Chapter 2
Semantic Nominalism: How I Learned to Stop Worrying and Love Universals

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Abstract Aldo Antonelli offers a novel view on abstraction principles in order to solve a traditional tension between different requirements: that the claims of science be taken at face value, even when involving putative reference to mathematical entities; and that referents of mathematical terms are identified and their possible relations to other objects specified. In his view, abstraction principles provide representatives for equivalence classes of second-order entities that are available provided the first- and second-order domains are in the equilibrium dictated by the abstraction principles, and whose choice is otherwise unconstrained. Abstract entities are the referents of abstraction terms: such referents are to an extent indeterminate, but we can still quantify over them, predicate identity or non-identity, etc. Our knowledge of them is limited, but still substantial: we know whatever has to be true no matter how the representatives are chosen, i.e., what is true in all models of the corresponding abstraction principles. This view is backed up by an "austere" conception of universals, according to which these are first-order objects, i.e., ways of collecting first-order objects. Antonelli thus claims that second-order logic does not import any novel ontological commitment beyond ontology of first-order, naturalistically acceptable, objects. Moreover, in the case of arithmetic, even if Hume's Principle is construed as described above, Frege's Theorem goes through unaffected, since there is nothing in its proof that depends on an account of the "true nature" of numbers. A viable construal of logicism is then given by the combination of semantic nominalism and a naturalistic conception of abstraction. [*Editors note*]

Keywords Abstraction principles and Frege's Theorem · Invariance and logicality · Arithmetic with Frege quantifiers · Semantic nominalism

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2.1 Introduction

When it comes to the question of the existence and proper status of *abstract entities* (such as numbers, propositions, functions, and even sets) philosophers have found that turning to science can often lead to conflicting conclusions. On the one hand, scientifically inspired empiricism poses the urgent question of our epistemic access to such abstract entities, given their non-spatio-temporally located and causally inert nature. The question was famously posed by Benacerraf (1973) and again by Field (1989): if all knowledge is at least partially—but ultimately—based on experience, then the specific nature of these abstract entities places them beyond the reach of human cognition. This is true at least on a causal theory of knowledge, as Hale points out in Hale (2001, p. 169), but even if somehow the causal theory is rejected it remains true that abstract entities, *as such*, and as ordinarily conceived, remain outside the purview of *physical* science.

On the other hand, scientific naturalism insists that the claims of science be taken at face value, even when they involve putative reference to the abstract entities of mathematics: scientific practice cannot be challenged on purely philosophical grounds. As Lewis (1991, p. 59) famously put it:

> I am moved to laughter at the thought of how *presumptuous* it would be to reject mathematics for philosophical reasons. How would *you* like the job of telling the mathematicians that they must change their ways, and abjure countless errors, now that *philosophy* has discovered that there are no classes?

Taking the statements of mathematical or natural science "at face value" requires, among other things, that we ascribe to them the logical form that they appear to have, that we construe the singular terms thus identified as referential, and supply the resulting sentences with Tarskian truth conditions. In particular, when scientific statements contain singular terms purportedly referring to abstract entities—numbers, sets, functions, etc.—the requirement enjoins us to identify appropriate referents for such terms and specify the relations in which such objects can stand to other objects, abstract or ordinary as they might be.

Lewis is right in pointing out the philosophers' *hubris* in dictating the mathematicians' ontology. But even if we adopt the naturalistic viewpoint, we still have to provide philosophically compelling answers to the questions pressed on us by Burgess and Rosen (2005):

(a) *The ontological question*: What are abstract entities?
(b) *The epistemological question*: How can we have knowledge of them?

Answers to the two questions are often inter-related in interesting ways, and sometimes, as noticed, even in tension with one another. Our aim in what follows is to resolve the tension by articulating an account that is naturalistic in taking the claims of mathematical science at face value, while at the same time addressing empiricist reservations about the special status of abstract entities.

2.2 Fregean Platonism

Platonism about abstract entities is, roughly speaking, the view that such entities are endowed with independent existence (independent, that is, of the cognitive processes of the human mind). While this form of realism is attractive for its ability to account for the role that abstract entities play in our scientific practice, a tension arises when trying to articulate a concurrent epistemological view. But Wright (1983) offered supporters of the realist approach a new and promising avenue for the introduction of abstract entities: one could view such entities as delivered by *abstraction principles*, with the consequence that one needed only to be conversant with an already familiar range of concepts in order competently to employ singular terms referring to such entities and to recognize truths about them.

Second-order abstraction principles take the form:

$$\alpha(F) = \alpha(G) \text{ if and only if } \mathscr{R}(F, G),$$

asserting that the α-abstract of F is the same as the α-abstract of G if and only if the two concepts F and G are related by the equivalence relation $\mathscr{R}$. The abstraction is second-order because it involves the assignment of abstracts to *concepts*. On the other hand, first-order abstraction principles assign abstracts to first-level objects (whatever those may be); such principles can be represented schematically as

$$\delta(x) = \delta(y) \text{ if and only if } R(x, y).$$

First-order abstraction principles provide an important contrast class, since they exhibit properties that are significantly different from the second-order case (more about which later). The main example of higher-order abstraction that will concern us here is the same one that was put forward by Wright as paradigmatic: Hume's Principle, introducing *numbers* as second-order abstracts of the equivalence relation $F \approx G$, which obtains when just as many objects fall under F as they fall under G. If Hume's Principle is indeed to allow a *non-problematic* way of introducing numbers, "just as many" is, crucially, to be spelled out without, in turn, appealing to the notion of number (using the second-order statement to the effect that there is a bijection between the F's and the G's).

Rosen is quite adamant about the advantages of introducing abstract entities, and numbers in particular, by means of abstraction principles. Unreflective forms of Platonism suffer from the "vaguely feudal" picture of two distinct realms of objects, the one inhabited by ordinary objects of everyday life, and one that is home to a "vast catalogue of abstract objects": on this basis, "[P]latonism can only sound insane" (Rosen 1993, p. 152). On the other hand, Fregean Platonism "promises to undermine the hopeless two-world-picture by identifying facts about abstract objects with facts about ordinary concrete things". This identification is in line with another feature of ordinary thought, the so-called "supervenience of the abstract", the idea that facts about abstract objects are fixed by the facts about concrete objects. To use Rosen's example: although *King Lear*, the play, is an abstract object distinct from each and all its concrete manifestations (books, performances, etc.), the facts about the play are determined by the array of concrete objects in the world. When applied, *mutatis mutandis*, to the case of mathematics, this insight gives a form of Platonism "without tears", which is

> the only serious attempt […] to establish [P]latonism while restoring the intelligibility of the supervenience of the abstract and thereby diffusing the pernicious effects of the [two-world] picture (Rosen 1993, p. 154).

Consider now the first-order abstraction principle for directions, introduced by Frege in §65 of *Grundlagen* (Frege 1980, p. 76), which is also extensively discussed by Rosen:

$$\mathsf{dir}(a) = \mathsf{dir}(b) \text{ if and only if } a \parallel b.$$

Fregean Platonists argue that the abstract entities thus introduced, *directions*, are completely unproblematic, in that anyone who already grasps the notion of parallelism of two lines will also, *eo ipso*, grasp the notion of direction. This claim is made precise by showing that the adjunction of the principle to a theory of parallelism is *conservative* over the base theory of parallelism, in that it does not allow the derivation of new facts expressible in the base theory. There is a sense in which adjoining the abstraction principle for directions to the base theory does not really tell us anything about line parallelism that we did not know before.

The abstraction principle for directions shares, in fact, such unproblematic nature with all other first-order abstraction principles. All such principles are conservative over the corresponding base theory. But to take first-order abstraction principles as representative of the larger class of second-order principles is both misleading and instructive. It is misleading because not all abstraction principles are as innocent as first-order ones (as Frege himself came to realize), but interesting since, if we accept the theoretical surplus that can accompany higher-order abstraction principles, then there is a lot to be learned from the comparison.

2.3 Semantic Nominalism

At the other end of the spectrum from unreflective Platonism, we find *eliminative nominalism*. Although nominalism dates back to medieval scholastics, its modern form originates with Goodman and Quine (1947), and it has among its contemporary proponents Field (1980) and Hellman (1989). Eliminative nominalism is engaged in a *reductive* program, advocating the elimination of abstract entities from scientific discourse, very much in the same way in which Weierstraß's (1854) arithmetization of analysis eliminated infinitesimals from the calculus. Such an elimination, when achieved, would then blunt standard *indispensability arguments* for abstract entities (first explicitly proposed by Putnam 1971). Such arguments are based on the notion that acceptance of a scientific theory carries along ontological commitment to all (and only?) those entities that are indispensable to it, but nominalization—if successful—would show that abstract entities are not indispensable.

It is fair to say that this form of reductive eliminativism fails, for various reasons, to achieve its goal in full generality, a case that has been made by Colyvan (2001) among others (but see also Panza and Sereni 2013). Colyvan in particular argues that it is not sufficient that abstract entities be purged from a scientific theory in order for us to drop our commitment to such entities; in addition one also needs to show that the resulting theory is *better* than the one it replaces along the usual metrics employed by scientists in the assessment of theories (simplicity, elegance, explanatory power, etc.). And this is a tall order that no form of eliminative nominalism has been able to fulfill.

Fregean Platonism, in particular, presents a particular challenge for the nominalist project, in that abstract terms introduced via abstraction principles are not eliminable. This is in fact the force of the so-called *Caesar's problem*: abstraction principles allow us to settle the truth value of identities between abstract terms by reducing them to expressions no longer containing reference to *abstracta* ("the number of F = the number of G" can be replaced by the second-order statement saying that there is a bijection of the F's onto the G's); but the principle is silent when it comes to identities between an abstract term and a singular term such as "Julius Caesar". In this case, Fregean Platonism can offer no recipe for the elimination of abstract terms. (Thus, the Caesar problem, far from being a shortcoming of the Platonistic strategy, offers a vigorous defence against attempts at nominalization).

If nominalism is to stand a chance *vis à vis* the naturalistic force of indispensability arguments, an important distinction is needed. Until now, we have grouped together the nominalistic rejection of abstract entities of all sorts: numbers, propositions, classes, etc. But in fact these entities are quite different from one another, and versions of nominalism differ according to which class of entities is targeted. Let us introduce a distinction between two different kinds of nominalism:

- *A-Nominalism* addresses the opposition between abstract and concrete objects.
- *U-Nominalism* addresses the opposition between universals and particulars.

A-Nominalism is only concerned with what kinds of *objects* there are, i.e., whether among the objects populating our ontology there are any that are *abstract* in the sense that they are non-spatio-temporally located and causally inert. One needs to be careful here not to collapse the abstract/concrete distinction onto the theoretical/observable distinction. Obviously objects that are spatio-temporally located need not be directly observable, as long as they are a kind of objects posited by natural science. But perhaps even more importantly, in line with Russell's mature logical atomism (Russell 1918–1919), there is no absolute notion as to what counts as an *object*. By "object" we understand any kind of entity that is *saturated* (in Frege's sense), an entity, that is, of the kind suitable for being the referents of a singular term. Pragmatic choices in language design and progress in conceptual analysis might, over time, change the range of entities that we regard as "saturated" in this sense.

A-Nominalists as characterized here deny that there are abstract objects; since it follows that all objects are spatio-temporally located, all objects are therefore firmly within the purview of natural science. A priori methods apply, of course, but any a priori methods as to the existence and properties of objects have to survive the special kind of push-back that only experience can provide. This stance is fully in line with a naturalistic conception in that it puts science first when it comes to questions regarding the existence and properties of objects.

U-Nominalism is concerned with the distinctions between particulars and universals, between what is traditionally characterized as a *this* and what is *one over many* instead. In Fregean terms, this is the distinction between saturated and unsaturated entities. For our purposes we will adopt a rather austere view of universals: we identify universals with entities in the type hierarchy over some domain D of objects; in particular collections of such entities or maps between such entities. Universals so conceived correspond to unsaturated entities in Fregean terms: predicates, or (more generally) functions. The view is austere because it attends to just one of the various functions of universals, foregoing for instance any consideration of their intensional nature, but it is sufficient for our purpose of developing a naturalistic account of abstract entities.

If we had to choose a label for the kind of account to be proposed, the choice would fall on "Semantic Nominalism", as the doctrine that advocates A-Nominalism but embraces universals under the austere conception. The term is introduced by Hale and Wright (2001, p. 352) for a view that they ultimately (but, we argue, prematurely) reject. Quine (1951) proposes a distinction between *ontology* and what Quine refers to, for lack of a better term, as *ideology*. Ontology is a doctrine as to what kind of *objects* are to be countenanced, whereas ideology deals with the same question within the realm of universals. We want to propose an account that is ontologically nominalistic but ideologically liberal, albeit in an "austere" way.

This is not the place to give a defense of A-Nominalism, as the empiricist motivations for such a position are well known, and a rejection of A-Nominalism in favor of unreflective Platonism would be, as Rosen pointed out, "insane". We will instead motivate a rejection of U-Nominalism by drawing on *semantic* and *logical* considerations. Lest the realists get too excited, though, we anticipate that the argument in the next section will only support a "thin" notion of universals, one which falls way short of any attempt at reification.

2.4 The Semantic Case for Universals

Humans are, to some extent and to varying degrees, conversant with the kind of austere universals identified above. We recognize them, apprehend them, reason about them, and use them to express facts about first-order objects; also, although not nearly as often as philosophers like to think, we occasionally explicitly assert their existence (a distinction between *expressing* and asserting the existence of a given higher-order entity can be found in Antonelli and May 2012). Much of the evidence for this claim is linguistic. Prior (1971) was perhaps the first to point out that ordinary language is replete with instances of quantification over austere universals of higher type; a similar case is also found in Higginbotham (1998, p. 3), who considers the sentence:

He's everything we wanted him to be,

which contains a quantifier over austere universals, viz., second-order properties. In a similar vein, Prior (1971, p. 48) proposes the following examples of *non-nominal quantification* (i.e., quantification over expressions of a syntactic category other than names):

I hurt him somehow;
He's something I am not—kind.

These sentences are obtained by existential generalization over adverbial and noun phrases, respectively (so that "somehow" represents a *third-order* quantifier):

I hurt him by treading on his toes;
He is kind, but I am not.

The claim that humans are conversant with austere universals and other higher-order entities needs to be qualified in at least two ways. The first, obviously, is that our facility with universals need not extend all the way up the type hierarchy: there is no sense in which humans have native proficiency with logics of order ω, and in fact natively such proficiency extends at most to the first few levels of the hierarchy. But even restricting ourselves to the first few levels—and this is the second qualification—we need to recognize that any "access" to such higher-order entities may (and probably must) be limited to some collection that falls short of the full domain of the appropriate type. For instance, when contemplating properties of objects, we implicitly restrict our attention to some salient collection of them, much in the same way in which when using ordinary first-order quantifiers we restrict their range to some implicitly given salient collection of objects.

We claim that such access is real access to austere universals nonetheless, witness the fact that it is not restricted to generic semantic proficiency with the expressions in question, but is in fact specific and detailed enough to provide validation of some characteristic inference patterns. Consider first the familiar case of ordinary inferences involving quantifiers other than the ones from the first-order predicate calculus:

Most of the students went to the lecture;
Most of the students passed the course;
Therefore, some student who went to the lecture passed the course.

A moment's reflection reveals, even to the untrained mind, that the inference is valid. However, a rigorous proof of this fact, while not difficult, is not completely trivial, as it requires some facility with the combinatorics of finite maps (in particular, the proof exceeds the inferential power of first-order logic, as the quantifier "most" is not first-order definable, see Peters and Westerståhl 2006). It is precisely this kind of proficiency with unsaturated entities (injective maps between finite sets) that bears witness to our apprehension of austere universals. Notice that no claim is made here as to the precise nature of this proficiency: it might well be that recognition of the validity of the argument is based on some geometric intuition, or it might be based on some rudimental formal system. Be that as it may, the upshot is that elementary facts about higher-order entities, even of some complexity, underpin our inferential ability.

Similar inferences are also available further up the type hierarchy; the following is adapted from Higginbotham (1998, p. 3):

All we expected him to be was honest, polite, and scholarly;
He is mostly what we expected him to be;
Therefore, he is either honest or polite.

Here again we see some elementary combinatorics involving higher-order entities, and many more examples could be found.

We should mention one more aspect of the dependence of our semantic proficiency upon access to higher-order entities. As argued in Antonelli (2013, 2015), the semantics of first-order quantifiers, including, but not limited to, the ordinary quantifiers "All" and "Some", has a higher-order dimension. When considered from the point of view of the theory of generalized quantifiers (and as adumbrated in Frege's *Grundgesetze*), a first-order quantifier is a third-order predicate, i.e., a predicate of predicates. According to this view, then, the existential quantifier $\exists$ is a predicate applying to all and only the non-empty subsets of the first-order domain D. But the quantifiers need not be interpreted as predicates over the full power set of the first-order domain, and we can allow first-order models with a non-standard second-order domain. If this is the case, then the semantics of first-order quantifiers is not really determinate until and unless a (possibly non-standard) second-order domain is specified alongside the first-order one. The technical details are given in Antonelli (2013), and some of the philosophical repercussions are explored in Antonelli (2015). But for now let us observe that while this is yet more evidence for the role played by higher-order entities, one particular consequence of the framework is that it allows for a clear analysis of issues of ontological commitment. In a recent article Florio and Linnebo (forthcoming) argue that it is only within the context of non-standard, i.e., Henkin models for higher-order logic that the question of ontological commitment can sensibly be asked: the same holds true of non-standard *first-order* models, of course, a fact that is not in the foreground until such models are given due consideration.

The remarks in this section thus have a bearing on the issue of the proper demarcation of logic and mathematics. The idea that at least parts of mathematical knowledge can be traced back to logic is not new, and in fact it figures prominently in several versions of logicism and neo-logicism. But rather than focusing on the logicist tradition originating with Frege, it is important to notice that human proficiency with universals as we have characterized them is front and center already in Dedekind's version of logicism:

> If we scrutinize closely what is done in counting an aggregate or number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing or to represent a thing by a thing, an ability without which no thinking is possible. (Dedekind 1988, p. 14)

Thus the ability to "let a thing correspond to a thing," is singled out by Dedekind as a fundamental function of the human mind, and this is an ability that is grounded in a facility with austere universals. Almost ninety years on, a similar point is made by Feferman:

> …when explaining the general notion of *structure* …we implicitly *presume as understood* the ideas of *operation* and *collection* …at each step we must make use of the unstructured notions of operation and collection to explain the structured notions to be studied. The *logical* and *psychological priority* if not primacy of the notions of operation and collection is thus evident. (Feferman 1977, p. 150)

Both Dedekind and Feferman emphasize the role that our ability to engage entities corresponding to the unsaturated parts of language plays in the development of mathematics, and on the basis of the semantic evidence provided, this seems right. But this does not settle the issue of which side of the boundary between logic and mathematics such ability is to be located: where does logic end and mathematics start? Those who question the very idea of such a boundary, or would like to move the boundary much further out from where it is usually found (as it is for many versions of logicism), are happy to say that at least a large part of mathematics just *is* (or is reducible to) logic.

The conception of austere universals, on the other hand, supports the claims that the ability to "let a thing correspond to a thing" and to collect such things in various ways, is indeed to be properly located within logic. On this view, second- and higher-order logic is, indeed, *logic*. But it is important to recognize at the same time that this conception of higher-order entities is ontologically *thin*: engagement with the universals does not add anything—any *thing*—to our ontology. This is a point forcefully made by the supporters of Plural Logic, a position that Florio and Linnebo (forthcoming) refer to as "Plural Innocence". But on our view there is nothing special discriminating plural quantifiers from *bona fide* higher-order quantifiers: if the former are ontologically innocent, so are the latter. *Pure* second-order logic is indeed an ontologically weak theory: all the second-order validities are true in the standard second-order model whose first-order domain comprises one and only one object (just like in the first-order case). It is this conception of universals as ethereal insubstantial entities (the "Will-o'-the-wisp" theory of universals, as it were) that underpins our logical abilities.

However, the option of *reification* of the universals is always open, and, in many ways, tempting. Universals are reified when they are identified, or made to correspond to, objects in the domain. For instance, this is the framework of modern set theory, in which first-order objects are related by $\in$, so as to mimic the way in which first- or higher-order entities "fall under" universals. There is of course no underestimating how immensely successful this enterprise has turned out to be. But it should also be clear that this kind of reification leads out of the domain of pure logic, and into the mathematical domain.

Not all attempts at reification are successful, however. Frege's Basic Law V is exactly one example: we cannot, on pain of inconsistency, assign objects to what Frege called first-level concepts (i.e., second-order universals) in such a way that concepts under which the same objects fall are assigned the same object. In order to escape the inconsistency one needs either to restrict the range of concepts to which objects are assigned, or to coarsen the equivalence between concepts, as is done, for instance, in Hume's Principle. This second strategy figures prominently in the next section.

2.5 The Abstraction Mystique

Fregean Platonism, which according to Rosen holds the best hope for a realistic account of abstract entities, including those of mathematics, rests on two components: the mathematical core of the view, supplemented by an extra-mathematical thesis concerning the special status of entities introduced by abstraction (in the case of arithmetic, by Hume's Principle).

The mathematical core of the view concerns the basic fact that positing an abstraction principle of the form:

$$\alpha(F) = \alpha(G) \text{ if and only if } \mathscr{R}(F, G),$$

simply asserts the existence of a mapping α from the second-order domain of Fregean concepts back into the first-order domain, in such a way that concepts related by $\mathscr{R}$ are mapped onto the same object. The crucial issue is the *coarseness* of $\mathscr{R}$, i.e., the size of the collection of the equivalence classes of concepts induced by $\mathscr{R}$. The coarser the relation $\mathscr{R}$, the easier it is to satisfy the principle. At one extreme we find the coarsest, i.e., the universal relation on the second-order domain: the induced abstraction operator maps all concepts onto one and the same object, and the corresponding abstraction principle is always satisfiable as long as the first-order domain is non-empty. At the other end of the spectrum is the finest relation, i.e., the identity, whose induced operator assigns to each concept a distinct object from the first-order domain. This is the relation employed in Frege's Basic Law V, and it is unsatisfiable over standard domains (i.e., second-order domains comprising the full power set of the corresponding first-order domain). Somewhere in between we find Hume's Principle, where the relation $\mathscr{R}$ holds between concepts F and G precisely when just as many objects fall under F as they fall under G. This is where interesting mathematics can take place: Hume's Principle allows the recovery of second-order arithmetic, provided that second-order logic is assumed in the background, a result that is justly celebrated as "Frege's Theorem". Moreover, as we assign objects to concepts as their *numbers*, i.e., as representatives of their respective equivalence classes, we obtain as a consequence that numbers themselves can be *counted*. This mathematical fact lies at the heart of the abstractionist approach to arithmetic: since there are $n + 1$ numbers less than or equal to n, Hume's Principle cannot be satisfied over finite domains.

Superimposed to the mathematical core is the *abstraction mystique*, i.e., the idea that abstraction principles are the preferred vehicle for the delivery of a special kind of objects—abstract entities, and in the case of Hume's Principle, *numbers*—which would be otherwise unattainable given their causal inefficacy and non-spatio-temporal location. This is in fact the point raised by Rosen (1993), when he characterizes Fregean Platonism as the best hope for mathematical realists, precisely because abstraction principles appear elegantly to dispose of the tension between the ontological problem and the epistemological problem raised by abstract entities, giving us a picture of such entities that is not "insane". But it is important to see that the mathematical core and the accompanying mystique are in fact independent, and that one can put the former to good use without embracing the latter.

Let us recall once again that mathematically, a second-order abstraction principle is just a functional mapping of second-order entities into the first-order domain respecting a given equivalence relation. In order to obtain the most general characterization of such mappings, non-standard second-order domains have to be allowed, i.e., domains that fall short of the full power-set of the first-order domain. Accordingly, by a *model* we understand a pair (D_1, D_2), where D_1 is any non-empty set and D_2 a collection of subsets of D_1 (the model also provides interpretations of the appropriate type for the extra-logical constants of the language). In practice, several closure conditions of the second-order domain are relevant, for instance:

- $D_2 = \mathscr{P}(D_1)$ (i.e., D_2 is standard);
- D_2 contains all definable subsets of D_1 (in some given background language $\mathscr{L}$).
- D_2 is closed under some class Π of permutations of D_1.

The last closure condition comes down to the following: given a permutation π of D_1 and a subset $X \in D_2$, the point-wise image of X under π, denoted by $\pi[X]$, is also in D_2. This is important, since closure under permutations is one of the features characterizing logical notions. Tarski (1986) indeed proposes this kind of invariance as a necessary and sufficient condition for logicality (a claim that has been variously disputed, see, e.g., Bonnay 2008); but all we need is the much weaker claim that invariance is a necessary condition for some higher-order notion to claim a logical character. Antonelli (2010b) introduces several different notions of invariance that might be taken to be germane for abstraction principles:

- Invariance of the equivalence *relation* $\mathscr{R}$;
- Invariance of the *operator* α;
- Invariance of the abstraction *principle*.

We say that the equivalence relation $\mathscr{R}$ is invariant if and only if for all $X \in D_2$, we have $\mathscr{R}(X, \pi[X])$ (assuming D_2 is closed under permutations). Similarly we say that the abstraction operator α is invariant if and only it commutes with π, i.e., $\pi(\alpha(X)) = \alpha(\pi[X])$. The invariance of $\mathscr{R}$ is indeed a desirable property, but it does not speak to the logical status of α. For abstraction to be correctly characterized as a logical operation, α would have to be invariant. However, the invariance of $\mathscr{R}$ is mostly incompatible with the invariance of α, and to make matters worse, no operator α satisfying Hume's principle is invariant (see Antonelli 2010b, propp. 6 and 7).

But abstraction is a form of reification, so it should not come as a surprise, given our discussion in the previous section, that it turns out to be mathematical rather than logical. However, the challenge remains for empirically-minded philosophers to make sense of abstraction in terms that do not postulate a separate realm of entities (regardless of whether they are logical or mathematical in character). The way out of the difficulty lies in the realization that reification does not by itself commit one to the acceptance of a separate realm. Reification as embodied by abstraction principles consists only in the positing of a correspondence of the appropriate kind between second-order entities and first-order objects. There is no concomitant assumption about the ultimate nature of those objects, no accompanying mystique. The objects targeted by the reification are drawn from the same first-order domain D constituted by the objects of natural science. When those objects are delivered as referents of abstract terms, all we need to know about them is that they function as representatives of equivalence classes of second-order concepts.

The characteristic mathematical role played by abstraction principles resides in laying down constraints on the cardinality of the second-order domain D_2 (which is allowed to be non-standard) *vis-à-vis* the cardinality of the first-order domain. The finer the equivalence relation $\mathscr{R}$ the more upward pressure is exerted on the cardinality of D_1, to the point that where $\mathscr{R}$ is maximally fine (i.e., when $\mathscr{R}$ is identity of extensionally given concepts), the corresponding abstraction principle is not satisfiable save on highly non-standard second-order domains, as Cantor's theorem tells us. But doesn't naturalism enjoin us to take the claims of (mathematical or natural) science at face value? And those claims very clearly regard abstract entities as first-order objects, as when we say that the number of the planets is eight, or that the speed of light equals $2.99792458 * 10^8$ meters per second. On this view, the issue of the special nature of *abstracta* simply does not arise—abstracta are simply ordinary objects recruited for a specific mathematical purpose.

One question that arises on the naturalistic view of abstraction concerns the advantages, if any, of postulating Hume's Principle as opposed to one of the many available versions of the Axiom of Infinity. For if the proper function of Hume's Principle is not to deliver a special class of entities, but only to guarantee that there are enough of them for the proof of Frege's Theorem to go through, why shouldn't we just postulate an appropriate Axiom of Infinity (to the effect that, say, a given binary relation R is serial, irreflexive, and transitive)? But the Axiom of Infinity does not provide for a way to identify, among the countably many objects, those that function as *numbers*, and it does not supply the means to *refer* to them, *quantify* over them, and *predicate* various properties of them (including identity and difference). And it is not even clear how to perform all of these linguistics tasks without accessing resources (set-theoretic or otherwise) that are not germane to arithmetic proper. In contrast, Hume's Principle delivers all of this in one fell swoop: not only does it force the domain to be infinite, by identifying representatives of equivalence classes of concepts, but it also allows us to refer to such representatives, *numbers*, while remaining within a properly arithmetical context.

When it come to matters of ontology, naturalism, properly understood, is the view that ontological commitment should extend to all entities required for the truth of scientific claims—*but no further*. It is this "no further" clause that naturalists tend to forget. In the case where scientific claims make reference to abstract entities, i.e., entities such as numbers that are the referents of abstract terms, all that naturalism requires is that we have a way of appropriately selecting first-order representatives for equivalence classes of higher-order entities. This is possible in the case of claims of (pure or applied) arithmetic as long as we have enough first-order entities to make such an assignment possible. Thus in such cases naturalism can be reconciled with the empiricist worldview. True, the naturalist is still committed by Hume's Principle to the existence of countably many first-order objects; but this is a commitment shared with the Fregean Platonist, who is thus no better off.

Burgess and Rosen (2005) go to great length to lay out the naturalistic claims faced by the empirically or nominalistically inclined philosopher. These claims can be summarized as follows:

(a) Standard mathematics is rich in theorems asserting the existence of mathematical objects.
(b) Existence theorems are accepted not just by *mathematical* standards, but by broader *scientific* standards (no empirical refutation).
(c) Philosophy cannot override mathematical or scientific standards of acceptability (this is the essence of naturalism).
(d) Hence, we are justified in believing, along with mathematicians and scientists, in the existence of the various mathematical objects.

Supporters of Semantic Nominalism can gladly embrace all of this claims, at least in the case of arithmetic, but also in all cases where the relevant mathematical objects can be introduced by means of abstraction principles.

2.6 Arithmetic with the Frege Quantifier

This section provides an outline of a formal theory of arithmetic, AFQ, which employs a cardinality quantifier F (the Frege quantifier) and a term-forming abstraction operator, Num, taking formulas as arguments (both devices are understood to bind a variable). AFQ provides an example of *non-reductive logicism*, i.e., a theory in which the standard of logicism is borne by the cardinality quantifier (expressing a logical notion, based on the arguments of Sect. 2.4). The theory still employs an abstraction principle (a form of Hume's Principle), but such a principle is characterized as extra-logical and mathematical. The details can be found in Antonelli (2010c), and more of the philosophical background in Antonelli (2010a).

Given some standard supply of extra-logical constants, the language of our theory comprises the two variable-binding devices already mentioned: the binary Frege quantifier, allowing for formulas of the form $\mathsf{F}x(\varphi(x), \psi(x))$, and the abstraction operator allowing for terms of the form $\mathsf{Num}\, x\, \varphi(x)$. The intended interpretation is that $\mathsf{F}x(\varphi(x), \psi(x))$ expresses (but does not assert) the fact that there are no more φ's than ψ's, and $\mathsf{Num}\, x\, \varphi(x)$ represents the number of the φ's.

A standard model $\mathfrak{M}$ for the language provides, just like for ordinary first-order logic, a non-empty domain D_1 of objects and interpretations for the extra-logical constants (predicate and function symbols) as well a function η from the power-set of D_1 into D_1 (η interprets the extra-logical abstraction operator). Satisfaction of a formula φ in $\mathfrak{M}$ by an assignment s to the variables is recursively defined as usual, with the two clauses:

- $\mathfrak{M} \models \mathsf{F}x(\varphi(x), \psi(x))[s]$ if and only the cardinality of $[\![\varphi]\!]^x_s$ is less than or equal to that of $[\![\psi]\!]^x_s$;
- $[\![\mathsf{Num}\, x\, \varphi(x)]\!]_s$ is defined to be $\eta([\![\varphi]\!]^x_s)$,

where $[\![\varphi]\!]^x_s$ is the extension of $\varphi(x)$ relative to s. Notice that several notions are expressible in the language, beginning with the ordinary quantifiers $\exists$ and $\forall$: $\exists x\varphi(x)$ can be represented by denying that the cardinality of (the extension of) φ is less than or equal to that of the empty set; and dually $\forall x\varphi(x)$ can be represented by the assertion that the cardinality of $\neg\varphi$ is less than or equal to that of the empty set. The expressive power of this language exceeds that of ordinary first-order logic, since we can say, for instance, that the extension of φ is Dedekind-finite by denying that there is a y such that the cardinality of φ is less than or equal to the cardinality of $\varphi(x) \wedge x \neq y$. It is convenient to introduce the abbreviations $\mathsf{Fin}\, x\, \varphi(x)$ for the latter statement, and $\mathsf{I}\, x\, (\varphi(x), \psi(x))$ for:

$$\mathsf{F}x\, (\varphi(x), \psi(x)) \wedge \mathsf{F}x\, (\psi(x), \varphi(x)),$$

which (by the Schröder-Bernstein Theorem) expresses that φ and ψ are equinumerous, i.e., they have the same cardinality. The symbol I is ordinarily used to represent Härtig's quantifier.

We now come to the extra-logical axioms of AFQ. The extra-logical constants comprise a 2-place relation $<$ and a 1-place predicate $\mathbb{N}$. A first group of axioms are definitional, they give us uniqueness conditions for various notions. Unsurprisingly, the first one is a version of Hume's Principle:

$$\mathsf{Num}\, x\, \varphi(x) = \mathsf{Num}\, x\, \psi(x) \leftrightarrow \mathsf{I}x\, (\varphi(x), \psi(x)). \qquad \textbf{Ax1}$$

The second axiom gives us a definition of $\leq$ ($x \leq y$ abbreviates $x < y \vee x = y$):

$$\mathsf{Num}\, x\, \varphi(x) \leq \mathsf{Num}\, x\, \psi(x) \leftrightarrow \mathsf{F}x\, (\varphi(x), \psi(x)). \qquad \textbf{Ax2}$$

The next axiom gives the definition of successor, modeled on Frege's original definition. The number of the ψ's is the successor of the number of the φ's if and only if there is a ψ such that there are just as many φ's as there are ψ's other than it:

$$\mathsf{Succ}\, x\, (\varphi(x), \psi(x)) \leftrightarrow \exists x(\psi(x) \wedge \mathsf{I}y(\varphi(y), \psi(y) \wedge y \neq x)). \qquad \textbf{Ax3}$$

The next two axioms are substantial, as opposed to purely definitional. The first one gives an implicit definition of the collection of natural numbers:

$$\forall y(\mathbb{N}(y) \leftrightarrow \mathsf{Fin}\, x(\mathbb{N}(x) \wedge x < y) \wedge y = \mathsf{Num}\, x\, (\mathbb{N}(x) \wedge x < y)). \qquad \textbf{Ax4}$$

The last axiom is a form of comprehension: for formulas $\theta(x, y)$, $\varphi(x)$ and $\psi(y)$, let $\theta : \varphi \xrightarrow{1-1} \psi$ say that θ defines an injection from the φ's into the ψ's:

$$\begin{aligned}\forall x\big[\varphi(x) \rightarrow \exists y\big(\psi(y) \wedge \theta(x, y) \wedge \\ \forall z(\psi(z) \wedge \theta(x, z) \rightarrow z = y) \wedge \\ \forall z(\varphi(z) \wedge \theta(z, y) \rightarrow z = x)\big)\big].\end{aligned}$$

Our last axiom then says that if there is a definable injection of the φ's into the ψ's then the cardinality of the former is less than or equal to that of the latter:

$$[\theta : \varphi \xrightarrow{1-1} \psi] \rightarrow \mathsf{F}x\,(\varphi(x), \psi(x)). \qquad \textbf{Ax5}$$

Ax5 is, of course, valid over standard models, so it is not needed as long we restrict our attention to such models. But Antonelli (2010c) also introduces a notion of *general model* for the language with the Frege quantifier, and then the role of this axiom in the proof of the Principle of Induction becomes apparent (see Antonelli and May 2012 for details).

The five axiom schemas of AFQ are enough to interpret first-order Peano Arithmetic (and perhaps more). In particular, one can prove the analogues of the Peano-Dedekind axioms:

(i) $\mathbb{N}(0)$: where 0 abbreviates $\mathsf{Num}\,x\,(x \neq x)$; i.e., 0 is a number.
(ii) For numbers p and q let $\mathsf{Succ}(p, q)$ abbreviate $\mathsf{Succ}(\mathbb{N}(x) \wedge x < p, \mathbb{N}(x) \wedge x < q)$; then every number has a unique successor.
(iii) Every number other than 0 is a successor (importantly this is provable without induction).
(iv) Succ is an injective function.
(v) $\big[\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))\big] \rightarrow \forall n \varphi(n)$ (requires **Ax5** over general models).

In particular, since Succ is an injection of $\mathbb{N}$ into itself, the theory proves that $\mathbb{N}$ is not Dedekind-finite.

2.7 A Naturalistic View of Abstraction

According to *Naturalism*, as we understand it here, the purview of natural science encompasses all there is—the furniture of the universe from the micro level to cosmological scale. Accordingly, there is nothing, no *thing*, that falls outside its purview. As a corollary, the philosopher has to accept the existence of any entities that are required for the proper functioning of science; this is the thrust of indispensability arguments. But an equally important, if somewhat less emphasized, corollary is that our ontological commitments should *not* extend beyond what is so required.

Our austere universals do not contravene to this second corollary, for they are not—*ex hypothesi*—first-order objects on a par with all the other first-order objects comprising the domain of natural science. Rather, they are ways to collect such objects according to this or that principle, or (in Dedekind's words) to "represent a thing by a thing". Various forms of reification of our universals do contravene the second corollary, for instance when issues are raised about their location or causal efficacy. But at least in some cases, i.e., those in which reification takes place through the positing of abstraction principles (arithmetic being the most prominent example), the resulting framework is compatible with the naturalistic outlook. Abstraction does take us outside of the domain of pure logic and deep into mathematical territory, but it does not do so by positing entities that fall outside the purview of science. This is not to imply, of course, that (say) numbers *as such* are spatio-temporally located or causally efficacious. On the naturalistic view of abstraction, numbers are whatever entities are selected as representatives of equivalence classes of concepts, and as we know the selection can be carried out in infinitely many ways, as long as we assume the existence of an infinite number of objects (there can be no arithmetic, failing this assumption). *Abstracta* are delivered as referents of abstraction terms, but no further claims are made as to their special nature. In this sense, if the eliminative nominalism of Field or Hellman is analogous with Weierstraß' elimination of infinitesimals, naturalistic abstraction is more in line with Hilbert's introduction of *ideal elements*.

The view combining naturalistic abstraction with semantic nominalism represents a natural equilibrium point between the competing demands of realism and anti-realism: naturalistic abstraction shares the advantages of Fregean Platonism, but without the accompanying mystique. In particular it meets the following requirements, which are commonly presented as the features that make a realistic outlook on abstract entities desirable:

(a) The left-hand sides of abstraction principles is taken at face value; there is no need to re-interpret identities between abstract terms as other than statements of objectual identity.
(b) The denotations of terms so introduced are available for quantification, identification (and differentiation), predication, and reference. In particular, numbers themselves can be counted, which is at the heart of the abstractionist account of arithmetic.
(c) No *additional* commitment to abstract entities is necessitated by the positing of abstraction principles *as such*—ontology is not to be settled through semantics, and mathematics is not logic—as long as the concomitant cardinality requirements are met.

This view, or something very similar, can be found at various junctions in the literature, only to be dismissed (with great effort) by both nominalists and realists. For instance, a similar position is taken up for consideration by Rosen, who points out that the proposed interpretation works for first-order abstraction:

> Whenever the Fregean introduces new singular terms by abstraction on an equivalence relation among ordinary particulars, it will be possible to interpret the new terms as referring to these ordinary things. (Rosen 1993, p. 174)

But the strategy fails at the higher-order, essentially because of the non-conservative nature of second-order abstraction, the primary example being Frege's doomed Basic Law V. We know of course that the failure of Basic Law V does not extend to Hume's Principle (provided we are ready to admit the existence of infinitely many objects), and even Basic Law V is satisfiable over non-standard domains.

For another example of how the strategy naturally arises, only to be dismissed, here is Dummett's assessment of the case of first-order abstraction:

> The stipulation that the direction of line *a* is to be the same as that of a line *b* just in case *a* is parallel to *b does not determine whether the direction of a line is itself a line or something quite different*: this contextual definition indeed has a solution, but it is far from unique. Even if the requirement were to be made that every direction should itself be a line, the stipulation would in no way determine which line any given direction was to be; it could, in fact, be any line whatever [...] The contextual definition might well be defended on the ground that we do not need to know anything about directions save what it tells us: as long as we know that the direction of *a* is the same as that of *b* just in case *a* is parallel to *b*, *we are quite indifferent* to what, specifically, the direction of *a* may be, or any other facts about it. Frege makes it plain in §66 [of *Grundlagen*] that this defence would not satisfy him at all ...(Dummett 1991, p. 126, emphases added)

The view entertained by Dummett in the above passage is very similar to the naturalistic conception of abstract entities. Once we have the notion of parallelism between two lines, we can pick any representatives of the equivalence classes—or indeed any distinct objects whatsoever—to play the role of directions. This is completely unproblematic in the case of first-order abstraction, for first-order abstraction principles are always satisfiable just in the way indicated by Dummett, and in fact they are conservative over the underlying theory (since any model of the underlying theory can be expanded to a model of the principles). The reason Frege would not embrace the definition, Dummett continues, is that:

> It is an inexcusable defect in a proposed definition of the direction-operator that it fails to tell us what, specifically, the direction of a given line is to be; and hence it must be replaced by an explicit definition that does tell us that. (Dummett 1991, *ibid.*)

Needless to say, the mathematics is unaffected by this "inexcusable defect," and neither are we. Things are somewhat different in the case of second-order abstraction (which is not conservative over the background theory), but the intuition has to remain the same. Abstraction principles provide representatives for equivalence classes of second-order entities; such a choice of representatives is possible when, and only when, there are no more equivalence classes in the second-order domain (which might be non-standard) than there are first-order objects. Such representatives will be available provided the first- and second-order domains are in the equilibrium dictated by the abstraction principles. But otherwise the choice of representatives is unconstrained. The proof of Frege's Theorem goes through unaffected on this construal of Hume's Principle, since *formally* there is nothing in the proof that depends on an account of the "true nature" of numbers.

We are now well positioned to solve the ontological problem, by answering the question, *What are abstract entities?*: They are the referents of abstraction terms. Such a referent is to an extent indeterminate, but we can still work with such terms, quantify over their referents, predicate identity or non-identity, etc. What about the epistemological question, *How do we know about abstract entities?* Our knowledge of them is limited, but still substantial. In particular, we know whatever has to be true no matter how the representatives are chosen, i.e., what is true in all models of the corresponding abstraction principles. We won't know anything about the special nature of the representatives—their spatio-temporal location, for instance. But we will know whatever follows from the positing of such representatives.

Before we proceed to the conclusion, we consider a particular objection to semantic nominalism developed by Hale and Wright, which deals with issue of *trans-world abstraction* (see Hale and Wright 2001, pp. 352ff). The argument then runs as follows. On the assumption that particular lines are chosen as directions:

(a) The direction of a line exists in all worlds where the line exists (abstract terms are referential).
(b) Assuming a domain of contingents, there is a world w where line a exists orientationally unchanged at the same time as all other lines (parallel to a in the real world) change their direction.
(c) Therefore at w, and hence at all worlds, $\mathsf{dir}(a) = a$ (a is the only possible representative of its equivalence class).
(d) Thus in the real world, given that abstract terms are rigid, $\mathsf{dir}(a) = a \neq b = \mathsf{dir}(b)$ even when $a \parallel b$.

Observe that the argument requires that the "representatives" be selected from their respective equivalence classes, while this is not in general a requirement of semantic nominalism; and indeed in the case of second-order abstraction the representatives are of a lower type altogether than the entities whose equivalence classes they represent. Similarly one might insist that (were we actually to embrace the framework of trans-world abstraction) representatives need not exist at the *same world* as the lines whose classes they represent (the direction of line a at world w might be identified with an object b—perhaps itself a line, perhaps not—at world w').

Thus, we are left with a viable construal of logicism, given by the combination of semantic nominalism and a naturalistic conception of abstraction. It is a form of logicism because cardinality notions, as embodied for instance in the Frege quantifier characteristic of the theory AFQ can be reasonably construed as logical notions. However, this kind of logicism is *non-reductive* in character: no reduction to Hume's Principle is needed in order to establish the logical nature of the notion of cardinality. On the contrary, Hume's Principle is regarded as properly mathematical, and not logical. At the same time we avoid the temptation proffered by the abstraction mystique, and we show how, at least in the case of arithmetic as reconstructed in AFQ (but hopefully more in general in the case of any theory whose abstract entities are delivered by an abstraction principle), one can develop an empirically acceptable theory of mathematical entities. Unsurprisingly, the mathematical development is independent of the philosophical construal of the abstraction principles and the ultimate nature of the objects so introduced.

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