Chapter 2
Continuum Mechanics in One Dimension

This chapter gives a short introduction to the continuum mechanics applied to the uni-axial stress state. Here we consider a rod to illustrate main ideas of continuum mechanics in a simple, transparent manner, without jungles of tensors. However, to show parallels to the three-dimensional theory we apply the notation of continuum mechanics. For example, we use F to designate the "deformation gradient", P to designate the Piola-Kirchhoff or engineering stress, etc. Three-dimensional equations will be discussed in Chap. 4.

The present chapter deals with basic equations of continuum mechanics applied to the theory of rods. A rod is a structural member with cross-section dimensions much less than the axial length. Rods can be subjected to different types of loadings including tension (compression), bending and torsion. A deformed configuration of a rod can be described by specifying the deformed rod axis, the actual cross-section area and triads of unit vectors to characterize the actual orientation of cross-sections. To define the deformed line only one coordinate is required. The problem to compute a deformed configuration for given loads is therefore one-dimensional.

Two approaches can be applied to formulate the theory of rods. The first one—called direct—considers a rod as a deformable line. The basic assumption is that every cross section behaves like a rigid body in the sense that translations and cross-section rotations are basic degrees of freedom for every point of the line. The mechanical interactions between two neighboring cross sections are (normal and/or shear) forces and (bending and/or twisting) moments. Basic balance equations of continuum mechanics are applied directly to the deformable line. Direct theories of rods are discussed in Altenbach et al. (2005, 2013), Antman (1995), Green et al. (1974b), Zhilin (2006) among others.

The second approach is based on equations of the three-dimensional continuum mechanics. With cross-section assumptions to the components of displacement vector and/or stress tensor, approximate one-dimensional equations for a rod can be derived, e.g. Green et al. (1974a).

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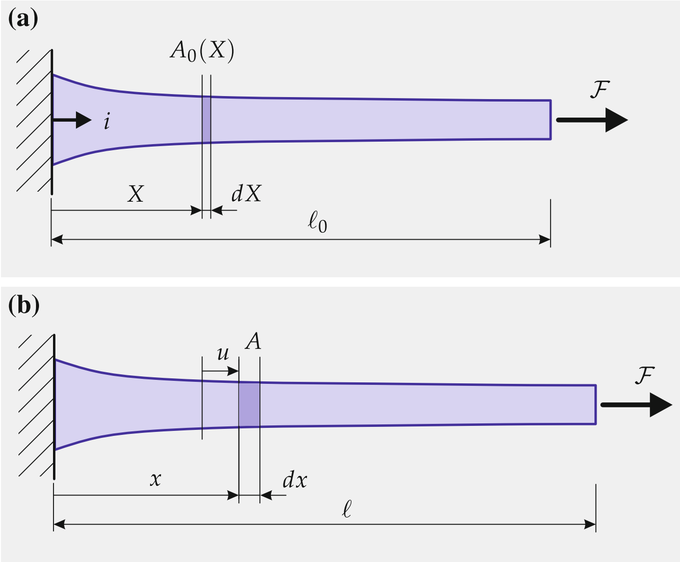

Fig. 2.1 A rod subjected to a tensile force. **a** Reference configuration, **b** actual configuration

In this section we assume that the rod is subjected to tensile (compressive) loading only. Figure 2.1 provides a sketch of a straight rod subjected to a tensile force. Let $\boldsymbol{i}$ be the unit vector designating the direction of the rod axis, Fig. 2.1a. To describe the positions of cross sections of the rod in the reference configuration the vector $\boldsymbol{R} = X\boldsymbol{i}$ with the coordinate coordinate X is introduced. The corresponding position in the actual configuration is defined by the vector $\boldsymbol{r} = x\boldsymbol{i}$ with the coordinate x.

2.1 Motion, Derivatives, and Deformation

The motion of the rod is defined by the following mapping

$$x = \Phi(X, t) \tag{2.1.1}$$

The basic problem of continuum mechanics is to find the function Φ for all of $0 \leq X \leq \ell_0$, for the given time interval $t_0 \leq t \leq t_n$ as well as for defined external loads and temperature. It is obvious that $X = \Phi(X, t_0)$. The displacement u is defined as it follows (Fig. 2.1b)

$$u = x - X \tag{2.1.2}$$

To analyze the motion it is useful to introduce the rates of change of Φ with respect to the reference coordinate X and time t. The deformation gradient F is defined as follows[1]

$$F = \frac{\partial \Phi}{\partial X} \tag{2.1.3}$$

The velocity field v is defined as follows

$$v = \frac{\partial \Phi}{\partial t} = \dot{u} \tag{2.1.4}$$

Within the one-dimensional theory the deformation gradient F is identical with the local stretch λ which is defined as

$$\lambda = \frac{\mathrm{d}x}{\mathrm{d}X}$$

$\mathrm{d}X$ and $\mathrm{d}x$ are line elements defined in the infinitesimal neighborhood of a cross section in the reference and actual configurations, respectively. The local strain ε can be defined as follows

$$\varepsilon = \frac{\mathrm{d}x - \mathrm{d}X}{\mathrm{d}X} = \lambda - 1 = \frac{\partial u}{\partial X} \tag{2.1.5}$$

If the material properties and the cross section area do not depend on X then the rod is called homogeneous. The stretch and the strain can be computed as follows

$$\lambda = \frac{\ell}{\ell_0}, \quad \varepsilon = \frac{\ell - \ell_0}{\ell_0} \tag{2.1.6}$$

The formulas (2.1.6) are applied to evaluate strains from experimental data of uniaxial tests. If a rod is non-homogeneous than local strains should be evaluated. In this case a strain gauge should be placed in a position along the rod to provide the local strain value. The length of the strain gauge should be small such that the measured strain could be assumed constant. Otherwise the measured strain would depend on the length of the strain gauge. A similar assumption can be applied to motivate Eq. (2.1.5)—the value of the tested line element $\mathrm{d}X$ should be "small enough" such that the strain over $\mathrm{d}X$ is constant. This is the basic idea behind the classical continuum mechanics—the notions of infinitesimal volume, area and line elements are introduced such that the quantities like density, stress, strain etc. can be assumed uniform over the considered elements.

Assuming the mapping Φ to be invertible one may introduce the inverse of the deformation gradient as follows

$$F^{-1} = \frac{\mathrm{d}X}{\mathrm{d}x} \tag{2.1.7}$$

[1]The deformation gradient is usually not introduced within the one-dimensional theory of rods. Here we introduce this and other quantities to explain basic ideas of continuum mechanics.

Let f be a field like density, displacement, stress, etc. f can be considered as a function of the coordinate X and time t. This is sometimes called Lagrangian description. Alternatively, one may refer f to the actual coordinate x and time. This kind of description is called spatial or Eulerian. The derivatives of a function f with respect to X and x can be specified as follows

$$\frac{\partial f}{\partial X} \equiv f'^0, \quad \frac{\partial f}{\partial x} \equiv f' \tag{2.1.8}$$

Between the derivatives the obvious relation exists

$$\frac{\partial f}{\partial X} = F\frac{\partial f}{\partial x} \qquad \Rightarrow \qquad f'^0 = Ff' \tag{2.1.9}$$

As the motion Φ is assumed invertible

$$X = \Phi^{-1}(x, t), \tag{2.1.10}$$

the material and the spacial descriptions are equivalent in the sense that if f is known as a function of X and t, one may use the transformation (2.1.10) to find

$$f(X, t) = g(x, t)$$

For example the density ρ can be as a function of the reference coordinate and time or the actual coordinate and time

$$\rho = f(X, t) = g(x, t)$$

With Eqs. (2.1.3) and (2.1.4) the derivative of the velocity with respect to the reference coordinate can be computed as follows

$$v'^0 = \dot{F} \tag{2.1.11}$$

With (2.1.9) the derivative of the velocity with respect to the actual coordinate is

$$v' = \dot{F}F^{-1} \tag{2.1.12}$$

Assuming that $f(x)$ is continuous for $a \leq x \leq b$ the fundamental theorem of integral calculus provides

$$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a) \tag{2.1.13}$$

If $f(x)$ has n jumps at points $x_k, k = 1, 2, \ldots, n$ within $a \leq x \leq b$ and $f'(x)$ is continuous between the jump points then

$$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a) + \sum_{k=1}^{n} [\![f(x_k)]\!], \quad [\![f(x_k)]\!] \equiv f(x_k^+) - f(x_k^-) \tag{2.1.14}$$

Assume that the velocity field is given as a function of the spatial coordinate and the time, i.e. $v(x, t)$. The material time derivative of a field $f(x, t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} f = \frac{\partial}{\partial t} f + v f' \tag{2.1.15}$$

2.2 Conservation of Mass

The mass of an infinitesimal part of the rod is

$$\mathrm{d}m = \rho A \mathrm{d}x = \rho_0 A_0 \mathrm{d}X, \tag{2.2.16}$$

where ρ and ρ_0 is the density in the actual and the reference configurations, respectively. With Eq. (2.1.7) the conservation of mass (2.2.16) takes the form

$$F \rho A = \rho_0 A_0 \tag{2.2.17}$$

Introducing the change in the volume

$$J = \frac{\mathrm{d}V}{\mathrm{d}V_0} = \frac{A \mathrm{d}x}{A_0 \mathrm{d}X} = \frac{A}{A_0} F, \tag{2.2.18}$$

where $\mathrm{d}V$ and $\mathrm{d}V_0$ are infinitesimal volume elements of the rod in the actual and reference configurations, respectively, the conservation of mass (2.2.16) yields

$$\frac{\rho_0}{\rho} = J \tag{2.2.19}$$

It is obvious that $J > 0$ and if $\rho = \rho_0$ one obtains $J = 1$.

2.3 Balance of Momentum

The momentum of an infinitesimal part of the rod is

$$\mathrm{d}p = v \mathrm{d}m = v \rho A \mathrm{d}x$$

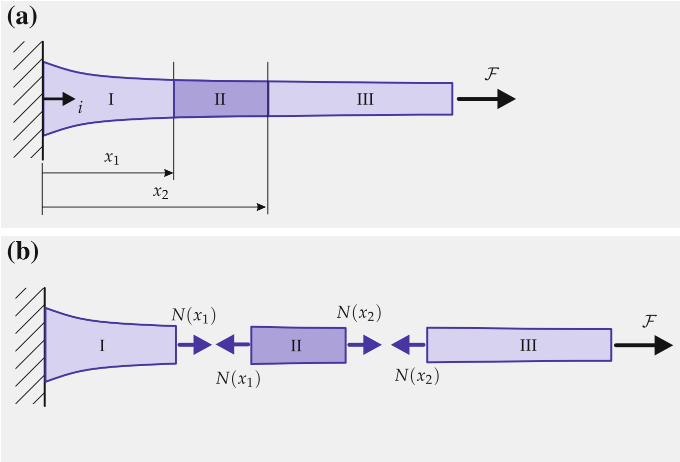

Fig. 2.2 Internal forces in a rod. **a** Rod in the actual configuration and two cutting planes, **b** free-body diagrams visualizing internal forces

Consider a part of the rod, for example, part II, Fig. 2.2b. The momentum for this part in the actual configuration is

$$p_{\mathrm{II}} = \int\limits_{x_1}^{x_2} v\rho A \mathrm{d}x \tag{2.3.20}$$

The balance of momentum or the first law of dynamics states that the rate of change of momentum of a body is equal to the total force acting on the body. To introduce the forces acting on the part II of the rod let us cut it by two cross sections with the coordinates x_1 and x_2. The parts I and III belong to the environment of the part II and the corresponding mechanical actions can be modeled by two forces: $\boldsymbol{N}_{\mathrm{I-II}}$—the action of the part I on the part II and $\boldsymbol{N}_{\mathrm{III-II}}$ the action of the part III on the part II. Similarly, the actions on the parts I and III can be introduced. For example, $\boldsymbol{N}_{\mathrm{II-I}}$ is the action of the part II on the part I. The following abbreviations can be introduced

$$\begin{aligned}
\boldsymbol{N}_{\mathrm{II-I}} &= \boldsymbol{N}_{(\boldsymbol{i})}(x_1) &&= N(x_1)\boldsymbol{i}, \\
\boldsymbol{N}_{\mathrm{I-II}} &= \boldsymbol{N}_{(-\boldsymbol{i})}(x_1) &&= -N(x_1)\boldsymbol{i}, \\
\boldsymbol{N}_{\mathrm{III-II}} &= \boldsymbol{N}_{(\boldsymbol{i})}(x_2) &&= N(x_2)\boldsymbol{i}, \\
\boldsymbol{N}_{\mathrm{II-III}} &= \boldsymbol{N}_{(-\boldsymbol{i})}(x_2) &&= -N(x_2)\boldsymbol{i}
\end{aligned} \tag{2.3.21}$$

With the free-body diagram presented in Fig. 2.2 the balance of momentum for the part II is[2]

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{x_1}^{x_2} v\rho A \mathrm{d}x = N(x_2) - N(x_1) \tag{2.3.22}$$

[2]Body forces like the force of gravity are not included here for the sake of brevity.

With the fundamental theorem of integral calculus (2.1.13)[3]

$$N(x_2) - N(x_1) = \int_{x_1}^{x_2} N' \mathrm{d}x \tag{2.3.23}$$

Applying the mass conservation equation (2.2.16) one may evaluate the rate of change of momentum as follows

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{x_1}^{x_2} v\rho A \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \int_{X_1}^{X_2} v\rho_0 A_0 \mathrm{d}X = \int_{x_1}^{x_2} \dot{v}\rho A \mathrm{d}x \tag{2.3.24}$$

With Eqs. (2.3.23) and (2.3.24) the integral form of the balance of momentum is

$$\int_{x_1}^{x_2} (\dot{v}\rho A - N')\mathrm{d}x = 0 \tag{2.3.25}$$

Equation (2.3.25) is valid for any part of the rod. Since x_1 and x_2 are arbitrary, the integral (2.3.25) is zero if

$$\rho A \dot{v} = N' \tag{2.3.26}$$

Multiplying both parts of Eq. (2.3.26) by F yields

$$F\rho A \dot{v} = F N' \tag{2.3.27}$$

With the conservation of mass (2.2.17) and the relation between the derivatives (2.1.9), Eq. (2.3.27) takes the following form

$$\rho_0 A_0 \dot{v} = N'^0 \tag{2.3.28}$$

2.4 Balance of Energy

The total energy E for any part of the rod is defined as the sum of the kinetic energy K and the internal energy U as follows

$$\mathrm{E} = \mathrm{K} + \mathrm{U}, \quad \mathrm{K} = \int_{x_1}^{x_2} \rho\mathcal{K} A \mathrm{d}x, \quad \mathrm{U} = \int_{x_1}^{x_2} \rho\mathcal{U} A \mathrm{d}x, \quad \mathcal{K} = \frac{1}{2}v^2, \tag{2.4.29}$$

[3]Here and in the following derivations we assume that N and other field variables are smooth functions. In the case of finite jumps one should apply Eq. (2.1.14) and introduce the jump conditions.

where $\mathcal{K}$ and $\mathcal{U}$ are densities of the kinetic and the internal energy, respectively. The energy balance equation or the first law of thermodynamics states that the rate of change of the energy of a body is equal to the mechanical power plus the rate of change of non-mechanical energy, for example heat, supplied into the body. The energy balance equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E} = \mathrm{L} + \mathrm{Q}, \tag{2.4.30}$$

where L is the mechanical power and Q is the rate of change of non-mechanical energy supply. The mechanical power of internal forces (2.3.21) is defined as follows

$$\mathrm{L} = N(x_2)v(x_2) - N(x_1)v(x_1) = \int_{x_1}^{x_2} (Nv)'\mathrm{d}x \tag{2.4.31}$$

The rate of change of energy supply through the cross sections of the parts I, II and III of the rod can be defined by analogy to Eqs. (2.3.21)

$$\begin{aligned}
\mathrm{Q}_{\mathrm{II-I}} &= Q_{(\boldsymbol{i})}(x_1) &&= -Q(x_1),\\
\mathrm{Q}_{\mathrm{I-II}} &= Q_{(-\boldsymbol{i})}(x_1) &&= Q(x_1),\\
\mathrm{Q}_{\mathrm{III-II}} &= Q_{(\boldsymbol{i})}(x_2) &&= -Q(x_2),\\
\mathrm{Q}_{\mathrm{II-III}} &= Q_{(-\boldsymbol{i})}(x_2) &&= Q(x_2)
\end{aligned} \tag{2.4.32}$$

The rate of change of the energy supply through the volume of the part II is

$$\mathrm{Q}_{V_{\mathrm{II}}} = \int_{x_1}^{x_2} \rho r A \mathrm{d}x,$$

where r is the density of the energy supply. The total rate of energy supply into the part II is

$$Q(x_1) - Q(x_2) + \int_{x_1}^{x_2} \rho r A \mathrm{d}x = \int_{x_1}^{x_2} (-Q' + \rho r A)\mathrm{d}x \tag{2.4.33}$$

With Eqs. (2.4.29), (2.4.31) and (2.4.33) the energy balance equation (2.4.30) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} (\rho\frac{1}{2}v^2 + \rho\mathcal{U})A\mathrm{d}x = \int_{x_1}^{x_2} (N'v + Nv' - Q' + \rho r A)\mathrm{d}x \tag{2.4.34}$$

With the mass conservation equation (2.2.16) the rate of change of the total energy can be evaluated as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2}(\rho\frac{1}{2}v^2+\rho\mathcal{U})A\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{X_1}^{X_2}(\rho_0\frac{1}{2}v^2+\rho_0\mathcal{U})A_0\mathrm{d}X \\ &= \int_{X_1}^{X_2}\left(\dot{v}v+\dot{\mathcal{U}}\right)\rho_0A_0\mathrm{d}X = \int_{x_1}^{x_2}\left(\dot{v}v+\dot{\mathcal{U}}\right)\rho A\mathrm{d}x\end{aligned} \tag{2.4.35}$$

The energy balance equation (2.4.34) takes the following form

$$\int_{x_1}^{x_2}\left[(\rho A\dot{v}-N')v+\rho A\dot{\mathcal{U}}-Nv'+Q'-\rho rA\right]\mathrm{d}x=0 \tag{2.4.36}$$

With the balance of momentum (2.3.26), Eq. (2.4.34) is simplified to

$$\int_{x_1}^{x_2}\left(\rho A\dot{\mathcal{U}}-Nv'+Q'-\rho rA\right)\mathrm{d}x=0 \tag{2.4.37}$$

As x_1 and $x_2 > x_1$ are arbitrary coordinates, the local (per unit length of the rod) form of the energy balance is

$$\rho A\dot{\mathcal{U}} = Nv' - Q' + \rho Ar \tag{2.4.38}$$

Multiplying both sides of (2.4.38) by F and using the conservation of mass (2.2.17) as well as the relation between the derivatives (2.1.9) provides the local form of the energy balance per unit length of the rod in the reference configuration

$$\rho_0A_0\dot{\mathcal{U}} = Nv'^0 - Q'^0 + \rho_0A_0r \tag{2.4.39}$$

2.5 Entropy Inequality

For historical overview of thermodynamics principles we refer to Ericksen (1998), Truesdell (1984), Müller (2007). The second law of thermodynamics states that the entropy production of a body is non-negative. This statement is given as the Clausius-Planck inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{S}-\frac{\mathrm{Q}}{T}\geq 0, \tag{2.5.40}$$

where S is the entropy and T is the absolute temperature. The entropy of the part II of the rod (Fig. 2.2) is defined as follows

$$S = \int_{x_1}^{x_2} \rho \mathcal{S} A \mathrm{d}x, \tag{2.5.41}$$

where $\mathcal{S}$ is the entropy density. With Eqs. (2.4.32) and (2.1.13)

$$\left(\frac{Q}{T}\right)_{II} = -\frac{Q(x_2)}{T(x_2)} + \frac{Q(x_1)}{T(x_1)} + \int_{x_1}^{x_2} \frac{\rho A r}{T} \mathrm{d}x = -\int_{x_1}^{x_2} \left[\left(\frac{Q}{T}\right)' - \frac{\rho A r}{T}\right] \mathrm{d}x \tag{2.5.42}$$

Inserting (2.5.41) and (2.5.42) into (2.5.40) provides the integral form of the entropy inequality

$$\int_{x_1}^{x_2} \left[\rho \dot{\mathcal{S}} A + \left(\frac{Q}{T}\right)' - \frac{\rho A r}{T}\right] \mathrm{d}x \geq 0 \tag{2.5.43}$$

Since x_1 and $x_2 > x_1$ are arbitrary the local form of the entropy inequality can be given as follows

$$\rho \dot{\mathcal{S}} A \geq -\left(\frac{Q}{T}\right)' + \frac{\rho r A}{T} \tag{2.5.44}$$

This is a one-dimensional version of the Clausius-Duhem inequality. Multiplying the both sides of (2.5.44) by T it can be formulated as follows

$$\rho \dot{\mathcal{S}} T A \geq -Q' + Q \frac{T'}{T} + \rho r A \tag{2.5.45}$$

2.6 Dissipation Inequality, Free Energy, and Stress

From the energy balance Eq. (2.4.38) it follows

$$\rho A r - Q' = \rho A \dot{\mathcal{U}} - N v' \tag{2.6.46}$$

Inserting Eq. (2.6.46) into the entropy inequality (2.5.45) yields the dissipation inequality

$$N v' - \rho A \dot{\mathcal{U}} + \rho \dot{\mathcal{S}} T A - Q \frac{T'}{T} \geq 0 \tag{2.6.47}$$

Dividing (2.6.47) by the cross section area provides the following local form of the dissipation inequality

$$\sigma v' - \rho \dot{\mathcal{U}} + \rho \dot{\mathcal{S}} T - q \frac{T'}{T} \geq 0, \quad \sigma = \frac{N}{A}, \quad q = \frac{Q}{A}, \tag{2.6.48}$$

where σ is called stress or true stress and q is the heat supply through the infinitesimal cross section. Introducing the Helmholtz free energy density $\Phi = \mathcal{U} - \mathcal{S}T$ the dissipation inequality (2.6.48) takes the following form

$$\sigma v' - \rho\dot{\Phi} - \rho\mathcal{S}\dot{T} - q\frac{T'}{T} \geq 0 \tag{2.6.49}$$

With Eq. (2.1.12) it follows that

$$\sigma\dot{F}F^{-1} - \rho\dot{\Phi} - \rho\mathcal{S}\dot{T} - q\frac{T'}{T} \geq 0 \tag{2.6.50}$$

Multiplying both sides of (2.6.47) by F and using the conservation of mass (2.2.17) as well as the relation between the derivatives (2.1.9), the local form of the dissipation inequality per unit length of the rod in the reference configuration can be obtained

$$Nv'^0 - \rho_0 A_0\dot{\mathcal{U}} + \rho_0 A_0\dot{\mathcal{S}}T - Q\frac{T'^0}{T} \geq 0 \tag{2.6.51}$$

Dividing by A_0 yields

$$Pv'^0 - \rho_0\dot{\mathcal{U}} + \rho_0\dot{\mathcal{S}}T - \tilde{q}\frac{T'^0}{T} \geq 0, \quad P = \frac{N}{A_0}, \quad \tilde{q} = \frac{Q}{A_0} \tag{2.6.52}$$

where P is the engineering stress and $\tilde{q}$ is the heat flow through the infinitesimal cross section of the rod in the reference state. In terms of the free energy the inequality takes the following form

$$Pv'^0 - \rho_0\dot{\Phi} - \rho_0\mathcal{S}\dot{T} - \tilde{q}\frac{T'^0}{T} \geq 0 \tag{2.6.53}$$

With Eq. (2.1.11) the velocity derivative can be replaced by the rate of the deformation gradient leading to

$$P\dot{F} - \rho_0\dot{\Phi} - \rho_0\mathcal{S}\dot{T} - \tilde{q}\frac{T'^0}{T} \geq 0 \tag{2.6.54}$$

Taking into account that the normal force is $N = PA_0 = \sigma A$ the following relation between the stress measures can be established

$$PA_0 = \sigma J A_0 F^{-1} \qquad \Rightarrow \qquad P = \sigma J F^{-1} \tag{2.6.55}$$

Similarly, with the heat flux $Q = qA = \tilde{q}A_0$

$$\tilde{q} = qJF^{-1} \tag{2.6.56}$$

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