Preface

The theory of automorphic representations of the group $G(\mathbb{A})$ of the adèle points of a reductive connected group G over a global field F, and that of admissible representations of the group $G(F_v)$ of points of a reductive connected group G over a local field F_v, are governed by a hypothetical reciprocity law, introduced by Langlands, that relates them to representations of a variant of the Galois group of the base field, named Weil or Weil-Deligne group, into the complex Langlands dual group ${}^LG$ of G.

This "principle of functoriality"—not touched upon in the present tome—suggests relations between such automorphic and admissible representations of different groups G. These relations have been termed liftings, correspondences, transfers, and are suggested by relations amongst the underlying dual groups.

For example, establishing lifting from GL(2) to GL$(n+1)$ corresponding to the irreducible n-dimensional representation Sym^n from the dual group GL$(2,\mathbb{C})$ to GL$(n+1,\mathbb{C})$ would imply the Ramanujan conjecture for GL(2). Some of these liftings, which are analytic implications of the principle, have been established by various techniques, using various invariants of the representations. The work of Jacquet-Langlands [JL70] showed that the Selberg trace formula [Se62] could give very complete results on the correspondence of representations of GL(2) and its inner forms, the multiplicative groups of quaternion algebras.

Deligne and Kazhdan [DK] then introduced a simple form of the trace formula, which applies to test functions—and representations—with two cuspidal components ("two" was reduced later to "one" in a form of the simple trace formula developed by Flicker and Kazhdan [FK88]), and established in [DKV84] the correspondence of representations between GL(n) and its inner forms, the multiplicative groups of simple algebras.

This work played the global trace formula against local analysis and used multiple induction arguments to prove not only the lifting of representations but also the existence of matching orbital integrals. The latter was previously considered to be a prerequisite for deriving lifting applications from the trace formula.

To remove this last constraint, and of course for other applications, Arthur developed the trace formula for a general test function, in a series of papers, over many years. In particular, Arthur put the trace formula in invariant form, namely, expressed all terms that appear in the formula as invariant distributions. This is necessary for comparison applications, as when comparing representations of different groups, only characters or orbital integrals can be related, and this is through a norm map relating conjugacy classes of elements in the two groups. In crude terms, only eigenvalues of elements can be related between two groups, not individual elements.

The present volume grew out of an attempt to study Arthur's work. We started in a course following Arthur's expository notes [Ar05], but quickly realized that to attempt to understand the theory we had to study the source articles. To make the subject more accessible, we decided to unite the main articles in one volume and rewrite them as one unit in a conventional way. Thus we cut many arguments into lemmas, propositions, and theorems, stated the claims before giving their proofs, uniformized the notation to make it easier to read (e.g., π on G but τ on M), and the like.

Thus, in Chapter 3, we explain Arthur's proof of the basic, noninvariant trace formula, following his early Duke and Compositio papers.

In Chapter 4, we explain Arthur's Annals and J. Funct. Anal. papers [Ar81] and [Ar89], which study the noninvariance of the terms in the basic trace formula and prepare the ingredients for setting up the invariant formula.

In Chapter 5, we explain Arthur's JAMS papers [Ar88.2, Ar88.3], where the invariant formula is finally developed. We quote some of Arthur's computations of contributions to the continuous spectrum from Amer. J. Math., [Ar82.I, Ar82.II], and of weighted orbital integrals from Duke [Ar88.1] and [Ar85, Ar86], where we felt we could not improve the exposition sufficiently to justify the increased volume. Thus Chapters 3–5 here give an almost complete attempt to develop the invariant trace formula in a form fit for applications.

To illustrate the use of this trace formula, in Chapter 6, we compare the invariant trace formulae for $G' = \mathrm{GL}(n)$ and its inner form G, for matching functions f' on $G'(\mathbb{A})$ and f on $G(\mathbb{A})$, thus functions with matching orbital integrals. This is already contained in [AC89, Chapter 2], as the secondary case, accompanying the main case of interest there: base change for $\mathrm{GL}(n)$. This marriage makes it hard to follow the inner forms case, so we decided to write it separately in our Chapter 6. This is after all the initial case of comparison; in principle it should be the simplest, and we thought consequently that it deserved its own full treatment.

The comparison of the two trace formulae is far from being simple. A key argument is a multiple induction process, reminiscent of Kazhdan's double induction argument using the simple trace formula.

Equipped with the comparison from Chapter 6, we set in Chapter 2 to prove the correspondence between $\mathrm{GL}(n)$ and its inner forms in general. This chapter is based on our course [F87.1] at Harvard 1986, where a different proof of the results of [DKV84] was given, based on using the then recent works [BDK86] on the trace Paley-Wiener theorem of Bernstein, Deligne, and Kazhdan, and on cuspidal geometry [Ka86.1] by Kazhdan.

The first section of Chapter 2 deals with the comparison of GL(n) and the anisotropic inner forms, multiplicative groups of a division algebra, following [F90.2], using the simple trace formula. This is the only "easy" case, so we bring it first. It was once considered difficult too, but [F90.2] observed that a finiteness result known at the time permitted a relatively easy proof. This finiteness result is now known for all inner forms of GL(n), but is not needed in the proof of the general case.

The second section of Chapter 2 discusses the asymptotic behavior of orbital integrals, following Shalika [Shal72], who used ideas of Harish-Chandra [HC70]. An ingredient here is the convergence of the orbital integrals on the unipotent orbits in G. There is a publication of Rao [Ra72] on this, and we sketch, toward the end of this Section 2, a proof of Deligne of this fact. This section ends with an elegant computation of Kottwitz [Ko88] of the orbital integral of the Euler-Poincaré function he introduced, which gives explicitly a pseudo-coefficient of the Steinberg representation.

In Section 24, we extend the correspondence from the *non-degenerate* case of square-integrable, tempered, standard, or relevant local representations—what follows on using purely the simple trace formula in Section 13, to a correspondence of local *unitarizable* representations from GL(n) to its inner forms, by purely local arguments of the type of [BZ76, BZ77, Ze80, Tc90], and the Langlands classification [BW80], following Badulescu [Ba08].

Section 25 uses the comparison of the invariant trace formulae of Chapter 6 to establish the full global correspondence, for cuspidal representations without any local constraint and also for residual representations, permitting to transfer multiplicity one and rigidity theorems from the known case of the split group GL(n) to that of the inner forms, where no global representation is generic; thus the theory of Whittaker models is not available, as well as establishing for the inner forms a description of the residual spectrum, analogous to that established by Moeglin and Waldspurger [MW89] for GL(n).

We postpone to the final Section 26 of Chapter 2 an account of the simple trace formula of Flicker and Kazhdan [FK88] that uses one cuspidal component and a second component regular—that leads to no constraints on lifting applications. The idea of regular functions was inspired by Deligne's conjecture on the validity of the Lefschetz fixed point formula for a correspondence on a variety over a finite field, provided it is sufficiently twisted by the Frobenius; see [F13, Fu97, Va07].

Chapter 2 was used for a course at Ohio State in 2014. Chapter 1 contains a statement of the results of Chapter 2 on the correspondence between GL(n) and its inner forms, as well as a summary of the statement of Arthur's invariant trace formula. Since the statement is so involved, it is not surprising that the proof and the development of the trace formula are so long. We hope this work would make Arthur's work more accessible. We note, however, that it, and in fact the entire theory of automorphic representations, is based on the theory of Eisenstein series, of Langlands [La66, La76]; see also [Ar79], best explained by Moeglin and Waldspurger [MW95], that remains a fundamental challenge.

This work was partially supported by grants from the Simons Foundation (#267097 and #317731 to Yuval Z. Flicker), as well as the Humboldt-Stiftung, TÜBITAK, and ERC AdG Grant 247049, while the author was a Schonbrunn professor at the Hebrew University. The author enjoyed discussions with and hospitality of Elmar Große-Klönne at HU Berlin, David Kazhdan at HU Jerusalem, Takayuki Oda at Tokyo U., Eric Opdam at Amsterdam, and Engin Özkan at Erzincan, during the preparation of this volume. This work greatly benefited from the collaboration of Michael Belfanti at Ohio State who attended the courses, took notes, improved the exposition, and was invited to coauthor this book, but declined.

The envisaged readership of this book consists of graduate students and researchers interested in the trace formula and its applications, especially to lifting problems. We hope it would simplify—in fact make it possible for such people to enter this subject.

Ariel, Israel Yuval Z. Flicker

Springer
http://www.springer.com/978-3-319-31591-1

Arthur's Invariant Trace Formula and Comparison of
Inner Forms
Flicker, Y.
2016, XI, 567 p. 3 illus., Hardcover
ISBN: 978-3-319-31591-1
A product of Birkhäuser Basel