

# Chapter 2

## Equations of Motion

### 2.1 Incompressible Flow

The equations of motion describing the flow in a fluid are based on the three laws of conservation of mass, momentum and energy. For a detailed formulation of these equations, one of the standard works in fluid mechanics should be studied (see literature list). Here we limit ourselves to a short derivation of these equations.

We start with a so-called *fluid element* whose volume is  $\delta V$ . This fluid element should be large compared to the molecules that make up the fluid (for example, it should be much larger than the mean free path  $\lambda_v$  of a gas) while simultaneously it should be small compared to the smallest dimensions of flow we are going to describe. The position of this fluid element is determined in a right-handed Cartesian coordinate system  $(x_1, x_2, x_3)$ . Using such a fluid element, the so-called *continuum hypothesis* allows us to define a pressure  $p$ , a velocity  $u_i$ , and a fluid density  $\rho$  at every point  $x_i$  and every instant in time  $t$ .

The fluid element moves through the medium with a velocity  $u_i$  at a given position of this element. For this reason, the fluid element is sometimes called a *material particle*. In the trajectory of this fluid element, alterations as a function of time are indicated by the so-called *material derivative*:  $D/Dt$ . In a Cartesian coordinate system this material derivative is defined as:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}. \quad (2.1)$$

The material derivative consists of two terms:  $\partial/\partial t$ , which is known as the *local derivative*, and  $u_j \partial/\partial x_j$ , which is called the *advection term*. This advection term describes the part of the material derivative due to transport in a velocity field varying in space. In the literature this contribution to the material derivative is also called *convection*. Here, however, we call it advection in order to distinguish it from *thermal convection* when a non-isothermal flow is dominated by effects due to variation in density.

As in Eq. (2.1) above, we make use of the *Einstein notation*. This means that a repeated index (as in  $u_i u_i$  for example) is summed over all directions of the coordinate system:  $u_i u_i \equiv u_1 u_1 + u_2 u_2 + u_3 u_3$ . An exception is when we use a *Greek index letter* (for example  $u_\alpha u_\alpha$ ; in that case we simply indicate the term *without* applying the summation).

Conservation of mass for the fluid element defined previously can now be described as

$$\frac{D(\rho \delta V)}{Dt} = 0. \quad (2.2)$$

This equation can be reduced to

$$\frac{1}{\delta V} \frac{D\delta V}{Dt} = -\frac{1}{\rho} \frac{D\rho}{Dt}. \quad (2.3)$$

When the following approximation holds, we call a flow incompressible:

$$\frac{1}{\rho} \frac{D\rho}{Dt} \approx 0. \quad (2.4)$$

This appears to be the case when the flow velocities are much smaller than the speed of sound  $a$  (i.e.,  $U \equiv |u_i| \ll a$ ).

In the remainder of this book we limit ourselves to incompressible flow. Conservation of mass then reduces to

$$\frac{1}{\delta V} \frac{D\delta V}{Dt} \equiv \frac{\partial u_i}{\partial x_i} = 0. \quad (2.5)$$

The derivation of this equation can be found in the problems section. The velocity field is thus shown to be *divergence-free* at all points. Eq. (2.5) is also known as the *continuity equation*.

Next we consider the conservation of momentum for our fluid element. According to Newton's second law it follows that:

$$\rho \delta V \frac{Du_i}{Dt} = F_i, \quad (2.6)$$

where  $F_i$  represents the net force acting on the fluid element. For  $F_i$  we can write

$$F_i = \left( \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j} \right) \delta V. \quad (2.7)$$

The first term of the equation above is called the *volume force*. In this case we equated this volume force to gravitation with gravitational acceleration  $g_i = (0, 0, -g)$ . The second term on the right-hand side of (2.7) is called the *surface force* with a *surface stress tensor*  $\sigma_{ij}$ . For incompressible flow and for a Newtonian fluid it follows that

$$\sigma_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (2.8)$$

where  $\delta_{ij}$  represents the *Kronecker- $\delta$*  symbol:  $\delta_{ij} = 0$  if  $i \neq j$ , and  $\delta_{ij} = 1$  when  $i = j$ . The surface stress tensor thus consists of two parts. The first term represents an isotropic pressure. The second term, which relates to deformations of the fluid element, is called the *shear stress*. In this term,  $\mu$  is known as *dynamic viscosity*. This is a material property, and thus only depends on the fluid. In the following we consider a Newtonian fluid and thus take  $\mu$  as a constant.

We note that  $\mu$  is often combined with the fluid density to form the *kinematic viscosity*  $\nu = \mu/\rho$ . Some representative values for  $\nu$  at standard atmospheric conditions are:  $1.5 \times 10^{-5} \text{ m}^2/\text{s}$  for air, and  $1.0 \times 10^{-6} \text{ m}^2/\text{s}$  for water.

Substituting (2.8) in (2.7) and then in (2.6) leads to the following equation for the conservation of momentum in an incompressible flow:

$$\rho \frac{Du_i}{Dt} \equiv \rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \rho g_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}. \quad (2.9)$$

Eqs. (2.9) (for  $i = 1, 2, 3$ ) are better known as the *Navier-Stokes equations*. Together with the continuity Eq. (2.5), they form a system of four equations containing five unknown variables:  $u_i$ ,  $p$  and  $\rho$ . We thus have to specify an additional constraint in order to solve the equations. For this we take  $\rho = \text{constant}$ , which means that the fluid is *homogeneous*. In other words: the density is reduced to a material constant (for air  $\rho \approx 1.2 \text{ kg m}^{-3}$ ; for water  $\rho \approx 1.0 \times 10^3 \text{ kg m}^{-3}$ ).

Another further simplification is possible when no free surface is present in our flow. In that case we can absorb the gravity term in (2.9) in the pressure term, which then is referred to as the *modified pressure*. The equations for the conservation of momentum are then reduced to

$$\rho \frac{Du_i}{Dt} \equiv \rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}. \quad (2.10)$$

Equations (2.5) and (2.10) form the basis for describing incompressible flow of a homogeneous fluid. We note that initial and boundary conditions have to be specified before a solution of the equations of motion can be found.

### 2.1.1 Problem

1. Consider a material line segment  $\delta L_x$  at  $x_i$ . The segment is oriented parallel to the  $x$ -axis. The change in length of this line segment follows from

$$\frac{D\delta L_x}{Dt} = u(x_i + \delta L_x) - u(x_i),$$

in which  $u$  represents the  $x$ -component of the velocity vector  $u_i$ . Apply a Taylor-series expansion to the right-hand side of the equation above. Next, consider the line segments  $\delta L_y$  and  $\delta L_z$ , which are oriented parallel to the  $y$ - and  $z$ -axes respectively and repeat the exercise. Now prove that

$$\frac{1}{\delta V} \frac{D\delta V}{Dt} = \frac{\partial u_i}{\partial x_i},$$

using  $\delta V = \delta L_x \delta L_y \delta L_z$ .

## 2.2 The Boussinesq Approximation

In Eqs. (2.5) and (2.10) we accounted for the influence of gravity on flow by absorbing it into the pressure term. We argued that this is only possible for a homogeneous fluid ( $\rho = \text{constant}$ ) in the absence of a free surface.

In practice, however, we are often confronted with flows in fluids where the density depends on position and time. These are called *heterogeneous fluids*. An example is the density variation that can occur due to temperature differences. Also a variable composition of the fluid can lead to variations of the density. An example of this would be the salt concentration in the ocean.

The dynamics of heterogeneous fluids are directly affected by gravity, in which case we speak of a *stratified flow*. The equations of motion for these flows are again based on the laws of conservation of mass, momentum and energy.

For the conservation of mass we use again the continuity equation

$$\frac{\partial u_i}{\partial x_i} = 0, \quad (2.11)$$

provided, of course, that the condition  $U \ll a$  is satisfied, as mentioned in the previous section. Also, we need to take into account the fact that, in a heterogeneous fluid, the density varies with height due to gravity. This is described by the *hydrostatic law*:  $\partial p / \partial z = -\rho g$ . We now introduce the *scale height* as a measure of the distance over which the density varies due to gravity. This scale height  $H$  is defined as

$$H = \left( -\frac{1}{\rho} \frac{\partial \rho}{\partial z} \right)^{-1} = \frac{p_0}{\rho g}. \quad (2.12)$$

where  $p_0$  is a reference pressure (for example the pressure at the surface for an atmospheric flow). It follows that (2.11) is only valid when  $L \ll H$ , where  $L$  is representative of the extent of the vertical motion in the flow.

Applying the conservation of momentum leads again to (2.9). However, the density  $\rho$  in (2.9) should now be seen as an unknown variable, which we have to compute as a function of position and time. This means that we need an additional equation. For this, we use a so-called *equation of state*, which reads, in general form

$$\rho = \rho(p, \theta), \quad (2.13)$$

where  $\theta$  represents temperature. We first limit ourselves to liquids for which the *compressibility modulus* ( $\partial\rho/\partial p$ ) is negligible. This means that  $\rho$  is only a function of temperature. Now we just need an additional equation describing the temperature, and for this we apply the equation for the *conservation of energy* to a fluid element. For a full discussion on the energy equation, one should consult one of the standard works on thermodynamics or fluid mechanics. It follows that, for a liquid, the energy equation can be approximated by the following equation for the temperature:

$$\frac{\partial\theta}{\partial t} + u_j \frac{\partial\theta}{\partial x_j} = \alpha \frac{\partial^2\theta}{\partial x_j^2}, \quad (2.14)$$

where  $\alpha$  is the *thermal diffusivity coefficient*. For air the value for  $\alpha$  is  $\sim 2.0 \times 10^{-5} \text{ m}^2\text{s}^{-1}$ , and for water  $\sim 1.4 \times 10^{-7} \text{ m}^2\text{s}^{-1}$ . The ratio between the material constants  $\nu$  and  $\alpha$  is referred to as the *Prandtl number*

$$Pr = \frac{\nu}{\alpha}. \quad (2.15)$$

For air we find  $Pr \approx 0.7$  and for water  $Pr \approx 8$ .

The set of Eqs. (2.11), (2.9) and (2.14) may now form a closed system, but is still too complicated to solve. This is why we need to simplify them by applying the so-called *Boussinesq approximation*. The first step of this approximation is the definition of a *reference state*:  $p_0$ ,  $\rho_0$  and  $T_0$ . This reference state has to obey the equations of motion for the fluid at rest

$$\theta_0 \equiv T_0 = \text{constant} \quad \text{and} \quad \frac{\partial p_0}{\partial z} = -\rho_0 g. \quad (2.16)$$

We now consider a flow with velocity  $u_i$ , pressure  $p_0 + p$ , density  $\rho_0 + \rho$ , and temperature  $T_0 + \theta$ . We impose the conditions that  $p/p_0 \ll 1$ ,  $\rho/\rho_0 \ll 1$ , and  $\theta/T_0 \ll 1$ . In other words,  $p$ ,  $\rho$  and  $\theta$  are small disturbances with respect to the reference state. Substitution in (2.9) leads to

$$(\rho_0 + \rho) \frac{Du_i}{Dt} = (\rho_0 + \rho) g_i - \frac{\partial}{\partial x_i} (p_0 + p) + \mu \frac{\partial^2 u_i}{\partial x_j^2}. \quad (2.17)$$

After multiplying this equation by  $1/\rho_0$ , and after substitution of the equation of motion (2.16) for  $p_0$ , we find

$$\frac{Du_i}{Dt} = \frac{\rho}{\rho_0} g_i - \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + \frac{\mu}{\rho_0} \frac{\partial^2 u_i}{\partial x_j^2}. \quad (2.18)$$

Here we used the condition  $\rho/\rho_0 \ll 1$  for the left-hand side of (2.17), so that  $(\rho_0 + \rho)Du_i/Dt \approx \rho_0 Du_i/Dt$ . However, this simplification is not applied to the term  $(\rho_0 + \rho)g_i$ . The physical background for this is that, for all flows considered here,  $g \gg |Du_i/Dt|$ .

Next, we apply a linearization of the equation of state (2.13) around  $\theta_0$ . Now we have

$$\frac{\rho}{\rho_0} = -\beta \frac{\theta}{T_0}, \quad (2.19)$$

where  $\beta$  is the *volumetric expansion coefficient*:

$$\beta = -\frac{T_0}{\rho_0} \frac{\partial \rho}{\partial \theta}.$$

One has to pay careful attention here! We cannot apply the linearization to the velocity term, that is,  $Du_i/Dt \approx \partial u_i/\partial t$ . The reason for this is that the velocity field is zero for the reference state. Hence, the velocity is never small.

The result of these calculations is a set of three equations, which are referred to as the *Boussinesq equations*

$$\frac{\partial u_i}{\partial x_i} = 0, \quad (2.20)$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\beta \frac{\theta}{T_0} g_i - \frac{1}{\rho_0} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2}, \quad (2.21)$$

$$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \alpha \frac{\partial^2 \theta}{\partial x_j^2}. \quad (2.22)$$

The set of equations above has, in principle, been derived for liquids. However, the energy Eq.(2.14) is also applicable to gases, provided that  $\theta$  is interpreted as the so-called *potential temperature*. In this way the compressibility of a gas with height is accounted for. The potential temperature is defined as the temperature of a fluid element with pressure  $p$  and temperature  $T$  when it is brought to a standard pressure  $p_*$  via an isentropic process. For an ideal gas it follows that

$$\theta = T \left( \frac{p_*}{p} \right)^{\frac{\kappa-1}{\kappa}}, \quad (2.23)$$

where  $\kappa = c_p/c_v$  is the ratio between the *specific heat at constant pressure*  $c_p$  and the *specific heat at constant volume*  $c_v$ . For air,  $c_p = 1005 \text{ J kg}^{-1} \text{ K}^{-1}$  and  $R (= c_p - c_v) = 287 \text{ J kg}^{-1} \text{ K}^{-1}$ , so that  $\kappa \approx 1.4$ .

It should be mentioned here that Eq. (2.23) is commonly known as the *Poisson equation* for an isentropic process, i.e. the entropy  $S$  of the fluid element remains constant during the process. It follows from this definition of potential temperature that  $\theta \sim S$ . The background for this is that the energy equation, using the concept of entropy, can be written in its most general form as

$$\frac{DS}{Dt} = Q_s$$

where  $Q_s$  represents all processes that increase entropy, such as molecular conduction. Simplification of this equation forms the fundamental basis for Eq. (2.14).

We can now calculate the difference between the normal temperature  $T$  and the potential temperature  $\theta$ . We take the atmosphere as an example, for which  $\kappa = 1.4$ . The standard pressure  $p_*$  is taken to be equal to 1000 mbar (or  $10^5$  Pa), which is approximately the pressure at ground level. If  $p$  does not vary too much from  $p_*$ , in other words  $T \approx \theta$  (that is, we limit ourselves to the lower layer of the atmosphere, which is the so-called *atmospheric boundary layer*), then substitution of (2.23) in the hydrostatic law (2.16) yields for  $p$

$$\frac{\partial \theta}{\partial z} = \frac{\partial T}{\partial z} + \gamma_d, \quad \text{with: } \gamma_d = \frac{g}{c_p}. \quad (2.24)$$

Here  $\gamma_d$  is the so-called *adiabatic temperature gradient*. For our atmosphere this is  $0.01^\circ\text{C m}^{-1}$ . This means that we obtain the potential temperature by multiplying the temperature in the atmosphere with  $\gamma_d z$  (where  $z$  is measured from the surface of the earth).

It is clear that the correction of the potential temperature matters only to the atmosphere; this term is therefore often seen in the meteorological literature. In a laboratory,  $z$  is virtually negligible with respect to the atmospheric scale height. This means that flows of liquids and gases are indistinguishable in the laboratory. That is why, from now on, we use the term ‘temperature’ for  $\theta$ , rather than ‘potential temperature’ (although one should be aware that in the atmosphere a correction should be applied to the term).

In addition, we also limit ourselves to *ideal gases*, that is:  $p = \rho RT$ . For such gases the volumetric expansion coefficient  $\beta$  is unity ( $\beta = 1$ ).

### 2.2.1 Problems

1. Compute the scale height of an isothermal atmosphere (meaning that  $T$  is constant with increasing height). We consider air to be an ideal gas:  $p = \rho RT$ , where  $R$  is the gas constant, which has a value for air of  $R = 287.04 \text{ m}^2 \text{ s}^{-2} \text{ K}^{-1}$ . Show that turbulence occurring in the atmospheric boundary layer (the lower few kilometers of the atmosphere) can be considered as an incompressible flow, while

compressibility is not negligible for turbulence related to thunderstorms (which are spread out over the lower 10 Km of the atmosphere).

2. Consider the atmospheric boundary layer at rest with an average temperature gradient  $dT_0/dz = \text{constant}$  as the initial state (the average potential temperature gradient is then equal to  $dT_0/dz + \gamma_d$ ). We consider the vertical motion of material air particles. Show that for small disturbances, neglecting molecular effects, it follows that for the vertical position  $z_p$  of an air particle

$$\frac{d^2 z_p}{dt^2} + N^2 z_p = 0,$$

where:

$$N^2 = \frac{g}{T_0} \left( \frac{dT_0}{dz} + \gamma_d \right) \equiv \frac{g}{T_0} \left( \frac{d\Theta_0}{dz} \right)$$

is called the *Brunt–Väisälä frequency*.

At  $t = 0$  we shift the position of the particle by  $z_p(0)$  relative to its initial position. Using the equation above it follows that for

- (a)  $dT_0/dz = -\gamma_d$  (or:  $d\Theta_0/dz = 0$ ) the particle will not move. The atmosphere is called *neutral*.
- (b)  $dT_0/dz > -\gamma_d$  (or:  $d\Theta_0/dz > 0$ ) the particle will start oscillating with frequency  $N$ . The atmosphere is called *stable*.
- (c)  $dT_0/dz < -\gamma_d$  (or:  $d\Theta_0/dz < 0$ ) the particle is unstable. The atmosphere is called *convective*.



**Fig. 2.1** Plume from a smokestack in a stable atmosphere



**Fig. 2.2** Definition of coordinate systems in the laboratory and in the atmosphere

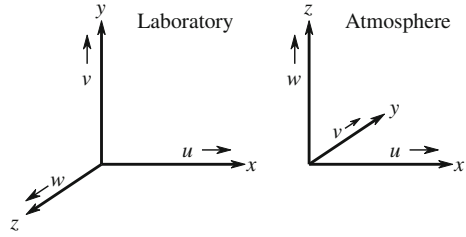


Fig. 2.1 shows a plume from a smokestack on a cold winter day; despite a strong wind there is little mixing due to the strong stratification of the atmosphere. The clear sky allows a strong radiation of heat, so that the temperature near the surface is lower than higher up in the atmosphere (i.e.,  $d\Theta_0/dz > 0$ ).

### 2.3 Coordinate System

As mentioned in Sect. 2.1, we primarily use a *Cartesian coordinate system*:  $(x_1, x_2, x_3)$  or  $(x, y, z)$ . However, two conventions for the vertical axis can be used.

In the so-called *laboratory coordinate system* the  $y$ -axis is taken vertical. However, for flows in the atmosphere and in the ocean, most often the  $z$ -axis is chosen as the vertical axis. Both coordinate systems, illustrated in Fig. 2.2, are used throughout this book.

Turbulence

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