in Section 6, let $R_{\mathbb{Z}}(G(F))$ denote the integral Grothendieck group of the category of smooth $G(F)$-modules, namely, the group of virtual representations of $G(F)$ of finite length (the free abelian group with basis $\Pi(G(F))$). Set $R(G(F)) = R_{\mathbb{Z}}(G(F)) \otimes \mathbb{C}$. Let $i_M^G : R(M(F)) \to R(G(F))$ be the induction homomorphism. As in Subsection 7.1, let $r_M^G : R(G(F)) \to R(M(F))$ be the coinvariants homomorphism. Let $X(G(F))$ be the group of unramified characters of $G(F)$. Then $X(G(F))$ acts naturally on $\Pi(G(F))$ and $R(G(F))$ by $\psi : \pi \mapsto \pi\psi$.

Recall that an unramified character of $F^\times$ has the form $x \mapsto |x|^s$, with $s \in \mathbb{C}/(2\pi i\mathbb{Z}/\ln q)$, since $|u| = 1$ for the units $u \in F^\times$, and $|\boldsymbol{\pi}| = q^{-1} = e^{-\ln q}$ for any generator $\boldsymbol{\pi}$ of the maximal ideal $(\boldsymbol{\pi})$ in the ring R of integers in F, where $q = R/(\boldsymbol{\pi})$ is the residual cardinality. Thus $X(F^\times) \simeq \mathbb{C}/(2\pi i\mathbb{Z}/\ln q) \simeq \mathbb{C}^\times$; the last isomorphism is via $z \mapsto q^z$.

The set $X(G(F))$ has a natural structure of a complex algebraic group, isomorphic to $(\mathbb{C}^\times)^d$, where $d = d(G(F)) = \dim X(G(F))$. As usual, fix a Haar measure $\mathrm{d}x$ on $G(F)$. We work in this section with the space $C_c^\infty(G(F))$ of smooth compactly supported functions on $G(F)$. Passing to the space $C_c^\infty(G(F), Z(F), \chi)$ as before is easy. Each function f in $C_c^\infty(G(F))$ defines a linear form $\beta_f : R(G(F)) \to \mathbb{C}$ by $\beta_f(\pi) = \operatorname{tr} \pi(f)$. The form $\beta = \beta_f$ satisfies the following two conditions:

(PW(1)) For any Levi subgroup M and irreducible $M(F)$-module τ, the function $\psi \mapsto \beta(i_M^G(\tau\psi))$ is a regular function on the complex algebraic variety $X(M(F))$.

(PW(2)) There exists an open compact subgroup K in $G(F)$ which dominates β, in the sense that β vanishes on each $G(F)$-module π which has no nonzero K-fixed vector.

Let $R^*(G(F)) = \operatorname{Hom}_{\mathbb{C}}(R(G(F)), \mathbb{C}) = \operatorname{Hom}(\Pi(G(F)), \mathbb{C})$ be the space of all linear forms on $R(G(F))$. A form $\beta : R(G(F)) \to \mathbb{C}$ is called *good* if it satisfies conditions (PW(1)) and (PW(2)). The form is called *trace* if $\beta = \beta_f$ for some $f \in C_c^\infty(G(F))$. We denote the spaces of good and trace forms by $F_{\text{good}} = F_{\text{good}}(G(F))$ and $F_{\text{tr}} = F_{\text{tr}}(G(F))$. We can now state the primary theorem of [BDK86]. See [F95] for a different proof and for a twisted analogue.

THEOREM 8.4 (Trace Paley-Wiener). *For every p-adic reductive group $G(F)$, we have $F_{\text{tr}} = F_{\text{good}}$.*

This theorem describes the image of the natural morphism $\mathrm{tr} : C_c^\infty(G(F)) \to R^*(G(F))$. As noted at the end of Section 5, Proposition 5.1 and Theorem B of [Ka86.2] imply that $\ker \mathrm{tr} = [C_c^\infty(G(F)), C_c^\infty(G(F))]$ for every local non-Archimedean field (of any characteristic).

8.5 Weyl Group

We continue to assume that a Levi subgroup M of G contains the fixed Levi component M_0 of the minimal parabolic subgroup P_0. Denote by W_M the quotient by M_0 of the normalizer $N(M_0, M)$ of M_0 in M. It is the Weyl group of M_0 in M. Let L be a Levi subgroup of G and let $W(M, L)$ denote a set of representatives in W_G, of minimal length, for $W_M \backslash W_G / W_L$. For every w in $W(M, L)$, put $M_w = M \cap wLw^{-1}$ and $L_w = w^{-1}Mw \cap L$.

We can now state the Geometric Lemma of [BZ77, (2.12)].

PROPOSITION 8.6 (Geometric Lemma). *For every ρ in $R(L(F))$, we have*

$$F(\rho) \overset{\text{defn}}{=} r_M^G \circ i_L^G(\rho) = \sum_{w \in W(M,L)} i_{M_w}^M \circ w \circ r_{L_w}^L(\rho).$$

COROLLARY 8.7. *For each Levi subgroup M of G, put $T_M = i_M^G \circ r_M^G : R(G(F)) \to R(G(F))$. Then*

(1) $T_L \circ i_M^G = \sum_{w \in W(L,M)} i_{M_w}^G \circ r_{M_w}^M$
(2) $T_L \circ T_M = \sum_w T_{M_w}$.

As before, $M_w = M \cap w^{-1}Lw$. The w range over $W(L, M)$.

PROOF. (1) We have that

$$i_L^G \circ r_L^G \circ i_M^G = \sum_w i_L^G \circ i_{L_w}^L \circ w \circ r_{M_w}^M = \sum_w i_{L_w}^G \circ w \circ r_{M_w}^M$$

is equal to the required expression, since $i_{L_w}^G \circ w = i_{M_w}^G$ by [BDK86, Lemma 5.4(iii)].

(2) We also have

$$T_L \circ i_M^G \circ r_M^G = \sum_w i_{M_w}^G \circ r_{M_w}^M \circ r_M^G = \sum_w i_{M_w}^G \circ r_{M_w}^G = \sum_w T_{M_w}.$$

∎

8.8 Dual Maps

We now proceed to establish the Proposition 8.2. Denote the pairing $R^*(G(F)) \times R(G(F)) \to \mathbb{C}$ by $(\beta, \pi) \mapsto \langle \beta, \pi \rangle$. Let

$$i_M^{G*} : R^*(G(F)) \to R^*(M(F))$$

and

$$r_M^{G*} : R^*(M(F)) \to R^*(G(F))$$

be the morphisms adjoint to i_M^G and r_M^G. Note that $\overline{C}_c^\infty(G(F))$ is a subspace of $R^*(G(F))$. The function f defines the form $\beta = \beta_f : \pi \mapsto \langle \beta, \pi \rangle = f_G(\pi) = \operatorname{tr} \pi(f)$. Put $\langle f, \pi \rangle$ for $\langle \beta, \pi \rangle$ in this case.

LEMMA. *For every $M(F)$, i_M^{G*} maps $\overline{C}_c^\infty(G(F))$ to $\overline{C}_c^\infty(M(F))$ and r_M^{G*} maps $\overline{C}_c^\infty(M(F))$ to $\overline{C}_c^\infty(G(F))$.*

PROOF. For f in $\overline{C}_c^\infty(G(F))$, f_M satisfies $\langle f_M, \tau \rangle = \langle f, i_M^G \tau \rangle$ for every τ in $R(M(F))$ by virtue of a standard formula for the character of an induced representation. By virtue of Proposition 5.1, we have $i_M^{G*} f = f_M$, as required. For the second part of the lemma, for every f^M in $\overline{C}_c^\infty(M(F))$, define a form $\beta = r_M^{G*}(f^M)$ in $R^*(G(F))$ by $\langle \beta, \pi \rangle = \langle f^M, r_M^G \pi \rangle$, for $\pi \in R(G(F))$. This is a good form, hence a trace form by the trace Paley-Wiener theorem. Namely, $r_M^{G*}(f^M)$ is a function in $\overline{C}_c^\infty(G(F))$, and the second part of the lemma follows. □

COROLLARY 8.9. *The homomorphisms $i_M^G : R(M(F)) \to R(G(F))$ and $r_M^G : R(G(F)) \to R(M(F))$ admit adjoints $i_M^{G*} : \overline{C}_c^\infty(G(F)) \to \overline{C}_c^\infty(M(F))$ and $r_M^{G*} : \overline{C}_c^\infty(M(F)) \to \overline{C}_c^\infty(G(F))$.*

A function f in $\overline{C}_c^\infty(G(F))$ is called *discrete* if $i_M^{G*} f = 0$ for all Levi subgroups $M \neq G$. By Proposition 5.1 the space of discrete functions in $\overline{C}_c^\infty(G(F))$ is $\overline{A}_c^\infty(G(F))$.

PROPOSITION 8.10 (Combinatorial Lemma). *For each proper Levi subgroup M of G, there is a rational number c_M such that*

$$f^d = f - \sum_{M \neq G} c_M r_M^{G*} \circ i_M^{G*}(f)$$

is discrete for every f in $\overline{C}_c^\infty(G(F))$.

PROOF. This is an analogue of Lemma 3.3 of [BDK86]. In [BDK86] a form β in $R^*(G(F))$ is called discrete if $i_M^{G*} \beta = 0$ for all $M \neq G$. Lemma 3.3 [BDK86] asserts that there are c_M such that for each β in $R^*(G(F))$ the form

$$\beta^d = \beta - \sum_{M \neq G} c_M r_M^{G*} \circ i_M^{G*}(\beta)$$

is discrete. But Lemma 8.8 asserts that if f lies in $\overline{C}_c^\infty(G(F))$, then f^d lies in $\overline{C}_c^\infty(G(F))$, not only in $R^*(G(F))$; hence it is in $\overline{A}^\infty(G(F))$, as asserted. □

THEOREM 8.11. *The space $\overline{C}_c^\infty(G(F))$ is the direct sum over a set of representatives for the conjugacy classes of Levi subgroups in G of $r_M^{G*}(\overline{A}_c^\infty(M(F)))$.*

PROOF. To show that $\overline{C}_c^\infty(G(F))$ is the sum of $r_M^{G*}(\overline{A}_c^\infty(M(F)))$, we assume by induction that this claim holds for every proper Levi subgroup M of G. Namely, we assume that for each $M \neq G$, and for each $L \subset M$, there is a rational number $c_{M,L}$ with the following property. Given f^M in $\overline{C}_c^\infty(M(F))$, there are $f^{M,L}$ in $\overline{A}_c^\infty(L(F))$ for each $L \subset M$, such that $f^M = \sum_{L \subset M} c_{M,L} r_L^{M*}(f^{M,L})$. Hence, given f in $\overline{C}_c^\infty(G(F))$ there are $f^{M,L}$ in $\overline{A}_c^\infty(L(F))$ for every $M \neq G$ and $L \subset G$ with $i_M^{G*} f = \sum_{L \subset G} c_{M,L} r_L^{M*}(f^{M,L})$. Using the Combinatorial Lemma Proposition 8.10, we conclude that

$$\begin{aligned} f &= f^d + \sum_{M \neq G} c_M r_M^{G*}(i_M^{G*}(f)) \\ &= f^d + \sum_{M \neq G} c_M r_M^{G*} \sum_{L \subset M} c_{M,L} r_M^{L*} f^{M,L} \\ &= f^d + \sum_{L \neq G} r_L^{G*} \left(\sum_M c_M c_{M,L} f^{M,L} \right), \end{aligned}$$

where M ranges over the $M \neq G$ which contain L, as required.

To prove that the sum is direct, note that if f^M lies in $\overline{C}_c^\infty(M(F))$, then by the Geometric Lemma 8.6 for each Levi subgroup L and ρ in $R(L(F))$, we have

$$\begin{aligned} \langle i_L^{G*} r_M^{G*} f^M, \rho \rangle &= \langle f^M, r_M^G \circ i_L^G(\rho) \rangle \\ &= \sum_{w \in W(M(F), L(F))} \langle f^M, i_{M_w}^M \circ w \circ r_{L_w}^L(\rho) \rangle. \end{aligned} \tag{8.11.1}$$

If f^M lies in $\overline{A}_c^\infty(M(F))$ and w contributes a nonzero term in the sum, then $M_w = M$, namely, $L \supset w^{-1} M w$. Consequently (8.11.1) is zero if L contains no conjugate of M. If L is conjugate to M, say $L = s^{-1} M s$ for some s in $W(M, L)$, then (8.11.1) is equal to $\langle f^M, s\rho \rangle$. Now, if $f = \sum_M r_M^{G*}(f^M)$ is zero, where the f^M lies in $\overline{A}_c^\infty(M(F))$, then choose L to be a minimal Levi subgroup (up to conjugation) for which $f^L \neq 0$ in this sum. Then $i_L^{G*}(f) = f^L$, and $f = 0$ implies that $f^L = 0$. This contradiction completes the proof of the theorem. □

8.12 *Proof of Proposition* 8.2

Given $M \neq G$ and f^M in $\overline{C}_c^{\infty}(M(F))$, since

$$\overline{C}_c^{\infty}(M(F)) = \oplus_{L \subset M} r_L^{M*}(\overline{A}_c^{\infty}(L(F)))$$

by Theorem 8.11, we may assume that $f^M = r_Y^{M*}(f^Y)$ for some Levi Y in M and f^Y in $\overline{A}_c^{\infty}(Y(F))$. We claim that the product of $f = r_M^{G*}f^M = r_Y^{G*}(f^Y)$ by a scalar which depends only on Y, M, and G has the properties required by the proposition. Indeed, as in (8.11.1), for each ρ in $R(Y(F))$, we have

$$\langle i_L^{G*} r_Y^{G*}(f^Y), \rho\rangle = \sum_{w \in W(Y,L)} \langle f^Y, i_{Y_w}^Y \circ w \circ r_{L_w}^L(\rho)\rangle$$

and it suffices to consider w with $L \supset w^{-1}Yw$, since f^Y lies in $\overline{A}_c^{\infty}(Y(F))$. Hence if L contains no conjugate of M, then the sum is empty and $i_L^{G*}(f) = f_L$ is zero, as required. If $L = M$, our sum becomes the sum over all w in $W(Y, M)$ with $w^{-1}Yw \subset M$ of $\langle f^Y, w \circ r_{M_w}^M(\rho)\rangle$. The condition (8.2.1) implies that each of the summands is equal to $\langle f^Y, r_Y^M(\rho)\rangle = \langle r_Y^{M*}(f^Y), \rho\rangle = \langle f^M, \rho\rangle$; hence $i_M^{G*}(f)$ is equal to f^M up to a multiple by the cardinality of the set of w in $W(Y, M)$ with $w^{-1}Yw \subset M$. The proposition follows. $\square$

Proposition 8.2 implies that a function f^M in $C_c^{\infty}(M(F))$ can be "lifted" to a function f in $C_c^{\infty}(G(F))$ with the "same" orbital integrals on the regular conjugacy classes of $G(F)$ which intersect $M(F)$. The orbital integrals of f are not necessarily zero on x in $G(F)$ whose conjugacy class does not intersect $M(F)$. However, we have the following:

COROLLARY 8.13. *Suppose that f^M has the property that $I^M(m, f^M)$ is supported on the set of m in $M(F)$ with $|\alpha(m)| \neq 1$ for every root of the split component of the center of M in N of the parabolic subgroup $P = MN$. Then f can be chosen to have the property that $I(x, f)$ is zero unless x is conjugate in $G(F)$ to an element of $M(F)$.*

PROOF. Let S_M denote the support of $I^M(f^M)$ in $M(F)$. Put $S = (S_M)^G = \{g^{-1}sg\, ;\, g \in G(F), s \in S_M\}$. Then S is open and closed in $G(F)$. Replace the f obtained in the proposition by its product with the characteristic function of S to obtain the function f of the corollary. $\square$

9 Stability

9.1 Stable Conjugacy

Continue to let F be a local non-Archimedean field of characteristic 0. Let G' be a quasisplit connected reductive linear algebraic group, defined over F. Recall that an F-group is called *quasisplit* if it has a Borel subgroup defined over F. By a Borel subgroup, we mean a minimal parabolic subgroup B over an algebraic closure $\overline{F}$ of F. The Levi subgroup of B is an F-torus, and G is called *split* if this F-torus is split, namely, isomorphic over F to a product $\mathbb{G}_m^r$ of r copies of the multiplicative group $\mathbb{G}_m$ over F. Two F-groups G_1 and G_2 are *F-forms* if they are isomorphic over $\overline{F}$. We recall in the next subsection the definition of inner forms and simply note here that given a reductive connected linear algebraic F-group G, there is a unique quasisplit F-group G' which is an inner form of G.

In the current work, we are interested mainly in the example of $G' = \mathrm{GL}(n)$ and its inner forms G. To put the problem in perspective, we comment first on a more general situation, where stable conjugacy does not reduce to conjugacy.

The stable conjugacy class of x in $G'(F)$ is defined in Subsection 3.6. We recall that two elements are stably conjugate if they are conjugate by an element of $G'(\overline{F})$. Let T be the centralizer of x in G'. We will only be interested in regular x, in which case T is an F-torus. The conjugacy classes within the stable class of x are parametrized by the (finite) set

$$B(T/F) = \ker[H^1(F,T) \to H^1(F,G')],$$

see also 3.6. If x and x' are stably conjugate, then $G'_x = \mathrm{Ad}(y)G'_{x'}$ is isomorphic to $G'_{x'}$ over $\overline{F}$. This allows us to transfer a differential form of maximal degree, yielding compatible Haar measure on $G'_x(F)$ and $G'_{x'}(F)$.

Let $\{\mathrm{Ad}(b)x\,;\, b \in B(T/F)\}$ be a set of representatives for the conjugacy classes within the stable conjugacy class of the regular element x of $G'(F)$.

DEFINITION. Let Ξ be a function on the regular conjugacy classes in $G'(F)$. The *stable* function Ξ^s associated with Ξ is defined by

$$\Xi^s(x) = \sum_b \Xi(\mathrm{Ad}(b)x).$$

It depends only on the stable conjugacy class of the regular x. In particular, for any f in $C_c^\infty(G'(F))$, we have the stable orbital integral $\Phi^s(\cdot,f)$ of f, and the normalized $I^s(\cdot,f)$.

9.2 Inner Twisting

The stable orbital integrals are introduced for purposes of comparison between the group G' and a reductive connected F-group G, such that the following holds. Let G and H be F-groups. An isomorphism $\psi : G \to H$ over $\overline{F}$ is called an *inner twisting* if for every τ in $\mathrm{Gal}(\overline{F}/F)$ there is g_τ in $G(\overline{F})$ such that $(\tau\psi)^{-1} \circ \psi = \mathrm{Ad}(g_\tau)$. If such ψ exists, then G and H are called *inner forms*. Suppose that G is an inner form of G' and fix an inner twisting $\psi : G \to G'$. Fix a maximal split torus A in G. It can be identified with a torus A' of G' via ψ. Each Levi subgroup M of G containing A corresponds by ψ to a Levi subgroup M' of G' containing A'. Fix a lattice L as in Subsection 7.2, so that $M(F)$ is of the form $M_\lambda(F) = G_\lambda(F)$ for some λ in L^-.

9.3 Norm Map

In every known comparison situation (base change, symmetric square or more generally symplectic and orthogonal groups in $\mathrm{GL}(n)$, metaplectic correspondence, inner twisting), there exists a map N which we call a *norm map*, with at least the following properties. The map N is a bijection, from a subset S'_0 of the set S' of stable conjugacy classes of regular elements in $G'(F)$, to a subset S_0 of the set S of stable conjugacy classes of regular elements in $G(F)$, such that the following properties hold:

(1) G'_x and G_{Nx} are inner forms.
(2) $Nx = \psi^{-1}(x)$ for x in $A'(F)$.
(3) x has a representative in $M'(F)$ if and only if Nx has a representative in $M(F)$.
(4) At least one of the subsets S_0, S'_0 is equal to S, S'.

We use (1) to relate measures on the two groups there. Fix a norm map N.

9.4 Matching Functions

We let $W(M, G) = N(M, G)/M$ be the quotient by M of the normalizer $N(M, G)$ of M in G. We define similarly $W(M', G')$. Given f^M in $C_c^\infty(M(F))$, let ${}^MI(f^M)$ be the conjugacy class function on the set of regular x in $G(F)$ which attains the value 0 unless (a conjugate of) x lies in $M(F)$ when we put

$$ {}^MI(x, f^M) = \sum_{w \in W(M,G)} I^M(wxw^{-1}, f^M). $$

Similarly, for ϕ^M in $C_c^\infty(M'(F))$, we define ${}^MI(x', \phi^M)$, for regular $x' \in G'(F)$. In particular, ${}^GI = I = I^G$. Recall that ${}^MI^s$ indicates the stable function on $G(F)$ associated with ${}^MI$.

DEFINITION. The functions ϕ^M in $C_c^\infty(M'(F))$ and f^M in $C_c^\infty(M(F))$ are called *matching* if the following hold:

(1) ${}^{M}I^{s}(s, f^{M})$ is zero for any s in $S \smallsetminus S_0$.
(2) ${}^{M}I^{s}(s, \phi^{M})$ is zero for any s in $S' \smallsetminus S'_0$.
(3) ${}^{M}I^{s}(s, \phi^{M})$ is equal to ${}^{M}I^{s}(Ns, f^{M})$ for all s in S'_0.

In the comparison of $G' = \mathrm{GL}(n)$ with its inner form G, the norm map associates a regular $x \in G(F)$ to $x' \in G'(F)$ which has the same characteristic polynomial. In the notations of this section: $Nx' = x$ for the x' coming from x. This relation defines not the elements but only the conjugacy classes x and x', and these coincide with the stable conjugacy classes.

10 Discrete Series

10.1 Central Exponents

Let F be a local non-Archimedean field of characteristic 0. Let G be a connected reductive F-group. Let G_s be the centralizer of s in G. Let π be a smooth $G(F)$-module of finite length. As noted in Subsection 3.3, it is admissible. By a *central exponent* of π with respect to a Levi subgroup M of G, we mean the central character of an irreducible constituent of the module π_N of coinvariants (defined in 7.1) of π with respect to any parabolic subgroup $P = MN$ with Levi component M.

10.2 Small Part

As in Subsection 7.2, let A be a maximal split torus in G and let L be a cocompact lattice in $A(F)$. Let B be a minimal parabolic subgroup of G containing A, and A^- the set of a in A with $|\alpha(a)| \leq 1$ for any α in the set Δ of roots of A in B, $L^- = L \cap A^-(F)$. To any semisimple t, we associate (using the action of the Weyl group) a in $A^-(F)$ (or λ in L^-) and a subset θ of Δ, consisting of the α with $\alpha(\lambda) = 1$.

10.3 Decay of Central Exponents

Given λ in L^-, consider the centralizer M_λ of λ in G and denote by $P_\lambda = M_\lambda N_\lambda$ the standard parabolic subgroup of G with Levi component M_λ. The center A_λ of M_λ lies in A. We say that the central exponent ω of π with respect to M_λ *decays* if $|\omega(a)| < 1$ for every a in $A_\lambda(F)$ with the following properties:

(1) $|\alpha(a)| \leq 1$ for any root α of A_λ in N_λ.
(2) $|\alpha(a)| < 1$ for some such α.

We define π to be *discrete series* if its central character is unitary, and its central exponents with respect to any proper Levi subgroup M_λ, where λ is any element in L^-, all decay.

10.4 Square Integrability

Harish-Chandra's criterion for square integrability (see [Cas, Theorem 4.4.6] or [Si80, Theorem 4.4.4]) asserts that π is a discrete-series representation in the sense above if and only if it is *square-integrable* π, in the sense that its matrix coefficients $f_{v,v'}(x) = \langle \pi'(x)v, v' \rangle$ are absolutely square-integrable functions on $G(F)/Z(F)$.

DEFINITION 10.5. We say that a discrete series $G'(F)$-module π' satisfies a *trace identity* if the following hold:

(1) There is a set $\{\pi\}$ which, for any open compact subgroup C in $G(F)$, contains only finitely many $G(F)$-modules with a C-fixed vector.
(2) There are positive integers $m(\pi)$ (depending on π') and a complex number c so that, for all matching ϕ in $C_c^\infty(G'(F))$ and f in $C_c^\infty(G(F))$, we have

$$c \operatorname{tr} \pi'(\phi) = \sum m(\pi) \operatorname{tr} \pi(f). \tag{10.5.1}$$

Assumption 10.6. We make the following assumption. For any proper Levi subgroup M, and any open compact subgroup C as in Proposition 7.5, there exists ϕ^M in $C_c^\infty(M'(F))$ matching the characteristic function of $C \cap M(F)$ in $C_c^\infty(M(F))$.

Our assumption is tantamount to the following. For any proper Levi subgroup M with center A_M contained in A, and any t_0 in $A_M(F)$, we have the following. There exists a function $\phi_{t_0}^M$ in $C_c^\infty(M'(F))$ matching the function $f_{t_0}^M$ in $C_c^\infty(M(F))$ defined in Subsection 7.4. Indeed, the function $f_{t_0}^M$ is obtained from the characteristic function of $C \cap M(F)$ on translating by the central element t_0 and multiplying by a scalar, so that $\phi_{t_0}^M$ can be obtained from ϕ^M on translating by the central element t_0 and multiplying by the same scalar.

11 Decay

PROPOSITION 11.1. *Suppose that the discrete-series $G'(F)$-module π' satisfies a trace identity* (10.5.1)*, and $G(F)$ satisfies Assumption* 10.6. *Then all π in* (10.5.1) *are discrete series $G(F)$-modules.*

PROOF. Let M be a proper Levi subgroup, C a compact open subgroup of $G(F)$ as in Proposition 7.5, t_0 in $A_M(F)$ such that $|\alpha(t_0)| \leq 1$ for all roots α of A_M in the unipotent radical of the standard parabolic subgroup with Levi component M, and $f_{t_0}^M$ the function of Proposition 7.5. Proposition 7.5 and Proposition 7.7 imply that the function f_{t_0} on $G(F)$ defined in Proposition 7.5, which is C-bi-invariant, satisfies $I(x, f_{t_0}) = {}^M I(x, f_{t_0}^M)$; hence

$$I^s(x, f_{t_0}) = {}^M I^s(x, f_{t_0}^M)$$

for all regular x in $G(F)$.

As noted following Proposition 7.5, the function f_{t_0} is C-bi-invariant. Hence $\operatorname{tr}\pi(f_{t_0}) \neq 0$ only for π with a nonzero C-invariant vector. By definition of the trace identity (10.5.1), there are only finitely many such π in (10.5.1). On the other hand, if ω_τ is the central character of the irreducible constituent τ of the $M(F)$-module π_N, then $\operatorname{tr}\pi(f_{t_0}) = \operatorname{tr}\pi_N(f_{t_0}^M)$ is a sum over τ of $\omega_\tau(t_0)\delta_P^{1/2}(t_0)n(\tau, C)$, where $n(\tau, C)$ is the nonnegative integral multiplicity of the trivial representation of C in τ (the dimension of the space of C-fixed vectors in τ).

Assumption 10.6 asserts that there exists a function $\phi_{t_0}^M$ in $C_c^\infty(M'(F))$ matching $f_{t_0}^M$. Proposition 7.7 asserts that there exists a function ϕ_{t_0} in $C_c^\infty(G'(F))$ with $I^s(x, \phi_{t_0}) = {}^M I^s(x, \phi_{t_0}^M)$ for all regular x in $G'(F)$. Hence the functions f_{t_0} on $G(F)$ and ϕ_{t_0} on $G(F)$ are matching. Since π' appears in the trace identity, it is clear that its character $\chi_{\pi'}$ is a stable function, depending only on the stable conjugacy class of x in $G'(F)$. Using the Weyl integration formula, we have

$$\operatorname{tr}\pi'(\phi) = \sum_T^s w(T)^{-1} \int_{T(F)/Z(F)} (\Delta\chi_{\pi'})(x) I^s(x, \phi_{t_0})\, \mathrm{d}(Nx).$$

The sum is over the stable conjugacy classes of F-tori in $G(F)$; $w(T)$ is the cardinality of the quotient $W(T) = N(T)/T$ by T of the group $N(T)$ of x in $G(\overline{F})$ such that $\mathrm{Ad}(x) : T(F) \to T(F)$, $t \mapsto xtx^{-1}$, is defined over F. Recall that

$$I^s(x, \phi_{t_0}) = {}^M I^s(x, \phi_{t_0}^M) = {}^M I^s(Nx, f_{t_0}^M).$$

Since t_0 lies in the center of $M(F)$, we have $M \subset M_{t_0}$. As we assumed that $|\alpha(t_0)| < 1$ for all roots α of A_M in N_M, we have $M = M_{t_0}$. But $f_{t_0}^M$ is supported on a small neighborhood of t_0. Hence $I^s(x, \phi_{t_0}) \neq 0$ implies that M_x^0 is equal to M. Proposition 7.3 now implies that we have

$$\sum_T w(T)^{-1} \int (\Delta_M \chi_{\pi'_N})(x) \cdot {}^M I^s(x, \phi_{t_0}^M)\, \mathrm{d}(Nx).$$

As t_0 lies in the center of $M(F)$, changing variables $x \mapsto t_0 x$, we obtain

$$\sum_{\tau' \text{ in } \pi'_N} \omega_{\tau'}(t_0) \sum_T w(T)^{-1} \int (\Delta_M \chi_{\tau'})(x) \cdot {}^M I'(x, \phi_1^M)\, \mathrm{d}(Nx) = \sum_{\tau'} \omega_{\tau'}(t_0) \operatorname{tr}\tau'(\phi_1^M),$$

with τ' ranging over the irreducible subquotients of π'_N. Here $\omega_{\tau'}$ is the central character of τ'.

We conclude from the trace identity that for any t_0 in the center of $M(F)$ with $M_{t_0} = M$ in the notation of 7.2, we have

$$\sum_{\tau'} c(\tau')\omega_{\tau'}(t_0) = \sum_{\tau} n(\tau)\omega_\tau(t_0).$$

The sum on the left ranges over the constituents τ' of π'_N; hence it is finite, since π', whence π'_N, is admissible. The $c(\tau')$ are complex numbers. On the right the sum is finite, depending on the compact open subgroup C, and the coefficients are positive, so that no cancelation may occur. Linear independence of characters (on the set of t_0 in $A_M(F)$ with $|\alpha(t_0)| < 1$ for the positive roots α) implies that for each τ there exists τ' with $\omega_\tau(t_0) = \omega_{\tau'}(t_0)$. Consequently the character ω_τ decays, where τ is any constituent of π_N. Here M is any proper Levi subgroup of G, and π is any $G(F)$-module with a nonzero C-fixed vector. Since any π has a nonzero C-fixed vector for a sufficiently small C, it follows that all π are discrete series, as required. □

REMARK 11.2. It is clear that if π' is assumed to be only tempered (see 13.6), then the above proof implies that the π of (10.5.1) are tempered.

12 Finiteness

We now continue with the situation and Assumption 10.6 of Section 11. We make two additional assumptions.

Assumption 12.1. Suppose that Φ^s is a stable function in $S_e(G'(F))$ (see Section 6). Then there exists Φ on $G(F)$ in $S_e(G(F))$ matching Φ^s.

Namely, we suppose that $\Phi^s(x) = \Phi^s(x')$ for all stably conjugate x, x' and assume the existence of a function $f \in A_c^\infty(G(F))$ with $\widetilde{\Phi}^s(Nx, f) = \Phi^s(x)$ on the regular set.

Assumption 12.2. For any f in $A_c^\infty(G(F))$, there exists a matching function ϕ in $A_c^\infty(G'(F))$.

Using these assumptions, we have the following:

PROPOSITION 12.3. *Suppose that the discrete series $G'(F)$-module π' satisfies a trace identity* (10.5.1). *Then the set of π is finite.*

PROOF. Note that Proposition 11.1 asserts that the π are all discrete series. To prove our proposition, note that by the trace identity (10.5.1), $\mathrm{tr}\,\pi'(\phi)$ depends only on f, namely, on the stable orbital integral of ϕ; hence the character $\chi_{\pi'}$ of π' on $G(F)$ is a stable function. Assumption 12.1 implies that there exists a finite linear combination of $G(F)$-modules π with complex coefficients $c(\pi)$ so that

$$\sum_{\pi} c(\pi)\chi_\pi(Nx) = \chi_{\pi'}(x)$$

for any elliptic regular x in $G'(F)$, and $\sum_\pi c(\pi)[\chi(\pi)](y) = 0$ for the elliptic regular y which are not norms. We may assume that all π here are tempered by [Ka86.1, Proposition 1.1].

Applying the Weyl integration formula, we deduce that

$$\operatorname{tr}\pi'(\phi) = \sum_T^s w(T)^{-1} \int (\Delta\chi_{\pi'})(x) I^s(x,\phi)\, \mathrm{d}x.$$

Only elliptic tori occur since we take ϕ in $A_c^\infty(G'(F))$. Further, we take ϕ so that it has a matching f, so that $I^s(x,\phi) = I^s(Nx,f)$. Replacing $\chi(\pi')$ by our linear combination $\sum_\pi c(\pi)\chi(\pi)$, we obtain

$$\sum_\pi c(\pi) \sum_T^s w(T)^{-1} \int (\Delta\chi_\pi)(x) I^s(x,f)\, \mathrm{d}x = \sum_\pi c(\pi) \operatorname{tr}\pi(f).$$

We deduce from (10.5.1) the identity $\sum_\pi c(\pi) \operatorname{tr}\pi(f) = \sum_\pi m(\pi) \operatorname{tr}\pi(f)$. On the left the sum is finite and consists of tempered π. On the right all π are discrete series. The identity holds for all f in $A_c^\infty(G(F))$ which have a matching function ϕ. So fix π_0 on the right. By [Ka86.1, Theorem K], there exists a pseudo-coefficient f_0 in $A_c^\infty(G(F))$ with $\operatorname{tr}\pi_0(f_0) = 1$ and $\operatorname{tr}\pi(f_0) = 0$ for any tempered irreducible π inequivalent to π_0. But Assumption 12.2 implies that f_0 has a matching function ϕ. Using our identity with $f = f_0$, we conclude that $m(\pi_0) = 0$ for all π_0 on the right which are not equivalent to any of the finitely many π on the left. Consequently, the set of π with $m(\pi) \neq 0$ is finite, as asserted. □

13 Simple Algebras

13.1 Invariants of Simple Algebras

Let F be a local field. Let G be an inner form of $\mathrm{GL}(n)$ over F. Thus G is the multiplicative group of a central simple F-algebra A. There is a central division algebra D over F of rank d with $A = M(m,D)$, $n = md$. Class field theory—see [We67]—associates with A an invariant $\operatorname{inv} A$ of the form i/d (modulo 1), with i prime to d, and $\operatorname{inv} A = \operatorname{inv} D$ independent of m. There exists a unique central simple F-algebra A of rank n with invariant i/d (modulo 1) (where $(i,d) = 1$ and d divides n). If $F = \mathbb{C}$, then $d = 1$. If $F = \mathbb{R}$, then $d = 1$ or 2. Otherwise, d can be any positive integer. We let $G' = \mathrm{GL}(n)$. Note that if $\overline{F}$ is an algebraic closure of F, then $G(\overline{F}) = \mathrm{GL}(n,\overline{F})$. This isomorphism is over $\overline{F}$.

13.2 Regular Conjugacy Classes

A conjugacy class γ in $G(F)$ is called *regular* if its characteristic polynomial p_γ has distinct roots (in some algebraic closure of F). If γ, δ are regular and $p_\gamma = p_\delta$, then $\gamma = \delta$. There is an embedding $\gamma \mapsto \gamma'$, defined by $p_{\gamma'} = p_\gamma$, of the set of regular conjugacy classes γ in $G(F)$ into the set of regular conjugacy classes γ' in $G'(F)$.

13.3 Grothendieck Group

Let $C_c^\infty(G(F))$ denote the convolution algebra of complex valued smooth compactly supported measures f on $G(F)$. These are all of the form fdg where dg is any fixed Haar measure on $G(F)$ and f is now a function. Put $R(G(F)) = R_{\mathbb{Z}}(G(F)) \otimes \mathbb{C}$, where $R_{\mathbb{Z}}(G(F))$ is the Grothendieck group of the category of smooth $G(F)$-modules. It is the free abelian group generated by the set $\Pi(G(F))$ of equivalence classes of irreducible such $G(F)$-modules. By [BZ76, Theorem 3.25] irreducible smooth $G(F)$-modules are admissible.

13.4 Character

If π is an admissible $G(F)$-module, then the convolution operator $\pi(f\,\mathrm{d}g)$, defined by $\int_{G(F)} f(g)\pi(g)\,\mathrm{d}g$, is of finite rank, and its trace is denoted by $\operatorname{tr}\pi(f)$. We often delete the measure $\mathrm{d}g$ from the notation, namely, think of f as a measure. There exists a complex valued conjugacy invariant smooth function $\chi = \chi_\pi$ on the regular set of $G(F)$ with $\operatorname{tr}\pi(f) = \int \chi(g)f(g)$ for any $f \in C_c^\infty(G(F))$ which is supported on the regular set of $G(F)$. It is called the *character* of π. It depends only on the image of π in $R(G(F))$. The characters of inequivalent irreducible $G(F)$-modules are linearly independent. In particular, $\chi \neq 0$ if $\pi \neq 0$ in $R(G(F))$. Harish-Chandra [HC78] showed that χ extends to a locally integrable function on $G(F)$.

13.5 Induction

Fix a minimal parabolic subgroup P_0 together with its Levi decomposition M_0N_0 in G. Denote by i_M^G the homomorphism $R(M(F)) \to R(G(F))$ of normalized, or unitary, induction, for any (standard) Levi subgroup M. Here "standard" means that M is the Levi subgroup containing M_0 of a parabolic subgroup P containing P_0. The map i_M^G is independent of the choice of the parabolic subgroup P with Levi component M, which is used in its definition.

13.6 Tempered Representations

An irreducible $G(F)$-module π whose central character ω_π is unitary is called *square-integrable* π or *discrete series* if it has a matrix coefficient which is square-integrable on $G(F)$ modulo its center $Z(F)$. In this case all of its matrix coefficients are square-integrable, and π embeds in $L^2(G, Z, \omega_\pi)$. An alternative definition is that all of its central exponents decay.

An irreducible $G(F)$-module π is called *tempered* if there exists a Levi subgroup $M(F)$ and a square-integrable (= discrete series) $M(F)$-module τ such that π is a subquotient (necessarily a direct summand) of $i_M^G \tau$.

Put $\nu(x) = |x|$ for $x \in F$, where $|\cdot|$ is the normalized valuation on F. Put $\nu(g) = \nu(\det g)$, where $\det g$ is the reduced norm of g in $G(F)$.

DEFINITION 13.7. *A $G(F)$-module π is called* relevant *if there is a Levi subgroup of $G(F)$ of the form $M = \prod_{i=1}^m (M_i \times M_i)$ or $M_0 \times M$, where the M_i, $0 \le i \le m$, are multiplicative groups of central simple F-algebras, and tempered $M_i(F)$-modules τ_i and distinct positive numbers $s_i < 1/2$ such that π is*

$$i_M^G \left(\prod_{i=1}^m (\tau_i \nu^{s_i} \times \tau_i \nu^{-s_i}) \right)$$

or

$$i_{M_0 \times M}^G \left(\tau_0 \times \prod_{i=1}^M (\tau_i \nu^{s_i} \times \tau_i \nu^{-s_i}) \right)$$

in $R(G(F))$.

THEOREM 13.8 (Local Theorem for Non-degenerate Representations).

(1) *Relevant representations of $G(F)$ are unitarizable and irreducible. In particular, a $G(F)$-module normalizedly (= unitarily) induced from a tempered one is irreducible.*
(2) *The relation $\chi'(\gamma') = (-1)^{n-m}\chi(\gamma)$ for all matching $\gamma \mapsto \gamma'$ regular conjugacy classes $\gamma \in G(F)$, $\gamma' \in G'(F)$, defines a bijection between the set of equivalence classes of square-integrable (resp. tempered, relevant) $G(F)$-modules π and the set of equivalence classes of square-integrable $G'(F)$-modules π' (resp. tempered, relevant, $G'(F)$-modules π' whose character χ' is nonzero on the set of regular γ' obtained from γ in $G(F)$).*

The bijection of (2) is called the Deligne-Kazhdan correspondence.

13.9 Global Invariants

Let F be a global field. Let G be an inner form of $G' = \mathrm{GL}(n)$ over F. Then G is the multiplicative group of a central simple F-algebra $A = M(m, D)$, for D a central division algebra over F of rank d, $n = md$. Class field theory—see [We67]—associates with A the sequence $\{\mathrm{inv}_v A = \mathrm{inv} A \otimes_F F_v\}$ of rational numbers modulo one which are almost all zero and whose sum is zero modulo one. Each such sequence $\{i_v/d_v\}$ determines, up to F-isomorphism, a unique division algebra D central over F and a unique simple algebra A of rank n central over F with these invariants, for any n which is divisible by d_v for all v. Let $G(\mathbb{A})$ be the group of $\mathbb{A}$-points of G, where $\mathbb{A}$ is the ring of adèles of F. Let Z (resp. Z') denote the center of G (resp. G'). Then $Z = Z'$ is the multiplicative group. Fix a unitary character ω of $Z(\mathbb{A})/Z(F) = \mathbb{A}^\times/F^\times$. For each place v of F, denote by F_v the completion of F at v and by ω_v the restriction of ω to $F_v^\times$.

13.10 Automorphic Representations

Let $L(G(F)\backslash G(\mathbb{A}))$ denote the space of slowly increasing, see [BJ79], functions ϕ on $G(F)\backslash G(\mathbb{A})$ with $\phi(zg) = \omega(z)\phi(g)$ for z in $Z(\mathbb{A})$. By $L^2(G(F)\backslash G(\mathbb{A}))$ we mean the space of ϕ on $G(F)\backslash G(\mathbb{A})$ with $\phi(zg) = \omega(z)\phi(g)$ for z in $Z(\mathbb{A})$ that are square-integrable on $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$. The group $G(\mathbb{A})$ acts on $L = L(G(F)\backslash G(\mathbb{A}))$ and on $L^2 = L^2(G(F)\backslash G(\mathbb{A}))$ by right translation. Thus $(r(g)\phi)(h) = \phi(hg)$. Any irreducible subquotient of L^2 is unitary (preserves a non-degenerate inner product). The irreducible subquotients of L are called *automorphic* $G(\mathbb{A})$-modules.

The space $L = L(G(F)\backslash G(\mathbb{A}))$ is the direct sum of the discrete spectrum $L_d = L_d(G(F)\backslash G(\mathbb{A}))$, which is the direct sum of all irreducible *sub*representations of L—these $G(\mathbb{A})$-modules are called "discrete spectrum" $G(\mathbb{A})$-modules, and the continuous spectrum $L_c = L_c(G(F)\backslash G(\mathbb{A}))$, which is a "continuous sum." The space L_d is also the discrete spectrum in L^2.

A *cuspidal* $G(\mathbb{A})$-module is an irreducible constituent of the subspace $L_0 = L_0(G(F)\backslash G(\mathbb{A}))$, which consists of the ϕ in $L(G(F)\backslash G(\mathbb{A}))$ with $\int_{N(F)\backslash N(\mathbb{A})} \phi(nx)\,\mathrm{d}n$ equals zero for every x in $G(\mathbb{A})$ and for the unipotent radical N of any proper parabolic subgroup of G over F. Each cuspidal ϕ is rapidly decreasing, hence absolutely square-integrable, on $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$. The space $L_0(G(F)\backslash G(\mathbb{A}))$ is a sub-$G(\mathbb{A})$-module of $L_d(G(F)\backslash G(\mathbb{A}))$.

Any cuspidal $G'(\mathbb{A})$-module π is non-degenerate, namely, each of its local components π_v has a Whittaker model. This means that for any additive nontrivial character $\psi_v : F_v \to \mathbb{C}^\times$, there is an embedding of the $G'(F_v)$-module π_v in the space $W(\psi_v)$ of right smooth functions $\varphi : G'(F_v) \to \mathbb{C}$ with $\varphi(ng) = \psi_v(\sum_i n_{i,i+1})\varphi(g)$ for $n = (n_{i,j})$ upper triangular unipotent matrix and $g \in G'(F_v)$, with $G'(F_v)$ action by right shifts.

Moreover such π is generic (has a global Whittaker model: same definition, but with global $\psi : \mathbb{A}/F \to \mathbb{C}^\times$, $\psi \neq 1$), and it occurs with multiplicity one in $L_0(G'(F)\backslash G'(\mathbb{A}))$. See [BZ76] and [PS79] for Whittaker models and [Shal74] or [PS79] for the multiplicity one theorem.

An irreducible admissible $G(\mathbb{A})$-module π decomposes as a restricted tensor product $\otimes_v \pi_v$ of irreducible admissible $G(F_v)$-modules π_v, almost all of which are unramified; see [Ft79].

An irreducible π_v is call unramified if its space contains a nonzero $K_v = G'(R_v)$-fixed vector ξ_v, necessarily unique up to a scalar. Here v is finite and R_v is the ring of integers in F_v.

The space of the irreducible π is the union over all sets S of valuations of F such that π_v is unramified for $v \notin S$, of $(\otimes_{v\in S} V_v) \otimes (\otimes_{v\notin S}\xi_v)$, where V_v is the space of π_v.

If $\pi'_1 = \otimes_v \pi'_{1,v}$ and $\pi'_2 = \otimes_v \pi'_{2,v}$ are cuspidal $G'(\mathbb{A})$-modules and $\pi'_{1,v} \simeq \pi'_{2,v}$ for almost all v, then $\pi'_{1,v} \simeq \pi'_{2,v}$ for all v by the rigidity theorem, also called the "strong multiplicity one theorem"; see [PS79]. All components of a cuspidal $G'(\mathbb{A})$-module are relevant by [Ze80, (9.7)] and, as noted above, unitarizable. Note that an irreducible π in L^2 is unitary (its space is complete in an invariant inner form). Its subspace π^∞ of smooth vectors is admissible and unitarizable. A representation π is called unitarizable if its space is dense in a unitary representation $\widehat{\pi}$.

13.11 Correspondence

Given G or D, there is a finite set S of places v of F such that for every v outside S the division algebra D *splits*, namely, $D \otimes_F F_v = M(d, F_v)$. We say that π_v *corresponds* to π'_v if $G_v \simeq G'_v$ (thus $v \notin S$) and $\pi_v \simeq \pi'_v$, or, more generally for arbitrary v, if π_v corresponds to π'_v by the Local Theorem 13.8. In this case we also say that π'_v comes from π_v. An irreducible $G(\mathbb{A})$-module $\pi = \otimes_v \pi_v$ *lifts*, or *corresponds*, to an irreducible $G'(\mathbb{A})$-module $\pi' = \otimes_v \pi'_v$ if π_v *corresponds* to π'_v for all v. We also say that π' comes from π , in this case. An automorphic $G(\mathbb{A})$-module which lifts to a cuspidal $G'(\mathbb{A})$-module will be called G'*-cuspidal*.

THEOREM 13.12 (Global Theorem for Generic Representations).

(1) *All local components of a G'-cuspidal $G(\mathbb{A})$-module are relevant.*
(2) *Each G'-cuspidal $G(\mathbb{A})$-module occurs in the discrete spectrum L_d of $L = L(G(F)\backslash G(\mathbb{A}))$ with multiplicity one.*
(3) *If $\pi_1 = \otimes_v \pi_{1,v}$ and $\pi_2 = \otimes_v \pi_{2,v}$ are G'-cuspidal $G(\mathbb{A})$-modules and $\pi_{1,v} \simeq \pi_{2,v}$ for almost all v, then $\pi_1 \simeq \pi_2$.*
(4) *Correspondence defines a bijection from the set of G'-cuspidal $G(\mathbb{A})$-modules $\pi = \otimes_v \pi_v$ to the set of cuspidal $G'(\mathbb{A})$-modules $\pi' = \otimes \pi'_v$ such that π'_v is obtained by the local correspondence for all v (in S). Every cuspidal $G(\mathbb{A})$-module is G'-cuspidal.*

REMARK. Part (1) is the motivation for the definition of "relevant" representations. Part (2) is called the "multiplicity one" theorem for the G'-cuspidal spectrum of G. Part (3) is called the "rigidity" theorem for the G'-cuspidal spectrum. Part (4) is called the Deligne-Kazhdan correspondence.

13.13 Remarks

The local theorem is proven below for F of characteristic 0. The positive characteristic case follows from [Ka86.2]. Theorem 13.12 is proven here only for the subset of the cuspidal $G'(\mathbb{A})$-modules π' with two cuspidal components, using the simple form of the trace formula of Corollary 4.4. We reduce "two" to "one" using the same version of the trace formula and "regular functions" in Section 26. This Corollary 4.4 applies to any test function $f = \otimes_v f_v$ which has a cuspidal component f_u, and at a second place u', the component $f_{u'}$ is *any* function whose orbital integrals vanish on the regular non-elliptic set. This $f_{u'}$ lies in the class $A_c^\infty(G(F_{u'}))$ of [Ka86.1]; see also 6.9, which is called the class of discrete functions in [BDK86] (see also 8.9). In particular, $f_{u'}$ can be taken to be a pseudo-coefficient of any square-integrable $G(F_{u'})$-module. Had we proven Corollary 4.4 only for f such that $f_{u'}$ is supported on the elliptic regular set, we would not have been able to prove Global Theorem 13.12 except in the special, more elementary case where the simple algebra underlying G is a division algebra which is a division algebra at some place. An elementary proof of this case is given in the first section.

14 Germs

14.1 Comparison

The goal is to compare representations of and orbital integrals on $G' = \mathrm{GL}(n)$ with the analogous objects associated to its inner form G. Note that there is no difference between conjugacy and stable conjugacy; if two elements of $G(F)$ are conjugate in $G(\overline{F})$, then they are conjugate in $G(F)$. This property fails for a general reductive connected F-group.

14.2 Matching

Let F be a local non-Archimedean field of characteristic 0. Fix Haar measures $\mathrm{d}x$ and $\mathrm{d}x'$ on $G(F)$ and $G'(F)$. Write $\gamma \mapsto \gamma'$ if γ, γ' are semisimple elements of $G(F)$ and $G'(F)$ with $p_\gamma = p_{\gamma'}$. If γ, γ' are regular, that is, have distinct eigenvalues, their centralizers in G, G' are tori T, T'. These tori are isomorphic if $\gamma \mapsto \gamma'$. In this case we take the Haar measures on $T(F)$ and $T'(F)$ to be equal. The orbital integral $\Phi(x,f)$ of a function f in $C_c^\infty(G(F))$ and its analogue for $G'(F)$ was defined in Subsection 2.4.

PROPOSITION. *For every f in $A_c^\infty(G(F))$, there exists f' in $A_c^\infty(G'(F))$, and for every such f', there is such f, so that $\widetilde{\Phi}(\gamma,f) = \widetilde{\Phi}(\gamma',f')$ for every elliptic regular γ and γ' with $p_\gamma = p_{\gamma'}$.*

PROOF. This follows from Proposition 6.11. □

This proposition proves assumptions 12.1 and 12.2 in the present case.

DEFINITION 14.3. The functions f in $C_c^\infty(G(F))$ and f' in $C_c^\infty(G'(F))$ are called *matching* if $\Phi(x,f) = \Phi(x',f')$ for all regular x' in $G'(F)$ and x in $G(F)$ with $p_x = p_{x'}$, and $\Phi(x',f') = 0$ for all regular x' in $G'(F)$ which do not come from $G(F)$.

We also state the following:

THEOREM 14.4. *For every f in $C_c^\infty(G(F))$, there exists f' in $C_c^\infty(G'(F))$, and for every f' in $C_c^\infty(G'(F))$ so that $\Phi(f')$ is zero at any regular x in $G'(F)$ which does not come from $G(F)$, there exists f in $C_c^\infty(G(F))$, so that f and f' are matching.*

This theorem will be proven by induction on the Levi subgroup of $G(F)$. Hence we now assume the validity of the theorem for every proper Levi subgroup M of G. Consequently, we can use Assumption 10.6 in our case. The proof is based on the correspondence theorem for tempered local representations. We will complete the proof in Section 22.

15 Comparison

15.1 Measures

Let F be global, $n = md$, $G = \mathrm{GL}(m,D)$ the multiplicative group of the $m \times m$ matrix algebra over the central division algebra D of dimension d^2 over F, and $G' = \mathrm{GL}(n)$. Note that $G(F_v) = \mathrm{GL}(m,D_v)$. By definition of inner forms, G and G' are isomorphic over an algebraic closure $\overline{F}$ of F. Using this isomorphism, we can transfer a differential form of maximal degree on G' rational over F to one on G. These define Haar measures $\mathrm{d}x_v$ and $\mathrm{d}'x_v$ on $G(F_v)$ and $G'(F_v)$ for all v, which we call compatible, and consequently we can choose compatible product measures $\mathrm{d}x = \otimes \mathrm{d}x_v$ and $\mathrm{d}'x = \otimes \mathrm{d}'x_v$ on $G(\mathbb{A})$, $G'(\mathbb{A})$.

15.2 Conjugacy Classes

There is a bijection from the set of conjugacy classes in $D^\times$ (over a local or global field), to the set of elliptic conjugacy classes in $\mathrm{GL}(d,F)$. Similarly, there is a bijection from the set of semisimple conjugacy classes in $G(F) = \mathrm{GL}(m,D)$ to the set of semisimple conjugacy classes in $G'(F) = \mathrm{GL}(n,F)$ with an elliptic representative in the Levi subgroup $\prod_i \mathrm{GL}(da_i,F)$, $\sum_i a_i = m$. Globally, if G ramifies at the finite set V of places of F, there is a bijection from the set of conjugacy classes of tori T in G over F into the set of conjugacy classes of tori T' in G' such that at each v in V the torus $T'(F_v)$ of $G'(F_v)$ is obtained from a F_v-torus $T(F_v)$. We choose compatible product measures $\mathrm{d}t = \otimes_v \mathrm{d}t_v$, $\mathrm{d}'t = \otimes_v \mathrm{d}'t_v$ on the matching tori $T(\mathbb{A})$, $T'(\mathbb{A})$, which are isomorphic over F.

15.3 Test Functions

We choose functions $f = \otimes_v f_v$ on $G(\mathbb{A})$ and $f' = \otimes_v f'_v$ on $G'(\mathbb{A})$ such that f_v and f'_v are matching for all v. In fact, for v outside V, the groups $G(F_v)$ and $G'(F_v)$ are isomorphic over F_v, and we take f_v, f'_v equal under this isomorphism. For almost all v, we take $f_v = f_v^0 = f'_v$. Corollary 14.2 and the inductive assumption of Theorem 14.4 show that there exist sufficiently many matching pairs in $C_c^\infty(G(F_v))$, $C_c^\infty(G'(F_v))$ for our purposes.

PROPOSITION 15.4. *If f and f' are matching and satisfy (each) the (three) requirements of Corollary 4.4, then $\sum \operatorname{tr} \pi'(f') = \sum m(\pi) \operatorname{tr} \pi(f)$. The sums range over the cuspidal spectra of $L^2(G'(F)\backslash G'(\mathbb{A}))$ and $L^2(G(F)\backslash G(\mathbb{A}))$.*

PROOF. This follows from Corollary 4.4. □

We used the multiplicity one theorem for $L_0^2(G'(F)\backslash G'(\mathbb{A}))$ to conclude that the multiplicities $m(\pi')$ on the left are equal to 1.

16 Existence

16.1 Pseudo-Coefficients

Let $G(F)$ be a reductive p-adic group and π_0 a square-integrable $G(F)$-module. A *pseudo-coefficient* of π_0 is a function f in $A_c^\infty(G(F))$ (see Section 6) with $\operatorname{tr} \pi_0(f) = 1$ and $\operatorname{tr} \pi(f) = 0$ for every tempered (irreducible) $G(F)$-module π inequivalent to π_0. If π_0 is cuspidal, then each of its (normalized) matrix coefficients is a pseudo-coefficient (in fact $\operatorname{tr} \pi(f) = 0$ if π is irreducible and inequivalent to π_0). In general, the existence of a pseudo-coefficient is proven in [Ka86.1, Theorem K]. See also [BDK86].

Let F be a global field. Fix a finite set V of non-Archimedean places. Fix three distinct non-Archimedean places w, u and u' outside V. Although more general variants of the following proposition can be proven, for simplicity we now assume that $G = \mathrm{GL}(n)$.

PROPOSITION 16.2. *Fix a cuspidal $G(F_u)$-module $\pi_{0,u}$. Let $\pi_{0,w}$ be a square-integrable $G(F_w)$-module. Then there exists a cuspidal $G(\mathbb{A})$-module $\pi = \otimes_v \pi_v$, such that the following properties hold:*

(1) $\pi_w \simeq \pi_{0,w}$.
(2) $\pi_u \simeq \pi_{0,u}$.
(3) *For each v in V the component π_v is Steinberg.*
(4) *$\pi_{u'}$ is square-integrable.*
(5) *π_v is unramified for each non-Archimedean place $v \neq u$, u', w outside V.*

PROOF. We use Corollary 4.4 with a function $f = \otimes_v f_v$, chosen to have the following properties.

- f_w is a pseudo-coefficient of $\pi_{0,w}$.
- f_u is a matrix coefficient of $\pi_{0,u}$.
- For each v in V, the component f_v is a pseudo-coefficient of the Steinberg $G(F_v)$-module.
- $f_{u'}$ is supported on the regular elliptic set in $G(F_{u'})$.
- At each finite $v \neq u, u', w$ outside V, we take spherical (K_v-bi-invariant) f_v, with $f_v = f_v^0$ for almost all v.

These components can be and are chosen so that $\Phi(x,f) \neq 0$ for some elliptic regular x in $G(F)$. Since the sum of (4.3.1) is finite, we can reduce the support of $f_{u'}$ so that the sum (4.3.1) consists of a single entry; hence it is nonzero. Hence there is a cuspidal π with $\mathrm{tr}\,\pi(f) \neq 0$. This π is generic; hence each of its local components π_v is non-degenerate. It is easy to check that π has the properties required by the proposition, using the following:

REMARK. A $G(F_v)$-module is called *elliptic* if its character is not identically zero on the regular elliptic set of $G(F_v)$. Theorem 9.7(b) of [Ze80] implies that every irreducible non-degenerate elliptic $G(F_v)$-module is square-integrable (in fact of a "generalized Steinberg" type).

The proposition follows. □

16.3 Hilbert-Schmidt Operators

For this subsection we consider a locally compact unimodular group H with center Z, ω a character of Z of absolute value one, and set $f^*(h) = \overline{f}(h^{-1})$. Let $L(H)$ denote the convolution $*$-algebra of complex valued functions on H with $f(zh) = \omega(z)^{-1} f(h)$, $h \in H$ and $z \in Z$, such that $|f(h)|^2$ is integrable on H/Z. For a unitary irreducible H-module π, put $\pi(f) = \int_{H/Z} f(h)\pi(h)\,\mathrm{d}h$. Suppose B is a dense $*$-closed subalgebra of $L(H)$, I is a set, $\{\pi_i\}_{i\in I}$ is a set of irreducible unitary pairwise inequivalent H-modules such that $\pi(f)$, $\pi_i(f)$ are Hilbert-Schmidt operators for all f in B, and $\|\cdot\|$ is the norm. Suppose that $\{c_i\}_{i\in I}$ is a set of nonnegative real numbers such that $\sum_i c_i\|\pi_i(f)\|^2$ is finite for all f in B. Then the remark in the second (proven in the third) paragraph in the proof of Lemma 16.1.1 (on page 251 of recent printing and page 496 in the original draft) of [JL70] asserts the following. For each positive ϵ, there exists f in B with $\|\pi(f)\| \neq 0$ and $\sum_i c_i\|\pi_i(f)\|^2 \leq \epsilon\|\pi(f)\|^2$.

LEMMA. *If $\{d_i\}_{i\in I}$ are complex numbers such that $\sum_i d_i \operatorname{tr}\pi_i(f*f^*)$ is absolutely convergent to zero for all f in B, then $d_i = 0$ for all i.*

PROOF. Note that $\operatorname{tr}\pi_i(f*f^*) = \|\pi_i(f)\|^2$. If $d_0 \neq 0$, there is f in B such that

$$\sum_{i\neq 0} |d_i| \operatorname{tr}\pi_i(f*f^*)$$

is bounded by $(1/2)|d_0| \operatorname{tr}\pi_0(f*f^*) \neq 0$, and we arrive at a contradiction. □

17 Isolation

Let F_w be a local non-Archimedean field of characteristic 0. Let G_w be the multiplicative group of the matrix algebra $M(m, D_w)$, where D_w is a central division algebra over F_w of rank d and invariant i/d (modulo one), with $(i,d) = 1$. Put $G' = \mathrm{GL}(n)$, $n = md$.

Recall the following standard notation. If S is any finite set of places of F, put $\pi^S = \otimes_v \pi_v$ and $f^S = \otimes_{v\notin S} f_v$. Put also $\pi_S = \otimes_{v\in S}\pi_v$ and $f_S = \otimes_{v\in S} f_v$. Denote by S_∞ the set of Archimedean places of F. Put $\mathbb{A}_f$ for the ring of adèles without Archimedean components.

PROPOSITION 17.1. *For every square-integrable $G'(F_w)$-module π'_w, there exist G_w-modules π_w and positive integers $m(\pi_w)$ such that, for all matching f'_w and f_w, we have*

$$(-1)^{n-m} \operatorname{tr}\pi'_w(f'_w) = \sum m(\pi_w) \operatorname{tr}\pi_w(f_w).$$

If C_w is an open compact subgroup of G_w, then the sum consists only of finitely many π_w with a nonzero C_w-invariant vector.

PROOF. Let F be a totally imaginary number field whose completion at some place w is our local field F_w. Choose a set V of $n - m + 1$ non-Archimedean places including w. We may assume that i is prime to $n = md$, since there are infinitely many primes in the arithmetic progression $\{i+kd;\ k \geq 0\}$. Choose a division algebra D central over F with the following invariants. At w it is i/d, it is i/n at each $v \neq w$ in V and 0 outside V. Take $G = D^\times$. Then $G(F_w)$ is our $G_w = \mathrm{GL}(m, D_w)$, where $\mathrm{inv}_w D_w = i/d$. Fix three distinct non-Archimedean places u, u', and u'' of F outside V, a cuspidal $G(F_u)$-module π_u and a matrix coefficient f_u of π_u. Choose a unitary irreducible $G(F_\infty)$-module π_∞. Using Lemma 16.3 with the dense $*$-closed subalgebra $B = C_c^\infty(G(F_\infty))$, we conclude from Proposition 15.4 that if $f'^\infty = \otimes f'_v$ and $f^\infty = \otimes_{v \notin S_\infty} f_v$ and f'_v, f_v are matching for all v, then

$$\sum \operatorname{tr} \pi'^\infty(f'^\infty) = \sum m(\pi) \operatorname{tr} \pi^\infty(f^\infty). \tag{17.1.1}$$

On the left, the sum ranges over all $G(\mathbb{A}_f)$-modules π'^∞ such that $\pi' = \pi'^\infty \otimes \pi_\infty$ is a cuspidal $G'(\mathbb{A})$-module with the cuspidal component $\pi'_u = \pi_u$ at u. On the right the sum is over the $G(\mathbb{A}_f)$-modules π^∞, whose component at u is the cuspidal π_u, so that $\pi = \pi^\infty \otimes \pi_\infty$ appears with positive multiplicity $m(\pi)$ in the (cuspidal) spectrum $L_0(G(F)\backslash G(\mathbb{A}))$ of $G(\mathbb{A})$.

Recall the following theorem of Harish-Chandra (see [BJ79]). This is just an adelic translation of a classical result of Siegel and others on the finiteness of the number of normalized modular forms with fixed level and weight.

LEMMA 17.2. *Let C be an open compact subgroup of $G(\mathbb{A}_f)$. Then there are only finitely many (irreducible) automorphic $G(F)$-modules π with a nonzero C-fixed vector and a given infinitesimal character at each Archimedean place (in particular with the fixed component π_∞ at S_∞).*

Let V' be the union of V and $\{u, u', u''\}$. Fix f_v and f'_v for v in V', and let $f_v = f'_v$ be a variable spherical ($K_v = G(\mathcal{O}_v)$-bi-invariant) function for the finite v outside V'. Lemma 17.2 implies that the sum in (17.1.1) are both finite. It is clear from the theory of the Satake transform that, given a finite set $\{\pi_{iv}\,;\, i \geq 0\}$ of irreducible unramified pairwise-inequivalent $G(F_v)$-modules, there exists a spherical function f_v with $\operatorname{tr} \pi_{iv}(f_v) = 0$ if $i \neq 0$ and $\operatorname{tr} \pi_{0v}(f_v) = 1$. We conclude that, given an irreducible $G(\mathbb{A}^{V'})$-module $\pi^{V'}$, for all matching $f_v, f_{v'}$ (v in V'), we have

$$\sum \operatorname{tr} \pi'_{V'}(f'_{V'}) = \sum m(\pi) \operatorname{tr} \pi_{V'}(f_{V'}). \tag{17.2.1}$$

On the left the sum is over the irreducible representations $\pi'_{V'}$ of $\prod_{v \in V'} G'(F_v)$ such that $\pi' = \pi'_{V'} \otimes \pi^{V'}$ is cuspidal. The component at u is our fixed cuspidal π_u. By the rigidity theorem of [JS81], there exists at most one such π'. We choose $\pi^{V'}$ so that π' of Proposition 16.2 appears on the left. On the right the sum is over the equivalence classes of irreducible $\pi_{V'}$ such that $\pi = \pi_{V'} \otimes \pi^{V'}$ is cuspidal, with multiplicity $m(\pi)$. The sum on the right is not finite, *a priori*.

Since f_u is a normalized coefficient of a cuspidal $G(F_u)$-module π_u, we have $\operatorname{tr} \pi_u(f_u) = 1$ and $\operatorname{tr} \pi'_u(f'_u) = 1$ for the π, π' which appear in (17.2.1). At each $v \neq w$ in V, let f_v be the function 1_v and f'_v a matching function on $G(F'_v)$. The function f'_v exists by Corollary 14.2. At such v, let π_v be the trivial $G(F_v)$-module and π'_v the Steinberg $G'(F_v)$-module. Then $\chi'_v(x') = (-1)^{n-1} \chi_v(x)$ on the elliptic regular set and $\operatorname{tr} \pi'_v(f'_v) = (-1)^{n-1}$. Moreover, if $v \neq w$ in V and π_v appears on the right of (17.2.1), then $\operatorname{tr}\pi_v(f_v)$ is 0 or 1. Since $(n-1)(n-m) \equiv (n-m)$ (modulo 2), we conclude that for all matching f_w, f'_w, and for all $f_{u''}$ that vanish on the singular set of $G_{u''}$, we have

$$(-1)^{n-m} \operatorname{tr}\pi'_w(f'_w) \operatorname{tr}\pi'_{u''}(f'_{u''}) = \sum m(\pi) \operatorname{tr}\pi_w(f_w) \operatorname{tr}\pi_{u''}(f_{u''}). \tag{17.2.2}$$

The sum is over an easily specified set of $(\pi_w, \pi_{u''})$. Note that G splits at u''; hence $f'_{u''} = f_{u''}$. Moreover, the place u'' is chosen so that $\pi'_{u''}$ is unramified. The π' of Proposition 16.2 is cuspidal; hence it has a Whittaker model, and $\pi'_{u''}$ is non-degenerate. Consequently, $\pi'_{u''}$ is equal to an irreducible representation which is induced from an unramified character of the upper triangular subgroup, by [Ze80, Theorem 9.7(b)].

Let $f'_{u''}$ be any function such that $\Phi(f'_{u''})$ is supported on the split regular set of $G(F_{u''})$, and its restriction to $A(F_{u''})$ is $A(\mathcal{O}_{u''})$-invariant. It is clear that if $I(t, f'_{u''}) \neq 0$, then the Levi subgroup M_t of Section 7 is A, so that $\operatorname{tr}\pi(f'_{u''}) = \operatorname{tr}\pi_N(f'_{u''N})$ for any irreducible $G'(F_{u''})$-module π, where N is the upper triangular unipotent group. The support of $I(f'_{u''})$ is an open closed set. Denote by θ its characteristic function, and replace $f'_{u''}$ by its product with θ. This does not change the value of the orbital integral, but assures the vanishing of the compactly (modulo center) supported $f'_{u''}$ on the singular set. Note that Theorem 4.2 of [BZ76] implies that if $\operatorname{tr}\pi_N(f'_{u''N}) \neq 0$, then π has a nonzero vector fixed by the first congruence subgroup, as it has Iwahori decomposition. By virtue of the Lemma 17.2, the sum of (17.2.2) is then finite, uniformly in the $f'_{u''}$ considered here. Hence we can apply linear independence of (finitely many) characters on $A_{u''}$. This, together with Frobenius reciprocity, implies that we may consider on the right only $\pi_{u''}$ which are subquotients of, hence equal to the irreducible unramified $\pi'_{u''}$. The first claim of the proposition follows. The last assertion of the proposition follows from Lemma 17.2. □

18 Correspondence

Let F be non-Archimedean. Let G be an inner form of $G' = \mathrm{GL}(n)$ over F, with $G(F) = \mathrm{GL}(m, D)$ for a division algebra D. We have an injection $x \mapsto x'$ of conjugacy classes from $G(F)$ to $G'(F)$. We denote the characters of the $G(F)$-module π and $G'(F)$-module π' by χ_π and $\chi_{\pi'}$ (or χ').

THEOREM 18.1. *The relation $\chi_{\pi'}(x') = (-1)^{n-m}\chi_\pi(x)$ for all matching regular conjugacy classes x, x' in $G(F)$, $G'(F)$ defines a bijection between the set of equivalence classes of square-integrable $G(F)$-modules π and such $G'(F)$-modules π' and between tempered $G(F)$-modules π and such $G'(F)$-modules π' whose character χ' is nonzero on the set of regular x' obtained from x in $G(F)$.*

We start the proof here. It is completed in the next Section 19. In this section we deal only with the square-integrable part of the theorem. It is used in the next section to study the general, tempered case.

Let π' be a square-integrable $G'(F)$-module. Proposition 17.1 (where we now omit the subscript w) establishes the existence of a Trace Identity (10.5.1) for the π'. By virtue of Proposition 14.2 and the induction assumption of Theorem 14.4 for $M \neq G$, the assumptions 10.6, 12.1 and 12.2 are valid. By Proposition 11.1 the π of the Trace Identity (10.5.1) are square-integrable, and by Proposition 12.3, there are only finitely many π in the sum. Since f is an arbitrary function on $G(F)$, we conclude an identity of characters

$$(-1)^{m-n}\chi_{\pi'}(x') = \sum_{\pi} m(\pi)\chi_{\pi}(x)$$

for regular matching classes $x \mapsto x'$. On the right the sum ranges over a finite set of square-integrable $G(F)$-modules π. Applying the orthonormality relations for square-integrable $G(F)$ and $G'(F)$-modules of [Ka86.1, Theorem K], we conclude from $1 = \sum_{\pi} m(\pi)^2$ that the sum consists of a single π with coefficient $m(\pi) = 1$.

18.2 Weyl Integration Formula

Recall the Weyl integration formula

$$\int_{G/Z} f(g)\mathrm{d}g = \int' \Phi(t,f)\mathrm{d}t, \quad \text{where} \quad \int' \text{ signifies } \quad \sum_{\{T\}}[W(T)]^{-1}\int_{T/Z}\Delta(t)^2.$$

Here the sum ranges over a set of representatives for the conjugacy classes of tori T in G, $[W(T)]$ denotes the cardinality of the Weyl group of T, and Δ is the Jacobian.

For conjugacy invariant functions χ, χ_1 on the elliptic set of G, let $\langle\chi,\chi_1\rangle$ be $\int' \chi(x)\chi_1(x)\mathrm{d}x$, where $\{T\}$ extends only over the conjugacy classes of the elliptic tori T.

REMARK 18.3. Another proof for the existence of a square-integrable π to match such π', without using the finiteness result of Proposition 12.3, yet using Proposition 14.2 or the assumptions 12.1, 12.2, is as follows. It is clear that some π appears in the sum of Proposition 17.1, since we can take f' to be a pseudo-coefficient of π' by Proposition 14.2. Fixing such π we take f in $A_c^\infty(G(F))$ with $\widetilde{\Phi}(f) = \chi_\pi$ on the elliptic regular set. It exists by [Ka86.1, Theorem K]. Then the sum of Proposition 17.1 is equal to $m(\pi)$. Then if f' is a matching function (which exists by Proposition 14.2), then

$$|\operatorname{tr}\pi'(f')|^2 = \left|\int' \chi'(x')\widetilde{\Phi}(x',f')\,\mathrm{d}x'\right|^2 \leq \int' |\chi'(x')|^2\,\mathrm{d}x' \int' |\chi(x)|^2\,\mathrm{d}x$$

by the Schwarz inequality. The integrals are taken only over the elliptic tori of $G(F)$ or $G'(F)$, and we use the fact that $\widetilde{\Phi}(x',f') = \widetilde{\Phi}(x,f) = \chi(x)$. Here χ, χ' are the characters of π and π'. By the orthonormality relations $\langle \chi_\pi, \chi_{\pi_1}\rangle = \delta(\pi,\pi_1)$ of [Ka86.1, Theorem K], we conclude that $m(\pi) \le 1$. As $m(\pi)$ is a positive integer, we conclude that $m(\pi) = 1$ and that the Schwarz inequality is an equality in our case, so that $\chi'(x') = c\widetilde{\Phi}(x',f') = c\widetilde{\Phi}(x,f) = c\chi(x)$ on the elliptic regular set, where c is a number with $|c| = 1$. Hence π is the only term in the sum, and $c = (-1)^{m-n}$.

In the opposite direction, given a square-integrable π, we take a pseudo-coefficient f in $A_c^\infty(G(F))$ of π, and a matching function f' in $A_c^\infty(G'(F))$. By the density Proposition 5.6 and the orthonormality relations of [Ka86.1], there exists a tempered elliptic, hence by [Ze80] square-integrable, $G'(F)$-module π', with $\operatorname{tr}\pi'(f') \ne 0$. By the orthonormality relations on $G(F)$, the $G(F)$-module matching π', whose existence was proven above, is our π.

18.4 Remark on Tempered Case

We have now completed the proof of that part of the theorem which concerns square-integrable π, π'. The extension to the case of any tempered π and π' follows once we establish in Section 19 below that any tempered $G(F)$-module is *equal* to an induced $G(F)$-module from a square-integrable module. This result is well known in the case of the split group G'. In its proof we use that part of the theorem proven above, for square-integrable modules.

REMARK 18.5. In particular, we completed the proof and hence can use the assertion of Theorem 18.1 in the case $m = 1$, namely, when G is the multiplicative group $D^\times$ of a division algebra D central over F. Indeed, all $G(F)$-modules in this case are square-integrable, and the image of the correspondence here is the set of elliptic tempered, hence square-integrable, $G'(F)$-modules. This case was independently established in Section 1.

19 Tempered

PROPOSITION 19.1. *Any (irreducible) elliptic tempered $G(F)$-module is square-integrable.*

See Remark in 16.2, Subsection 13.6, and Subsection 10.4 for the respective definitions.

PROOF. Suppose that the character χ of π is nonzero on the elliptic regular element y. Let f be the characteristic function of a small neighborhood of y (modulo $Z(F)$), where χ is constant. It is clear from the Weyl integration formula that

$$\langle \chi, \widetilde{\Phi}(f)\rangle = \int \chi(x)f(x)\,\mathrm{d}x \ (= \operatorname{tr}\pi(f))$$

for x in the elliptic set of $G(F)$. For our f and χ, we have $\langle \chi, \widetilde{\Phi}(f)\rangle \neq 0$. Since f is supported on the regular set, there is a matching f', with $\widetilde{\Phi}(x,f) = \widetilde{\Phi}(x',f')$ on the elliptic set. As f' lies in $A_c^\infty(G'(F))$, there is a matching function φ on the multiplicative group $D(F)$ of a division algebra of dimension n^2 central over F. Since $D(F)$ is compact modulo its center $Z(F)$, there are only finitely many $D(F)$-modules σ_i, with characters χ_{σ_i}, and complex numbers c_i, so that $\widetilde{\Phi}(x,\varphi) = \sum c_i\chi_{\sigma_i}(x)$ on the regular x in $D(F)$. If χ_i' are the characters of the $G'(F)$-modules π_i' that correspond to the σ_i, then $\widetilde{\Phi}(x',f') = \sum c_i\chi_i'(x')$ on the elliptic regular set. Since the π_i' are square-integrable, they correspond to square-integrable $G(F)$-modules π_i with characters χ_i. Hence $\widetilde{\Phi}(x,f) = \sum c_i\chi_i(x)$ on the regular elliptic set. Then $\langle \chi, \widetilde{\Phi}(f)\rangle = \sum c_i\langle\chi,\chi_i\rangle$. Since this is nonzero, we have $\langle\chi,\chi_i\rangle \neq 0$ for some i. But the orthonormality relations for square-integrable $G(F)$-modules of [Ka86.1, Theorem K] imply that π is equivalent to the square-integrable π_i, as required. □

PROPOSITION 19.2. *Suppose I is a $G(F)$-module normalizedly induced from a square-integrable $M(F)$-module, where M is a Levi component of a proper parabolic subgroup. Then I is irreducible.*

In particular, the same conclusion holds when the representation of $M(F)$ is tempered.

PROOF. Let $I = i_M^G\tau$ where τ is a square-integrable $M(F)$-module. Suppose that I contains the irreducible representation π. We shall first show that I is a multiple of the irreducible representation π. By induction, we assume this assertion for all proper Levi subgroups of G.

Suppose that π does not lie in the space $R_I(G)$, see Subsection 6.7, which is spanned over $\mathbb{C}$ by the properly induced representations. By [Ka86.1, Theorem D], π is elliptic. Since π is also tempered, Proposition 19.1 implies it is square-integrable. The uniqueness theorem [BW80, Theorem 2.10] part of the Langlands classification implies that $\pi = \tau$, $M = G$ an so $I = \pi$ as required.

Suppose π does lie in $R_I(G)$. Then by [Ka86.1, Proposition 1.1], there are finitely many proper Levi subgroups L_i, irreducible tempered $L_i(F)$-modules τ_i and complex numbers α_i so that in the Grothendieck group $R(G(F))$, we have

$$\pi = \sum_i \alpha_i i_{L_i}^G\tau_i.$$

Since τ_i is tempered, there is a unique (up to conjugacy) pair (σ_i, R_i), where R_i is a Levi subgroup of L_i and σ_i is a square-integrable $R_i(F)$-module, such that τ_i is a direct summand of $i_{R_i}^{L_i}\sigma_i$. Since $L_i \neq G$, the induction assumption implies that there is a positive integer β_i^{-1} such that $i_{R_i}^{L_i}\sigma_i = \beta_i^{-1}\tau_i$. Hence

$$\pi = \sum_i \alpha_i\beta_i i_{R_i}^G\sigma_i.$$

Since π is a constituent of $I = i_M^G \tau$, where τ is square-integrable, the expression for π and the uniqueness part of the Langlands classification implies that either $I = i_{R_i}^G \sigma_i$ or I and $i_{R_i}^G \sigma_i$ are not relatives in the terminology of [Ka86.1]. Hence $\pi = \alpha I$, where α is the sum of $\alpha_i \beta_i$ over the i where $I = i_{R_i}^G \sigma_i$, as required. Thus $I = \alpha^{-1}\pi$, where α^{-1} is a positive integer.

It remains to show that I is irreducible. For that we use the work of [Si80]. First consider the case where M is of rank one; thus $M = M' \times M''$, with $M' = \mathrm{GL}(a, D)$, $M'' = \mathrm{GL}(b, D)$. Then [Si80, Theorem 2.5.8, p. 99] implies that I is irreducible unless $a = b$, in which case its composition series has length bounded by the order of the Weyl group $W(A)$ of [Si80, p. 100], which is two. But if I is the direct sum of k copies of an irreducible, its commuting algebra has dimension k^2, which is at least 4, unless k=1.

Next we consider the general case. We shall express the $M(F)$-module τ as a product of square-integrables on $\mathrm{GL}(n_i, D)$'s. Thus we may, upon rearranging the factors, assume that

$$\tau = (\tau_1 \times \cdots \times \tau_1) \times \cdots \times (\tau_s \times \cdots \times \tau_s)$$

where each square-integrable $M_i(F)$-module τ_i, with $M_i = \mathrm{GL}(n_i, D)$, occurs t_i times and τ_i, τ_j are inequivalent if $i \neq j$. Then $n = \sum_i n_i t_i$. Put $t_0 = 0$. The center of

$$M = (M_1 \times \cdots \times M_1) \times \cdots \times (M_s \times \cdots \times M_s)$$

is

$$A = A_1^{t_1} \times \cdots \times A_s^{t_s},$$

where A_i is the center of M_i. Let $W(A)$ be the product of the symmetric groups S_{t_i}. The Harish-Chandra commuting algebra theorem [Si80, 5.5.3] asserts that the commuting algebra of I is spanned by the intertwining operators $R(w)$, $w \in W(A)$, subject to the relations $R(1) = 1$ and $R(ww') = R(w)R(w')$. Hence it is generated by the $R(s(i))$, where $s(i)$ is a reflection of the form $(i, i+1)$, $t_{j-1} < i < t_j$. However, the operator $R(s(i))$ is induced (recall the induction is a functor) from the intertwining operator of the representation induced from τ on $M = M'_1 \times \cdots \times M'_m$ (the M'_i are the M_i, relabeled) to

$$G_i = M'_1 \times \cdots \times M'_{i-1} \times X_i \times M'_{i+2} \times \cdots \times M'_m$$

and $X_i = \mathrm{GL}(2n_j, D)$ if $M'_i = \mathrm{GL}(n_j, D) = M'_{i+1}$. It follows from the rank one case that $R(s(i))$ is a scalar. Hence the commuting algebra of I consists of scalars, which proves that I is irreducible, as required. □

Remark 1. (i) We do not discuss here the normalization of intertwining operators.
(ii) It is possible to complete the proof of irreducibility above by further analyzing the proof of [Si80, Chapter II]. This proof does not require the commuting algebra theorem of [Si80, Chapter V], but we do not give it here.

20 Irreducibility

20.1 Induction

Let τ_i be cuspidal M_i-modules, where $M_i = \mathrm{GL}(n_i, D)$ and $\sum_{i=1}^m n_i = n$. Let ν be the character $\nu(x) = |\det(x)|$ of M_i—where by det we mean determinant on $\mathrm{GL}(n_i)$ followed by reduced norm on the division algebra D. Let s_i $(1 \le i \le m)$ be real numbers. Put $\mathbf{s} = (s_1, \ldots, s_m)$. Let $I = i_M^G(\tau(\mathbf{s}))$ be the $G(F)$-module obtained by induction from the $M = \prod_i M_i$-module $\tau(\mathbf{s}) = \prod_{1 \le i \le m} \tau_i \nu^{s_i}$. If P is a parabolic with Levi M, then the module of coinvariants I_P of I with respect to P consists, by [BZ76, (2.12)] (see the Geometric Lemma 8.6), of composition factors of the form $\tau_\alpha(\mathbf{s}) = \prod_i \tau_{\alpha(i)} \nu^{s_{\alpha(i)}}$, where α ranges over the symmetric group S_m on m letters.

DEFINITION 20.2. (i) The (cuspidal) *support* of a subquotient π of I is the set of $M(F)$-modules $\tau_\alpha(\mathbf{s})$ which are constituents of π_P.

(ii) The representation π is called *multiplicity free* if each $\tau_\alpha(\mathbf{s})$ occurs in π_P at most once.

(iii) A reflection in S_m of the form $r(i) = (i, i+1)$ is called *admissible* if $|s_{i+1} - s_i| \neq 1$ or τ_{i+1} is inequivalent to τ_i. This term depends on $\tau(\mathbf{s})$.

PROPOSITION 20.3. (i) *If $m = 2$ and $|s_1 - s_2| \neq 1$ or the cuspidal τ_1, τ_2 are inequivalent, then $I = i_M^G(\tau_1 \nu^{s_1} \times \tau_2 \nu^{s_2})$ is irreducible.*

(ii) *The support of π is invariant under the action of the set of admissible reflections.*

PROOF. (i) By Proposition 19.2, which deals with the tempered case $s_1 = s_2$, we may assume that $s_1 \neq s_2$, hence $s_1 > s_2$ without loss of generality. The module of coinvariants of I with respect to the parabolic subgroup of type (n_1, n_2) has two exponents, one increasing and one decaying. If I is reducible, then its composition series has length two by [Si80, Theorem 2.5.8], since the τ_i are cuspidal. One of the constituents has the decaying exponent and hence is square-integrable by Harish-Chandra's criterion, [Si80, (4.4.4)], [Cas, (4.4.6)], quoted in Subsection 10.4. But this square-integrable should correspond by Theorem 18.1 to a square-integrable constituent of the representation $I' = i_M^G(\tau_1' \nu^{s_1} \times \tau_2' \nu^{s_2})$ corresponding to I. As I' is irreducible by [BZ76], (i) follows.

(ii) Suppose that $\tau(\mathbf{s})$ lies in the support of π. We have to show that so does $\tau_{r(i)}(\mathbf{s}) = \tau_1 \nu^{s_1} \times \cdots \times \tau_{i+1} \nu^{s_{i+1}} \times \tau_i \nu^{s_i} \times \cdots$. For that we consider the parabolic subgroup Q of type $(n_1, \ldots, n_{i-1}, n_i + n_{i+1}, n_{i+2}, \ldots)$, and its standard Levi subgroup L. Since $\pi_P = (\pi_Q)_{L \cap P}$, there is an irreducible $L(F)$-module ε in the composition series of π_Q such that $\varepsilon_{L \cap P}$ contains $\tau(\mathbf{s})$. But part (i) implies that if $|s_{i+1} - s_i| \neq 1$ or τ_i, τ_{i+1} are inequivalent, then $\varepsilon_{L \cap P}$ contains also $\tau_{r(i)}(\mathbf{s})$, and (ii) follows.

□

PROPOSITION 20.4. (i) *If π is a multiplicity-free subquotient of I, and the set of admissible transpositions acts transitively on the support of π, then π is irreducible.*

(ii) *Suppose that σ_1 and σ_2 are square-integrable and $|s| < 1/2$. Then*

$$I = i_M^G(\sigma_1 \nu^s \times \sigma_2 \nu^{-s})$$

is irreducible.

REMARK. Part (ii) here sharpens (i) of Proposition 20.3.

PROOF. (i) This is clear by (ii) of Proposition 20.3 and the fact that each subquotient of π has a nonzero subquotient $\tau_\alpha(\mathbf{s})$ in its module of coinvariants π_P.

(ii) Since σ_1 and σ_2 are square-integrable on $\mathrm{GL}(n_i, D)$, they correspond by Theorem 18.1 to such representations σ_1' and σ_2' on the split groups $\mathrm{GL}(dn_i, F)$. The σ_i' are classified by [BZ76] for G': a square-integrable representation σ' of $G' = \mathrm{GL}(n, F)$, $n = md$, is the unique subrepresentation of the representation induced to G' from a cuspidal representation

$$\left(\left(\tau', \frac{m'-1}{2}\right), \left(\tau', \frac{m'-3}{2}\right), \dots, \left(\tau', \frac{1-m'}{2}\right)\right)$$

—where we put (τ', s) for $\tau'\nu^s$—on a Levi subgroup of type $(d', \dots, d')$ (m' times). Thus τ' is a cuspidal representation of $\mathrm{GL}(d', F)$. Since $n = m'd' = md$, we gather these representations in segments of length least common multiply of d and d', which is de, $e = d'/(d, d')$. Denote by τ the cuspidal representation of $\mathrm{GL}(e, D)$ corresponding to the square-integrable subrepresentation of the $\mathrm{GL}(ed, F)$-module induced from

$$\left(\left(\tau', \frac{e'-1}{2}\right), \left(\tau', \frac{e'-3}{2}\right), \dots, \left(\tau', \frac{1-e'}{2}\right)\right), \qquad e' = d/(d, d').$$

We conclude that there exist cuspidal τ_1 and τ_2 on $\mathrm{GL}(e_1, D)$ and $\mathrm{GL}(e_2, D)$, with e_i dividing n_i, thus $m_i = n_i/e_i$ is an integer, and $n = dn_1 + dn_2$, such that the *support* of I consists of all $m_1 + m_2$ tuples $(a_{\alpha(i)})$ obtained from

$$(a_i) = \left(\left(\tau_1, \frac{m_1-1}{2} + s\right), \left(\tau_1, \frac{m_1-3}{2} + s\right), \dots, \left(\tau_1, \frac{1-m_1}{2} + s\right); \left(\tau_2, \frac{m_2-1}{2} - s\right), \dots, \right) \tag{20.4.1}$$

on permuting by α in $S_{m_1+m_2}$ which satisfies $\alpha(i) < \alpha(j)$ if $i < j \leq m_1$ or $m_1 < i < j$. We may assume that $s \neq 0$ by Proposition 19.2. This set is multiplicity free, and the set of admissible transpositions act transitively if (i) τ_1 is inequivalent to τ_2; when $\tau_1 = \tau_2$, if (ii) $m_1 - m_2$ is even, as $2|s| < 1$; or

(iii) $m_1 - m_2$ is odd, unless $|s| = 1/4$. Hence the proposition follows from part (i) for π being I, except that we have to deal with the case when $\tau_1 = \tau_2$ (and $m_1 - m_2$ is odd, $|s| = 1/4$). In this case we use the notation (20.4.1) for vectors in the support, omitting the reference to τ_1, τ_2, namely, from now on we deal with the case $\tau_1 = \tau_2$.

By a *segment* we mean a vector (c_i) of real numbers with $c_i - c_{i+1} = 1$ for all i. The center of the segment $(c_1, \ldots, c_m)$ is $(c_1 + c_m)/2$. The vector (c_i) is called an *L-vector* if it has a partition $(\mathbf{b}_j)$ into segments $\mathbf{b}_j = (b_{ij})$, whose centers are nondecreasing. The description of tempered representations of G' by [BZ76], transferred to G by Theorem 18.1 and Proposition 19.2, together with the classification theorems of [BW80, IV§2] asserts that each irreducible $G(F)$-module has (at least one) L-vector in its support. But it is easy to check that the support of our I, namely, the set of $(a_{\alpha(i)})$ obtained from the (a_i) of (20.4.1), contains only one L-vector. Hence I is irreducible, as required. □

COROLLARY 20.5. *Given any irreducible τ_1, τ_2 and σ_1, σ_2, s as in (ii) of Proposition 20.4, $i_M^G(\tau_1 \times \sigma_1 \nu^s \times \sigma_2 \nu^{-s} \times \tau_2)$ is equal to $i_M^G(\tau_1 \times \sigma_2 \nu^{-s} \times \sigma_1 \nu^s \times \tau_2)$. In particular, one of them is unitarizable if and only if the other is.*

PROOF. This follows from induction in stages, since $i_M^G(\sigma_1 \nu^s \times \sigma_2 \nu^{-s})$ is irreducible, hence equal to $i_M^G(\sigma_2 \nu^{-s} \times \sigma_1 \nu^s)$. □

21 Unitarity

21.1 Relevance

Recall (Definition 13.7) that a $G(F)$-module π is called *relevant* is there is M as in (1) and τ_i as in (2) such that π is as in (3).

(1) A Levi subgroup M of G of the form $M_0 \times \prod_{i=1}^m (M_i \times M_i)$, or of the form $\prod_{i=1}^M (M_i \times M_i)$, where M_i is a multiplicative group of a simple algebra for each i.
(2) Irreducible tempered M_i-modules τ_i, $0 \le i \le m$.
(3) Distinct positive numbers $s_i < 1/2$, $1 \le i \le m$, such that π is equivalent to $i_M^G(\tau)$ or $i_M^G(\tau_0 \times \tau)$ and τ is the $\prod_{i=1}^m (M_i(F) \times M_i(F))$-module $\prod_{i=1}^m (\tau_i \nu^{s_i} \times \tau_i \nu^{-s_i})$.

The motivation for this definition is the fact that each component of any cuspidal automorphic $G(\mathbb{A})$-module is unitarizable and non-degenerate, hence relevant by [Be84, Lemma 8.9] and [Ze80, Theorem 9.7(b)].

In this section we complete the proof of the Local Theorem 13.8 and show that a relevant $G(F)$-module is irreducible and unitarizable.

Let U be a finite dimensional complex vector space.

LEMMA 21.2. *Let $\langle\cdot,\cdot\rangle_s$ be a family of non-degenerate Hermitian forms on U depending continuously on a parameter s in a connected set. If $\langle\cdot,\cdot\rangle_s$ is positive definite for some value of s, then it is positive definite for all s.*

PROOF. The set of s where $\langle\cdot,\cdot\rangle_s$ is positive definite is open, and also closed, since s ranges over a connected set. □

PROPOSITION 21.3. *Let σ be square-integrable and $s \in \mathbb{R}$, $-1/2 < s < 1/2$. Then the $G(F)$-module $I(s) = i_M^G(\sigma\nu^s \times \sigma\nu^{-s})$ is unitarizable.*

PROOF. Let V_s be the space of the representation $I(s)$. As a space V_s is independent of s, but the action of G does depend on s. Since $I(s)$ is irreducible, it is equivalent to its contragredient (see [BZ76]) $I(s)' = i_M^G(\sigma\nu^{-s}, \sigma\nu^s)$. Note that the square-integrable σ is self-contragredient, so $\sigma' = \sigma$, $(\sigma\nu^s)' = \sigma\nu^{-s}$, and $i(\tau)' = i(\tau')$. The choice of an isomorphism $I(s) \to I(s)' = I(-s)$, which is unique up to a scalar, determines an Hermitian inner product $\langle\cdot,\cdot\rangle$ on V_s, which is non-degenerate. We can choose the isomorphism, or the inner product $\langle\cdot,\cdot\rangle_s$, to vary continuously with s. For each compact open congruence subgroup C in K, the isomorphism $V_s \to V_{-s}$ determines an isomorphism from the space V_s^C of C-fixed vectors in V_s, to the dual $(V_s^C)' = (V_s')^C = V_{-s}^C$. For each C we obtain a continuous family $\langle\cdot,\cdot\rangle_{s,C}$ of non-degenerate Hermitian inner product, which varies continuously with the parameter s in $-1/2 < s < 1/2$.

Now the tempered $I(0)$ is unitarizable, being normalizedly induced from a unitarizable representation $\sigma \times \sigma$. Hence $\langle\cdot,\cdot\rangle_{s,C}$ is positive definite at $s = 0$. Consequently it is positive definite for all s with $-1/2 < s < 1/2$ and for all C by Lemma 21.2. As V_s is the union of V_s^C over all C, we conclude that $\langle\cdot,\cdot\rangle_s$ is positive definite for all s; hence $I(s)$ is unitarizable. □

COROLLARY 21.4. *Let σ_i, $1 \le i \le m$, be square-integrable. Let s_i be positive numbers, $1 \le i \le k$, $k \le m$, with $s_i < 1/2$. Let σ denote the product $\prod_{i=1}^k(\sigma_i\nu^{s_i} \times \sigma_i\nu^{-s_i}) \times \prod_{j=k+1}^m \sigma_j$. Then $I = i_M^G\sigma$ is unitarizable for any choice of a parabolic subgroup or, equivalently, for any order of the factors $\sigma_i\nu^{s_i}$, $\sigma_i\nu^{-s_i}$, σ_j.*

PROOF. The representation I is independent of the choice of factors by Corollary 20.5. Let $M = \prod_{i=1}^k(M_i \times M_i) \times \prod_j M_j$ be the Levi subgroup from which we induce. If $M_i = \mathrm{GL}(n_i, D)$, put $L_i = \mathrm{GL}(2n_i, D)$, and $L = \prod_i L_i \times \prod_j M_j$. Since

$$i_M^G(\sigma) = i_L^G i_M^L \sigma,$$

and $i_M^L\sigma$ is unitarizable by Proposition 21.3, we conclude that I is unitarizable. □

THEOREM 21.5. *Suppose that the τ_i, $0 \le i \le m$, are irreducible and tempered and s_i are distinct positive numbers with $s_i < 1/2$. Put $\tau = \prod_{i=1}^m(\tau_i\nu^{s_i} \times \tau_i\nu^{-s_i})$. Then the induced representations $I = i_M^G(\tau)$ and $i_M^G(\tau_0 \times \tau)$ are irreducible.*

PROOF. We induce from the Levi $M = M_0 \times \prod(M_i \times M_i)$. Here $M_0 = \mathrm{GL}(n_0, D)$ for $n_0 \ge 0$, and $n_0 = 0$ means M_0 does not appear in M. There exists a parabolic subgroup P with Levi subgroup M such that the vector determined by

$(0; s_1, -s_1; s_2, -s_2; \ldots)$ lies in the positive Weyl chamber (in the Lie algebra of the diagonal subgroup) determined by P. Consequently I has a unique quotient J; see [BW80, IV, (4.6)]. On the other hand, Corollary 21.4 implies that I is unitarizable. As each constituent of a unitarizable representation is a direct summand of it, the unique quotient J must be I itself, and we conclude that I is irreducible. □

21.6 End of Proof of Local Theorem 13.8

It is now clear that by parabolic induction Theorem 18.1 extends to hold also for relevant, not only tempered, $G(F)$ and $G'(F)$-modules. This completes the proof of the Local Theorem 13.8. □

REMARK 21.7. The result of Proposition 19.2 and Theorem 18.1 are due to [DKV84] and that of 21.5 to [F87.1]. These theorems are proven here for local fields of characteristic zero. The positive characteristic case follows from [Ka86.2].

22 Induction

It remains to complete the proof of Theorem 14.4, which asserts the existence of matching orbital integrals, using the correspondence for tempered representations Theorem 18.1. In the proof of Theorem 18.1, we used the induction assumption of Theorem 14.4, namely, the statement of Theorem 14.4 for all proper Levi subgroups. Our aim is to show that for any f in $C_c^\infty(G(F))$, there exists a matching f' in $C_c^\infty(G'(F))$, and for any suitable f' in $C_c^\infty(G'(F))$ (thus $\Phi(x', f') = 0$ for any regular x' not obtained from x in $G(F)$), there is a matching f in $C_c^\infty(G(F))$.

Note we have the following:

LEMMA 22.1. *For every f there exists f' with* $\operatorname{tr}\pi(f) = \operatorname{tr}\pi'(f')$ *for all corresponding tempered π, π', and* $\operatorname{tr}\pi'(f') = 0$ *for the tempered π' which are not obtained by the correspondence.*

PROOF. Given f we define the function $f'_{G'}$ on the space of tempered π' by $f'_{G'}(\pi') = \operatorname{tr}\pi(f)$ if π corresponds to π' by Theorem 18.1 and by $f'_{G'}(\pi') = 0$ if the character of π' is zero on the set of regular x' obtained from x. Then $f'_{G'}$ is in the space F_{good} of [BDK86] or [F95]. Thus it is a trace function by the trace Paley-Wiener Theorem 8.4. Namely, there is an f' with $f'_{G'}(\pi') = \operatorname{tr}\pi'(f')$ for all tempered π', as required.

The same argument implies the existence of f for a given suitable f'. □

PROPOSITION 22.2. *Suppose that f and f' satisfy* $\operatorname{tr}\pi(f) = \operatorname{tr}\pi'(f')$ *for all corresponding tempered π and π', and* $\operatorname{tr}\pi'(f') = 0$ *for the tempered π' not obtained by the correspondence. Then f, f' are matching.*

PROOF. By induction on the Levi subgroup M of the parabolic subgroup $P = MN$ of G. Denote by P', M', N' the corresponding objects of G'. Let δ_P be the modulus homomorphism on P. Thus $\mathrm{d}(ab) = \delta_P(a)\mathrm{d}b$, for a, b in $P(F)$, for any right Haar measure $\mathrm{d}b$ on $P(F)$. For a in the center $A(F)$ of $M(F)$ we have $\delta_P(a) = \prod |\alpha(a)|$. The product ranges over all roots of A in N. As usual, we put

$$f_N(m) = \delta_P(m)^{1/2} \int_K \int_N f(k^{-1}mnk)\,\mathrm{d}n\,\mathrm{d}k.$$

Here K is a maximal compact subgroup of $G(F)$ with $G(F) = KP(F)$. For any m in $M(F)$ regular in $G(F)$, we have $I(m,f) = I^M(m,f_N)$, where $I(x,f) = \Delta(x)\Phi(x,f)$ and

$$\Delta(x) = |\prod_{i<j}(x_i - x_j)^2/x_ix_j|^{1/2}$$

if x has distinct eigenvalues x_i. The orbital integral I^M is the analogous object defined for M. Analogous notations are employed in the case of G'. Further, we note that if $\pi = i_M^G(\tau)$ is the $G(F)$-module normalizedly induced from the $M(F)$-module τ, then $\operatorname{tr}\pi(f) = \operatorname{tr}\tau(f_N)$ by a standard evaluation of the character of an induced representation. Consequently, if τ, τ' are corresponding tempered $M(F)$ and $M'(F)$, we have $\operatorname{tr}\tau(f_N) = \operatorname{tr}\tau'(f'_N)$, and $\operatorname{tr}\tau'(f'_N) = 0$ for the tempered τ' not obtained from any τ. By induction we have $I(x',f') = I^M(x',f'_N) = I^M(x,f_N) = I(x,f)$ for the regular x' in $M'(F)$ which come from x, and $I(x',f') = 0$ for the regular x' in $M'(F)$ which do not come from $G(F)$. It remains to show the proposition for elliptic regular x, x'.

Choose matching elliptic regular y, y'. Let U' be a sufficiently small compact neighborhood of y', and f'_0 a function on $G'(F)$, supported near $y'Z(F)$, whose orbital integrals $\widetilde{\Phi}(f'_0)$ is the characteristic function of $Z(F)U'^{G'}$. Let f_0 be a matching function on $G(F)$. It exists by Corollary 6.12. Now, $\widetilde{\Phi}(f'_0)$ is a finite linear combination of the characters of square-integrable π'_i with coefficients c_i, by [Ka86.1, Theorem K]. Then $\widetilde{\Phi}(f_0)$ is the corresponding combination of the characters of the π_i which correspond to the π'_i. Since U' is small, the Weyl integration formula implies that $\int_{T'(F)/Z(F)} I(t',f'_0)I(t',f')\,\mathrm{d}t'$ is equal to $\sum c_i \operatorname{tr}\pi'_i(f')$. Here T' is the centralizer of y' in G'. The assumption of our proposition implies that this is equal to $\sum c_i \operatorname{tr}\pi_i(f)$. But this is $\int_{T(F)/Z(F)} I(t,f_0)I(t,f)\,\mathrm{d}t$. We take U' to be so small that both $I(t',f'_0)$ and $I(t,f_0)$ are constant on U'. The desired equality $I(y,f) = I(y',f')$ now follows from the choice of f_0 and f'_0, which guarantees that $I(t,f_0) = F(t,f'_0)$. $\square$

23 Cuspidal Global Correspondence

23.1 Injection of Conjugacy Classes

Let F be a global field. We continue to let G, G' be as before. We have $G'(\mathbb{A}) = \mathrm{GL}(n, \mathbb{A})$, $G(\mathbb{A}) = \mathrm{GL}(m, D_{\mathbb{A}})$, where $D_{\mathbb{A}}$ denotes the adèle points of a division algebra D of dimension d^2 central over F and $n = md$. We let $D(F_v)$ denote the F_v-points of D. Then $G(F_v) = \mathrm{GL}(m_v, D(v))$, where $D(v)$ is a division algebra of dimension d_v^2 over F_v, and $n = m_v d_v$. We have $G(F_v) \simeq G'(F_v)$ outside of a finite set of places V. We have an injection $x \mapsto x'$ of conjugacy classes from $G(\mathbb{A})$ to $G'(\mathbb{A})$ and from $G(F_v)$ to $G'(F_v)$ for all v. It is a bijection for v outside V, but it is not surjective for v in V, the set where D is ramified.

23.2 Local Correspondence

Recall that an irreducible admissible $G(F_v)$-module π_v is said to *correspond* (or lift) to a $G'(F_v)$-module π'_v if their characters χ_v, χ'_v are related by $(-1)^{n-m}\chi'_v(x') = \chi_v(x)$ for all regular matching x, x'. At v outside V we have $m_v = n$ and this relation amounts to $\pi_v \simeq \pi'_v$. Our Local Theorem 13.8 asserts that the map $\pi_v \mapsto \pi'_v$ induces an embedding of the set of (equivalence classes of) tempered (resp. relevant) $G(F_v)$-modules as a subset of the set of tempered (resp. relevant) $G'(F_v)$-modules.

23.3 Global Correspondence

A $G(\mathbb{A})$-module $\pi = \otimes_v \pi_v$ is said to (quasi-) correspond to a $G'(\mathbb{A})$-module $\pi' = \otimes_v \pi'_v$ if π_v corresponds to π'_v for (almost) all v. Results about global lifting depend on the form of the trace formula which is available. Here we use only Corollary 4.4. It implies, on using transfer of orbital integrals (Theorem 14.4), that any discrete spectrum (automorphic) $G(\mathbb{A})$-module π, whose components at two places u, u' lift to cuspidal $G'(F_u)$ and square-integrable $G'(F_u)$-modules, quasi-lifts to an automorphic (necessarily cuspidal) $G'(\mathbb{A})$-module with cuspidal component at u, an elliptic component at u', and components π'_v with characters χ'_v which are not identically 0 on the set of regular classes χ' obtained from x in $G(F_v)$ for all v in V, is a quasi-lift of a discrete spectrum $G(\mathbb{A})$-module. Note that u, u' are not required to be in or out of V.

23.4 Multiplicity One and Rigidity Theorems

Since an automorphic π' with a cuspidal component is cuspidal, multiplicity one and rigidity theorems for the cuspidal spectrum of $L(G'(F)\backslash G'(\mathbb{A}))$ imply that the discrete spectrum quasi-correspond π' of π is unique if it exists. We shall now deal with the notion of correspondence, rather than quasi-correspondence, and conclude the uniqueness of π too, thereby obtaining multiplicity one and rigidity-type theorems for discrete spectrum $G(\mathbb{A})$-modules.

THEOREM 23.5. *Suppose that π' is an automorphic $G'(\mathbb{A})$-module with cuspidal components at two places u and u', and components π'_v whose characters are not identically zero on the set of the x' which come from $G(F_v)$ for all v in V. Then there exists a unique automorphic $G(\mathbb{A})$-module π which quasi-corresponds to π'. Moreover, π corresponds to π'.*

PROOF. The condition at u implies that π' is cuspidal. Hence it has a Whittaker model, and its components are all non-degenerate and unitarizable. Hence, by [Ze80, Theorem 9.7(b)], each π'_v is relevant. The Local Theorem 13.8 implies that π'_v corresponds to a relevant $G(F_v)$-module $\widetilde{\pi}_v$. The identity of Proposition 15.4, say in the form (17.2.1), with π' as the only term on the left and with a sufficiently large but finite set S of places of F (S depends on π'), implies that

$$\prod_{v\in S} \operatorname{tr} \widetilde{\pi}_v(f_v) = \prod_{v\in S} \operatorname{tr} \pi'_v(f'_v) = \sum_{\pi} m(\pi) \prod_{v\in S} \operatorname{tr} \pi_v(f_v)$$

for *all* functions f_v on $G(F_v)$ (v in S). The "generalized linear independence" of characters in Lemma 17.2 implies that the sum on the right consists of a single summand π with $m(\pi) = 1$, and the theorem follows. □

24 Complements on Local Representations

24.1 Non-Degenerate Case

Having used the trace formula in a simple form, we were able to describe the correspondence for global cuspidal representations with some local restrictions and for local representations that may occur as their local components. These local representations, in the case of $G' = \mathrm{GL}(n)$, are non-degenerate.

Our next aim is to extend the global correspondence from the context of the cuspidal spectrum to that of the entire discrete spectrum. For this we need to use the invariant trace formula and its comparison between G and G'. The complement of the cuspidal spectrum in the discrete spectrum is the residual spectrum. The representations that appear in the residual spectrum of G' were described by Moeglin and Waldspurger [MW95]. The components of these residual spectrum representations are nontempered and degenerate. However, they are unitarizable of a special type, described by [MW95].

The global correspondence will be seen to be an injection from the discrete spectrum of the inner form $G(\mathbb{A})$ to that of the quasisplit $G'(\mathbb{A})$. However, it is not compatible with the local correspondence from $G(F_v)$ to $G'(F_v)$, since the correspondence is not surjective.

The problem can be seen already in the initial case, considered by Jacquet and Langlands [JL70], of $G' = \mathrm{GL}(2)$ and its inner form G over F. In the local case, the correspondence maps the trivial representation of $G(F_v)$ to the special, or Steinberg, representation of $G'(F_v)$. The character relation $\chi_{\pi'_v}(g') = -\chi_{\pi_v}(g)$ is satisfied. The sign is -1 to the power $m(d-1) = 1$ ($m = 1$, $d = 2$). A non-one-dimensional representation π corresponds to a cuspidal π', and π_v corresponds to π'_v for all v. The one-dimensional π corresponds to the one-dimensional π' in the sense that $\pi_v = \pi'_v$ at all places where G splits. At the finite set of places v where G ramifies, the components of π' are one-dimensional. They are not the square-integrables mentioned above. We have the character relation $\chi_{\pi'_v}(g') = \chi_{\pi_v}(g)$, which differs from the previous relation by a sign. The explanation is that the trivial and the square-integrable are the constituents of an induced on $\mathrm{GL}(2, F_v)$ which does not come from the correspondence from any representation of the inner form, so its character is zero on the (elliptic) g' which come from g in the inner form. Consequently we need to define the correspondence from the split group to its inner form, and the sign in the character relation has to change.

This extends to the general case of $\mathrm{GL}(n)$ and its inner forms. The local theory is sufficiently developed to express the values of the characters of the local components of the global residual representations in terms of those of the non-degenerate ones that appear in the correspondence, and we shall simply review and summarize these local results, in order to complete the comparison of the unrestricted trace formulae.

24.2 Langlands Classification

We start with recalling the Langlands classification in our context. Let π be an irreducible representation of $G = \mathrm{GL}(m, D)$. Then [BZ76] asserts that there exists a standard (diagonal) Levi subgroup $M = \prod_{i=1}^{r} \mathrm{GL}(m_i, D)$ of G, and a cuspidal irreducible τ_i on $\mathrm{GL}(m_i, D)$ ($1 \le i \le r$) such that π is a constituent of $i_M^G(\tau_1 \times \cdots \times \tau_r)$. The unordered multiset (set with multiplicities) $\{\tau_1, \ldots, \tau_r\}$ of cuspidal representations is called the *cuspidal support* of π, or simply the *support* of π.

Next let σ_i be quasi-square-integrable irreducible representations of $\mathrm{GL}(m_i, D)$; thus there are real exponents e_i and square-integrable σ_i^0 with $\sigma_i = \nu^{e_i}\sigma_i^0$. Let α be a permutation in the symmetric group S_r of $\{1, \ldots, r\}$ such that $e_{\alpha(i)}$ decrease with i. Put $M_\alpha = \prod_{i=1}^{r} \mathrm{GL}(m_{\alpha(i)}, D)$ and $\sigma_\alpha = \sigma_{\alpha(1)} \times \cdots \times \sigma_{\alpha(r)}$ and $\sigma = \sigma_1$. The Langlands classification, [BW80, Si80], asserts that $i_{M_\alpha}^G \sigma_\alpha$ has a unique irreducible quotient π, denoted $L(\sigma)$ or $L(i_M^G \sigma)$, called the *Langlands quotient* of $i_M^G \sigma$. Every irreducible π is so obtained. The unordered multiset $\{\sigma_1, \ldots, \sigma_r\}$ is uniquely determined by π (if $L(\sigma) = L(\sigma')$, $\sigma' = \sigma'_1 \times \cdots \sigma'_{r'}$, then $r' = r$ and there is $\beta \in S_r$ with $n'_i = n_{\beta(i)}$, $\sigma'_i = \sigma_{\beta(i)}$). It is called the *quasi-square-integrable support* of π. The representations $i_M^G \sigma$ are called *standard representations*.

The set of standard representations of G makes a basis of the Grothendieck group of admissible representations of G. Another basis is the set (of equivalence classes) $\mathrm{Irr}G$ of irreducible representations of G, and $i_M^G\sigma \mapsto L(i_M^G\sigma)$ is a bijection between these bases. Write $\pi_1 < \pi$ for $\pi_1 \in \mathrm{Irr}G$ if $\pi_1 \neq \pi$, $\pi = L(i_M^G\sigma)$ and π_1 is a constituent of $i_M^G\sigma$. This is a partial order relation on $\mathrm{Irr}G$. Note that if $\pi_1 < \pi$, then the cuspidal support of π_1 coincides with that of π.

As in [Ze80], we often write $\tau_1 \times \cdots \times \tau_r$ for $i_M^G(\tau_1 \times \cdots \times \tau_r)$. Then $L(\sigma_1 \times \sigma_2)$ is a constituent of $L(\sigma_1) \times L(\sigma_2)$ with multiplicity one; see [Ze80, Proposition 8.4] and [Tc90, Proposition 2.3]. Since any irreducible constituent of $L(\sigma_1) \times L(\sigma_2)$ is also a constituent of $\sigma_1 \times \sigma_2$, any irreducible constituent π of $L(\sigma_1) \times L(\sigma_2)$ other than $L(\sigma_1 \times \sigma_2)$ has $\pi < L(\sigma_1 \times \sigma_2)$. In particular, if $L(\sigma_1) \times L(\sigma_2)$ is reducible, then it has at least two different constituents.

24.3 Generalized Steinberg Representations

The quasi-square-integrable representations σ of $G = \mathrm{GL}(m, D)$ can be listed as follows. Consider first the case of $G' = \mathrm{GL}(n, F)$, where $d = \mathrm{rk}\, D$ is 1, following [Ze80]. Suppose $n = kr$ for positive integers k, r and τ' is a cuspidal representation of $\mathrm{GL}(k, F)$. Then the G-module $\tau' \times \tau'\nu \times \cdots \times \tau'\nu^{r-1}$ induced from $M = \mathrm{GL}(k, F) \times \cdots \times \mathrm{GL}(k, F)$ has a unique quotient σ', denoted $Z(\tau', r)$ or $Z(\tau' \times \tau'\nu \times \cdots \times \tau'\nu^{r-1})$. It is quasi-square-integrable; every quasi-square-integrable irreducible G'-module is so obtained (and is sometimes called generalized Steinberg). A set $S' = \{\tau', \nu\tau', \ldots, \nu^{r-1}\tau'\}$, where τ' is a cuspidal $\mathrm{GL}(k, F)$-module, is called a *segment of length r with end $\nu^{r-1}\tau'$*.

24.4 Segments

To extend this to $G = \mathrm{GL}(m, D)$, note that a standard parabolic subgroup M of G is determined by a sequence $(m_1, \ldots, m_r)$ of positive integers whose sum is m. Its split form is a standard Levi subgroup M' of $G' = \mathrm{GL}(n, F)$, $n = md$, associated with the sequence $(n_1, \ldots, n_r)$, $n_i = m_i d$. The standard Levis M' of G' of type $(n_1, \ldots, n_r)$ with $d|n_i$ for all i is said to come from G. Given a cuspidal $\mathrm{GL}(k, D)$-module τ, denote by σ' the corresponding quasi-square-integrable $\mathrm{GL}(kd, F)$-module. Then $\sigma' = Z(\tau', r')$ for some integer $r' > 0$ and cuspidal $\mathrm{GL}(kd/r', F)$-module τ'. We say that r' is the *Zelevinsky length* $z(\tau)$ of τ. Put $\nu_\tau = \nu^{z(\tau)}$. Then the $G = \mathrm{GL}(kr, D)$-module $i_M^G(\tau \times \nu_\tau\tau \times \cdots \times \nu_\tau^{r-1}\tau)$ induced from $M = \mathrm{GL}(k, D)^r$ has a unique irreducible quotient representation σ, denoted $Z(\tau, r)$. This σ is quasi-square-integrable G-module. Every quasi-square-integrable G-module is so obtained. We put $\sigma = Z(\tau, r)$ and set $z(\sigma) = z(\tau)$. The numbers k, r, and the cuspidal τ are uniquely determined by σ. For this extension of [Ze80], see [Tc90]. A set $S = \{\tau, \nu_\tau\tau, \ldots, \nu_\tau^{r-1}\tau\}$ is called a *segment of length r and end $\nu_\tau^{r-1}\tau$*.

24.5 Multi-Segments

In summary, to a quasi-square-integrable $\mathrm{GL}(m, D)$-module σ, we can associate a segment X. A *multisegment* X is a multiset of segments; its multiset of ends is denoted by $E(X)$.

If π is an irreducible $G = \mathrm{GL}(m, D)$-module, the multiset of the segments of the representations of the quasi-square-integrable support of π is a multisegment, denoted by $X(\pi)$. This $X(\pi)$ determines π. The sum of the representations in $X(\pi)$ is the cuspidal support of π.

Two segments S_1, S_2 are *linked* if $S_1 \cup S_2$ is a segment not equal to S_1 or S_2. Linked S_1, S_2 are *adjacent* if $S_1 \cap S_2 = \varnothing$. If S_1, S_2 are linked segments in a multisegment X, define a multisegment Y to be obtained from X by *elementary operations* if Y is

$$X \cup \{S_1 \cup S_2\} \cup \{S_1 \cap S_2\} - \{S_1, S_2\}$$

if S_1, S_2 are not adjacent (thus $S_1 \cap S_2 \neq \varnothing$) or

$$X \cup \{S_1 \cup S_2\} - \{S_1, S_2\}$$

if S_1, S_2 are adjacent ($S_1 \cap S_2 = \varnothing$). We then write $Y < X$. This extends by transitivity to a partial order relation on the set of multisegments of G. By [Ze80, Theorem 7.1] for $G' = \mathrm{GL}(n, F)$ and [Tc90] for $G = \mathrm{GL}(m, D)$, we have

PROPOSITION 24.6. *Let π_1, π_2 be irreducible G-modules. Then $\pi_1 < \pi_2$ if and only if $X(\pi_1) < X(\pi_2)$.*

Define the *length* $l(X)$ of a multisegment X to be the maximum of the lengths of the segments in X. For an irreducible π, put $l(\pi) = l(X(\pi))$.

PROPOSITION 24.7. (i) *If Y is obtained from X by an elementary operation, then $l(X) \leq l(Y)$ and $E(Y) \subseteq E(X)$. Thus l is decreasing on the set $\mathrm{Irr}\, G$ of irreducible representations of G.*

(ii) *Let $\pi_1 \in \mathrm{Irr}\ \mathrm{GL}(m, D)$, $\pi_2 \in \mathrm{Irr}\ \mathrm{GL}(k, D)$. If for all $S_1 \in X(\pi_1)$ and $S_2 \in X(\pi_2)$ the segments S_1 and S_2 are not linked, then $\pi_1 \times \pi_2$ is irreducible.*

PROOF. Part (i) is easy. Part (ii) is a result of [Ze80] for G' extended to G by [Tc90]. □

If τ is a cuspidal $\mathrm{GL}(k, D)$-module, the *line* generated by τ is $\{\nu_\tau^r \tau\ ;\ r \in \mathbb{Z}\}$; it is equal to the line of $\nu_\tau^r \tau$ for any integer r. An irreducible π of $G = \mathrm{GL}(m, D)$ is called *rigid* if all representations in the cuspidal support of π lie on the same line.

COROLLARY 24.8. *Let π be an irreducible G-module. Let $\mathrm{supp}(\pi)$ be the set of irreducibles in the cuspidal support of π. Let $L_1, \ldots, L_r$ be the set of lines which have nonempty intersection with $\mathrm{supp}(\pi)$. Then up to permutation, π has a unique presentations as $\pi_1 \times \cdots \times \pi_r$ where π_i are rigid and $\mathrm{supp}(\pi_i) \subset D_i$ for all i $(1 \leq i \leq r)$.*

We say that $\pi_1 \times \cdots \times \pi_r$ is the *standard decomposition* of π as a product of rigid representations.

24.9 Opposite Correspondence

We defined a correspondence $C' : R(G) \to R(G')$ by a character relation. It maps irreducible square-integrable (resp. tempered, resp. standard, resp. relevant) G-modules to such G'-modules. Since the standard representations make a basis of the Grothendieck group, C' extends to an injection on all of $R(G)$ by $C' : i_M^G \sigma \mapsto i_{M'}^{G'}(C'(\sigma))$, where σ is a quasi-square-integrable G-module. The character relation commutes with induction. Let us define a map $Q : \mathrm{Irr}(G) \to \mathrm{Irr}(G')$ by $L(i_M^G \sigma) \to L(i_{M'}^{G'}(C'(\sigma)))$, using the fact that each irreducible is a Langlands quotient of a standard representation. If $\pi_1 < \pi_2$, then $Q\pi_1 < Q\pi_2$. Define $\pi_1 << \pi_2$ if $Q\pi_1 < Q\pi_2$. Then $\pi_1 < \pi_2$ implies $\pi_1 << \pi_2$, but in principle there can be irreducible π_1, π_2 with $Q\pi_1 < Q\pi_2$ without $\pi_1 < \pi_2$.

Define a correspondence $C : R(G') \to R(G)$ in the opposite direction to be the $\mathbb{Z}$-morphism with $C(i_{M'}^{G'}\sigma') = 0$ if $i_{M'}^{G'}\sigma'$ is not in the image of the correspondence C', and $C(i_{M'}^{G'}\sigma') = i_M^G \sigma$ if $\sigma' = C'(\sigma)$, where σ' is quasi-square-integrable. Then C is a surjective group homomorphism, and it is the unique map from $R(G')$ to $R(G)$ satisfying the character relation $\chi_{\pi'}(g') = (-1)^{m(d-1)} \chi_{C(\pi')}(g)$ whenever $g' \in G'$ and $g \in G$ are regular with equal characteristic polynomials. It commutes with induction. We have

$$C(Q(\pi)) = \pi + \sum_{\{\pi_j\,;\,\pi_j << \pi\}} b_j \pi_j$$

for irreducibles π_j and integers b_j.

Definition. A virtual G'-module π' in $R(G')$ is a *d-lift* if $C(\pi') \neq 0$; thus $\chi_{\pi'}(g') \neq 0$ for a regular $g' \in G'$ which comes from a $g \in G$. Thus the characteristic polynomial of g' is a product of irreducible factors of degrees divisible by d. Hence π' being a d-lift depends only on d, not on D. An induced $\pi' = i(\tau_1' \times \cdots \times \tau_r')$ is a d-lift if and only if each τ_j' is a d-lift.

24.10 Langlands Quotients

Recall that $i(\tau' \times \nu\tau' \times \cdots \times \nu^{r-1}\tau')$ has a unique irreducible quotient $Z(\tau', r)$, which is quasi-square-integrable when τ' is an irreducible cuspidal $\mathrm{GL}(k, F)$-module. It is unitarizable if and only if $\tau_u' = \nu^{\frac{r-1}{2}}\tau'$ is unitarizable. Thus $i(\nu^{-\frac{r-1}{2}}\tau_u' \times \nu^{1-\frac{r-1}{2}}\tau_u' \times \cdots \times \nu^{\frac{r-1}{2}}\tau_u')$ has a unique irreducible quotient $Z_u(\tau_u', r)$ which is square-integrable when τ_u' is unitarizable. We write $Z_u(\tau', r)$ only when τ' is unitarizable.

When σ' is a square-integrable representation of $\mathrm{GL}(k, F)$, write $u(\sigma', r)$ for the Langlands quotient $L(\prod_{j=1}^{r} \nu^{\frac{r+1}{2}-j}\sigma')$. It is an irreducible representation of $\mathrm{GL}(rk, F)$.

For s $(0 < s < \frac{1}{2})$, the representation $\pi(u(\sigma', r), s) = i(\nu^s u(\sigma', r) \times \nu^{-s} u(\sigma', r))$ is irreducible by Proposition 24.7(ii).

PROPOSITION 24.11 (Bernstein, Tadic). *Let U' be the set of all representations $u(\sigma', r)$ and $\pi(u(\sigma', r), s)$, where σ' is any square-integrable $\mathrm{GL}(k, F)$-module, $r, k \in \mathbb{Z}_{>0}$. Then any product of members of U' is irreducible and unitarizable, and any unitarizable G'-module π' is such a product. The nonordered multiset of the factors in the product is uniquely determined by π'.*

24.12 Unitarizable Representations

Let τ be a cuspidal irreducible $\mathrm{GL}(k, D)$-module. It corresponds to a quasi-square-integrable $\mathrm{GL}(kd, F)$-module $\sigma' = C'(\tau)$, of the form $Z(\tau', r)$ for some cuspidal $\mathrm{GL}(kd/r, F)$-module τ' and divisor r of kd. Put $z(\tau) = r$ and $\nu_\tau = \nu^{z(\tau)}$.

The representations $i\left(\prod_{j=0}^{r} \nu_\tau^j \tau\right)$ has a unique irreducible quotient σ, which is a quasi-square-integrable $\mathrm{GL}(kr, D)$-module, denoted by $Z(\tau, r)$. By [Tc90] every quasi-square-integrable σ is so obtained, and we put $z(\sigma)$ for $z(\tau)$.

Then $\sigma = Z(\tau, r)$ is unitarizable if and only if $\tau_u = \nu_\tau^{\frac{r-1}{2}} \tau$ is unitarizable, and we write $\sigma = Z_u(\tau_u, r)$. For a square-integrable σ, write $u(\sigma, r)$ for the Langlands quotient $L\left(\prod_{j=1}^{r} \nu_\tau^{\frac{r+1}{2}-j} \tau\right)$. Also put $\overline{u}(\sigma, r)$ for the Langlands quotient $L\left(\prod_{j=1}^{r} \nu^{\frac{r+1}{2}-j} \tau\right)$. Both are irreducible unitarizable representations of $\mathrm{GL}(kr, D)$. For s $(0 < s < \frac{1}{2})$, put $\pi(u(\sigma, r), s) = \nu_\tau^s u(\tau, r) \times \nu_\tau^{-s} u(\tau, r)$. It is an irreducible representation of $\mathrm{GL}(2kr, D)$. There are the relations

$$\overline{u}(\sigma, rz(\sigma)) = \prod_{j=1}^{z(\sigma)} \nu^{j-\frac{z(\sigma)+1}{2}} u(\sigma, r)$$

and for all integers m $(1 \leq m < z(\sigma))$:

$$\overline{u}(\sigma, rz(\sigma) + m) = \left(\prod_{j=1}^{m} \nu^{j-\frac{m+1}{2}} u(\sigma, r+1)\right) \times \left(\prod_{j=1}^{z(\sigma)-m} \nu^{j-\frac{z(\sigma)-m+1}{2}} u(\sigma, r)\right).$$

The second factor is erased if $r = 0$. These products are irreducible as the segments in the quasi-square-integrable support of any two factors are not linked.

24.13 Hermitian Representations

Let π be an irreducible $\mathrm{GL}(m,D)$-module. Denote by $h(\pi)$ the Hermitian conjugate of π. This is an irreducible $\mathrm{GL}(m,D)$-module with a non-degenerate sesquilinear form $\langle\cdot,\cdot\rangle$ on $\pi\times h(\pi)$. Then π is said to be *Hermitian* if it is equivalent to $h(\pi)$. Unitarizable representations are Hermitian. Let $H=\{\sigma_j\,;\,1\le j\le k\}$ be a multiset of quasi-square-integrable representations of $G(m_j,D)$. Let $h(H)$ be the multiset $\{h(\sigma_j)\,;\,1\le j\le k\}$. For $s\in\mathbb{R}$, $h(\nu^s\pi)=\nu^{-s}h(\pi)$.

PROPOSITION 24.14. (i) *If H is the quasi-square-integrable support of an irreducible* $\mathrm{GL}(m,D)$*-module π, then $h(H)$ is this support of $h(\pi)$. In particular, π is Hermitian iff $h(H)=H$.*

(ii) *Let π_j be irreducible unitarizable* $\mathrm{GL}(m_j,D)$*-modules such that $i(\pi_j\times\pi_j)$ is irreducible. Then $i\left(\prod_{j=1}^k\pi_j\right)$ is irreducible.*

24.15 Permutations

Let σ be a square-integrable $\mathrm{GL}(m,D)$-module. It corresponds to a square-integrable $\mathrm{GL}(n,F)$-module $\sigma'=C'(\sigma)$. Then $\sigma=Z_u(\tau,k)$ for a cuspidal unitarizable $\mathrm{GL}(m/k,D)$-module τ for a $k|m$. This τ corresponds to a square-integrable $C'(\tau)$ of the form $Z_u(\tau',z(\sigma))$ for some cuspidal unitarizable $\mathrm{GL}(\frac{md}{kz(\sigma)},F)$-module τ'. Then $\sigma'\in Z_u(\tau',k')$ where $k'=kz(\sigma)$. In particular, the cuspidal support of σ' can be found from that of σ, since the correspondence commutes with taking the module of coinvariants.

Let W_r^k be the set of permutations w of $\{1,\dots,r\}$ with $w(j)+k\ge j$ for all j $(1\le j\le r)$. Let R be the group of permutations w of $\{1,\dots,r'\}$, $r'=rz(\sigma)$, with $z(\sigma)|w(j)-j$ for all j $(1\le j\le r')$.

PROPOSITION 24.16. (i) *We have $C(u(\sigma',r'))=\overline{u}(\sigma,r')$.*

(ii) *The induced representation $i(\overline{u}(\sigma,r')\times\overline{u}(\sigma,r'))$ is irreducible.*

(iii) *We have the character formula*

$$\overline{u}(\sigma,r')=\nu^{-\frac{k'+r'}{2}+\frac{z(\sigma)-1}{2}}\sum_{w\in R\cap W_{r'}^{k'}}\mathrm{sgn}(w)\prod_{j=1}^{r'}Z\left(\nu^j\tau,\frac{w(j)-j}{z(\sigma)}+k\right).$$

(iv) *The induced $i(u(\sigma,r)\times u(\sigma,r))$ is irreducible, and $\pi(u(\sigma,r),s)$ is unitarizable for all s $(0<s<\frac12)$.*

(v) *Suppose $\sigma\in Z_u(\tau,k)$ for a unitarizable cuspidal representation τ. Then*

$$u(\sigma,r)=\nu_\sigma^{-\frac{r+k}{2}}\sum_{w\in W_r^k}\mathrm{sgn}(w)\prod_{j=1}^{r}Z(\nu_\sigma^j,w(j)+k-j).$$

This is Theorem 3.2 and Corollary 3.6 of [Ba08], extending a result of Tadic when $D = F$ (and so $z(\sigma) = 1$).

24.17 Indices

Starting with a cuspidal unitarizable representation of $\mathrm{GL}(q, F)$, let $\sigma'_k = Z_u(\tau', k)$ be the associated square-integrable $\mathrm{GL}(kq, F)$-module, and similarly put $\sigma'_r = Z_u(\tau', r)$. Denote $d/(d, q)$ by z. It is the smallest natural number with $d|zq$. Assume $d|n = rkq$, thus $z|rk$. Proposition 3.7 of [Ba08] asserts that (i) if $d|kq$, thus $z|k$, then $C(\sigma'_k) = \sigma_k$ is a square-integrable $\mathrm{GL}(\frac{kq}{d}, D)$-module and z is $z(\sigma_k)$ and $C(u(\sigma'_k, r)) = \overline{u}(\sigma_k, r)$. (ii) If d divides neither kq nor rq, thus z does not divide k and r, then $C(u(\sigma'_k, r)) = 0$. If $d|rq$, thus $z|r$, then $C(\sigma'_r) = \sigma_r$ is a square-integrable $\mathrm{GL}(\frac{rq}{d}, D)$-module, we have $z = z(\sigma_r)$, and $C(u(\sigma'_k, r))$ is computed in [Ba08, subsection 3.3]. Using this, [Ba08, Proposition 3.9] concludes the following. Recall that the correspondence $C : R(G') \to R(G)$ is defined by a character relation.

Denote by U the set of representations $u(\sigma, r)$ and $\pi(u(\sigma, r), s)$, where $0 < s < \frac{1}{2}$, σ is a square-integrable $\mathrm{GL}(k, F)$-module, and $k, r \geq 1$.

PROPOSITION 24.18. (i) *The representations in U are all irreducible and unitarizable.*

(ii) *If $\pi_1, \ldots, \pi_r \in U$, then the induced $i(\pi_1 \times \cdots \times \pi_r)$ is irreducible and unitarizable.*

(iii) *If π' is an irreducible unitarizable* $\mathrm{GL}(md, F)$*-module, then $C(\pi')$ is* 0 *or it is a unitarizable irreducible representation π of* $\mathrm{GL}(m, D)$*, or it is $-\pi$ for such a π.*
Write $|C|(\pi') = \pi$ in the second case.

(iv) *If π is a unitarizable irreducible* $\mathrm{GL}(m, D)$*-module, and $i(\pi \times \pi)$ is irreducible, then π is induced from a product of representations in U.*

(v) *Let $\prod U$ be the set of induced representations $i(\pi_1 \times \cdots \times \pi_r)$, $\pi_i \in U$. It is a set of irreducible unitarizable representations including the $\overline{u}(\sigma, r)$. It is stable under the Zelevinsky involution—see* [Ze80]*—$\iota : R(G) \to R(G)$.*

(vi) *If π', a unitarizable representation of* $\mathrm{GL}(md, F)$*, is a lift, then $|C|(\pi') \in \prod U$.*

24.19 All Components

Finally we note that [Ze80] implies that a non-degenerate irreducible unitarizable $\mathrm{GL}(n, F)$-module π' has the form $\pi' = i(\nu^{e_1}\sigma'_1 \times \cdots \times \nu^{e_q}\sigma'_q)$ where the σ'_j are square-integrable and $e_j \in (-\frac{1}{2}, \frac{1}{2})$. Then for every $r > 0$ in $\mathbb{Z}$ the induced $i(\nu^{\frac{r+1}{2}-1}\pi' \times \cdots \times \nu^{\frac{r+1}{2}-r}\pi')$ is a standard representation. Denote its Langlands quotient by $L(\pi', r)$. Then $L(\pi', r) = i(\nu^{e_1}u(\sigma'_1, r) \times \cdots \times \nu^{e_q}u(\sigma'_q, r))$, and it is unitarizable as so was π'. Note that π' is tempered if and only if all e_j are 0.

Now all local components of any cuspidal representation are non-degenerate, so [MW95] implies that all components of a discrete spectrum representation of $\mathrm{GL}(n, \mathbb{A})$ are of this form, $L(\pi', r)$. When do they map by C to a nonzero representations?

To answer this question, write $\sigma_j' = Z_u(\tau_j', k_j) = Z(\nu^{-\frac{k_j-1}{2}}\tau_j' \times \cdots \times \nu^{\frac{k_j-1}{2}}\tau_j')$ with cuspidal unitarizable $\mathrm{GL}(p_j, F)$-modules τ_j'. Put $J = \{j \in \mathbb{Z};\ 1 \le j \le q, d|p_j k_j\}$. Let $z(\pi', d)$ be the least natural number z such that $d|zp_j$ for all $j \in \{1, \dots, q\} - J$. Then $C(L(\pi', r)) \neq 0$ if and only if $C(u(\sigma_j', r)) \neq 0$ for all j $(1 \le j \le q)$ if and only if $z(\pi', d)|r$ by 24.17. In this case $C(L(\pi', r)) = i\left(\prod_{j=1}^{q} \nu^{e_j} C(u(\sigma_j', r))\right)$.

25 Complete Global Correspondence

25.1 Local Lifting

Using the comparison of the trace formulae for arbitrary matching test functions, achieved in Chapter 6 below, a more complete correspondence theorem can be obtained, without any conditions at any local place, and for the entire discrete spectrum, rather than only for the cuspidal representations with cuspidal components. A form of such a global theorem is as follows.

Let F be a number field. Put $G = \mathrm{GL}(m, D)$ where D is a central division algebra over F of rank d. Put $G' = \mathrm{GL}(n, F)$, $n = md$, $G_v' = G'(F_v)$. Put $G_v = G(F_v) = \mathrm{GL}(m_v, D_v)$, where $D \otimes_F F_v = M(m_v', D_v)$, D_v is a division algebra of rank d_v central over F_v and $m_v' d_v = d$, $m_v = mm_v'$.

Recall that we defined, in Subsection 24.9, the local correspondence C_v' : $R(G_v) \to R(G_v')$ and its partial inverse $C_v : R(G_v') \to R(G_v)$, by means of a character relation on all regular elements, in such a way that each irreducible G_v-module π_v corresponds to a G_v'-module π_v'. Also we defined—in Proposition 24.18(iii)—a correspondence $|C_v|$ from the set $\mathrm{Irr}_u(G_v')$ of irreducible unitarizable representations of G_v' to the union of $\{0\}$ and the analogous set $\mathrm{Irr}_u(G_v)$ for G_v. It coincides with the correspondence C_v up to a sign. If π_v corresponds to π_v' up to a sign—thus $|C_v|(\pi_v') = \pi_v$—we also say that π_v' *comes from* π_v. If π_v' is a unitarizable irreducible d_v-lift, namely, its character $\chi_{\pi_v'}$ is nonzero at a regular g' in G_v' whose characteristic polynomial equals that of some $g \in G_v$, then $|C_v|(\pi_v') \neq 0$, and it is an irreducible G_v-module, by Proposition 24.18(vi).

For an irreducible representation π' of $G'(\mathbb{A})$, we say that π' comes locally from $G(\mathbb{A})$ if π_v' comes from G_v for all v (namely, comes from some π_v on G_v). By Proposition 24.18(vi), this is the same as saying that π' is a D-lift, in the sense that π_v' is a d_v-lift for all $v \in V$.

Let V be the set of places of F where D ramifies.

THEOREM 25.2. *There exists a unique map* C' *from the discrete spectrum* $\mathrm{DS}(G(\mathbb{A}))$ *to the discrete spectrum* $\mathrm{DS}(G'(\mathbb{A}))$ *such that* $C'(\pi) = \pi'$ *is defined* *by $\pi'_v \simeq \pi_v$ for all $v \notin V$. It is injective. If $C'(\pi) = \pi'$, then $|C_v|(\pi'_v) = \pi_v$ for all $v \in V$. If π' is cuspidal, then $C_v(\pi'_v) = \pi_v$ for all $v \in V$. The latter means that the character relation $\chi_{\pi'_v}(g') = (-1)^{(m_v-1)d_v}\chi_{\pi_v}(g)$ holds for all regular $g \in \mathrm{GL}(m_v, D_v)$ and $g' \in \mathrm{GL}(m_v d_v, F_v)$ with equal characteristic polynomials.*

The global map C' is injective. Its image is the set of discrete spectrum π' which come locally from $G(\mathbb{A})$, equivalently π'_v is a d_v-lift for all $v \in V$, namely, $\chi_{\pi'_v}$ is nonzero at some regular $g'_v \in G'_v$ coming from G_v. Thus if π' is a discrete spectrum representation of $G'(\mathbb{A})$, and $\chi_{\pi'_v}(g') \neq 0$ for a regular g' coming from G_v, then π'_v comes from G_v, for all $v \in V$.

Multiplicity one theorem holds for $G(\mathbb{A})$. Thus each irreducible representation in the discrete spectrum of $G(\mathbb{A})$ occurs with multiplicity one.

Rigidity theorem, also called "strong multiplicity one theorem", holds for $G(\mathbb{A})$. Thus if π_1, π_2 occur in the discrete spectrum of $G(\mathbb{A})$ and $\pi_{1,v} \simeq \pi_{2,v}$ for almost all v, then $\pi_{1,v} \simeq \pi_{2,v}$ for all v.

If π is in the discrete spectrum of $G(\mathbb{A})$, then $\pi_v \in \prod U_v$ for all $v \in V$.

25.3 Trace Formula

Let F be a number field. Let $F_\infty^\times$ be $\prod_{v \in S_\infty} F_v^\times$, product over the Archimedean places of F, embedded in $\mathbb{A}^\times$ as the idèles with component 1 at the non-Archimedean places. Thus $F_\infty^\times \subset \mathbb{A}^\times = Z(\mathbb{A})$, where Z is the center of $\mathrm{GL}(n)$. Let ω be a unitary character of $F_\infty^\times$. Let D be a division algebra of rank d central over F. Put $G = \mathrm{GL}(m, D)$. Let $\mathcal{L}$ denote the set of F-Levi subgroups of G which are standard, namely, contain the (maximal split torus in the) center of the diagonal group of G. Write the expression 6.(3.4.1) in the form which appears in 6.(16.11.1) as

$$I_{t,\omega,\mathrm{disc}}(f) = \sum_{M \in \mathcal{L}} |W_0^M|/|W_0^G| \sum_{s \in W(\mathfrak{a}_M)_{\mathrm{reg}}} |\det(s-1)_{\mathfrak{a}_M^G}|^{-1} \operatorname{tr} \mathcal{M}_M^G(s,0) r_{M,t}(0,f). \tag{25.3.1}$$

Recall that $r_{M,t}(0,f)$ is $r_{Q,t}(0,f)$, where Q is any parabolic with Levi M, the induced representation of $G(\mathbb{A})$ obtained from the subrepresentation of $M(\mathbb{A})$ in $L^2(M(F)\backslash M(\mathbb{A}))$ which decomposes into a discrete sum of irreducible representations τ of $M(\mathbb{A})$ which transform under $F_\infty^\times$ by ω, and the imaginary part of the Archimedean infinitesimal character of τ has norm t, where $t \in \mathbb{R}_{\geq 0}$. The other terms are $|W_0^M|$, the cardinality of the Weyl group of M, so $|W_0^G|$ is that of G. Then $\mathfrak{a}_M^G = \mathfrak{a}_M/\mathfrak{a}_G$, where $\mathfrak{a}_M = \mathrm{Hom}(X(M)_F, \mathbb{R})$ and $X(M)_F$ is the lattice of F-rational characters of M. Further, $W(\mathfrak{a}_M)$ is the Weyl group of $\mathfrak{a}_M$, and $W(\mathfrak{a}_M)_{\mathrm{reg}}$ is the set of s in $W(\mathfrak{a}_M)$ with $\det(s-1)_{\mathfrak{a}_M^G} \neq 0$. Finally, $\mathcal{M}_M^G(s,0) = \mathcal{M}(s,0)$ is the intertwining operator associated to s at the point 0. It intertwines the representations $i_M^G\tau$ and $i_{sM}^G s\tau$, where τ is a representation of M.

The identity 6.(16.11.1) asserts that for matching test functions $f = \otimes_v f_v$ on $G(\mathbb{A})$ and $f' = \otimes_v f'_v$ on $G'(\mathbb{A})$, $G' = \mathrm{GL}(n)$, $n = md$, the expression $I_{t,\omega,\mathrm{disc}}(f)$ for G is equal to the analogous expression $I_{t,\omega',\mathrm{disc}}(f')$.

25.4 Weyl Groups

The set $W(\mathfrak{a}_M)_{\mathrm{reg}}$ is empty unless M is $\mathrm{GL}(r_i, D)$ along the diagonal, $1 \le i \le k$, with r_i independent of i, up to conjugation. Suppose M is then the Levi subgroup $\mathrm{GL}(r,D) \times \cdots \times \mathrm{GL}(r,D)$, with k factors and $m = kr$. Then $W(\mathfrak{a}_M)$ is isomorphic to the symmetric group S_k and $W(\mathfrak{a}_M)_{\mathrm{reg}}$ is the set of k-cycles. Then $W(\mathfrak{a}_M)_{\mathrm{reg}}$ consists of $(k-1)!$ elements. For every $s \in W(\mathfrak{a}_M)_{\mathrm{reg}}$, we have $|\det(s-1)_{\mathfrak{a}_L^G}| = k$. Further, $|W_0^M| = (r!)^k$ and $|W_0^G| = m!$. Thus the coefficient of the term indexed by M in $I_{t,\omega,\mathrm{disc}}$ is $(r!)^k/m!k$.

If M_1 is conjugate to M, the contribution of M_1 to the sum is equal to that of M. If M_1 is conjugate to M, and it contains the diagonal torus, then the center of M_1 is contained in the (center of the) diagonal torus. But M_1 is the centralizer of its center. Hence the number of M_1 is the number of non-ordered partitions of $\{1, \ldots, m\}$ into k subsets of cardinality r. This number is $\binom{m}{m-r}\binom{m-r}{m-2r}\binom{m-2r}{m-3r}\cdots\binom{2r}{r}/k!$, which is $m!/(r!)^k k!$.

25.5 Rewriting the Formula

We can then rewrite the formula as follows. For $k|m$, let M_k be the Levi subgroup of G, diagonal with k blocks of size $r = m/k$. Then

$$I_{t,\omega,\mathrm{disc}}(f) = \sum_{k|m} \frac{1}{k!k} \sum_{s \in W(\mathfrak{a}_{M_k})_{\mathrm{reg}}} \mathrm{tr}\, \mathcal{M}_{M_k}^G(s,0) r_{M_k,t}(0,f).$$

For any M_k, the $(k-1)!$ elements $s \in W(\mathfrak{a}_{M_k})_{\mathrm{reg}}$ give the same contribution to the sum. Thus if s denotes now the cycle $(1, 2, \ldots, k)$, we get

$$I_{t,\omega,\mathrm{disc}}(f) = \sum_{k|m} k^{-2}\, \mathrm{tr}\, \mathcal{M}_{M_k}^G(s,0) r_{M_k,t}(0,f).$$

25.6 Intertwinging Operators

To describe the operator $\mathcal{M}_{M_k}^G(s,0) r_{M_k,t}(0,f)$, note that a discrete spectrum representation τ of $M_k = \mathrm{GL}(r,D) \times \cdots \times \mathrm{GL}(r,D)$ has the form $\tau_1 \times \cdots \times \tau_k$, where each τ_i is a discrete spectrum representation of $\mathrm{GL}(r, D(\mathbb{A}))$. The symmetric group

S_k permutes the entries of the sequence $(\tau_1, \ldots, \tau_k)$. Denote by $\mathrm{Fix}(\tau)$ the subgroup which fixes $(\tau_1, \ldots, \tau_k)$. Let $\mathrm{Rep}(\tau)$ be a set of representatives in S_k for $S_k/\mathrm{Fix}(\tau)$. The subspace $V_\tau = \otimes_{\alpha \in \mathrm{Rep}(\tau)} \tau_{\alpha(1)} \times \cdots \times \tau_{\alpha(k)}$ of $r_{M_k,t}$ is invariant under the operator $\mathcal{M}^G_{M_k}(s,0)$. If $\tau_i \neq \tau_j$ for some $i \neq j$, then this operator acts on each $\tau_{\alpha(1)} \times \cdots \times \tau_{\alpha(k)}$ without fixed point, and its trace on V_τ is zero. Thus contributions to the formula are only from $\tau = \tau_1 \times \cdots \times \tau_1$, $\mathrm{Fix}(\tau) = S_k$ and $\mathrm{Rep}(\tau) = \{1\}$, so

$$I_{t,\omega,\mathrm{disc}}(f) = \sum_{\pi \in \mathrm{DS}(m,t,\omega)} m(\pi)\,\mathrm{tr}\,\pi(f) + \sum_{1<k|m} k^{-2} \sum_{\tau \in \mathrm{DS}(m/k,t/k,\omega_k)} m(\tau)^k\,\mathrm{tr}\,\mathcal{M}^G_{M_k}(s,0)i(\tau^k,0;f).$$

Here $\mathrm{DS}(m,t,\omega)$ is the set of discrete spectrum representations τ of $\mathrm{GL}(m, D(\mathbb{A}))$ such that τ transforms via ω on $F_\infty^\times$ and the norm of the imaginary part of its infinitesimal character is $t \geq 0$. The coefficient $m(\tau)$ is the multiplicity of the irreducible τ in the discrete spectrum of $\mathrm{GL}(m, D(\mathbb{A}))$. The same notation of course applies to $\mathrm{DS}(m/k, t/k, \omega_k)$, where ω_k is a character of $F_\infty^\times$ with $\omega_k^k = \omega$. By τ^k we mean $\tau \times \cdots \times \tau$ (k times).

25.7 Discrete Part of Trace Formula

The same formula holds of course when $D^\times = \mathbb{G}_m$, the multiplicative group; thus $d = 1$, for $G' = \mathrm{GL}(n)$. In this case the multiplicities $m(\tau)$ are 1 by the multiplicity one theorem [Shal74] and its extension to the residual spectrum by [MW89]. In this case

$$I_{t,\omega,\mathrm{disc}}(f') = \sum_{\pi' \in \mathrm{DS}'(n,t,\omega)} \mathrm{tr}\,\pi'(f') + \sum_{1<k|n} k^{-2} \sum_{\tau \in \mathrm{DS}'(n/k,t/k,\omega_k)} \mathrm{tr}\,\mathcal{M}^{G'}_{M'_k}(s,0)i(\tau'^k,0;f').$$

We write DS' for DS of the split group.

Then 6.(16.11.1) asserts that for matching f on $G(\mathbb{A})$, $G = \mathrm{GL}(m,D)$, and f' on $G'(\mathbb{A})$, $G' = \mathrm{GL}(n)$, $n = md$, we have

$$I_{t,\omega,\mathrm{disc}}(f') = I_{t,\omega,\mathrm{disc}}(f).$$

PROPOSITION 25.8. *Assume that the multiplicity one theorem holds for all* $\mathrm{GL}(m', D)$*,* $m' < m$*. Then*

$$\sum_{\pi' \in \mathrm{DS}'_D(n,t,\omega)} \mathrm{tr}\,\pi'(f') + \sum_{1<k|m} k^{-2} \sum_{\tau' \in \mathrm{DS}'_D(n/k,t/k,\omega_k)} \mathrm{tr}\,\mathcal{M}^{G'}_{M'_k}(s,0)i(\tau'^k,0;f')$$
$$= \sum_{\pi \in \mathrm{DS}(m,t,\omega)} m(\pi)\,\mathrm{tr}\,\pi(f) + \sum_{1<k|m} k^{-2} \sum_{\tau \in \mathrm{DS}(m/k,t/k,\omega_k)} \mathrm{tr}\,\mathcal{M}^G_{M_k}(s,0)i(\tau^k,0;f)$$

where DS'_D *signifies the subset of* τ' *all of whose components* τ'_v *comes from* τ_v *in* $G(F_v)$*.*

PROOF. The $m(\tau)^k$ in the expression right of $=$ is replaced with 1 by the induction assumption. As for the first expression, left of =, associated with G', let k divide $n = md$, let $\tau \in \mathrm{DS}'(md/k, t/k, \omega_k)$, and let f, f' be matching functions. We need to show that if k does not divide m, or if $k|m$ and τ'_v does not come from τ_v on $G(F_v)$ for all v, then $\operatorname{tr} \mathcal{M}^{G'}_{M'_k}(s,0) i(\tau'^k, 0; f')$ is zero.

Suppose then that k does not divide m. Then d does not divide md/k. Since d is the least common multiple of the integers d_v, there is a place u such that d_u does not divide md/k. But then τ'_u does not come from τ_u, and τ'^k_u does not come from the correspondence. Now the operator $\mathcal{M}^{G'}_{M'_k}(s,0)$ acts as a scalar on the irreducible $i(\tau'^k)$. So $\operatorname{tr} i(\tau'^k_u, 0; f'_u) = 0$, as required. □

Using the proposition we prove the theorem, by induction on m. If $m = 1$ the identity of the proposition reduces to

$$\sum_{\pi' \in \mathrm{DS}'_D(d,t,\omega)} \operatorname{tr} \pi'(f') = \sum_{\pi \in \mathrm{DS}(1,t,\omega)} m(\pi) \operatorname{tr} \pi(f) \tag{25.8.1}$$

for all matching functions f, f'. Consider a discrete spectrum representation π'_0 which occurs on the left. Denote by V the finite set of places where D ramifies. Write π'_0 as $\pi'_{0V} \otimes \pi'^V_0$, accordingly. The standard argument of generalized linear independence of characters used in this chapter implies that for matching f_v, f'_v $(v \in V)$ we have

$$\sum_{\{\pi' ; \pi'^V = \pi'^V_0\}} \operatorname{tr} \pi'_V(f'_V) = \sum_{\{\pi ; \pi^V = \pi'^V_0\}} m(\pi) \operatorname{tr} \pi_V(f_V). \tag{25.8.2}$$

The sum on the left consists of the discrete spectrum representations π' of $\mathrm{GL}(d, \mathbb{A})$ with $\pi'^V = \pi'^V_0$.

By multiplicity one theorem and rigidity theorem for the cuspidal spectrum of $\mathrm{GL}(d)$, extended to the discrete noncuspidal spectrum by [MW89], the only contribution to the left side is π'_0. Since π'_{0v} comes from π_{0v} for all $v \in V$, we can write the term on the left as $\prod_{v \in V} \operatorname{tr} \pi_{0v}(f_v)$. Now on the right there are only finitely many discrete spectrum π with component π'^V_0 outside V by the argument involving L-functions explained in Section 1. Even without this argument, generalized linear independence of characters used in this chapter implies that the sum on the right reduces to a single term, say π_0, with $m(\pi_0) = 1$. Thus the discrete spectrum π'_0 on $\mathrm{GL}(d, \mathbb{A})$ corresponds to the discrete spectrum π_0 on $D(\mathbb{A})^\times$ with the same t and character ω on $F^\times_\infty$.

25.9 Lifting in First Case

Conversely, starting with $\pi_0 \in \mathrm{DS}(1, t, \omega)$ which appears in the sum on the right of (25.8.1) we obtain once again the identity (25.8.2), with π'^V_0 replaced by π^V_0. If the left side is 0 (its sum is empty), since f_V is arbitrary, we would obtain a contradiction (it suffices to take f_V to be the characteristic function of a small neighborhood of the identity); we need not appeal to the argument that the sum on the right is finite, which was used in Section 1 since we could not use arbitrary test functions f_V at that stage. Then there exists π_0' on the left with component $\pi_0'^V = \pi_0^V$, and as in the first part of this discussion of the case $m = 1$, the discrete spectrum π_0' exists and is unique, and it corresponds to the discrete spectrum π_0; both have the same infinitesimal character, hence t, and character ω on $F_\infty^\times$. Moreover, multiplicity one and rigidity theorems hold on the discrete spectrum of $D(\mathbb{A})^\times$.

25.10 Induction

To prove the theorem for $m > 1$, we can assume it by induction for all $m' < m$, having just established it for $m' = 1$. We then use the identity of the proposition. Note that the τ which appear on the right are discrete spectrum, in particular unitary, hence $i_{M_k}^G(\tau^k, 0)$ is irreducible, and the intertwining operator $\mathcal{M}_{M_k}^G(s, 0)$ is a scalar $\lambda(\tau)$ on the unit circle in $\mathbb{C}^\times$, as $\mathcal{M}$ is unitary. The same holds in particular for the left side; thus $\mathcal{M}_{M_k'}^{G'}(s, 0)$ is $\lambda(\tau')$ on $i_{M_k'}^{G'}(\tau'^k, 0)$ of absolute value one. By induction each discrete spectrum τ in $\mathrm{DS}(m/k, t/k, \omega_k)$, where $1 < k|m$, corresponds to a $\tau' \in \mathrm{DS}_D'(m/k, t/k, \omega_k)$, so we can write the identity of the proposition as

$$\sum_{\pi' \in \mathrm{DS}_D'(n,t,\omega)} \operatorname{tr} \pi'(f') + \sum_{1<k|m} k^{-2} \sum_{\tau' \in \mathrm{DS}_D'(n/k,t/k,\omega_k)} (\lambda(\tau') - \lambda(\tau \to \tau')) \operatorname{tr} i(\tau'^k, 0, f')$$
$$= \sum_{\pi \in \mathrm{DS}(m,t,\omega)} m(\pi) \operatorname{tr} \pi(f). \tag{25.10.1}$$

Here $\lambda(\tau \to \tau')$ is $\lambda(\tau)$ for the $\tau \in \mathrm{DS}(m/k, t/k, \omega_k)$ which corresponds to τ' by induction, as $1 < k|m$.

25.11 Trace Identity

As in the case $m = 1$, fix a discrete spectrum π_0' in $\mathrm{DS}_D'(n, t, \omega)$. The standard argument of generalized linear independence of characters implies that for matching f_v, f_v' $(v \in V)$, we have

$$\sum_{\{\pi' ; \pi'^V = \pi_0'^V\}} \operatorname{tr} \pi_V'(f_V') + \sum_{1<k|m} \frac{\lambda(\tau') - \lambda(\tau \to \tau')}{k^2} \operatorname{tr} i(\tau_V'^k, 0, f_V')$$
$$= \sum_{\{\pi ; \pi^V = \pi_0'^V\}} m(\pi) \operatorname{tr} \pi_V(f_V). \tag{25.11.1}$$

The sum on the left ranges over the π' in $\mathrm{DS}'_D(n, t, \omega)$ with $\pi'^V = \pi_0'^V$. By the rigidity theorem for $\mathrm{GL}(n)$, the only term in this sum is that of π_0'. The other sum on the left ranges over the discrete spectrum $\tau' \in \mathrm{DS}'_D(n/k, t/k, \omega_k)$, for $1 < k|m$, with $i(\tau'^k, 0)^V \simeq \pi_0'^V$.

25.12 Residual Spectrum

Moeglin and Waldspurger [MW89] classified the discrete spectrum of $\mathrm{GL}(n, \mathbb{A})$. Let τ' be a cuspidal representation of $\mathrm{GL}(k, \mathbb{A})$. Then the induced representation $i(\nu^{\frac{r+1}{2}-1}\tau' \times \cdots \times \nu^{\frac{r+1}{2}-r}\tau')$ from the Levi of type $(k, \dots, k)$ (k appears r times, $n = kr$) has a unique irreducible constituent π' which is a discrete spectrum of $\mathrm{GL}(n, \mathbb{A})$, denoted $\mathrm{MW}(\tau', r)$. Note that for all v the τ_v' is non-degenerate, hence irreducible standard, and when written as induced from quasi-square-integrables with decreasing central exponents, π_v' is the Langlands quotient $L(\tau_v', r)$ of $i(\nu_v^{\frac{r+1}{2}-1}\tau_v' \times \cdots \times \nu_v^{\frac{r+1}{2}-r}\tau_v')$. Every discrete spectrum representation π' of $\mathrm{GL}(n, \mathbb{A})$ is of the form $\mathrm{MW}(\tau', r)$ for uniquely determined τ' and r, and π' is cuspidal if and only if $r = 1$.

LEMMA 25.13. *Let τ_i $(1 \le i \le r)$ be discrete spectrum of* $\mathrm{GL}(k_i, \mathbb{A})$ *with $k_1 + \cdots + k_r = n$, and τ_i' $(1 \le i \le r')$ discrete spectrum of* $\mathrm{GL}(k_i', \mathbb{A})$, *$k_1' + \cdots + k_{r'}' = n$, such that for almost all finite places v the local components of the irreducible induced $i(\tau)$, $\tau = \tau_1 \times \cdots \times \tau_r$, and $i(\tau')$, $\tau' = \tau_1' \times \cdots \times \tau_{r'}'$, are equivalent. Then $r' = r$ and $(\tau_1, \dots, \tau_r)$ equals $(\tau_1', \dots, \tau_{r'}')$ up to permutation.*

PROOF. By [JS81, Theorem 4.4], the cuspidal support of the automorphic representations $i(\tau)$ and $i(\tau')$ is equal. If σ is a cuspidal representation of $\mathrm{GL}(k, \mathbb{A})$ for some integer k, we name the set $\{\nu^j\sigma\ ;\ j \in \mathbb{Z}\}$ a line, and the set $\{\nu^{j+\frac{1}{2}}\sigma\ ;\ j \in \mathbb{Z}\}$ a shifted line. By [MW95], the irreducibles in the cuspidal support of τ_i or τ_i' lie in a line or a shifted line. Separating the supports, we reduce the question to that where there is a line or a shifted line L such that the irreducibles in the cuspidal support of all τ_i and τ_i' lie in L. Namely, there is a cuspidal σ such that $\tau_i = \mathrm{MW}(\sigma, a_i)$ for all i $(1 \le i \le r)$ and $\tau_j' = \mathrm{MW}(\sigma, b_j)$ for all j $(1 \le j \le r')$ where either all a_i, b_j are even, or odd. We claim that the cuspidal support of $i(\tau)$ and $i(\tau')$ determines the τ_i (and τ_i') up to permutation. If the a_i are odd this follows from the following claim. The case where the a_i are even simpler.

If X is a multiset of integers that can be written as a union with multiplicities of sets of the form $x = \{-r, -r+1, -r+2, \dots, r-1, r\}$, then X determines the sets x.

Indeed, if $m : \mathbb{Z} \to \mathbb{Z}_{\ge 0}$ is the multiplicity map, thus $m(a)$ is the multiplicity of a in X, then $m(a)$ is also the number of the sets x containing a. If $a \ge 1$ and x contains a, then it contains also $a - 1$, so m is decreasing on $\mathbb{Z}_{\ge 0}$, and the number of the sets $\{-r, -r+1, \dots, r-1, r\}$ in X is given by $m(r) - m(r+1)$. □

25.14 Reduced Identity

We conclude that the sum over $1 < k|m$ and τ' in the last displayed equation, over the discrete spectrum τ' with $i(\tau'^k, 0)^V \simeq \pi_0'^V$ contains at most one element.

Now the π_0' and the possible τ', or rather $i(\tau'^k, 0)$ which may occur on the left side of the last displayed equation come locally from representations of the inner form $G = \mathrm{GL}(m, D)$, we deduce from generalized linear independence of characters that the term $i(\tau'^k, 0)$ cannot contribute, since its coefficient $(\lambda(\tau') - \lambda(\tau \to \tau'))/k^2$ is less than 1/2 in absolute value, in particular not an integer. Hence $\lambda(\tau') = \lambda(\tau \to \tau')$, and we are left with the identity

$$\sum_{\{\pi'\,;\,\pi'^V = \pi_0'^V\}} \operatorname{tr} \pi_V'(f_V') = \sum_{\{\pi\,;\,\pi^V = \pi_0'^V\}} m(\pi) \operatorname{tr} \pi_V(f_V). \tag{25.14.1}$$

Linear independence of characters, as used earlier in this chapter, implies that the right side is not zero and consists of a single term indexed by some π_0 with $\pi_0^V = \pi_0'^V$, and its multiplicity $m(\pi_0)$ is 1.

25.15 Multiplicity One

Similarly, fixing a discrete spectrum π_0 in $\mathrm{DS}(m, t, \omega)$, we cannot have that both sums over π' with $\pi'^V = \pi_0^V$ and τ' with $i(\tau'^k)^V = \pi_0^V$ are empty. By rigidity the sum over π' has at most one entry, and the argument above shows that $\lambda(\tau') = \lambda(\tau \to \tau')$, so that the sum over τ' is zero. This implies that there is a nonzero entry in the sum over π', denote it by π_0', and that π_0 corresponds to π_0' and the multiplicity $m(\pi_0)$ of π_0 in the cuspidal spectrum of $\mathrm{GL}(m, D)$ is one.

25.16 End of Proof of Global Theorem

In fact, we can use equation (25.11.1) with $\pi_0' = i(\tau'^k)$ for any $k > 1$, $k|m$, and a discrete spectrum τ'. Using Proposition 7.5 we obtain an analogous identity for the cuspidal support of $i(\tau'^k)$. Once again, since the coefficient $(\lambda(\tau') - \lambda(\tau \to \tau'))/k^2$ is less than one in absolute value, while all other terms have integral coefficients, our generalized linear independence of characters implies that $\lambda(\tau \to \tau') = \lambda(\tau')$ for all discrete spectrum τ of $\mathrm{GL}(m/k, D(\mathbb{A}))$ which we know correspond to a discrete spectrum τ' of the split form of the group. Hence the sum over k $(1 < k|m)$ and τ' can be removed from (25.11.1).

From this, namely, from (25.14.1), we conclude that for each discrete spectrum π, there is a unique discrete spectrum π' with $\pi_v' \simeq \pi_v$ for all $v \notin V$, such that $\operatorname{tr} \pi_v'(f_v') \neq 0$ for some f_v' matching some f_v, for all $v \in V$. Since (25.14.1) holds for all matching f'_v and f_v ($v \in V$), using again our generalized linear independence of characters, using Proposition 7.5 and the finiteness argument of Section 12, we conclude that each component π'_v of π' comes from G_v ($v \in V$). Conversely, starting from a discrete spectrum π' in (25.14.1), we see that $\operatorname{tr} \pi'_V(f'_V) \neq 0$ for f'_V matching f_V implies, by the same argument, that π' comes from a discrete spectrum π, in particular each of its components at $v \in V$ comes from G_v. This completes the proof of the global theorem. $\square$

25.17 Refinement

Let $M = \mathrm{GL}(m_1, D) \times \cdots \times \mathrm{GL}(m_r, D)$ be a standard Levi subgroup of $\mathrm{GL}(m, D)$. A quasi-cuspidal, or a quasi-discrete spectrum, representation of $M(\mathbb{A})$ is a representation $\tau = \nu^{a_1}\tau_1 \times \cdots \times \nu^{a_r}\tau_r$ where $a_j \in \mathbb{R}$ and τ_j are cuspidal, or discrete spectrum, on $\mathrm{GL}(m_j, D(\mathbb{A}))$. Such $\tau' = \nu^{a_1}\tau'_1 \times \cdots \times \nu^{a_r}\tau'_r$ on $M'(\mathbb{A})$ will be said to *come locally* from $M(\mathbb{A})$ if all τ'_j come locally from $\mathrm{GL}(m_j, D(\mathbb{A}))$; thus τ'_{jv} comes from $\mathrm{GL}(m_j, D(F_v))$ for all $v \in V$.

Let τ' be a cuspidal representation of $\mathrm{GL}(k, \mathbb{A})$. As described in Subsection 24.19, each component τ'_v is generic and unitarizable so has the form $\tau'_v = i\left(\nu_v^{e_{1,v}}\sigma'_{1,v} \times \cdots \times \nu_v^{e_{q_v,v}}\sigma'_{q_v,v}\right)$ with $e_{jv} \in (-1/2, 1/2)$ and square-integrable $\sigma'_{jv} \in Z_u(\tau'_{jv}, k_{jv})$, where the τ'_{jv} are cuspidal unitarizable $\mathrm{GL}(p_{jv}, F_v)$-modules. Put $J_v = \{j \in \mathbb{Z};\ 1 \leq j \leq q_v, d_v | p_{jv}k_{jv}\}$. Let $z(\tau'_v, d_v)$ be the smallest integer $z_v > 0$ with $d_v | z_v p_{jv}$ for all $j \in \{1, \ldots, q_v\} - J_v$. As in 24.19, $C(L(\tau'_v, r_v)) \neq 0$ if an only if $z(\tau'_v, d_v) | r_v$.

PROPOSITION 25.18. *Let τ' be a cuspidal representation of* $\mathrm{GL}(k, \mathbb{A})$. *Let $z(\tau', D)$ be the least common multiple of the $z(\tau'_v, d_v)$, $v \in V$. Then* $\mathrm{MW}(\tau', r)$ *comes locally from* $\mathrm{GL}(kr/d, D(\mathbb{A}))$ *if and only if $z(\tau', D)$ divides r. Further, the discrete spectrum representation $\pi = C(\mathrm{MW}(\tau', z(\tau', D)))$ of* $\mathrm{GL}(kz(\tau', D)/d, \mathbb{A})$, *corresponding to the discrete spectrum* $\mathrm{MW}(\tau', z(\tau', D))$ *on* $\mathrm{GL}(kz(\tau', D), \mathbb{A})$, *is cuspidal. In particular, a cuspidal $G'(\mathbb{A})$-module can only be obtained from a cuspidal $G(\mathbb{A})$-module.*

PROOF. The first claim follows from the definition of $z(\tau', D)$ and 24.19. For the further claim, suppose π is not cuspidal. Then there is a quasi-cuspidal representation θ of $M(\mathbb{A})$, where M is a proper standard Levi subgroup of $G = \mathrm{GL}(kz(\tau', D)/d, D)$, such that π is a constituent of the representation induced to $\mathrm{GL}(kz(\tau', D)/d, D(\mathbb{A}))$ from θ. Then θ corresponds to $\theta' = C'(\theta)$. This $\theta' = C'(\theta)$. This θ' is a quasi-square-integrable $M'(\mathbb{A})$-module; M' is the proper Levi subgroup of $\mathrm{GL}(kz(\tau', D))$ corresponding to M, coming locally from $M(\mathbb{A})$. By [JS81, Theorem 4.4], θ' has the same cuspidal support as $\mathrm{MW}(\tau', z(\tau', D))$. Then θ' is a quasi-square-integrable representation of the smaller group $M'(\mathbb{A})$, coming from $M(\mathbb{A})$. This contradicts the minimality of $z(\tau', D)$.

The last, "in particular," claim is that where $z(\tau', D) = 1$, thus $d_v | p_{jv} k_{jv}$ for all j ($1 \leq j \leq q_v$) and $v \in V$. Then $\mathrm{MW}(\tau', z(\tau', D))$ is the cuspidal τ', and the corresponding $\pi = C(\tau')$ is cuspidal, rather than just discrete spectrum. □

DEFINITION 25.19. The cuspidal representation $\pi = C(\mathrm{MW}(\tau', z(\tau', D)))$ will be called *basic* cuspidal. Put $z(\pi)$ for $z(\tau', D)$, and ν_π for $\nu^{z(\tau', D)}$. Let $M = \mathrm{GL}(m_1, D) \times \cdots \times \mathrm{GL}(m_r, D)$ be a standard Levi subgroup of $G = \mathrm{GL}(m, D)$. A *basic* quasi-cuspidal representation of M is $\nu^{a_1}\tau_1 \times \cdots \times \nu^{a_r}\tau_r$ where $a_j \in \mathbb{R}$ and τ_j is a basic cuspidal representation of $\mathrm{GL}(m_j, D(\mathbb{A}))$.

PROPOSITION 25.20. (i) *Let τ be a basic cuspidal representation of* $\mathrm{GL}(k, D(\mathbb{A}))$. *The induced representation* $i\left(\nu_\tau^{\frac{r+1}{2}-1}\tau \times \cdots \times \nu_\tau^{\frac{r+1}{2}-r}\tau\right)$ *has a unique quotient π, denoted* $\mathrm{MW}(\tau, r)$. *It is in the discrete spectrum of* $\mathrm{GL}(rk, D(\mathbb{A}))$. *Every discrete spectrum representation π of* $\mathrm{GL}(m, D(\mathbb{A}))$ *is of this form, where r and τ are uniquely determined by π. The discrete spectrum π is basic cuspidal if and only if $r = 1$. Put $\pi = \mathrm{MW}(\tau, r)$. Then $C'(\tau) = \mathrm{MW}(\tau', z(\tau', D))$ if and only if $C'(\pi) = \mathrm{MW}(\tau', rz(\tau', D))$.*

(ii) *Let M_1, M_2 be standard Levi subgroups of $G = \mathrm{GL}(m, D)$, and τ_1, τ_2 basic quasi-cuspidal representations of $M_1(\mathbb{A})$, $M_2(\mathbb{A})$. Let V' be a finite set of places of F including the Archimedean places and the finite places where τ_1 and τ_2 are not both unramified. If the unramified subquotients of the induced representations $i(\tau_{1v})$ and $i(\tau_{2v})$ of G_v are equivalent for all $v \notin V'$, then (M_1, τ_1') is conjugate to (M_2, τ_2').*

(iii) *Let π be an automorphic representation of $G(\mathbb{A})$, $G = \mathrm{GL}(m, D)$. Then there is a standard Levi subgroup M of G and a basic quasi-cuspidal representation τ of $M(\mathbb{A})$ such that π is a constituent of the induced representation $i_M^G\tau$. Then the pair (M, τ) is uniquely determined by π up to conjugation.*

Part (i) generalizes [MW89] from the case $G' = \mathrm{GL}(n)$ to that of $\mathrm{GL}(m, D)$, that is, from the case of $d = 1$ to that of $d \geq 1$. Part (ii) coincides with [JS81, Theorem 4.4] when $d = 1$.

PROOF. (i) Write $C'(\tau)$ as $\mathrm{MW}(\tau', z(\tau', D))$. The discrete spectrum $\mathrm{MW}(\tau', rz(\tau', D))$ comes locally from $\mathrm{GL}(krz(\tau', D), D(\mathbb{A}))$ by Proposition 25.18. We claim that $C(\mathrm{MW}(\tau', rz(\tau', D))$ is a constituent of $i\left(\nu_\tau^{\frac{r+1}{2}-1}\tau \times \cdots \times \nu_\tau^{\frac{r+1}{2}-r}\tau\right)$. It suffices to show that $|C|_v(\mathrm{MW}(\tau', rz(\tau', D))_v)$ in a constituent of $\nu_{\tau_v}^{\frac{r+1}{2}-1}\tau_v \times \cdots \times \nu_{\tau_v}^{\frac{r+1}{2}-r}\tau_v$ for all v in V. It suffices to show that the quasi-square-integrable support of $|C|_v(\mathrm{MW}(\tau', rz(\tau', D))_v)$ is the union of such support of the representations $\nu_{\tau_v}^{\frac{r+1}{2}-j}\tau_v$.

As in 24.19 we write the non-degenerate τ_v' as induced $i(\nu^{e_1}\sigma_1' \times \cdots \times \nu^{e_{q_v}}\sigma_{q_v}')$, where σ_j' are square-integrable and $e_j \in (-1/2, 1/2)$. For simplicity we write σ_j' for σ_{jv}'. We have seen that

$$\tau_v = |C|_v(L(\tau_v', z(\tau', D))) = \prod_{j=1}^{q_v} \nu^{e_j}|C|_v(u(\sigma_j', z(\tau', D)))$$

and

$$|C|_v(L(\tau'_v, rz(\tau', D))) = \prod_{j=1}^{q_v} \nu^{e_j}|C|_v(u(\sigma'_j, rz(\tau', D))).$$

If σ'_j comes from σ_j, then $|C|_v(u(\sigma'_j, z(\tau', D))$ is $\overline{u}(\sigma_j, z(\tau', D))$, and $|C|_v(u(\sigma'_j, rz(\tau', D))$ is $\overline{u}(\sigma_j, rz(\tau', D))$. The quasi-square-integrable support of $\overline{u}(\sigma_j, z(\tau', D))$ is the union of the quasi-square-integrable support of $\nu^{(\frac{r+1}{2}-j)z(\tau',D)}\overline{u}(\sigma_j, z(\tau', D))$ $(1 \leq i \leq r)$. If σ'_j is not in the image of the correspondence (see reference in 24.17), a similar discussion applies. The conclusion is that $i\left(\nu_\tau^{\frac{r+1}{2}-1}\tau \times \cdots \times \nu_\tau^{\frac{r+1}{2}-r}\tau\right)$ has a subquotient which is in the discrete spectrum. Rigidity theorem for $\mathrm{GL}(m, D(\mathbb{A}))$ implies that there is no other subquotient that is in the discrete spectrum.

Now let π be in the discrete spectrum of $G(\mathbb{A})$, $G = \mathrm{GL}(m, D)$. We claim that it is obtained as in the last paragraph. Write $C'(\pi) = \mathrm{MW}(\tau', r)$. Then $z(\tau', D)|r$ since $\mathrm{MW}(\tau', q)$ comes locally from $G(\mathbb{A})$, by Proposition 25.18. Put $\theta = C(\mathrm{MW}(\tau', z(\tau', D)))$. It is basic cuspidal. Then we have that $\pi = \mathrm{MW}(\theta, q/z(\tau', D))$. Rigidity theorem for $\mathrm{GL}(md, \mathbb{A})$ implies that τ' and q are uniquely determined by π. Hence $r = q/z(\tau', D)$ and θ are determined by π. Finally, π is basic cuspidal if and only if $q = z(\tau', D)$, if and only if $r = 1$.

(ii) Write $\tau'_1 = C'(\tau_1)$ as a product $\otimes_{j=1}^{q_1}\nu^{a_j}\mathrm{MW}(\alpha_j, z(\alpha_j, D))$ and $\tau'_2 = C'(\tau_2)$ as the product $\otimes_{j=1}^{q_2}\nu^{b_j}\mathrm{MW}(\beta_j, z(\beta_j, D))$, where α_j and β_j are cuspidal. The representations of $\mathrm{GL}(md, \mathbb{A})$ induced from τ'_1 and τ'_2 have equivalent unramified components at all places outside $V' \cup V$. By [JS81, Theorem 4.4], they have the same quasi-cuspidal support. But α_j and β_j are cuspidal. From the formulae for τ'_1 and τ'_2, it follows that the multisets $\{(a_j, \alpha_j)\}$ and $\{(b_j, \beta_j)\}$ are equal. Hence the products representing τ'_1 and τ'_2 are equal up to permutation.

(iii) Existence follows from part (i) and uniqueness from part (ii).

□

The appendix to [Ba08], by Neven Grbac, shows—upon further analyzing normalizing factors of intertwining operators and poles of Eisenstein series on the inner form $G = \mathrm{GL}(m, D)$—that all cuspidal representations of $G(\mathbb{A})$ are basic. Consequently in Proposition 25.20, we may erase the word "basic."

26 One Cuspidal Place

The Selberg trace formula is of unquestionable value for the study of automorphic forms and related objects. In principal it is a simple and natural formula, generalizing the Poisson summation formula, relating traces of convolution operators with orbital integrals. This section is motivated by the belief that such a fundamental and natural relation should admit a *simple and short* proof in key and useful cases. This is accomplished here for test functions with a single discrete component and another component which is spherical and "sufficiently admissible" with respect to the other components. The resulting trace formula is then used to sharpen and extend the (metaplectic correspondence and the) simple algebras correspondence, of automorphic representations, to the context of automorphic forms with a *single* cuspidal component, over any global field. We deal with the extension of these theorems to the context of all automorphic representations in the following chapters. In the previous sections, a simple form of the trace formula was developed for test functions with two discrete components; this was used to establish these correspondences for automorphic forms with two discrete components. The notion of "sufficiently admissible" spherical functions has its origins in Drinfeld's study of the reciprocity law for GL(2) over a function field, and our form of the trace formula is analogous to Deligne's conjecture on the fixed point formula in étale cohomology, for a correspondence which is multiplied by a sufficiently high power of the Frobenius, on a separated scheme of finite type over a finite field, first published in [FK87.1], and proven by Fujiwara [Fu97] and Varshavsky [Va07]. Our trace formula can be used (see the announcement [FK87.1], and in more detail in [F13]), to prove the Ramanujan conjecture for automorphic forms with a cuspidal component on GL(n) over a function field and to reduce the reciprocity law for such forms to Deligne's conjecture. Similar techniques are used in [F90.1] to establish base change for GL(n) in the context of automorphic forms with a single cuspidal component. They can be used to give short and simple proofs of rank one lifting theorems for *arbitrary* automorphic forms; see [F90.1] for base change for GL(2), [F06] for base change for U(3), and for the symmetric square lifting from SL(2) to PGL(3).

Let F be a global field, $\mathbb{A}$ its ring of adèles and $\mathbb{A}_f$ the ring of finite adèles, G a connected reductive algebraic group over F with center Z. The group $G(F)$ of F-rational points on G is discrete in the adèle group $G(\mathbb{A})$ of G. Put $G' = G/Z$. The quotient $G'(F)\backslash G'(\mathbb{A})$ has finite volume with respect to the unique (up to scalar multiple) Haar measure dg on $G'(\mathbb{A})$. Fix a *unitary* complex-valued character ω of $Z(F)\backslash Z(\mathbb{A})$. For any place v of F, let F_v be the completion of F at v. If F_v is non-Archimedean, let R_v denote its ring of integers. For almost all v, the group $G(F_v)$ is defined over R_v, quasisplit over F_v, split over an unramified extension of F_v, and $K_v = G(R_v)$ is a maximal compact subgroup. For an infinite set of places (of positive density) u of F, the group $G(F_u)$ is split (over F_u). A fundamental system of open neighborhoods of 1 in $G(\mathbb{A})$ consists of the set $\prod_{v\in V} H_v \times \prod_{v\notin V} K_v$, where V is a finite set of places of F and H_v is an open subset of $G(F_v)$ containing 1.

Let $L^2(G(F)\backslash G(\mathbb{A}))$ denote the space of all complex-valued functions ϕ on $G(F)\backslash G(\mathbb{A})$ which satisfy $\phi(zg) = \omega(z)\phi(g)$ ($z \in Z(\mathbb{A})$, $g \in G(\mathbb{A})$) and are square-integrable on $G'(F)\backslash G'(\mathbb{A})$. The group $G(\mathbb{A})$ acts on $L^2(G(F)\backslash G(\mathbb{A}))$ by right translation: $(r(g)\phi)(h) = \phi(hg)$. The representation is unitary since ω is unitary. The function ϕ in $L^2(G(F)\backslash G(\mathbb{A}))$ is called *cuspidal* if for each proper parabolic subgroup P of G over F with unipotent radical N, we have

$$\int_{N(F)\backslash N(\mathbb{A})} \phi(ng)\,\mathrm{d}n$$

for any g in $G(\mathbb{A})$. Let $L_0^2(G(F)\backslash G(\mathbb{A}))$ denote the space of cuspidal functions in $L^2(G(F)\backslash G(\mathbb{A}))$, and r_0 the restriction of r to $L_0^2(G(F)\backslash G(\mathbb{A}))$. The space $L_0^2(G(F)\backslash G(\mathbb{A}))$ decomposes as a direct sum with finite multiplicities of invariant irreducible unitary $G(\mathbb{A})$-modules called *cuspidal* G-modules.

Let f be a complex-valued function on $G(\mathbb{A})$ with $f(g) = \omega(z)f(zg)$ for z in $Z(\mathbb{A})$, which is supported on the product of $Z(\mathbb{A})$ and a compact open neighborhood of 1 in $G(\mathbb{A})$, smooth as a function on the Archimedean part $G(F_\infty)$ of $G(\mathbb{A})$, and bi-invariant by an open compact subgroup of $G(\mathbb{A}_f)$. Fix Haar measures $\mathrm{d}g$ on $G'(F_v)$ for all v, such that the product of the volumes $|K_v/Z_v \cap K_v|$ converges. Then $\mathrm{d}g = \otimes_v \mathrm{d}g_v$ is a measure on $G'(\mathbb{A})$. The convolution operator

$$r_0(f) = \int_{G'(\mathbb{A})} f(g) r_0(g)\, \mathrm{d}g$$

is of trace class. Its trace is denoted by $\operatorname{tr} r_0(f)$. Then

$$\operatorname{tr} r_0(f) = \sum_\pi m(\pi) \operatorname{tr} \pi(f), \tag{26.0.1}$$

where the sum is over all equivalence classes of cuspidal representations π of $G(\mathbb{A})$, and $m(\pi)$ denotes the multiplicity of π in $L_0^2(G(F)\backslash G(\mathbb{A}))$. Each π here is unitary, and the sum is absolutely convergent.

The Selberg trace formula is an alternative expression for (26.0.1). To introduce it we recall the following:

DEFINITION 26.1. Denote by $H_\gamma = \{h \in H;\, h\gamma = \gamma h\}$ the centralizer of an element γ in a group H. A semisimple element γ in $G(F)$ is called *elliptic* if $G'(\mathbb{A})_\gamma / G'(F)_\gamma$ has finite volume. It is called *regular* if $G'(\mathbb{A})_\gamma$ is a torus, and *singular* otherwise. Let γ be an elliptic element of $G(F)$. The *orbital integral* of f at γ is defined to be

$$\Phi(\gamma, f) = \int_{G'(\mathbb{A})/G'(F)_\gamma} f(g\gamma g^{-1})\, \mathrm{d}g.$$

Similarly, for any place v of F, the element γ of $G(F_v)$ is called *elliptic* if $G'(F_v)_\gamma$ has finite volume, and *regular* if $G(F_v)_\gamma$ is a torus. If γ is an element of $G(F)$ and there is a place v of F such that γ is elliptic (resp. regular) in $G(F_v)$, then γ is elliptic (resp. regular). The orbital integral of f_v at γ in $G(F_v)$ is defined to be

$$\Phi(\gamma, f_v) = \Phi(\gamma, f_v\, \mathrm{d}_\gamma) = \int_{G'(F_v)/G'(F_v)_\gamma} f_v(g\gamma g^{-1})\, \frac{\mathrm{d}g}{\mathrm{d}_\gamma}.$$

It depends on the choice of a Haar measure d_γ on $G'(F_v)_\gamma$.

Let $\{\phi_\alpha\}$ be an orthonormal basis for the space $L_0^2(G(F)\backslash G(\mathbb{A}))$. The operator $r_0(f)$ is an integral operator on $G'(\mathbb{A})$ with kernel $K_f^0(x,y) = \sum_{\alpha,\beta}(r(f)\phi_\alpha(x))\overline{\phi}_\beta(y)$. The operator $r(f)$ is an integral operator on $G'(\mathbb{A})$ with kernel $K_f(x,y) = \sum_\gamma f(x^{-1}\gamma y)$ with γ in $G'(F)$. If G is *anisotropic*, namely, $G'(F)\backslash G'(\mathbb{A})$ is compact, then $L_0^2(G(F)\backslash G(\mathbb{A})) = L^2(G(F)\backslash G(\mathbb{A}))$ and $r = r_0$. Since $K_f^0(x,y) = K_f(x,y)$ is smooth in both x and y, we integrate over the diagonal $x = y$ in $G'(\mathbb{A})$, change the order of summation and integration as usual, and obtain the Selberg trace formula in the case of compact quotient, as follows:

PROPOSITION 26.2. *If G is anisotropic, then for every function f on $G(\mathbb{A})$ as above, we have*

$$\sum_\pi m(\pi)\,\mathrm{tr}\,\pi(f) = \sum_{\{\gamma\}} \Phi(\gamma,f). \tag{26.2.1}$$

The sum on the left is as in (26.0.1). The sum on the right is finite. It ranges over the conjugacy classes of elements in $G'(F)$.

REMARK 26.3. If G is anisotropic, then each element γ in $G(F)$ is elliptic.

For a general group G, we introduce the following:

DEFINITION 26.4. The function f is called *discrete* if for every x in $G(\mathbb{A})$ and γ in $G(F)$, we have $f(x^{-1}\gamma x) = 0$ unless γ is elliptic regular.

Changing again the order of summation and integration as usual, we obtain the

PROPOSITION 26.5. *If f is discrete, then*

$$\int_{G'(\mathbb{A})}\left(\sum_{\gamma\in G'(F)} f(x^{-1}\gamma x)\right) dx = \sum_{\{\gamma\}} \Phi(\gamma,f). \tag{26.5.1}$$

The sum on the right is finite. It ranges over the set of conjugacy classes of elliptic regular elements in $G'(F)$.

REMARK 26.6. It is well known that the sum on the right is finite. For a proof see [FK87.2] when $G = \mathrm{GL}(n)$ and Proposition 4.2 in general.

DEFINITION 26.7. The function f is called *cuspidal* if for every x, y in $G(\mathbb{A})$ and every proper F-parabolic subgroup P of G, we have

$$\int_{N(\mathbb{A})} f(xny)\,\mathrm{d}n = 0,$$

where N is the unipotent radical of P.

When f is cuspidal, the convolution operator $r(f)$ factorizes through the projection to the cuspidal spectrum $L_0^2(G(F)\backslash G(\mathbb{A}))$; it is of trace class with $\operatorname{tr} r(f) = \operatorname{tr} r_0(f)$, and $K_f(x,y) = K_f^0(x,y)$. We obtain the

COROLLARY 26.8. *If f is cuspidal and discrete, then the equality* (26.2.1) *holds. The sum on the left is as in* (26.0.1). *The sum on the right is as in* (26.5.1).

For some applications we need to replace the requirement that f be discrete by a requirement on the orbital integrals of f (but not on f itself). The purpose of this section is to present such a requirement and apply the resulting trace formula to extend some global lifting theorems, such as those of [FK87.2].

Fix a non-Archimedean place u of F such that $G(F_u)$ is split, and the component ω_u of ω at u is unramified (namely trivial on the multiplicative group $R_u^\times$ of R_u).

DEFINITION 26.9. A complex-valued compactly supported modulo center function f_u on $G(F_u)$ is called *spherical* function if it is K_u-bi-invariant. Let $\mathbb{H}_u$ be the convolution algebra of such functions. Of course $\mathbb{H}_u$ is empty unless the central character ω_u is unramified.

For any maximal (proper) F_u-parabolic subgroup $P(F_u) = M(F_u)N(F_u)$ of $G(F_u)$, where $N(F_u)$ is the unipotent radical of $P(F_u)$ and $M(F_u)$ is a Levi subgroup, define an $F_u^\times$-valued character $\alpha_{P(F_u)}$ of $M(F_u)$ by $\alpha_{P(F_u)}(m) = \det(\operatorname{ad}(m)|\mathfrak{n})$, where $\mathfrak{n}$ is the Lie algebra of $N(F_u)$ and $\operatorname{ad}(m)|\mathfrak{n}$ denotes the adjoint action of m in $M(F_u)$ on $\mathfrak{n}$. Let $\nu_u : F_u^\times \to \mathbb{Z}$ be the normalized additive valuation. Let $A(F_u)$ be a maximal split torus in $G(F_u)$. For any nonnegative integer n, let $A(F_u)^{(n)}$ be the set of a in $A(F_u)$ such that $|\nu_u(\alpha_{P(F_u)}(a))| < n$ for some maximal F_u-parabolic subgroup $P(F_u)$ containing $A(F_u) \subset G(F_u)$.

DEFINITION 26.10. A spherical function f_u is called *n-admissible* if the orbital integral $\Phi(a,f_u)$ is zero for every regular a in $A(F_u)^{(n)}$.

Let $\mathbb{A}^u$ denote the ring of F-adèles without u-component. Write $f = f_u f^u$ if f is a function on $G(\mathbb{A})$, f_u is a function on $G(F_u)$, f^u is a function on $G(\mathbb{A}^u)$, and $f(x,y) = f_u(x) f^u(y)$ for x in $G(F_u)$ and y in $G(\mathbb{A}^u)$. We choose the place u such that the central character ω is unramified at u.

THEOREM 26.11. *Let f^u be a function on $G(\mathbb{A}^u)$ which is compactly supported modulo $Z(\mathbb{A}^u)$ and vanishes on the $G(\mathbb{A}^u)$-orbit of any singular γ in $G(F)$. Then there exists a positive integer $n_0 = n_0(f^u)$ such that for every spherical n_0-admissible function f_u, there is a function f_u' on $G(F_u)$ such that $\Phi(x,f_u') = \Phi(x,f_u)$ for all regular x in $G(F_u)$ and $f' = f_u' f^u$ is discrete.*

PROOF. For every maximal f-parabolic subgroup P of G and every place $v \neq u$ of F, there exists a nonnegative integer $C_{v,P}$ which depends on f^u, with $C_{v,P} = 0$ for almost all v, such that if γ lies in a Levi subgroup M of P and $f^u(x^{-1}\gamma x) \neq 0$ for some x in $G(\mathbb{A}^u)$, then

$$|\nu_v(\alpha_P(\gamma))| \leq C_{v,P}. \tag{26.11.1$_v$}$$

Put $C_{u,P} = \sum_{v \neq u} C_{v,P}$. Since γ is rational in $G(F)$, the product formula $\sum_v \nu(\alpha_P(\gamma)) = 0$ on $F^\times$ implies that the inequality ($26.11.1_v$) remains valid also for $v = u$. Choose $n_0 > C_{u,P}$ for all (of the finitely many conjugacy classes of) P. Let f_u be any spherical n_0-admissible function. Put $f = f_u f^u$. Since a discrete and compact set is finite (see the proof of Proposition 4.2), there are only finitely many rational conjugacy classes γ in $G'(F)$ such that f is not zero on the $G'(\mathbb{A})$-orbit of γ. Note that f is zero on the $G(\mathbb{A})$-orbits of all singular γ in $G(F)$ by assumption. Let γ_i $(1 \leq i \leq m)$ be a set of representatives for the regular non-elliptic rational conjugacy classes in $G(F)$ such that f is nonzero on their $G(\mathbb{A})$-orbits. Since γ_i is non-elliptic, it lies in a Levi subgroup M_i of a maximal parabolic subgroup P_i of G. Since f_u is n_0-admissible, the relation $\Phi(\gamma_i, f_u) \neq 0$ implies that $|\nu_u(\alpha_{P_i}(\gamma_i))| > n_0$. This contradicts $(26.11.1)_u$. Hence $\Phi(\gamma_i, f_u) = 0$ for all i. Let S_i denote the characteristic function of the complement in $G(F_u)$ of a sufficiently small open-closed neighborhood of the orbit of γ_i in $G(F_u)$. Since γ_i is regular non-elliptic, we may and do take S_i to be one on the elliptic set of $G(F_u)$. Put $f'_u = f_u \prod_{i=1}^m S_i$. Then f'_u is zero on the orbit of γ_i $(1 \leq i \leq m)$, and $\Phi(\gamma, f'_u) = \Phi(\gamma, f_u)$ for every regular γ in $G(F_u)$. Since $f' = f'_u f^u$ vanishes on the $G(\mathbb{A})$-orbit of each rational γ in $G(F)$ which is not elliptic regular, the theorem follows. □

Since both sides of (26.2.1) are invariant distributions, we conclude the immediate

COROLLARY 26.12. *Suppose that $f = f_u f^u$ is a cuspidal function which vanishes on the $G(\mathbb{A})$-orbit of every singular γ in $G(F)$, and f_u is a spherical n_0-admissible function with $n_0 = n_0(f^u)$. Then the equality* (26.2.1) *holds, where the sum on the left is as in* (26.0.1)*, while the sum on the right is as in* (26.5.1).

DEFINITION 26.13. A $G(F_u)$-module π_u is called *unramified* if it has a nonzero K_u-fixed vector.

For applications such as those given in Theorem 26.18 below, we need to show that the set of n-admissible functions is sufficiently large in the following sense.

THEOREM 26.14. *Let $\{\pi_i\,;\, i \geq 0\}$ be a sequence of inequivalent unitarizable unramified $G(F_u)$-modules, and c_i complex numbers, such that $\sum_i c_i \operatorname{tr} \pi_i(f_u)$ is absolutely convergent for every spherical f_u. Suppose that there is a positive integer n_0 such that $\sum_i \operatorname{tr} \pi_i(f_u) = 0$ for all n_0-admissible f_u. Then $c_i = 0$ for all i.*

PROOF. Delayed to the end of this section. □

REMARK 26.15. The notion of n-admissible functions is suggested by Drinfeld [Dr77], at least in the case of $G = \mathrm{GL}(2)$. For a general G, the corollary is a representation theoretic analogue of Deligne's conjecture on the Grothendieck-Lefschetz fixed-point formula for the trace of a finite flat correspondence on a separated scheme of finite type over a finite field, which is multiplied by a sufficiently high power of the Frobenius morphism, proven by Fujiwara [Fu97] and Varshavsky [Va07]. This analogy in more detail in the work [F13] on the geometric Ramanujan conjecture for $\mathrm{GL}(n)$ (see also [FK87.1]).

In the proofs of Theorem 26.14 and Theorem 26.18 below, we shall use some results concerning unramified representations and spherical functions (see [Car79]) and regular functions. These will be recalled now in order to be able to give an uninterrupted exposition of the proof of Theorem 26.18.

Let G be a split p-adic reductive group with minimal parabolic subgroup $B = AN$, where N is the unipotent radical of B and the Levi subgroup A is a maximal (split) torus. Let $X^* = X^*(A)$ be the lattice of rational characters on A and let $X_* = X_*(A)$ be the dual lattice. If A^0 is the maximal compact subgroup of A, then $X_* \simeq A/A^0$. Let $T = X^*(\mathbb{C})$ denote the complex torus $\mathrm{Hom}(X_*, \mathbb{C}^\times)$. The Weyl group W of A in G acts on A, X^*, X_*, and T. Each t in T defines a unique $\mathbb{C}^\times$-valued character of B which is trivial on N and on A^0. The G-module $I(t) = i_B^G(\delta^{1/2}t)$ normalizedly induced from the character t of B is unramified and has a unique unramified irreducible constituent $\pi(t)$. We have $\pi(t) \simeq \pi(t')$ if and only if $t' = wt$ for some w in W. The map $t \mapsto \pi(t)$ is a bijection from the variety T/W to the set of unramified irreducible G-modules. Put $t(\pi)$ for the t associated with such a π. Let α_i $(1 \leq i \leq m)$ be a set of simple (with respect to N) roots in the vector space $X^* \otimes \mathbb{R} = \mathrm{Hom}(X_*, \mathbb{R})$ and α_i the corresponding character of A, defined as usual by $\alpha_i(a) = \mathrm{ad}(a)|\mathfrak{n}_i$, where $\mathrm{ad}(a)$ denotes the adjoint action of A on the Lie algebra $\mathfrak{n}_i$ of the root subgroup N_i of α_i in N. Denote by $\alpha_i^\vee$ $(1 \leq i \leq m)$ the corresponding set of coroots in the dual space $X_* \otimes \mathbb{R}$, and by $\alpha_i^\vee$ the corresponding set of characters of the torus $T = X^*(\mathbb{C}) = \mathrm{Hom}(X_*, \mathbb{C}^\times)$, defined as usual by $\alpha_i^\vee(\exp T) = \exp\langle \alpha_i^\vee, T\rangle$ for all T in $X^* \otimes \mathbb{C} = \mathrm{Hom}(X_*, \mathbb{C})$. Here $\langle \cdot, \cdot \rangle$ is the pairing between X_* and X^*. There exists $q = q(G) > 1$ such that if π is (irreducible, unramified and) unitarizable, then $q^{-1} < |\alpha_i^\vee(t)| < q$ for all $1 \leq i \leq m$ and the complex conjugate $\bar{t}$ of t is equal to wt^{-1} for some w in W.

If f is a spherical function, the value of the normalized orbital integral $F(a,f) = \Delta(a)\Phi(a,f)$ at a regular a in A depends only on the W-orbit of the image x of a in X_*; it is denoted by $F(x,f)$. Let $\mathbb{C}[X_*]^W$ be the algebra of W-invariant elements in the group ring $\mathbb{C}[X_*]$. The Satake transform $f \mapsto f^\vee = \sum_{x \in X_*} F(x,f)x$ defines an algebra isomorphism from the convolution algebra $\mathbb{H}$ of spherical functions, to $\mathbb{C}[X_*]^W$. For each x in X_*, let $f(x)$ be the element of $\mathbb{H}$ with $f(x)^\vee = \sum_{w \in W} wx$. Then $f(x)$ is n_0-admissible if $|\nu\alpha_P(w(a(x)))| \geq n_0$ for every w in W and parabolic subgroup P containing A; $a(x)$ is an element of A which corresponds to x under the isomorphism of A/A^0 with X_* fixed above. We have $\mathrm{tr}(\pi(t))(f) = \mathrm{tr}(i(t))(f) = f^\vee(t)$ for every f in $\mathbb{H}$ and t in T, where $f^\vee(t) = \sum_{x \in X_*} F(x,f)t(x)$.

Definition 26.16. Consider x in X_* with $\nu(a(x)) \neq 0$ for each root α of A on N. A complex-valued locally constant function f with $f(zg)\omega(z) = f(g)$ for all g in G and z in Z which is compactly supported modulo Z is called *x-regular* if $f(g)$ is zero unless there is z in Z such that zg is conjugate to an element a in A whose image in X_* is x, in which case the normalized orbital integral $F(g,f)$ is equal to $\omega(z)^{-1}$. If f is x-regular, then we denote it by f_x. A *regular* function is a linear combination with complex coefficients of x-regular functions.

REMARK 26.17. (1) Any regular function vanishes on the singular set. In fact, it is supported on the regular split set by definition.

(2) If π is an admissible G-module with central character ω, then the normalized module π_N of coinvariants [BZ76] is an A-module. Its character is denoted by χ_{π_N}. If f_x is an x-regular function, then a simple application of the Weyl integration formula and the theorem of Deligne-Casselman [Cas77] implies that

$$\operatorname{tr}\pi(f_x) = [W]^{-1}\int_{A/Z}(\Delta\chi_{\pi_N})(a)F(a,f_x)\,\mathrm{d}a.$$

If $\operatorname{tr}\pi(f_x)$ is nonzero, then by Frobenius reciprocity, there exists t in T such that π is a constituent of $I(t)$ and a subset $W(\pi,t)$ of W such that

$$\operatorname{tr}\pi(f_x) = \sum_{w\in W(\pi,t)} t(wx).$$

(3) Each constituent of $I(t)$, including π, has a nonzero vector fixed by the action of an Iwahori subgroup (see [Bo76, (4.7)] for the case of a reductive group, [FK87.2, §17] for the case of the metaplectic groups, [F11] for an extension to the tame algebra, and[MP96] for an extension to level zero representations).

(4) Regular functions play a crucial role in the study of orbital integrals of spherical functions. See [F87.2].

We shall now use the corollary, Theorem 26.14, and the results concerning spherical and regular functions, to extend the global correspondence results of [DKV84] and [F87.1] which deal with cuspidal representations of inner forms of $\mathrm{GL}(n)$. The definitions and proofs which are not given in the following discussion are detailed in these references. Put $G' = \mathrm{GL}(n)$. Let G be the multiplicative group of a central simple F-algebra of rank n. The cuspidal G'-module $\pi' = \otimes_v \pi'_v$ and the cuspidal G-module $\pi = \otimes_v \pi_v$ are called *corresponding* if π'_v and π_v correspond for each place v of F, where the notion of local correspondence is defined by means of the character relations; see Theorem 13.8. Fix a non-Archimedean place u'. Let A be the set of equivalence classes of cuspidal G'-modules π with a cuspidal component at u', such that each component of π' is obtained by the local correspondence. Let A be the set of equivalence classes of cuspidal G-modules π whose component at u' corresponds to a cuspidal $G(F_{u'})$-module. Then $\pi_{u'}$ is necessarily cuspidal.

THEOREM 26.18. *The correspondence defines a bijection between the sets A' and A. The multiplicity of each π of A in the cuspidal spectrum $L_0^2(G(F)\backslash G(\mathbb{A}))$ is one.*

REMARK 26.19. (1) This has been shown already for the subset of π' in A' with two cuspidal components and the corresponding subset of A.

(2) Theorem 26.18 can be extended from the context of A', A to the context of all cusp forms on G', G by known techniques, as described in Section 25 and the following chapters. It will be interesting to establish such an extension by *simple* means.

PROOF. Fix corresponding cuspidal $G'(F_{u'})$ and $G(F_{u'})$-modules $\pi'_{u'}$ and $\pi_{u'}$, and matrix coefficients $f'_{u'}$ and $f_{u'}$ thereof. These functions are matching, see Section 14, namely, have matching orbital integrals. For any functions $f'^{u'}$ on $G'(\mathbb{A}^{u'})$ and $f^{u'}$ on $G(\mathbb{A}^{u'})$, the functions $f' = f'_{u'}f'^{u'}$ and $f = f_{u'}f^{u'}$ are cuspidal; see Lemma 4.1. Fix two distinct non-Archimedean places u and u'' of F, other than u', with sufficiently large residual characteristics. Put $S = \{u, u', u''\}$. Objects associated to $\mathbb{A}^S$, the adèles without S-components, will be denoted with a superscript S. Let f^S be any function on $G(\mathbb{A}^S)$ and $f_{u''}$ any regular function on $G(F_{u''})$. Let f'^S be a matching function on $G'(\mathbb{A}^S)$ and $f'_{u''}$ a matching regular function on $G'(F_{u''})$. Put $f'^u = f'^S f_{u'} f_{u''}$ and $f^u = f^S f_{u'} f_{u''}$. Put $n_0 = \max\{n_0(f'^u), n_0(f^u)\}$. Let f_u and f'_u be matching spherical n_0-admissible functions. Since $f'_{u''}$ and $f_{u''}$ are zero on the singular set, the functions $f' = f'_u f'^u$ and $f = f_u f^u$ are zero on the $G'(\mathbb{A})$ and $G(\mathbb{A})$-orbits of any singular element γ in $G'(F)$ and $G(F)$, respectively. Hence they are discrete. Since f' and f are matching, the right sides of the trace formulae (26.2.1) for G' and for G, namely, $\sum \Phi(\gamma', f')$ and $\sum \Phi(\gamma, f)$ (see [FK87.2, §4]) are equal. By the corollary to Theorem 26.11, the left sides are equal, namely, $\sum m(\pi') \operatorname{tr} \pi'(f') = \sum m(\pi) \operatorname{tr} \pi(f)$. By virtue of the choice of $f'_{u'}$ and $f_{u'}$, the π' and π are cuspidal, with the cuspidal components $\pi'_{u'}$ and $\pi_{u'}$ at u'. Hence $m(\pi) = 1$ (by multiplicity one theorem for the cuspidal representations of $\mathrm{GL}(n)$), and each component π'_v of π' is relevant; see Section 19. Since $\operatorname{tr} \pi'_v(f'_v) \neq 0$ for f'_v matching an f_v, and π'_v is relevant, the main local correspondence theorem, Section 23, implies that π'_v corresponds to some $\pi_v(\pi'_v)$, for each v. Since f'_u and f_u are spherical, if $\operatorname{tr} \pi'_u(f'_u)$ and $\operatorname{tr} \pi_u(f_u)$ are nonzero, then π'_u and π_u are unramified, and so is $\pi_u(\pi'_u)$. We write our equality in the form

$$\sum_{\pi_u} \left(\sum_{\pi^u} m(\pi) \operatorname{tr} \pi^u(f^u) - \sum_{\pi'^u} m(\pi') \operatorname{tr} \pi^u(\pi'^u)(f^u) \right) \operatorname{tr} \pi_u(f_u) = 0.$$

The sum on π_u is over equivalence classes of unramified unitarizable $G(F_u)$-modules. The sum on π^u is over the equivalence classes of $G(\mathbb{A}^u)$-modules such that $\pi = \pi_u \otimes \pi^u$ appears in (26.2.1). The sum over π'^u is over those $\pi'^u = \otimes_{v \neq u} \pi_v$ such that there is a cuspidal $\pi' = \otimes \pi'_v$ with $\pi_v = \pi_v(\pi'_v)$ for all v. Since all the sums and products in the trace formula are absolutely convergent, and all the representations which appear there are unitarizable, Theorem 26.2.1 implies that the sums inside parentheses are equal for each π_u. We write this identity in the form

$$\sum_{\pi^{u,u''}} \left(\sum_{\pi_{u''}} m(\pi) \operatorname{tr} \pi_{u''}(f_{u''}) - \sum_{\pi_{u''}} m(\pi') \operatorname{tr}(\pi_{u''}(\pi'_{u''}))(f_{u''}) \right) \operatorname{tr} \pi^{u,u''}(f^{u,u''}) = 0.$$

The sum on $\pi^{u,u''}$ is over equivalence classes of irreducible $G(\mathbb{A}^{u,u''})$-modules. The sum on $\pi_{u''}$ is over irreducible $G(F_{u''})$-modules such that $\pi^u \pi_{u''} \pi^{u,u''}$ appears in the sum on π^u above. The sum on $\pi_{u''}$ is over those such that the resulting π^u occurs in the sum above. Since the function f^S is arbitrary, all sums here are absolutely convergent and all representations are unitarizable, a standard argument of linear independence of characters implies the sums in the parentheses are equal for every $\pi^{u''} = \pi_u \pi^{u,u''}$.

We now use the fact that $f_{u''}$ is an arbitrary regular function. If $\operatorname{tr} \pi_{u''}(f_{u''}) \neq 0$, then $\pi_{u''}$ has a nonzero vector fixed by an Iwahori subgroup. Hence the first sum on $\pi_{u''}$ is finite by a theorem of Harish-Chandra; see [BJ79], which asserts that there are only finitely many cuspidal $G(\mathbb{A})$-modules with fixed infinitesimal character and fixed ramification at all finite places. The second sum on $\pi_{u''}$ consists of at most one term, by the rigidity theorem for cuspidal $G(\mathbb{A})$-modules.

Recall that $\operatorname{tr} \pi_{u''}(f_{u''})$ is a linear combination of characters (of the form $t \mapsto t(wx)$, where t lies in $T = \{(z_i) \in \mathbb{C}^{\times n};\ \prod_i z_i = 1\}$, and $x = (x_i)$ varies over $X_* = \mathbb{Z}^n/\mathbb{Z}$, and $(z_i)(wx) = \prod_i z_{w(i)}^{x_i}$). Applying linear independence of finitely many characters, it is clear that the first sum in parentheses is empty if the second is empty and that $m(\pi) = 1$ and $\operatorname{tr} \pi_{u''}(f_{u''}) = \operatorname{tr} \pi'_{u''}(f'_{u''})$ for all matching regular $f'_{u''}$ and $f_{u''}$ otherwise. Since the Hecke algebras of $G'(F_{u''})$ and $G(F_{u''})$ with respect to an Iwahori subgroup are isomorphic, we conclude that $\pi'_{u''}$ and $\pi_{u''}$ correspond, and Theorem 26.18 follows. □

We proceed to the proof of Theorem 26.14. Fix $q \geq 1$. Let $t' = T'(q)$ be the set of t in T with $\bar{t} = wt^{-1}$ for some w in W (w depends on t) and $q^{-1} \leq |\alpha^\vee(t)| \leq q$ for every root α of A on N. The quotient $\widetilde{T} = \widetilde{T}(q)$ of T' by W is a compact Hausdorff space. Let $\mathbb{C}(\widetilde{T})$ be the algebra of complex-valued continuous functions on $\widetilde{T}$. Let n_0 be a nonnegative integer. The element x of X_* is called *n_0-admissible* if $|\nu\alpha_P(a(x))| \geq n_0$ for every maximal parabolic subgroup P of G. This condition means that there are finitely many walls, determined by the α_P, in the lattice X_*, such that x is called n_0-admissible if it is sufficiently far (the distance depends on n_0) from these walls. The function $P_x(t) = \sum_{w \in W} t(wx)$ is a function on $\widetilde{T}$ which depends only on the image of x in X_*/W. Note that $f'(x)^\vee = P_x$, and in particular $\operatorname{tr}(\pi'(t))(f'_x) = P_x(t)$. Let $C(n_0)$ be the $\mathbb{C}$-span of all $P_x(t)$ with n_0-admissible x. It is a subspace of $\mathbb{C}(\widetilde{T})$, but it is not multiplicatively closed, unless $n_0 = 0$. An element of $\mathbb{C}(\widetilde{T})$ is called *n_0-admissible* if it lies in $C(n_0)$.

LEMMA 26.20. *The space $C(0)$ is dense in $\mathbb{C}(\widetilde{T})$.*

PROOF. This follows from the Stone-Weierstrass theorem, since the space $\widetilde{T}$ is compact and Hausdorff and $C(0)$ is a subalgebra of $\mathbb{C}(\widetilde{T})$ which separates points and contains the scalars and the complex conjugate of each of its elements. □

Theorem 26.14 follows from the special case where $G = \mathrm{GL}(n)$ and $c_i(t) = 0$ for all i in the following proposition. The general form with nonzero $c_i(t)$ is used in [F06] when $G = \mathrm{GL}(3)$ to give a short and simple proof of the trace formulae identity for the base-change lifting from $U(3)$ to $\mathrm{GL}(3, E)$ for an arbitrary test function.

PROPOSITION 26.21. *Fix $n_0 \geq 0$. Fix also a sequence t_i of elements of $\widetilde{T}$, a sequence of complex numbers c_i, a sequence $\widetilde{T}_j$ of compact submanifolds of $\widetilde{T}$, and a sequence $c_j(t)$ of complex-valued functions on $\widetilde{T}_j$ which are measurable with respect to a bounded measure dt on $\widetilde{T}_j$. Suppose that*

$$\beta = \sum_i |c_i| + \sum_j \sup_{t\in\widetilde{T}_j} |c_j(t)| + \sum_J \int_{\widetilde{T}_j} |c_j(t)|\,|dt|$$

is finite and that for any n_0 admissible x in X_* we have

$$\sum_i c_i P_x(t_i) = \sum_j \int_{\widetilde{T}_j} c_j(t) P_x(t)\,|dt|. \tag{26.21.1}$$

Then $c_i = 0$ for all i.

PROOF. We begin with a definition. Let ϵ be a positive number. The points t and t' in $\widetilde{T}$ are called *ϵ-close* if there are representatives τ and τ' of t and t' in T' such that $|\alpha^\vee(\tau) - \alpha^\vee(\tau')| < \epsilon$ for every root $\alpha^\vee$ on T (=coroot on X_*). Denote by $\widetilde{T}_\epsilon(t)$ the ϵ-neighborhood of t in $\widetilde{T}$. The quotient by ϵ of the volume of $\widetilde{T}_\epsilon(t)$ is bounded uniformly in ϵ.

Suppose that $c_0 \neq 0$. Multiplying by a scalar we assume that $c_0 = 1$. The lemma implies that for every $\epsilon > 0$, there is $P = P_\epsilon$ in $C(0)$ with $P(t_0) = 1$, $|P(t)| \leq 2$ for all t in $\widetilde{T}$, and $|P(t)| < \epsilon$ unless t is ϵ^2-close to t_0. Such a polynomial P is called below an ϵ-approximation of the delta function at t_0, or simply a *"delta function"* at t_0. Since β is finite, for every $\epsilon > 0$, there exists $N > 0$ such that

$$\sum_{i>N} |c_i| + \sum_{j>N} \int_{\widetilde{T}_j} |c_j(t)|\,|\mathrm{d}t| < \epsilon.$$

Take $\epsilon = 1/4(1+\beta)$. Substituting P for P_x in (26.21.1), if $n_0 = 0$, then we obtain a contradiction to the assumption that $c_0 = 1$. Hence the proposition is proven in the case of $n_0 = 0$. It remains to deal with a general n_0.

Let x be an n_0-admissible element of X_*. Put $k' = 2\max_P |\nu\alpha_P(a(x))|$. For any x' in X_*, the element $x+k'x'$ is n_0-admissible. Since $P_x(t)P_{x'}(t^{k'}) = \sum_{w\in W} P_{x+k'wx'}(t)$, we have that (26.21.1) applies with $P_x(t)$ replaced by $P_x(t)P_{x'}(t^{k'})$. For a fixed x (and k'), x' is arbitrary. Replacing q by $q^{k'}$ in the definition of $\widetilde{T}$, we argue as in the previous paragraph and conclude that for every $r \geq 0$ we have

$$\sum_i c_i P_x(t_i) = 0. \tag{26.21.2}$$

Here the sum ranges over all i with $t_i^{k'} = t_r^{k'}$ (equality in $\widetilde{T}$). Take $r = 0$. We conclude that the equality (26.21.2) holds also for any n_0-admissible x, provided that the sum ranges over the set I of all i for which there is $k = k(i)$ with $t_i^k = t_0^k$. It remains to prove the following

LEMMA 26.22. *Suppose that c_i is a sequence of complex numbers such that $\beta = \sum_i |c_i|$ is finite, and t_i are elements of T' whose images in $\widetilde{T} = T'/W$ are distinct, such that for each i there is $k = k(i)$ with $t_i^k = t_0^k$. If $\sum_i c_i P(t_i) = 0$ for every n_0-admissible P, then $c_i = 0$ for all i.*

PROOF. We may and do assume that $c_0 = 1$ in order to derive a contradiction. If $\eta = 1/4(1+\beta)$ there is $N > 0$ such that $\sum_{i>N} |c_i| < \eta$, and a W-invariant polynomial $P(t) = \sum_x b(x)P_x(t)$ with $P(t_0) = 1$, $|P(t)| \leq 2$ on T' and $|P(t_i)| < \eta$ for i $(1 \leq i \leq N)$. This P is a "delta function", and if $n_0 = 0$, then we are done. If $n_0 \neq 0$, then the "delta function" P is not necessarily n_0-admissible. Our aim is to replace P by an n_0-admissible "delta function" on multiplying P with a suitable admissible polynomial Q which (depends on P and) attains the value one at t_0, while remaining uniformly bounded (by $2[W]$) at each t_i $(i \geq 1)$. For this purpose note that our assumption (that for each i, there is k with $t_i^k = t_0^k$) implies that t_i/t_0 lies in the maximal compact subgroup of T for all i. Hence for every x in X_*, the absolute value $|t_i(x)|$ of the complex number $t_i(x)$ is independent of i. Take any one-admissible μ in X_*, such that $|t_i(\mu)| \geq |t_i(w\mu)|$ for all w in W. Then $|P_\mu(t_i^s)| \leq [W]|t_0(m)|^s$ for every positive integer s, and for all i. Put $u_w = t_0(w\mu)/|t_0(w\mu)|$ $(w \in W)$, and

$$s_0 = 2n_0 + 2\max\{|\nu\alpha_P(a(x))|\,;\text{ all } P \supset A, \text{all } x \text{ with } b(x) \neq 0\}.$$

For every $\epsilon > 0$ there is $s > s_0$ such that $|u_w^s - 1| < \epsilon$ for all w in W, and the choice of a sufficiently small ϵ guarantees that $P_\mu(t_0^s)| \geq 1/2|t_0(\mu)|^s$. Hence the W-invariant polynomial $Q_s(t) = P_\mu(t^s)/P_\mu(t_0^s)$ on T' satisfies $Q_s(t_0) = 1$ and $|Q_s(t_i)| \leq 2[W]$ for all i. The polynomial $Q(t) = P(t)Q_s(t)$ lies in $C(n_0)$; hence it satisfies the relation $\sum_i c_i Q(t_i) = 0$. Since Q is a delta function at t_0, we obtain a contradiction to the assumption that $c_0 \neq 0$. This proves the lemma and completes the proof of Theorem 26.14. □

□

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http://www.springer.com/978-3-319-31591-1

Arthur's Invariant Trace Formula and Comparison of Inner Forms
Flicker, Y.
2016, XI, 567 p. 3 illus., Hardcover
ISBN: 978-3-319-31591-1
A product of Birkhäuser Basel