

Chapter 2

Linear Massless/Massive Gauge Theories

In this chapter we propose to study the massive and massless theories of spin-1 and spin-2 fields through several approaches, each one of them providing a complementary viewpoint. As already mentioned in the introduction, the notions of degree of freedom and of dynamical field are not equivalent in non-local field theory. It is therefore important to first understand their equivalence in local field theory, and especially gauge theory, where not all fields propagate. We will thus see, in many different ways, how the field content splits into dynamical and non-dynamical fields and how this is related to the degrees of freedom of the theory. This will then allow us to understand the spectrum of non-local gauge theories, without making any confusion between the constraints that are due to non-locality and the ones that are due to gauge symmetry. Finally, this analysis will also bring us useful by-products that will allow us to construct linear non-local massive spin-2 gauge theories.

Although our main interest is in gravity and thus the spin-2 field, the spin-1 case will be very helpful in facilitating our intuition and argumentation. Indeed, it shares many properties with the spin-2 case, but at the same time has less fields, thus simplifying our analysis. On top of this, the spin-1 theory stands as exceptional, regarding some important properties, when compared with higher spin theories $s \geq 2$. Thus, the study of the spin-1 case will turn out to be essential in contrasting with some peculiarities of the spin-2 case.

For the kinetic term of the theory, in each case, we will consider the only one that is stable, i.e. the one that exhibits the highest gauge symmetry. These are the kinetic terms of electrodynamics and of linearized GR. For the mass terms however we will consider the most general quadratic Lorentz-invariant potential, which in the case of the spin-2 field usually activates a ghost mode. Indeed, that ghost will be a recurrent subject in this thesis, so it is important that we include these actions as well in our study. Moreover, considering this general case will lead us to the definition of projectors that are going to be very useful for constructing a genuinely non-local ghost-free theory. This chapter is based on, and extends, the following papers [1, 2].

2.1 Technical Preliminaries

2.1.1 Inverse Differential Operators

In this chapter we will consider only spatially localized fields, that is, fields which tend to zero sufficiently fast at infinity and which can therefore be represented by their spatial Fourier transform. On this space of fields the operator $\Delta - m^2$, where $\Delta \equiv \partial_i \partial_i$ is the Laplacian, is negative-definite, as is obvious in its Fourier representation. It has therefore zero kernel when acting on fields whose values and first spatial derivatives tend to zero at spatial infinity. This means that it admits a unique (right and left) inverse $(\Delta - m^2)^{-1}$, and actually a unique power $(\Delta - m^2)^\kappa$ for $\kappa \in \mathbb{R}$, which can again be obtained through its Fourier representation. These operators commute among themselves and with spatial derivatives.

These nice properties do not generalize to the Klein–Gordon operator $L \equiv \square - m^2$ because it has a non-trivial kernel, the vector space generated by the plane-wave solutions (see Appendix A.2.2 for detailed properties). It therefore admits more than one right-inverse $LL^{-1} = \text{id}$ and no left-inverse in general. The space of inverses is parametrized by the elements of the kernel since any two inversions are related by a homogeneous solution

$$L[L^{-1}(\phi) - L'^{-1}(\phi)] = 0. \quad (2.1.1)$$

Thus, if one picks a L^{-1} once and for all, all other inversions are found by adding a homogeneous solution, as we know from calculus. Here we will denote by “ L^{-1} ” the inverses of L that are also $\mathbb{R}$ -linear operators

$$L^{-1}(\alpha\phi + \beta\phi') = \alpha L^{-1}\phi + \beta L^{-1}\phi' \quad \alpha, \beta = \text{const} \in \mathbb{R}, \quad (2.1.2)$$

which must be contrasted with the general inverse operator which is affine

$$L_{\text{gen.}}^{-1}(\phi) = L^{-1}\phi + \psi, \quad L\psi = 0, \quad (2.1.3)$$

with ψ independent of ϕ . The operators L^{-1} can then be represented by the convolution with a Green’s distribution

$$(L^{-1}\phi)(x) = \int d^D y G(x, y) \phi(y), \quad L_x G(x, y) = \delta^{(D)}(x - y), \quad (2.1.4)$$

which by Poincaré covariance must be of the form $G(x, y) = G(x - y)$. The quantity iG is also called a “propagator” depending on the context. The different choices of L^{-1} now correspond to the different time boundary conditions of $G(x)$, which in turn correspond to the time boundary conditions of $(L^{-1}\phi)(x)$.¹

¹Given the set of fields we consider, the spatial boundary conditions are zero at infinity.

Two Green's functions are of particular relevance for physics on flat space-time, the retarded one in classical field theory and the Feynman one in perturbative QFT. Imposing trivial initial conditions

$$\lim_{x^0 \rightarrow -\infty} G(x) = 0, \quad \lim_{x^0 \rightarrow -\infty} \partial_{x^0} G(x) = 0, \quad (2.1.5)$$

gives the retarded propagator

$$G_r(x) = \lim_{\epsilon \rightarrow 0^+} \int \frac{d^D k}{(2\pi)^D} \frac{\exp(i\eta_{\mu\nu} k^\mu x^\nu)}{(k^0 + i\epsilon)^2 - \vec{k}^2 - m^2}, \quad (2.1.6)$$

while imposing no positive-frequency ingoing waves and no negative-frequency outgoing waves

$$\lim_{x^0 \rightarrow -\infty} G(x) = \int \frac{d^d k}{(2\pi)^d} \int_{-\infty}^0 \frac{dk^0}{2\pi} a(k) \exp[i\eta_{\mu\nu} k^\mu x^\nu], \quad (2.1.7)$$

$$\lim_{x^0 \rightarrow +\infty} G(x) = \int \frac{d^d k}{(2\pi)^d} \int_0^\infty \frac{dk^0}{2\pi} a(k) \exp[i\eta_{\mu\nu} k^\mu x^\nu], \quad (2.1.8)$$

gives the Feynman propagator

$$G_F(x) = \lim_{\epsilon \rightarrow 0^+} \int \frac{d^D k}{(2\pi)^D} \frac{\exp(i\eta_{\mu\nu} k^\mu x^\nu)}{-k^2 - m^2 + i\epsilon}. \quad (2.1.9)$$

Indeed, by writing (2.1.4) in Fourier space, and using the converging contour integrals with the residue theorem, we get that $L^{-1}\phi$ obeys the above mentioned boundary/initial conditions in each respective case. The domains of definition of the corresponding operators L_r^{-1} and L_F^{-1} are the fields obeying the same boundary conditions as G in each respective case. On their respective domains of definition, both operators commute with partial derivatives and are also left-inverses.² In practice the L_r^{-1} may act after some derivatives, in which case it is convenient to have a stronger condition for its applicability. At the bottom of Appendix A.2.2 we provide such a condition which we call “having finite past”. Loosely speaking, it amounts to ϕ being non-zero only after a finite time.

The retarded Green's function arises in situations where one wants to solve a sourced equation

$$L\phi = J, \quad \phi = \int d^D y G_r(x - y) J(y), \quad (2.1.10)$$

²See Appendices A.3.2 and A.3.3 where we show this for $\square_r^{-1}$ in real space and on arbitrary globally hyperbolic space-times. It can also be worked-out in Fourier space for both $\square_r^{-1}$ and $\square_F^{-1}$, since if the Fourier representation gives a finite result, i.e. if the operators are defined, then it is obvious that they commute with the derivatives and are also left-inverses.

in a causal way, i.e. such that $\phi(x)$ depends only on $J(x')$ with x' in the past light-cone of x . This is indeed the case as we can see by the real space representation in $D = 4$ given in Eq. (A.2.22) of Appendix A.2.2. Flipping the sign of ϵ in (2.1.6) amounts to flipping the sign of x^0 , after having redefined $k^0 \rightarrow -k^0$, so this gives us the advanced propagator G_a which is supported on the future light-cone and is thus anti-causal. We thus have

$$G_r(-x^0, \vec{x}) = G_a(x^0, \vec{x}), \quad (2.1.11)$$

while $G_r(x^0, \vec{x})$ is symmetric under the individual sign flip of spatial arguments. In perturbative QFT it is rather the Feynman propagator which is relevant because it is the one that arises in the computation of the scattering amplitudes. More precisely, it represents the particles of ϕ which mediate the interaction between sources J at different space-time points. To see this one can invoke the corresponding action

$$S = \lim_{\epsilon \rightarrow 0^+} \int d^D x \left[\frac{1}{2} \phi (\square - m^2 + i\epsilon) \phi - \phi J \right], \quad (2.1.12)$$

which has been regularized with an ϵ factor that ensures the convergence of the corresponding path integral. Thus, unitarity of e^{iS} forces upon us this choice for the sign of ϵ . We then have that by integrating-out ϕ

$$\int_{\mathcal{B}} D\phi e^{iS} \sim \exp \left[\frac{i}{2} \int d^D x J G_F J \right]. \quad (2.1.13)$$

Differentiating twice (2.1.12) with respect to the source one gets that the Feynman propagator is the two-point function

$$\langle 0 | \hat{\phi}^\dagger(k) \hat{\phi}(k) | 0 \rangle = -\frac{i}{k^2 + m^2 - i\epsilon}. \quad (2.1.14)$$

Actually, this path integration has been performed a bit formally since we have not specified its boundary conditions $\mathcal{B}$. However, these are already fixed for consistency reasons and there are several instructive ways to see this that will be useful for us at some point later on. First, note that the path integral is dominated by the classical solutions, which in this case are given by free wave-packets at infinity (where $J = 0$) with dispersion relation

$$k^0 = \pm (\sqrt{m^2 + \vec{k}^2} - i\epsilon). \quad (2.1.15)$$

Thus, positive-frequency modes diverge at past infinity, while negative-frequency modes diverge at future infinity. This means that the only boundary conditions for which the path integral makes sense around classical solutions are the Feynman ones (2.1.8), i.e. only negative-frequency waves at past infinity and only positive-frequency waves at future infinity. Conversely, if one imposes these boundary conditions but sets $\epsilon = 0$, then the result of integrating-out ϕ is the Feynman propagator.

One can also understand these boundary conditions from the point of view of the canonical quantization. One simply needs

$$\langle 0|T \dots |0\rangle \sim \int D\phi \dots e^{iS[\phi]}, \quad (2.1.16)$$

where $|0\rangle$ is the vacuum state at past infinity and $\langle 0|$ is the one at future infinity. We then have that $a|0\rangle = 0$, where a is the free annihilation operator corresponding to the amplitude of the modes with positive frequency, while $\langle 0|a^\dagger = 0$, where $a^\dagger$ is the creation operator corresponding to the amplitude of the modes with negative frequency.

Finally, note that since $G_F(k)$ is a function of k^2 , we have that $G_F(x)$ is symmetric under the individual flip of any of its arguments, so it is symmetric under time-reversal in particular. As a consequence it has both retarded $\sim \theta(x^0 - y^0)$ and advanced $\sim \theta(y^0 - x^0)$ parts. This is expected because in a scattering process the information of the whole interval $t \in]-\infty, \infty[$ is required, so that for finite t the dependence is acausal.

2.1.2 Degrees of Freedom, Dynamical and Non-dynamical Fields

In non-local theories the question of degrees of freedom of a theory can be a subtle issue, so it is important that we define clearly the words we will be using. The number of degrees of freedom of a field theory, denoted by N_f , is the number of initial field configurations that we are free to choose in order to evolve the system uniquely in time. In the theories we are going to study below we will find two types of fields. The “dynamical” (or “radiative”) ones are those obeying a second-order equation in time

$$(\square - m^2)\phi = J, \quad (2.1.17)$$

while the “non-dynamical” (or “non-radiative”) ones are those that obey a purely spatial differential equation

$$(\Delta - m^2)\phi = J. \quad (2.1.18)$$

In the dynamical case (2.1.17) the solution for a ϕ which is solely excited by J takes the form (2.1.10). This means that, by measuring ϕ at some x , one can deduce some information about the excitations of J at some other x' (as long as x is in the future light-cone of x'). We thus say that the field “propagates” the information of the source. This is how one can gain information about a distant object, by detecting the waves it emits in some *dynamical* field. Going even further, this is how two “sources” at different space-time points are going to interact through the “force” mediated by ϕ . Note that this scenario does not focus on the initial conditions that would have been given to ϕ . These are actually trivial since ϕ is solely excited by the source.

Thus, the forces that are present in the theory correspond to the dynamical fields, independently of whether these are degrees of freedom or not. Finally, since the dynamical fields induce poles in the propagator, and “propagate” the information of sources, one can equivalently refer to them as “propagating” fields.

In the non-dynamical case (2.1.18) the equation seems to be in conflict with relativity since it is not Lorentz invariant and implies an action at a distance, i.e. ϕ reacts instantaneously to the source J . As we will see however, in these cases, either ϕ will not be physically observable (gauge-dependent), or it will itself be a spatially non-local functional of the fundamental fields. In the latter case the measurement of ϕ is spatially non-local to begin with and can thus not be performed at a single time, so there is no contradiction with relativity. In that case, the information of the source does not propagate but is instead communicated simultaneously, to an unphysical or non-local field. Thus, non-dynamical fields do not allow us to gain local information on the source’s dynamics nor do they mediate any interaction.

Now, in the dynamical case, we have that one needs to provide the initial conditions $\phi(t_i, \vec{x})$ and $\dot{\phi}(t_i, \vec{x})$ on $\mathbb{R}^d$ in order to evolve the field in time, so that it corresponds to $N_f = 2$. In the non-dynamical case we have that the field is totally determined by the source at every time and, in particular, at the initial condition surface, so that $N_f = 0$. In the dynamical case the solutions for $J = 0$ are linear superpositions of plane-waves, whose vector space is isomorphic to the initial data space, while in the non-dynamical case the source-free solution is $\phi = 0$.

It therefore seems obvious that, if one denotes the number of dynamical fields by N_d , then $N_f = 2N_d$.³ This appears as a trivial statement in local field theory, but does not hold at all for non-local theories. It is thus important to stress in advance that the notion of dynamical field and degree of freedom should be considered separately.

2.2 Standard Lagrangian Approach

2.2.1 Spin 1

Massive

So let us start by considering the case of massive electrodynamics, that is, the Proca action

$$S \equiv \int d^D x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu + A_\mu j^\mu \right], \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (2.2.1)$$

³If the dynamical equations were of order n in the time-derivatives, this would give $N_f = nN_d$.

where j^μ is a conserved external source, i.e. $\partial_\mu j^\mu = 0$, and the mass parameter m breaks the U(1) gauge symmetry⁴

$$\delta A_\mu = -\partial_\mu \theta. \quad (2.2.2)$$

The equations of motion are

$$\partial_\mu F^{\mu\nu} - m^2 A^\nu = -j^\nu, \quad (2.2.3)$$

and taking the divergence one gets

$$m^2 \partial_\mu A^\mu = 0, \quad (2.2.4)$$

so we can rewrite them as

$$(\square - m^2) A_\mu = -j_\mu, \quad \partial_\mu A^\mu = 0. \quad (2.2.5)$$

Thus, as soon as $m \neq 0$, and therefore the gauge symmetry is lost, the usual Lorentz gauge condition of massless electrodynamics $\partial_\mu A^\mu = 0$ appears as the scalar part of the equations of motion. The latter along with the $\mu = 0$ components of the Klein–Gordon equation imply that A_0 is non-dynamical

$$(\Delta - m^2) A_0 = \partial_i \dot{A}_i - j_0, \quad \dot{A}_0 = \partial_i A_i, \quad (2.2.6)$$

and that its initial conditions are totally determined in terms of the ones of A_i and j_0 . We are then left with

$$(\square - m^2) A_i = -j_i, \quad (2.2.7)$$

that is, d unconstrained fields transforming in the vector representation of $\text{SO}(d)$ and obeying a massive Klein–Gordon equation. This amounts to $N_f = 2N_d = 2d$ degrees of freedom, corresponding to the initial conditions of A_i and $\dot{A}_i$. In $d = 3$ this gives $N_d = 3$.

Massless

In the case where $m = 0$, we have the gauge symmetry (2.2.2), so the Lorentz gauge $\partial_\mu A^\mu = 0$ can be reached by performing a gauge transformation, the result being

⁴In realistic cases where j^μ is also made of fundamental fields, the argument that the $m = 0$ action is gauge-invariant because $\partial_\mu j^\mu = 0$ no longer holds. Indeed, conservation equations can only hold for some field configurations, namely the on-shell ones, whereas a symmetry should hold for *all* field configurations in the action. There are then two possibilities. Either the $A_\mu j^\mu$ term corresponds to non-minimal couplings to other fields through $F_{\mu\nu}$, in which case it is itself gauge-invariant, or it emerges through minimal couplings that involve the covariant derivative $\nabla \equiv \partial - iA$, in which case its variation is compensated by a non-trivial variation of A -independent terms. Then, because of that gauge symmetry, by Noether's theorem for local symmetries we have that $\partial_\mu j^\mu = 0$ on-shell. In the massive case, if the matter sector is unchanged, then we still have a global U(1) symmetry and it is thus Noether's theorem for global symmetries which implies $\partial_\mu j^\mu = 0$.

again (2.2.6) and (2.2.7), but with $m = 0$. Now however these equations have a residual gauge symmetry given by the gauge parameters satisfying $\square\theta = 0$. To see what we can do with it, we can consider the general solution of the divergence of (2.2.7)

$$\partial_i A_i = \phi^{\text{hom}} - \square_r^{-1} \partial_i j_i, \quad (2.2.8)$$

where ϕ^{hom} is a homogeneous solution $\square\phi^{\text{hom}} = 0$. Remember that for the action of $\square_r^{-1}$ to be defined the source $\partial_i j_i$ must have finite past. Using the residual gauge transformation on that equation we get

$$\partial_i A_i - \Delta\theta = \phi^{\text{hom}} - \square_r^{-1} \partial_i j_i. \quad (2.2.9)$$

It is thus possible to cancel ϕ^{hom} by choosing

$$\theta = -\Delta^{-1} \phi^{\text{hom}}, \quad (2.2.10)$$

so that $\partial_i A_i$ is totally determined by the source and its initial conditions are thus fixed. The degrees of freedom are therefore the $N_f = 2(d - 1)$ components of the transverse part A_i^t , i.e. $\partial_i A_i^t = 0$, and its first derivatives.

It may appear however that the longitudinal part $\partial_i A_i$ is still a dynamical field, since it obeys a dynamical equation $\square\partial_i A_i = \partial_i j_i$, even though it does not correspond to a degree of freedom. This would be in contradiction with $N_f = 2N_d$. As it turns out, this is only an artefact of our choice of gauge, which is the natural one from the point of view of the massive theory, since then $\partial_\mu A^\mu = 0$ holds continuously with $m \rightarrow 0$. Indeed, one can always introduce a $\square_r^{-1} j$ term in the gauge parameter to make it appear as a source of a gauge-dependent component. We can therefore choose a different gauge to start with, such as the one which precisely eliminates the longitudinal mode

$$\partial_i A_i = 0. \quad (2.2.11)$$

This choice is more natural from the Hamiltonian point of view, as we will see soon. The equation of motion of A_0 then reads

$$\Delta A_0 = -j_0, \quad (2.2.12)$$

and we have that the divergence of the equation of A_i is automatically satisfied. We thus have that the initial conditions of both A_0 and $\partial_i A_i$ are fixed *and* that these fields are non-dynamical. We can therefore conclude that in the massless theory we have indeed $N_f = 2N_d = 2(d - 1)$, which for $d = 3$ gives $N_d = 2$.

2.2.2 Spin 2

Let us now consider linearized GR along with the most general quadratic potential

$$\begin{aligned}
S &= \int d^D x \left[-\frac{1}{2} \partial_\mu h_{\nu\rho} \partial^\mu h^{\nu\rho} + \partial_\mu h^{\mu\rho} \partial^\nu h_{\nu\rho} - \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{1}{2} \partial_\mu h \partial^\mu h \right. \\
&\quad \left. - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - (1 + \alpha) h^2) + h_{\mu\nu} T^{\mu\nu} \right] \\
&\equiv \int d^D x \left[\frac{1}{2} h_{\mu\nu} \mathcal{E}^{\mu\nu\rho\sigma} h_{\rho\sigma} - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - (1 + \alpha) h^2) + h_{\mu\nu} T^{\mu\nu} \right],
\end{aligned} \tag{2.2.13}$$

where $\mathcal{E}$ is known as the ‘‘Lichnerowicz operator’’

$$\mathcal{E}^{\mu\nu\rho\sigma} \equiv (\eta^{\mu(\rho} \eta^{\sigma)\nu} - \eta^{\mu\nu} \eta^{\rho\sigma}) \square - \eta^{\mu(\rho} \partial^{\sigma)} \partial^\nu - \eta^{\nu(\rho} \partial^{\sigma)} \partial^\mu + \eta^{\mu\nu} \partial^\rho \partial^\sigma + \eta^{\rho\sigma} \partial^\mu \partial^\nu. \tag{2.2.14}$$

The Fierz–Pauli theory corresponds to the choice $\alpha = 0$. Here $T^{\mu\nu}$ is some external conserved source $\partial_\mu T^{\mu\nu} = 0$ and the mass term breaks the following linear gauge symmetry

$$\delta h_{\mu\nu} = -\partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \tag{2.2.15}$$

The equations of motion are

$$(\square - m^2) h_{\mu\nu} - \eta_{\mu\nu} (\square - (1 + \alpha) m^2) h - \partial_\mu \partial^\rho h_{\rho\nu} - \partial_\nu \partial^\rho h_{\rho\mu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma h_{\rho\sigma} + \partial_\mu \partial_\nu h = -T_{\mu\nu}, \tag{2.2.16}$$

their divergence is

$$m^2 (\partial^\mu h_{\mu\nu} - (1 + \alpha) \partial_\nu h) = 0, \tag{2.2.17}$$

and their trace is

$$(D - 2) (\partial_\mu \partial_\nu h^{\mu\nu} - \square h) + ((1 + \alpha) D - 1) m^2 h = -T. \tag{2.2.18}$$

Taking the double divergence, we can simplify the trace equation to

$$(D - 2) \alpha \square h + ((1 + \alpha) D - 1) m^2 h = -T, \tag{2.2.19}$$

which is a dynamical equation for h only when $\alpha \neq 0$. In the following sections we will see that in this case the kinetic term of h has the wrong sign with respect to the rest of the fields, so that h is a ghost. For the moment, we can already say that if $\alpha = -(D - 1)/D$ then h is massless, because the mass term solely depends on the traceless part $h_{\mu\nu} - \eta_{\mu\nu} h/D$, while if $\alpha < -(D - 1)/D$ then h is also a tachyon. In particular, for $\alpha = -1/2$ it is a tachyon with mass $-m^2$. On the other hand, if $\alpha = 0$, then (2.2.19) becomes an algebraic equation for h and the latter gets totally determined by the source

$$h = -\frac{1}{dm^2} T, \tag{2.2.20}$$

so that it is no longer a degree of freedom, nor a dynamical field. Another peculiarity of this choice for α is that the double divergence of the equation of motion, i.e. the divergence of (2.2.17), is gauge-invariant.⁵ This suggests that, although $m \neq 0$, there is some kind of leftover gauge symmetry in the equations of motion, in contrast with massive electrodynamics where both the equation and its divergence are not gauge-invariant. However, the equations of motion are not invariant under any gauge transformation (2.2.15), even a pure-scalar one $\xi_\mu = \partial_\mu \theta$. We will understand this point better in the following sections, when we will have the appropriate technology at our disposal. For the moment we can simply note the interesting fact that for a pure-scalar transformation

$$\delta h_{\mu\nu} = -2\partial_\mu \partial_\nu \theta, \quad (2.2.21)$$

the action varies by

$$\delta S \sim \theta(\partial_\mu \partial_\nu h^{\mu\nu} - (1 + \alpha)\square h) + \alpha(\square\theta)^2, \quad (2.2.22)$$

so for $\alpha = 0$ this is proportional to the divergence of (2.2.17) and therefore vanishes for on-shell $h_{\mu\nu}$ configurations.⁶ Thus, if $h_{\mu\nu}$ is a solution, then

$$S[h_{\mu\nu}] = S[h_{\mu\nu} - 2\partial_\mu \partial_\nu \theta]. \quad (2.2.23)$$

Clearly, something special happens at the Fierz–Pauli point $\alpha = 0$, although it is not a gauge symmetry. So let us start with this $\alpha = 0$ case.

Massive $\alpha = 0$

Using (2.2.20) and (2.2.17) the system of equations simplifies to

$$(\square - m^2)h_{\mu\nu} = -T_{\mu\nu} + \frac{1}{d} \left[\eta_{\mu\nu} T - \frac{\partial_\mu \partial_\nu}{m^2} T \right], \quad (2.2.24)$$

$$\partial_\mu h^{\mu\nu} - \partial^\nu h = 0, \quad (2.2.25)$$

$$h = -\frac{1}{dm^2} T. \quad (2.2.26)$$

The latter allows us to fix h_{00}

$$h_{00} = h_{ii} + \frac{1}{dm^2} T. \quad (2.2.27)$$

The i component of (2.2.25), along with the $0i$ component of (2.2.24), fix h_{0i}

$$(\Delta - m^2)h_{0i} = \partial_j \dot{h}_{ij} - T_{0i}, \quad \dot{h}_{0i} = \partial_j h_{ij} + \frac{1}{dm^2} \partial_i T. \quad (2.2.28)$$

⁵It is actually the linearization of the Ricci scalar.

⁶This corresponds to the well-known fact that, when $h_{\mu\nu}$ takes the form $h_{\mu\nu} = \partial_\mu \partial_\nu \phi$ for some function ϕ , the Fierz–Pauli mass term is a total derivative. The generalization of this property to terms of cubic and higher order in $\partial_\mu \partial_\nu \phi$ gives rise to the Galileon family of operators [3].

Using the 0 component of (2.2.25) and the trace of the spatial part of (2.2.24), we get

$$(\Delta - m^2)h_{ii} = \partial_i \partial_j h_{ij} - T_{00}, \quad \dot{h}_{ii} = \partial_i h_{i0}. \quad (2.2.29)$$

We finally split h_{ij} into its trace h_{ii} and traceless part $\tilde{h}_{ij}$, and isolate h_{00} , h_{0i} , h_{ii} in the above equations

$$\left(\frac{d-1}{d} \Delta - m^2\right) h_{00} = \partial_i \partial_j \tilde{h}_{ij} - T_{00} + \frac{1}{d} T - \frac{d-1}{d^2 m^2} \Delta T, \quad (2.2.30)$$

$$(\Delta - m^2) h_{0i} = \partial_j \tilde{h}_{ij} - T_{0i} + ((d-1) \Delta - dm^2)^{-1} \partial_i (\partial_j \partial_k \tilde{h}_{jk} - \partial_j T_{0j}), \quad (2.2.31)$$

$$(\Delta - m^2) \dot{h}_{0i} = (\Delta - m^2) \left(\partial_j \tilde{h}_{ij} + \frac{1}{dm^2} \partial_i T \right) + \frac{1}{d} \partial_i (\partial_j \partial_k \tilde{h}_{jk} - T_{0j}), \quad (2.2.32)$$

$$\left(\frac{d-1}{d} \Delta - m^2\right) h_{ii} = \partial_i \partial_j \tilde{h}_{ij} - T_{00}, \quad (2.2.33)$$

$$(\Delta - m^2) \dot{h}_{ii} = \partial_i \partial_j \tilde{h}_{ij} - \partial_i T_{0i} + ((d-1) \Delta - dm^2)^{-1} \Delta (\partial_j \partial_k \tilde{h}_{jk} - \partial_j T_{0j}), \quad (2.2.34)$$

so the corresponding initial conditions are determined by the ones of $\tilde{h}_{ij}$ and $T_{\mu\nu}$ and these fields are non-dynamical. We are thus left with only $\tilde{h}_{ij}$ being unconstrained, obeying a massive Klein–Gordon equation (the spatial traceless part of (2.2.24))

$$(\square - m^2) \tilde{h}_{ij} = -\tilde{T}_{ij} - \frac{1}{dm^2} (\partial_i \partial_j - \frac{1}{d} \delta_{ij} \Delta) T, \quad (2.2.35)$$

and transforming in the tensor representation of $SO(d)$. We thus have $N_f = 2N_d = D^2 - D - 2$, which in $D = 4$ gives $N_d = 5$.

Massive $\alpha \neq 0$

Let us now move on to the $\alpha \neq 0$ case. As we have seen already in the $\alpha = 0$ case, the equations for the $h_{\mu\nu}$ components can easily become lengthy in the process of spotting the non-dynamical fields and their precise form is not particularly illuminating. This is even worse here because of the undetermined α parameter. We therefore propose to simply sketch the procedure for generic α and then give the precise equations for the case $\alpha = -1/2$ which is considerably simpler. In the subsequent sections where the method of analysis will be more suited, we will treat the generic case explicitly to see that it is not qualitatively different from $\alpha = -1/2$.

So let us sketch the procedure for the generic case. In the $\alpha = 0$ case, the trace equation eliminated h_{00} , so we were able to use the divergence equation to eliminate h_{0i} and h_{ii} . Here, since the trace is dynamical, we have that either h_{00} , or h_{ii} will remain dynamical. More precisely, using the 0 component of (2.2.17) and the appropriate combination of the 00 component of (2.2.16) and (2.2.19), we find non-dynamical equations for h_{00} and $\dot{h}_{0i}$ which fix the initial conditions in terms of the

ones of h_{ii} and h_{0i} . We can then use the $0i$ component of (2.2.17) along with the $0i$ component of (2.2.16) to do the same for h_{0i} . We are then left with the ij component of (2.2.16) in an appropriate combination with the 00 component and (2.2.19), which yield dynamical equations for the unconstrained fields h_{ij} . This therefore corresponds to $N_f = 2N_d = D^2 - D$, or $N_d = 6$ when $D = 4$, i.e. the trace h_{ii} (or equivalently the Lorentz trace $h \equiv h_{ii} - h_{00}$) is now part of the dynamical spectrum.

In particular, for $\alpha = -1/2$, Eqs. (2.2.16), (2.2.17) and (2.2.19) can be brought to the simple form

$$(\square - m^2)\bar{h}_{\mu\nu} = -T_{\mu\nu}, \quad \partial_\mu \bar{h}^{\mu\nu} = 0, \quad (2.2.36)$$

where

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h. \quad (2.2.37)$$

Using the second equation along with the 0μ component we get

$$(\Delta - m^2)\bar{h}_{0i} = \partial_j \dot{\bar{h}}_{ij} - T_{0i}, \quad \dot{\bar{h}}_{0i} = \partial_j \bar{h}_{ij}, \quad (2.2.38)$$

and

$$(\Delta - m^2)\bar{h}_{00} = \partial_i \partial_j \bar{h}_{ij} - T_{00}. \quad (2.2.39)$$

These fields are thus non-dynamical and their initial conditions are fixed in terms of the ones of h_{ij} and $T_{\mu\nu}$. We are thus left with h_{ij} obeying

$$(\square - m^2)\bar{h}_{ij} = -T_{ij}, \quad (2.2.40)$$

so $N_f = 2N_d = D^2 - D$, and in particular $N_d = 6$ for $D = 4$.

Massless

Let us now pass to the $m = 0$ case. First remember that (2.2.17) is a possible choice of gauge only if $\alpha \neq 0$, since otherwise its divergence is gauge-invariant. We therefore have that the $m = 0$ case follows from the massive $\alpha \neq 0$ case by simply setting $m \rightarrow 0$, although now (2.2.17) is obtained by a gauge transformation, as in the spin-1 theory. We work in the gauge corresponding to $\alpha = -1/2$ so that our equations are (2.2.36) with $m = 0$. Again, as in electrodynamics, there is a residual gauge symmetry given by the gauge parameters that satisfy $\square \xi^\mu = 0$. The divergence and trace of the spatial part of (2.2.36) read

$$\square \partial_j \bar{h}_{ij} = -\partial_j T_{ij}, \quad \square \bar{h}_{ii} = -T_{ii}, \quad (2.2.41)$$

and their solutions take the form

$$\partial_j \bar{h}_{ij} = \phi_i^{\text{hom}} - \square_r^{-1} \partial_j T_{ij}, \quad \bar{h}_{ii} = \phi^{\text{hom}} - \square_r^{-1} T_{ii}, \quad (2.2.42)$$

where $\square\phi_i^{\text{hom}} = 0$ and $\square\phi^{\text{hom}} = 0$ are homogeneous solutions. We can then perform a residual gauge transformation

$$\partial_j \bar{h}_{ij} - \Delta \xi_i - \partial_i \dot{\xi}_0 = \phi_i^{\text{hom}} - \square_r^{-1} \partial_j T_{ij}, \quad \bar{h}_{ii} + (d-2) \partial_i \xi_i - d \dot{\xi}_0 = \phi^{\text{hom}} - \square_r^{-1} T_{ii}, \quad (2.2.43)$$

and we see that we can kill the homogeneous solutions with the choice

$$\dot{\xi}_0 = -\frac{1}{2(d-1)} \left[(d-2) \Delta^{-1} \partial_i \phi_i^{\text{hom}} + \phi^{\text{hom}} \right], \quad (2.2.44)$$

$$\xi_i = -\Delta^{-1} \left[\phi_i^{\text{hom}} + \frac{1}{2(d-1)} \partial_i \left((d-2) \Delta^{-1} \partial_j \phi_j^{\text{hom}} + \phi^{\text{hom}} \right) \right]. \quad (2.2.45)$$

Therefore, $\partial_j \bar{h}_{ij}$ are $\bar{h}_{ii}$ are fully determined by the source and thus carry no degrees of freedom. The only unconstrained components are the spatial transverse-traceless part $\bar{h}_{ij}^{\text{tt}}$, i.e. $\partial_j \bar{h}_{ij}^{\text{tt}} = 0$ and $\bar{h}_{ii}^{\text{tt}} = 0$, whose equation of motion is the spatial transverse-traceless part of (2.2.36)

$$\square \bar{h}_{ij}^{\text{tt}} = -T_{ij}^{\text{tt}}, \quad (2.2.46)$$

and correspond to $N_f = d^2 - d - 2$ degrees of freedom. As in the spin-1 case, the fact that h_{ii} and $\partial_i h_{ij}$ are apparently dynamical is a gauge artefact. By starting all over again but rather considering the gauge

$$\partial_i \bar{h}_{ij} = 0, \quad (2.2.47)$$

we find indeed that they both obey non-dynamical equations and thus have that $N_f = 2N_d$, with $N_d = 2$ in the $D = 4$ case.

2.3 Canonical Formalism

The most rigorous way to perform the degree of freedom count and to study the stability of a theory is through the canonical formalism (see for instance [4–8] for the case of massive gravity). It is also the most suited way to unambiguously see that $N_f = 2N_d$ for gauge theories. Here we assume that the reader has the basic knowledge of constrained Hamiltonian systems, i.e. Dirac's algorithm, weak equality,⁷ first/second class constraint terminology⁸ etc.⁹

⁷Weak equality “ $\approx$ ” holds for “ $=$ up to the addition of terms that are zero on the constrained hypersurface”.

⁸A constraint is “first class” if its Poisson bracket with any other constraint and the Hamiltonian is weakly zero. A constraint that is not first class is called “second class”.

⁹See for instance [9] for details on this formalism.

2.3.1 Spin 1

Massive

Since A_0 has no kinetic term $\sim \dot{A}_0^2$ in (2.2.1)

$$S = \int d^D x \left[\frac{1}{2} \dot{A}_i^2 - \frac{1}{4} F_{ij} F_{ij} - \frac{1}{2} m^2 A_i^2 + A_i j_i - \dot{A}_i \partial_i A_0 + \frac{1}{2} (\partial_i A_0)^2 + \frac{1}{2} m^2 A_0^2 - A_0 j_0 \right], \quad (2.3.1)$$

we Legendre transform only with respect to $\dot{A}_i$. The conjugate momenta (the electric field) read

$$E_i \equiv \frac{\partial L}{\partial \dot{A}_i} = \dot{A}_i - \partial_i A_0, \quad (2.3.2)$$

so that the action in canonical form is

$$S = \int d^D x [E_i \dot{A}_i - \mathcal{H}[E_i, A_i, A_0]], \quad (2.3.3)$$

where

$$\mathcal{H}[E_i, A_i, A_0] = \frac{1}{2} E_i^2 + \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} m^2 A_i^2 - A_i j_i - A_0 (\partial_i E_i - j_0) - \frac{1}{2} m^2 A_0^2, \quad (2.3.4)$$

is the Hamiltonian density. Since A_0 is an auxiliary non-dynamical field, it can be integrated-out in order to provide a clearer picture of the dynamics, i.e. it can be replaced with the solution of its own equation of motion

$$\mathcal{H}[E_i, A_i] = \frac{1}{2} E_i^2 + \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} m^2 A_i^2 + \frac{1}{m^2} (\partial_i E_i)^2 - A_i j_i - \frac{2}{m^2} \partial_i E_i j_0 + \mathcal{O}(j^2). \quad (2.3.5)$$

It is then clear that we have $N_f = 2N_d = 2d$ degrees of freedom forming two vectors A_i and E_i under $\text{SO}(d)$.

Massless

In the case $m = 0$, we have to go back to (2.3.4) and observe that A_0 becomes a Lagrange multiplier enforcing the Gauss constraint

$$\mathcal{G} \equiv \partial_i E_i - j_0 = 0. \quad (2.3.6)$$

We now enter Dirac's constraint formalism so let us define the Poisson bracket

$$\{\mathcal{O}, \mathcal{O}'\} \equiv \int d^d x \left[\frac{\delta \mathcal{O}}{\delta A_i} \frac{\delta \mathcal{O}'}{\delta E_i} - \frac{\delta \mathcal{O}'}{\delta A_i} \frac{\delta \mathcal{O}}{\delta E_i} \right], \quad (2.3.7)$$

and let us also smear the phase space functions of interest

$$A[f] \equiv \int d^d x f_i A_i, \quad E[g] \equiv \int d^d x g_i E_i, \quad G[A_0] \equiv \int d^d x A_0 \mathcal{G}, \quad H \equiv \int d^d x \mathcal{H}, \quad (2.3.8)$$

so that time-evolution is given by¹⁰

$$\dot{\mathcal{O}} = -\{H, \mathcal{O}\} + \partial_t \mathcal{O}. \quad (2.3.9)$$

We then have that, for a conserved source, $\mathcal{G}$ is first class

$$\dot{G}[A_0] = 0, \quad \{G[A_0], G[A'_0]\} = 0 \quad (2.3.10)$$

so A_0 is not determined by the equations of motion and $G[A_0]$ generates abelian gauge transformations on phase space

$$\delta A[f] = -\{G[A_0], A[f]\} = -\int d^d x f_i \partial_i A_0, \quad \delta E[g] = -\{G[A_0], E[g]\} = 0, \quad (2.3.11)$$

which for A_i and E_i translate into

$$\delta A_i = -\partial_i A_0, \quad \delta E_i = 0. \quad (2.3.12)$$

This implies that $\partial_i A_i$ is pure-gauge, the simplest example being the Coulomb gauge $\partial_i A_i = 0$. Along with $\mathcal{G} = 0$, we thus get that the longitudinal parts of A_i and E_i are non-dynamical, leaving only the transverse parts as the $N_f = 2(d - 1)$ degrees of freedom of the theory. Moreover, here we can clearly see why $N_f = 2N_d$. Indeed, the fields with constrained initial conditions A_0 and $\partial_i A_i$ appear as a Lagrange multiplier A_0 , which is thus totally arbitrary and in fact represents the gauge parameter, and a canonical couple $\partial_i A_i, \partial_i E_i$ subject to a constraint (spatial differential equation) and a gauge transformation on phase space. Thus both A_0 and $\partial_i A_i$ are non-dynamical and thus $N_f = 2N_d$. Finally, note that in both the massive and massless cases, the quadratic part of the (gauge-fixed for $m = 0$) Hamiltonian is positive definite, so these theories are stable.

2.3.2 Spin 2

Massive $\alpha \neq 0$

Since the $h_{0\mu}$ components have no kinetic term in (2.2.13), we first remove all time-derivatives that act upon them by integrating by parts

¹⁰Note that the second term here is needed because $\mathcal{O}$ can depend on the source which has its own time-dependence. The ∂_t operator will of course not act on the smearing fields f_i, g_i and A_0 .

$$\begin{aligned}
S = \int d^D x \Bigg[& \frac{1}{2} (\dot{h}_{ij}^2 - \dot{h}_{ii}^2) - \frac{1}{2} (\partial_i h_{jk})^2 + (\partial_i h_{ij})^2 - \partial_i h_{ij} \partial_j h_{kk} + \frac{1}{2} (\partial_i h_{jj})^2 \\
& - 2 \dot{h}_{ij} \partial_i h_{j0} + 2 \dot{h}_{ii} \partial_j h_{j0} + \partial_i h_{ij} \partial_j h_{00} - \partial_i h_{jj} \partial_i h_{00} + 2 \partial_{[i} h_{j]0} \partial_i h_{j0} \\
& - \frac{1}{2} m^2 (h_{ij}^2 - (1 + \alpha) h_{ii}^2 + 2(1 + \alpha) h_{00} h_{ii} - 2 h_{0i}^2 - \alpha h_{00}^2) \\
& + h_{00} T_{00} - 2 h_{0i} T_{0i} + h_{ij} T_{ij} \Bigg], \tag{2.3.13}
\end{aligned}$$

and then Legendre transform only with respect to $\dot{h}_{ij}$. The conjugate momenta read

$$\pi_{ij} \equiv \frac{\partial L}{\partial \dot{h}_{ij}} = \dot{h}_{ij} - \delta_{ij} \dot{h}_{kk} - 2 \partial_{(i} h_{j)0} + 2 \delta_{ij} \partial_k h_{k0}, \tag{2.3.14}$$

and the inversion gives

$$\dot{h}_{ij} = \pi_{ij} - \frac{1}{d-1} \delta_{ij} \pi_{kk} + 2 \partial_{(i} h_{j)0}, \tag{2.3.15}$$

so that the action in canonical form reads

$$S = \int d^D x \left[\pi_{ij} \dot{h}_{ij} - \mathcal{H}[h_{ij}, \pi_{ij}, h_{00}, h_{0i}] \right], \tag{2.3.16}$$

and the Hamiltonian density is

$$\begin{aligned}
\mathcal{H}[h_{ij}, \pi_{ij}, h_{00}, h_{0i}] \equiv & \frac{1}{2} \left(\pi_{ij}^2 - \frac{1}{d-1} \pi_{ii}^2 \right) + \frac{1}{2} (\partial_i h_{jk})^2 - (\partial_i h_{ij})^2 + \partial_i h_{ij} \partial_j h_{kk} - \frac{1}{2} (\partial_i h_{jj})^2 \\
& + \frac{1}{2} m^2 (h_{ij}^2 - (1 + \alpha) h_{ii}^2 - 2 h_{0i}^2 - \alpha h_{00}^2) - h_{ij} T_{ij} \\
& + h_{00} (\partial_i \partial_j h_{ij} - \Delta h_{ii} + (1 + \alpha) m^2 h_{ii} - T_{00}) + 2 h_{0i} (\partial_j \pi_{ij} + T_{0i}). \tag{2.3.17}
\end{aligned}$$

We see that h_{0i} is an auxiliary field that appears quadratically whatever the value of α , so we can integrate it out as we did for A_0 in the spin-1 case to get

$$\begin{aligned}
\mathcal{H}[h_{ij}, \pi_{ij}, h_{00}] = & \frac{1}{2} \left(\pi_{ij}^2 - \frac{1}{d-1} \pi_{ii}^2 \right) + \frac{1}{m^2} (\partial_i \pi_{ij})^2 + \frac{1}{2} (\partial_i h_{jk})^2 - (\partial_i h_{ij})^2 + \partial_i h_{ij} \partial_j h_{kk} \\
& - \frac{1}{2} (\partial_i h_{jj})^2 + \frac{1}{2} m^2 (h_{ij}^2 - (1 + \alpha) h_{ii}^2 - \alpha h_{00}^2) \\
& + h_{00} (\partial_i \partial_j h_{ij} - \Delta h_{ii} + (1 + \alpha) m^2 h_{ii} - T_{00}) - h_{ij} T_{ij} + \frac{2}{m^2} \partial_i \pi_{ij} T_{0j} + \mathcal{O}(T^2). \tag{2.3.18}
\end{aligned}$$

Now, for $\alpha \neq 0$ we have that h_{00} is also a quadratic auxiliary field, so we can integrate it out as well

$$\begin{aligned} \mathcal{H}[h_{ij}, \pi_{ij}] = & \frac{1}{2} \left(\pi_{ij}^2 - \frac{1}{d-1} \pi_{ii}^2 \right) + \frac{1}{m^2} (\partial_i \pi_{ij})^2 + \frac{1}{2} (\partial_i h_{jk})^2 - (\partial_i h_{ij})^2 + \partial_i h_{ij} \partial_j h_{kk} - \frac{1}{2} (\partial_i h_{jj})^2 \\ & + \frac{1}{2m^2 \alpha} \left[\partial_i \partial_j h_{ij} - \Delta h_{ii} + (1 + \alpha) m^2 h_{ii} \right]^2 + \frac{1}{2} m^2 (h_{ij}^2 - (1 + \alpha) h_{ii}^2) \\ & - h_{ij} T_{ij} - \frac{1}{m^2 \alpha} (\partial_i \partial_j h_{ij} - \Delta h_{ii} + (1 + \alpha) m^2 h_{ii}) T_{00} + \frac{2}{m^2} \partial_i \pi_{ij} T_{0j} + \mathcal{O}(T^2). \end{aligned} \quad (2.3.19)$$

To see the instability in this setting we can harmonically decompose π_{ij}

$$\pi_{ij} \equiv \frac{1}{d} \delta_{ij} \pi + \left(\partial_i \partial_j - \frac{1}{d} \delta_{ij} \Delta \right) l + \partial_i v_j + t_{ij}, \quad \partial_i v_i = t_{ii} = 0, \quad \partial_i t_{ij} = 0, \quad (2.3.20)$$

and trade l for the more convenient variable

$$\Pi \equiv \pi + (d-1) \Delta l, \quad (2.3.21)$$

to get that the part of $\mathcal{H}$ which is quadratic in π_{ij} in the scalar sector reads

$$\mathcal{H}_{\mathcal{O}(\pi_{\text{scal}}^2)} = \frac{1}{2d(d-1)} \Pi^2 + \frac{1}{d^2 m^2} (\partial_i \Pi)^2 - \frac{1}{d(d-1)} \pi \Pi. \quad (2.3.22)$$

We see that $\mathcal{H}$ is thus not positive-definite, or that by completing the square there is a negative-definite term. Since this occurs at the level of the conjugate momenta, we have that the corresponding mode is a ghost. The degrees of freedom are the h_{ij} and π_{ij} components, that is, a total of $N_f = 2N_d = d^2 + d = D^2 - D$.

Massive $\alpha = 0$

So let us go back to (2.3.18) and move on to the $\alpha = 0$ case where h_{00} becomes a Lagrange multiplier enforcing the constraint

$$\mathcal{C}_t \equiv \partial_i \partial_j h_{ij} - (\Delta - m^2) h_{ii} - T_{00} = 0. \quad (2.3.23)$$

Defining the Poisson bracket

$$\{\mathcal{O}, \mathcal{O}'\} \equiv \int d^d x \left[\frac{\delta \mathcal{O}}{\delta h_{ij}} \frac{\delta \mathcal{O}'}{\delta \pi_{ij}} - \frac{\delta \mathcal{O}'}{\delta h_{ij}} \frac{\delta \mathcal{O}}{\delta \pi_{ij}} \right], \quad (2.3.24)$$

and the smeared observables

$$h[f] \equiv \int d^d x f_{ij} h_{ij}, \quad \pi[g] \equiv \int d^d x g_{ij} \pi_{ij}, \quad \mathcal{C}_t[h_{00}] \equiv \int d^d x h_{00} \mathcal{C}_t, \quad H \equiv \int d^d x \mathcal{H} \quad (2.3.25)$$

we get that $\mathcal{C}_t$ is second class (using (2.3.9) and for a conserved source)

$$\dot{\mathcal{C}}_t[h_{00}] = - \int d^d x h_{00} \mathcal{C}' \equiv \mathcal{C}'[h_{00}], \quad \mathcal{C}' \equiv \partial_i \partial_j \pi_{ij} + \frac{1}{d-1} m^2 \pi_{ii} + \partial_i T_{0i}, \quad (2.3.26)$$

so it is a priori not conserved under time evolution. To repair this, we must therefore consider $\mathcal{C}'$ as an additional (secondary) constraint and append a term $q \mathcal{C}'$ to the total Hamiltonian density $\mathcal{H}$

$$\mathcal{H} \rightarrow \mathcal{H} + q \mathcal{C}', \quad (2.3.27)$$

with q a Lagrange multiplier. Now, demanding that $\mathcal{C}'$ be conserved fixes h_{00}

$$\dot{\mathcal{C}}'[q] \sim \mathcal{C}_t + dm^2(h_{00} - h_{ii}) - T \approx 0, \quad (2.3.28)$$

so we can choose

$$h_{00} = h_{ii} + \frac{1}{dm^2} T, \quad (2.3.29)$$

which is nothing but (2.2.20), and the Hamiltonian density now reads

$$\begin{aligned} \mathcal{H}[h_{ij}, \pi_{ij}, q] &= \frac{1}{2} \left(\pi_{ij}^2 - \frac{1}{d-1} \pi_{ii}^2 \right) + \frac{1}{m^2} (\partial_i \pi_{ij})^2 \\ &\quad + \frac{1}{2} (\partial_i h_{jk})^2 - (\partial_i h_{ij})^2 + \frac{1}{2} (\partial_i h_{jj})^2 + \frac{1}{2} m^2 (h_{ij}^2 + h_{ii}^2) \\ &\quad + q \mathcal{C}' - h_{ij} T_{ij} - h_{ii} T_{00} + \frac{1}{dm^2} \mathcal{C}_t T + \frac{2}{m^2} \partial_i \pi_{ij} T_{0j} + \mathcal{O}(T^2). \end{aligned} \quad (2.3.30)$$

We must finally demand that $\mathcal{C}_t$ be conserved under time-evolution with this new Hamiltonian

$$\dot{\mathcal{C}}_t \sim \mathcal{C}' - \frac{dm^4}{d-1} q \approx 0, \quad (2.3.31)$$

which ends up fixing $q = 0$. The resulting Hamiltonian preserves the constraints $\mathcal{C}_t, \mathcal{C}'$ under time-evolution as long as they are imposed on the initial conditions. These constraints kill precisely the degree of freedom which causes the instability in the $\alpha \neq 0$ case and make the Hamiltonian positive-definite. Indeed, by performing a harmonic decomposition of π_{ij} (2.3.20) and also h_{ij}

$$h_{ij} \equiv \frac{1}{d} \delta_{ij} \phi + \left(\partial_i \partial_j - \frac{1}{d} \delta_{ij} \Delta \right) \lambda + \partial_{(i} \beta_{j)} + \tau_{ij}, \quad \partial_i \beta_i = \tau_{ii} = 0, \quad \partial_i \tau_{ij} = 0, \quad (2.3.32)$$

we can actually solve the constraints explicitly and get that they relate the traces to the longitudinal parts

$$\left(\frac{d-1}{d}\Delta - m^2\right)\phi = \frac{d-1}{d}\Delta^2\lambda - T_{00}, \quad (2.3.33)$$

$$\left(\frac{d-1}{d}\Delta + m^2\right)\pi = -(d-1)\left(\frac{d-1}{d}\Delta^2 l + \partial_i T_{i0}\right). \quad (2.3.34)$$

Actually, as in the $\alpha \neq 0$ case, one can use more convenient combinations instead of the longitudinal modes

$$\Phi \equiv \Delta\lambda - \phi, \quad \Pi \equiv \pi + (d-1)\Delta l, \quad (2.3.35)$$

so that, using the constraints, we can express ϕ, λ, π, l in terms of Φ and Π . We get

$$\phi = -\frac{1}{m^2}\left[\frac{d-1}{d}\Delta\Phi - T_{00}\right], \quad \Delta\lambda = -\frac{1}{m^2}\left[\frac{d-1}{d}\Delta\Phi - m^2\Phi - T_{00}\right]. \quad (2.3.36)$$

and

$$\pi = -\frac{d-1}{m^2}\left[\frac{1}{d}\Delta\Pi + \partial_i T_{i0}\right], \quad \Delta l = \frac{1}{m^2}\left[\frac{1}{d}\Delta\Pi + \frac{1}{d-1}m^2\Pi + \partial_i T_{i0}\right]. \quad (2.3.37)$$

and the action reads

$$S = \int d^D x \left[\frac{1}{d}\Pi\dot{\Phi} + \frac{1}{2}\partial_i v_j \partial_i \dot{\beta}_j + t_{ij}\dot{\tau}_{ij} - \mathcal{H} \right], \quad (2.3.38)$$

where

$$\mathcal{H} = \mathcal{H}_{\text{scalar}} + \mathcal{H}_{\text{vector}} + \mathcal{H}_{\text{tensor}}, \quad (2.3.39)$$

with¹¹

$$\begin{aligned} \mathcal{H}_{\text{scalar}} &= \frac{1}{2d(d-1)}\Pi^2 + \frac{d-1}{2d}\Phi(m^2 - \Delta)\Phi - \frac{1}{dm^2}\Pi\Delta q + \frac{d-1}{d}\Phi(\rho - \Delta\sigma) \\ &\quad + \frac{d-1}{dm^2}\Phi\Delta\left(p + \frac{d-1}{d}\Delta\sigma\right) + \mathcal{O}(T^2) \\ \mathcal{H}_{\text{vector}} &= \frac{1}{4m^2}\partial_i v_j (m^2 - \Delta)\partial_i v_j + \frac{1}{2}m^2(\partial_i \beta_j)^2 + \frac{1}{m^2}v_i \Delta q_i + \frac{1}{2}\beta_i \Delta\sigma_i + \mathcal{O}(T^2) \\ \mathcal{H}_{\text{tensor}} &= \frac{1}{2}t_{ij}^2 + \frac{1}{2}\tau_{ij}(m^2 - \Delta)\tau_{ij} - \tau_{ij}\sigma_{ij} + \mathcal{O}(T^2). \end{aligned} \quad (2.3.40)$$

The quadratic part is positive definite and the theory is thus stable, with $N_f = 2N_d = d^2 + d - 2 = D^2 - D - 2$ degrees of freedom. These correspond to Φ, β_i, τ_{ij} and their conjugate momenta.

¹¹The harmonic variables of the source $\rho, p, q, \sigma, q_i, \sigma_i, \sigma_{ij}$ are defined in (2.4.18) and the conservation equation in terms of them reads (2.4.36).

Massless

We can finally proceed to the massless case where h_{0i} becomes a Lagrange multiplier as well. We must therefore go back to (2.3.17) with $m = 0$ and define the constraint imposed by h_{0i} as

$$\mathcal{C}_i \equiv \partial_j \pi_{ij} - T_{0i}, \quad \mathcal{C}_s[h_{0i}] \equiv \int d^d x h_{0i} \mathcal{C}_i. \quad (2.3.41)$$

Observe that the secondary constraint $\mathcal{C}'$ of Fierz–Pauli theory (2.3.26) actually reduces to $\partial_i \mathcal{C}_i$ in the $m \rightarrow 0$ limit. The only non-trivial Poisson bracket arises in

$$\dot{\mathcal{C}}_i[h_{00}] = -\mathcal{C}_s[\partial_i h_{00}] \approx 0, \quad (2.3.42)$$

so the system is now first class and the $h_{0\mu}$ are not determined by the equations of motion. Rather, they serve as the gauge parameters of the gauge transformations generated by $\mathcal{C}_t$ and $\mathcal{C}_s$ on phase space

$$\delta h[f] = -\{\mathcal{C}_t[h_{00}], h[f]\} - \{\mathcal{C}_s[h_{0i}], h[f]\} = -\int d^d x f_{ij} \partial_i h_{j0}, \quad (2.3.43)$$

$$\delta \pi[g] = -\{\mathcal{C}_t[h_{00}], \pi[g]\} - \{\mathcal{C}_s[h_{0i}], \pi[g]\} = \int d^d x \pi_{ij} (\partial_i \partial_j - \delta_{ij} \Delta) h_{00}, \quad (2.3.44)$$

which for h_{ij} and π_{ij} imply

$$\delta h_{ij} = -\partial_i h_{j0}, \quad \delta \pi_{ij} = (\partial_i \partial_j - \delta_{ij} \Delta) h_{00}. \quad (2.3.45)$$

These can be used to fix the gauge to

$$\partial_i h_{ij} = 0, \quad \pi_{ii} = 0, \quad (2.3.46)$$

which, along with $\mathcal{C}_t = 0$ and $\mathcal{C}_i = 0$, imply that h_{ij} and π_{ij} are both transverse-traceless. The degrees of freedom of the theory are thus $N_f = 2N_d = d^2 - d - 2$. The Hamiltonian density in this gauge is positive-definite

$$\mathcal{H} = \frac{1}{2} \pi_{ij}^2 + \frac{1}{2} (\partial_i h_{jk})^2 + \mathcal{O}(T), \quad (2.3.47)$$

so the theory is stable.

We can now note that the combinations Φ and Π defined in (2.3.35), that were used in the treatment of the massive theory, are actually invariant under (2.3.45).¹² We thus have that the scalar sector of the FP theory $\alpha = 0$, once the second class

¹²This will become clear in the next section where we will deduce the transformations of the harmonic variables under the gauge symmetry.

constraints are solved, becomes invariant under the transformations generated by the *massless constraints* C_t, C_s . It is important to note however that not all of these constraints appear in the massive theory and, for those who do, they are second class for $m \neq 0$. This means they do not correspond to gauge symmetries, since there is no totally undetermined field playing the role of the gauge parameter. Thus, by discussing the transformations of the scalars in the massive theory we are actually comparing objects in two different theories.

Nevertheless, the fact that these modes are gauge-invariant is again a property of the scalar sector of FP theory alone, since this is not the case of the vector sector, where v_i is gauge-invariant but β_i is not, and it is also not the case in massive electrodynamics. Moreover, it is also not the case for the ghost scalar when $\alpha \neq 0$, so this has all the characteristics of the issue that was discussed in the previous section: on the FP point $\alpha = 0$ there is something that looks like a gauge symmetry but that is actually not.

Another interesting feature we can already see here is the vDVZ discontinuity of the $\alpha = 0$ theory. Indeed, sending $m \rightarrow 0$ in (2.3.40) effectively neutralizes the vector modes but the scalar mode remains, that is, one more dynamical field that in the $m = 0$ case.

It seems that the use of harmonic variables has helped our understanding of this apparent symmetry issue and has generally made the dynamics of the theory more transparent. Unfortunately, in the canonical formalism the action is a bit too crowded because of the presence of the conjugate momenta, so this is still not the optimal way to understand the theory. We therefore now propose to use harmonic variables, but in the Lagrangian formulation.

2.4 Harmonic Formalism

In this section we consider the d -harmonic decomposition, but at the level of the Lagrangian formulation. This will allow us to explore the above mentioned “residual gauge symmetry” of Fierz–Pauli theory, but it will also make the dynamical structure of the theory more transparent. Moreover, this formalism is also easily applicable in the case of a de-Sitter background. It will thus allow us to understand in a different language a number of results in the literature on the degrees of freedom of massive gravity over de-Sitter. This section is based on original work from our group [2].

To briefly introduce the harmonic decomposition, let us start by noting that at each space-time point x , the field components form irreducible representations of $SO(d)$, i.e. $A_0(x)$ is a $SO(d)$ -scalar, $A_i(x)$ a $SO(d)$ -vector, the traceless part of $h_{ij}(x)$ is a $SO(d)$ -tensor and so on. If we now consider the full group of isometries of d -dimensional space, i.e. the Euclidean group $ISO(d)$ of rotations and translations, then a field is no longer seen as an infinite collection of independent $SO(d)$ representations, but as a finite collection of irreducible representations of $ISO(d)$. For instance, we have that inside of A_i there hides a scalar under $SO(d)$, that is, $\partial_i A_i$. We can therefore split A_i into its scalar part $\partial_i A_i$ and its transverse vector part A_i^t , obeying $\partial_i A_i^t = 0$,

which obviously do not mix under translations, nor under rotations since the latter commute with ∂_i . For tensors one can analogously decompose the traceless part of h_{ij} into a scalar, a transverse vector and a transverse-traceless tensor.

We will therefore refer to “ d -scalars”, “ d -vectors” and “ d -tensors” for these irreducible representations of $\text{ISO}(d)$, while the irreducible representations of $\text{SO}(d)$ will be referred to as “ $\text{SO}(d)$ -vectors” and “ $\text{SO}(d)$ -tensors”. Note that d -vectors and d -tensors are thus automatically transverse. The basic advantage of the harmonic decomposition in our analysis lies in the following fact: the massless dynamical fields form the highest possible irreducible representation of $\text{ISO}(d)$, while the massive ones form the highest possible representation of $\text{SO}(d)$. This formalism is thus ideal for observing the activation of modes by mass.

2.4.1 Spin 1

We start by splitting A_i and j_i into longitudinal and transverse parts

$$A_0 \equiv \psi, \quad A_i \equiv \partial_i \lambda + \beta_i, \quad \partial_i \beta_i = 0, \quad (2.4.1)$$

$$j_0 \equiv -\rho, \quad j_i \equiv \partial_i \sigma + \sigma_i, \quad \partial_i \sigma_i = 0, \quad (2.4.2)$$

with the inverse map being

$$\lambda = \Delta^{-1} \partial_i A_i, \quad \beta_i = P_{ij} A_j, \quad (2.4.3)$$

and so on for j_i , where P_{ij} is the projector on the subspace of d -vector fields (transverse $\text{SO}(d)$ -vectors)

$$P_{ij} \equiv \delta_{ij} - \partial_i \Delta^{-1} \partial_j, \quad P_i^k P_k^j = P_i^j, \quad \partial_i P_{ij} = 0. \quad (2.4.4)$$

Note that the harmonic variables are therefore spatially non-local combinations of the original fields. In terms of the harmonic variables the gauge transformation (2.2.2) reads

$$\delta \psi = -\dot{\theta}, \quad \delta \lambda = -\theta, \quad \delta \beta_i = 0, \quad (2.4.5)$$

so that β_i is gauge-invariant, while λ and ψ can combine to form the gauge-invariant combination

$$\Psi \equiv \psi - \dot{\lambda}. \quad (2.4.6)$$

On the other hand, current conservation $\partial_\mu j^\mu = 0$ translates into

$$\dot{\rho} = -\Delta \sigma, \quad (2.4.7)$$

and this equation will be implicitly used every time some $\sim \dot{\rho}$ term appears. We then get that the action (2.2.1) can be written as

$$S = \int d^D x \left[-\frac{1}{2} \partial_\mu \beta_i \partial^\mu \beta_i - \frac{1}{2} m^2 \beta_i \beta_i + \frac{1}{2} \partial_i \Psi \partial_i \Psi + \frac{1}{2} m^2 (\psi^2 - \partial_i \lambda \partial_i \lambda) + \rho \Psi + \beta_i \sigma_i \right], \quad (2.4.8)$$

where we consider λ, β_i, ψ as the independent fields, while Ψ is just a shorthand notation for the combination $\psi - \dot{\lambda}$. Here one might be tempted to use Ψ as an independent field instead of ψ or λ , but should refrain from doing so. This is because Ψ depends on time-derivatives of the original fields A_μ . This in turn implies that the initial conditions of Ψ are not determined, since they require the knowledge of the initial value of $\dot{\lambda}$. So keep in mind that one can only consider field redefinitions that preserve the initial data for the Cauchy problem to remain well-posed.

Massless

This subtlety being mentioned, the first thing to notice in the above action is that for $m = 0$ it is explicitly gauge-invariant since it depends only on the gauge-invariant quantities Ψ and β_i . The latter obeys a massless Klein–Gordon equation

$$\square \beta_i = -\sigma_i, \quad (2.4.9)$$

and thus constitutes the $2(d - 1)$ degrees of freedom of the theory. The equation of motion of ψ is the Poisson equation

$$\Delta \Psi = \rho, \quad (2.4.10)$$

while the equation of motion of λ is the time-derivative of it. Pay attention to the way in which gauge-invariance neutralizes the longitudinal mode λ in this setting. The latter *does* have a kinetic term $\sim \dot{\lambda}^2$ in the action, which would naively make it dynamical, but the fact that it enters only through the combination Ψ and that the latter ultimately obeys a purely spatial equation makes λ non-dynamical. Therefore, in the massless case, it turns out that we *can* effectively consider Ψ as an independent variable and vary the action with respect to it because the initial conditions of λ are pure-gauge so the initial data of Ψ *are* defined. This will no longer be true in the massive theory.

Until now, the spatial differential equations we obtained always concerned gauge-dependent fields, so that we did not need to worry about questions of instantaneous response to a source. Here, we are witnessing an equation that involves only spatial derivatives for Ψ , which is a gauge-invariant variable. As anticipated in Sect. 2.1.2, we see however that Ψ is a spatially non-local functional of the original fields, so that it cannot be measured instantaneously to begin with.

Massive

Turning on the mass $m \neq 0$, we first see that the gauge-invariant variables are not sufficient to describe the mass term since the latter breaks the gauge symmetry. This means that the equation of λ will not be implied by the one of ψ any more and therefore that its time-derivatives will now make it a dynamical field. The equation of motion of β_i is now a massive Klein–Gordon equation

$$(\square - m^2)\beta_i = -\sigma_i, \quad (2.4.11)$$

while the ones of ψ and λ read

$$(\Delta - m^2)\psi = \rho + \Delta\dot{\lambda}, \quad (\partial_t^2 + m^2)\lambda = \sigma + \dot{\psi}. \quad (2.4.12)$$

Then, isolating $\dot{\psi}$ in the latter and plugging the result in the time-derivative of the equation of ψ , we get

$$(\square - m^2)\lambda = -\sigma, \quad (2.4.13)$$

so λ corresponds to the additional 2 degrees of freedom one gets when $m \neq 0$. On the other hand, solving for $\ddot{\lambda}$ in its own equation of motion and plugging the result in the time-derivative of the equation of ψ , we get that ψ is non-dynamical and that its initial conditions are totally determined by the ones of the other fields

$$(\Delta - m^2)\psi = \rho + \Delta\dot{\lambda}, \quad \dot{\psi} = \Delta\lambda. \quad (2.4.14)$$

One could therefore integrate-out ψ , and redefine the longitudinal modes by a spatially non-local operation¹³

$$\tilde{\lambda} \equiv \sqrt{\Delta(\Delta - m^2)^{-1}} \lambda, \quad \tilde{\sigma} \equiv \sqrt{\Delta(\Delta - m^2)^{-1}} \sigma, \quad (2.4.15)$$

to find a spatially local action for the dynamical fields only

$$S = \int d^D x \left[-\frac{1}{2} \partial_\mu \beta_i \partial^\mu \beta_i - \frac{1}{2} m^2 \beta_i \beta_i + \beta_i \sigma_i + m^2 \left(-\frac{1}{2} \partial_\mu \tilde{\lambda} \partial^\mu \tilde{\lambda} - \frac{1}{2} m^2 \tilde{\lambda}^2 + \tilde{\lambda} \tilde{\sigma} \right) + \mathcal{O}(j^2) \right]. \quad (2.4.16)$$

Now only the dynamical fields appear in the action. This was already the case in the canonical formalism after having integrated-out A_0 , but the advantage here is that the action is analytic in m^2 so that one can study the $m \rightarrow 0$ limit unambiguously. Note that the new dynamical mode one gets in the massive case (here λ or $\tilde{\lambda}$) is not gauge-invariant, as one would expect in a massive theory, and disappears in the $m \rightarrow 0$ limit.

Finally, observe that, after having eliminated the non-dynamical field, the dynamical ones come with the Klein–Gordon kinetic terms, even though they are not representations of the Lorentz group. This is because Poincaré invariance implies the standard relativistic dispersion relation $E^2 = m^2 + \vec{p}^2$ for the dynamical fields.

¹³Note that the square-root is real because $\Delta(\Delta - m^2)^{-1}$ is positive-definite as it can be seen by using its Fourier representation.

2.4.2 Spin 2

We start by defining the harmonic variables

$$\begin{aligned} h_{00} &\equiv \psi, \\ h_{0i} &\equiv \partial_i \chi + \chi_i, \end{aligned} \quad (2.4.17)$$

$$\begin{aligned} h_{ij} &\equiv \frac{1}{d} \delta_{ij} \phi + \left(\partial_i \partial_j - \frac{1}{d} \delta_{ij} \Delta \right) \lambda + \partial_{(i} \beta_{j)} + \tau_{ij}, \\ T_{00} &\equiv \rho, \\ T_{0i} &\equiv -\partial_i q - q_i, \end{aligned} \quad (2.4.18)$$

$$T_{ij} \equiv \delta_{ij} p + \left(\partial_i \partial_j - \frac{1}{d} \delta_{ij} \Delta \right) \sigma + \partial_{(i} \sigma_{j)} + \sigma_{ij},$$

where

$$\partial_i \chi_i = \partial_i \beta_i = \tau_{ii} = \partial_i q_i = \partial_i \sigma_i = \sigma_{ii} = 0, \quad \partial_i \tau_{ij} = \partial_i \sigma_{ij} = 0, \quad (2.4.19)$$

while the inverse relation is

$$\chi = \Delta^{-1} \partial_i h_{0i}, \quad (2.4.20)$$

$$\chi_i = P_{ij} h_{0j}, \quad (2.4.21)$$

$$\phi = h_{ii}, \quad (2.4.22)$$

$$\lambda = -\frac{1}{d-1} \Delta^{-1} [h_{ii} - d \Delta^{-1} \partial_i \partial_j h_{ij}], \quad (2.4.23)$$

$$\beta_i = 2 \Delta^{-1} P_{ij} \partial_k h_{jk}, \quad (2.4.24)$$

$$\tau_{ij} = P_{ijkl} h_{kl}, \quad (2.4.25)$$

and so on for the components of $T_{\mu\nu}$, where P_{ijkl} is the projector on the subspace of d -tensors (transverse-traceless $SO(d)$ -tensors)

$$P_{ij}{}^{kl} \equiv P_{(i}{}^k P_{j)}{}^l - \frac{1}{d-1} P_{ij} P^{kl}, \quad P_{ij}{}^{nm} P_{nm}{}^{kl} = P_{ij}{}^{kl}, \quad \partial_i P_{ijkl} = 0, \quad P_{iikl} = 0. \quad (2.4.26)$$

Decomposing the gauge parameter as well

$$\xi_0 = A, \quad \xi_i = \partial_i B + B_i, \quad \partial_i B_i = 0, \quad (2.4.27)$$

we get that the gauge transformation (2.2.15) reads

$$\delta \psi = -2 \dot{A}, \quad (2.4.28)$$

$$\delta \chi = -A - \dot{B}, \quad (2.4.29)$$

$$\delta \chi_i = -\dot{B}_i, \quad (2.4.30)$$

$$\delta\phi = -2\Delta B, \quad (2.4.31)$$

$$\delta\lambda = -2B, \quad (2.4.32)$$

$$\delta\beta_i = -2B_i, \quad (2.4.33)$$

$$\delta\tau_{ij} = 0, \quad (2.4.34)$$

so one can form the following independent gauge-invariant combinations

$$\Psi \equiv \psi - 2\dot{\chi} + \ddot{\lambda}, \quad \Phi \equiv \Delta\lambda - \phi, \quad \Xi_i \equiv \chi_i - \frac{1}{2}\dot{\beta}_i, \quad (2.4.35)$$

known as “Bardeen potentials” [10], of which Φ is already known from the previous section. Finally, the conservation equation $\partial_\mu T^{\mu\nu} = 0$ gives

$$\dot{\rho} = -\Delta q, \quad \dot{q} = -p - \frac{d-1}{d} \Delta\sigma, \quad \dot{q}_i = -\frac{1}{2} \Delta\sigma_i, \quad (2.4.36)$$

and, as in the spin-1 case, these will be implicitly used whenever we have a time-derivative acting on a source component in the subsequent computations. The action (2.3.13) in terms of these variables reads

$$\begin{aligned} S = \int d^D x \Big[& \frac{d-1}{d} \left(-\frac{1}{2} \dot{\Phi}^2 + \frac{d-2}{2d} (\partial_i \Phi)^2 + \partial_i \Phi \partial_i \Psi \right) + (\partial_i \Xi_j)^2 - \frac{1}{2} \partial_\mu \tau_{ij} \partial^\mu \tau_{ij} \\ & - \frac{1}{2} m^2 \left(\frac{d-1}{d} (-\Phi^2 + 2\Phi\Delta\lambda) + 2(\Delta\lambda - \Phi)\psi - \alpha(\Phi + \psi - \Delta\lambda)^2 \right. \\ & \left. - 2(\partial_i \chi)^2 - 2\chi_i^2 + \frac{1}{2} (\partial_i \beta_j)^2 + \tau_{ij}^2 \right) + \Psi\rho - \Phi p + 2\Xi_i q_i + \tau_{ij} \sigma_{ij} \Big]. \end{aligned} \quad (2.4.37)$$

As in the case of electrodynamics, note that for $m = 0$ only gauge-invariant quantities appear, thus making the symmetry manifest. Again, we cannot consider Ψ and Ξ_i as independent variables with respect to which we could vary the action because they contain time-derivatives of the original fields and their initial conditions are thus not defined. This is however not the case of Φ , so we choose to consider $\psi, \chi, \Phi, \lambda, \chi_i, \beta_i, \tau_{ij}$ as the independent fields, while Ψ and Ξ_i are mere shorthand notations.

Massless

So let us start with the massless case $m = 0$ and compute the equations of motion. In the d -scalar sector, the ones of ψ and Φ read

$$\Delta\Phi = \frac{d}{d-1} \rho, \quad \ddot{\Phi} - \frac{d-2}{d} \Delta\Phi - \Delta\Psi = \frac{d}{d-1} p, \quad (2.4.38)$$

respectively, while the ones of χ and λ are the first and second time-derivative of the former. To simplify the second equation we note that by taking the double time-derivative of the first one and using (2.4.36) we get

$$\Delta \ddot{\Phi} = \frac{d}{d-1} \ddot{\rho} = -\frac{d}{d-1} \Delta \dot{q} = \frac{d}{d-1} \Delta \left(p + \frac{d-1}{d} \Delta \sigma \right), \quad (2.4.39)$$

so that the equation of Φ actually gives

$$\Delta \Psi = -\frac{d-2}{d-1} \rho + \Delta \sigma. \quad (2.4.40)$$

In the vector sector we have the equation of motion of χ_i

$$\Delta \Xi_i = q_i, \quad (2.4.41)$$

and the one of β_i which is its time-derivative, while finally for the tensor modes

$$\square \tau_{ij} = -\sigma_{ij}. \quad (2.4.42)$$

Therefore, the Bardeen variables Φ , Ψ , Ξ_i are physical but non-dynamical fields, thus leaving the $d^2 - d - 2$ components of τ_{ij} and $\dot{\tau}_{ij}$ as the only degrees of freedom/dynamical fields of the theory.

Massive

Let us now turn on the mass $m \neq 0$ in which case the equations of motion of χ , λ , β_i are no longer implied by the ones of Φ , ψ and χ_i . Since there are a lot of variables now, it is not particularly illuminating to work at the level of the equations of motion. Rather, we can do directly as we did in the end of the spin-1 case, that is, to integrate-out at the level of the action the manifestly non-dynamical modes, i.e. those without time-derivatives. For notational simplicity, we will consider the d -scalar, d -vector and d -tensor sectors separately. As far as the last two are concerned, the procedure and properties are exactly analogous to the ones of the d -scalar and d -vector modes in massive electrodynamics. For the d -tensor sector there is nothing to do, we simply have that it becomes massive

$$S_{\text{tensor}} = \int d^D x \left[-\frac{1}{2} \partial_\mu \tau_{ij} \partial^\mu \tau_{ij} - \frac{1}{2} m^2 \tau_{ij}^2 + \tau_{ij} \sigma_{ij} \right]. \quad (2.4.43)$$

For the d -vector sector the non-dynamical field is χ_i , so integrating it out in (2.4.37) and using the spatially non-local redefinition

$$\tilde{\beta}_i \equiv \sqrt{\Delta(\Delta - m^2)^{-1}} \beta_i, \quad \tilde{\sigma}_i \equiv \sqrt{\Delta(\Delta - m^2)^{-1}} \sigma_i, \quad (2.4.44)$$

we get the spatially local action

$$S_{\text{vector}} = \frac{m^2}{2} \int d^D x \left[-\frac{1}{2} \partial_\mu \tilde{\beta}_i \partial^\mu \tilde{\beta}_i - \frac{1}{2} m^2 \tilde{\beta}_i^2 + \tilde{\beta}_i \tilde{\sigma}_i \right]. \quad (2.4.45)$$

As for the longitudinal mode in the spin-1 case, we have that the d -vector mode activated by the mass β_i is not gauge-invariant and smoothly disappears in the $m \rightarrow 0$ limit. The novel feature in the spin-2 case, as already anticipated in the previous sections, lies in the d -scalar sector. We can start by integrating-out χ to get

$$\begin{aligned} S_{\text{scal.}} = \int d^D x \Big[& \frac{d-1}{d} \left(-\frac{1}{2} \dot{\Phi}^2 - \frac{d-1}{dm^2} (\partial_i \dot{\Phi})^2 + \frac{d-2}{2d} (\partial_i \dot{\Phi})^2 + \partial_i \Phi \partial_i (\psi + \ddot{\lambda}) \right) \\ & - \frac{1}{2} m^2 \left(\frac{d-1}{d} (-\Phi^2 + 2\Phi \Delta \lambda) + 2(\Delta \lambda - \Phi) \psi - \alpha (\Phi + \psi - \Delta \lambda)^2 \right) \\ & + \psi \rho - \Phi p + \frac{2(d-1)}{dm^2} \Phi \Delta \left(p + \frac{d-1}{d} \Delta \sigma \right) + \Delta \lambda \left(p + \frac{d-1}{d} \Delta \sigma \right) + \mathcal{O}(T^2) \Big]. \end{aligned} \quad (2.4.46)$$

At this point, it is convenient to trade ψ for the trace

$$h \equiv h_\mu^\mu = -\psi - \Phi + \Delta \lambda, \quad (2.4.47)$$

in which case we have

$$\begin{aligned} S_{\text{scal.}} = \int d^D x \Big[& \frac{d-1}{d} \left(-\frac{1}{2} \dot{\Phi}^2 - \frac{d-1}{dm^2} (\partial_i \dot{\Phi})^2 - \frac{d+2}{2d} (\partial_i \dot{\Phi})^2 - \partial_i \Phi \partial_i h - \Delta \Phi (\ddot{\lambda} + \Delta \lambda) \right) \\ & - \frac{1}{2} m^2 \left(\frac{d+1}{d} \Phi^2 + 2(\Delta \lambda)^2 - \frac{2(d+1)}{d} \Phi \Delta \lambda + 2h(\Phi - \Delta \lambda) - \alpha h^2 \right) \\ & - h \rho - \Phi(p + \rho) + \frac{2(d-1)}{dm^2} \Phi \Delta \left(p + \frac{d-1}{d} \Delta \sigma \right) + \Delta \lambda \left(\rho + p + \frac{d-1}{d} \Delta \sigma \right) + \mathcal{O}(T^2) \Big]. \end{aligned} \quad (2.4.48)$$

We next integrate-out h , i.e. we solve the equation of motion of h

$$h = -\frac{1}{\alpha m^2} \left(\frac{d-1}{d} \Delta \Phi - m^2 \Phi + m^2 \Delta \lambda - \rho \right) \equiv -\frac{1}{\alpha m^2} G, \quad (2.4.49)$$

plug it back inside the action. Choosing the above defined G and $\tilde{\Phi} \equiv \Phi + G/m^2$ as the independent fields instead of $\{\Phi, \lambda\}$ we get a diagonal action

$$\begin{aligned} S_{\text{scal.}} = \frac{d-1}{d} \int d^D x \Big[& -\frac{1}{2} \partial_\mu \tilde{\Phi} \partial^\mu \tilde{\Phi} - \frac{1}{2} m^2 \tilde{\Phi}^2 - \tilde{\Phi}(\rho - \Delta \sigma) \\ & + \frac{1}{m^4} \left(\frac{1}{2} \partial_\mu G \partial^\mu G - \frac{1+d(1+1/\alpha)}{2(d-1)} m^2 G^2 - \frac{m^2}{d-1} G(\rho - dp) \right) + \mathcal{O}(T^2) \Big]. \end{aligned} \quad (2.4.50)$$

Note that G is proportional to the on-shell trace h , so in particular it is a Lorentz scalar on-shell. As a check, we can compare its equation of motion

$$\left(\square + \frac{1 + d(1 + 1/\alpha)}{d - 1} m^2 \right) G = -\frac{m^2}{d - 1} (\rho - dp), \quad (2.4.51)$$

with (2.2.19), using (2.4.49), and see that they match exactly. We have thus shown what we had claimed in Sect. 2.2, i.e. it is indeed the trace which is the unstable mode and, more precisely, it is a ghost with mass

$$m_{\text{ghost}}^2 = \frac{1 + d(1 + 1/\alpha)}{d - 1} m^2. \quad (2.4.52)$$

Unlike the case of the d -scalar sector or massive electrodynamics, here the action is non-analytic in both m^2 and α , if our fields are combinations of the $h_{\mu\nu}$ that are analytic in these parameters. In the $\alpha \rightarrow 0$ limit, with m kept fixed, we see that m_{ghost} diverges, while the coupling to the source remains constant, so we effectively have $G = 0$ and thus also $\tilde{\Phi} = \Phi$. It is also instructive to see how this condition appears when working directly at the $\alpha = 0$ point. So let us go back at the level of (2.4.53) where now h is a Lagrange multiplier. Integrating it out will therefore result in fixing another field, which we choose to be λ . The equation of motion of h is then simply $G = 0$ and, plugging this inside the action we are indeed left with¹⁴

$$S_{\text{scal.}} = \frac{d - 1}{d} \int d^D x \left[-\frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2 - \Phi(\rho - \Delta\sigma) + \mathcal{O}(T^2) \right]. \quad (2.4.53)$$

At this stage we can make a series of remarks. First, we have now reached in this formalism the same conclusion we did in the previous section, namely, that the scalar degree of freedom in FP theory is gauge-invariant and it does not go away in the $m \rightarrow 0$ limit. Second, what we have gained here with respect to the canonical formalism is a clearer picture of the whole (m^2, α) plane provided by (2.4.50). Indeed, we are now able to see more clearly the fact that the vDVZ discontinuity as $m \rightarrow 0$ arises only for $\alpha = 0$. If we go back to (2.4.50) and take $m \rightarrow 0$ with $\alpha \neq 0$ fixed, we get

$$\tilde{\Phi} \equiv \Phi + \frac{1}{m^2} G \rightarrow \frac{1}{m^2} G, \quad (2.4.54)$$

so that the kinetic terms in the action cancel out and we retrieve the same number of degrees of freedom as in the massless theory. We now understand that the vDVZ

¹⁴After deriving this result we were informed by S. Deser (private communication) that a similar form was obtained in an old and little known paper [11]. Interestingly enough, this paper appeared in 1966, that is 14 years before the introduction of gauge-invariant variables by Bardeen [10] in cosmological perturbation theory.

discontinuity and the fact that the FP d -scalar mode is gauge-invariant are intimately related features. Indeed, in the $m \rightarrow 0$ limit we retrieve the gauge symmetry so that only gauge-invariant combinations can survive. For instance, in the case of electrodynamics, we have that the longitudinal mode must disappear since it is not gauge-invariant. Here, for $\alpha \neq 0$ we have that the d -scalar modes are not gauge-invariant and must thus disappear in the $m \rightarrow 0$ limit. On $\alpha = 0$ however, since Φ survives the $m \rightarrow 0$ limit, it must be gauge-invariant.

Hidden Gauge Symmetry

It is now the appropriate moment for discussing the fact that FP theory seems to have something that looks like a symmetry but which is not quite it. This was the novel result of our paper [2], where (2.4.53) was derived, and we have therefore elaborated on the physical significance of Φ being gauge-invariant.

Already from Sect. 2.2.2 we know that, for on-shell configurations $h_{\mu\nu}$, the action is invariant under gauge transformations of the form $\xi_\mu = \partial_\mu \theta$ (see Eq. (2.2.23)), but not the equations of motion. The fact that this holds for on-shell configurations is equivalent to the fact that here some equations of motion have to be used, i.e. the ones of the non-dynamical fields, in order to get the invariance. The additional information we gain here with respect to Sect. 2.2.2 is that the sector of the equations of motion which corresponds to the dynamical field Φ is also gauge-invariant, a feature which is not visible when we work with $h_{\mu\nu}$. Moreover, the transformation considered in Sect. 2.2.2 was one-dimensional, whereas here we have two gauge parameters in the d -scalar sector of ξ_μ (2.4.27), that is, A and B . Trading the former, for $\tilde{A} \equiv A - \dot{B}$, we can write the corresponding gauge parameter (2.4.27)

$$\xi_\mu = \tilde{A} \delta_\mu^0 + \partial_\mu B, \quad (2.4.55)$$

so that B corresponds to the θ parameter considered in Sect. 2.2.2, while A parametrizes the additional transformation under which (2.4.53) is invariant.

Now observe that, if we perform the gauge transformation (2.4.55) at the level of the original action, we get a new action that depends on the gauge parameters

$$S_{A,B}[h_{\mu\nu}] = S[h_{\mu\nu}] + \Delta S[h_{\mu\nu}, A, B], \quad (2.4.56)$$

where $\Delta S \neq 0$ for general $h_{\mu\nu}$, so that this is not a gauge symmetry. If we decompose $h_{\mu\nu}$ harmonically, we have that $S_{A,B}$ will correspond to S with $\psi, \chi, \Phi, \lambda$ replaced by

$$\psi - 2\dot{A}, \quad \chi - A - \dot{B}, \quad \Phi, \quad \lambda - 2B, \quad (2.4.57)$$

respectively. Then, since these are simply the original variables that have been shifted, integrating-out the non-dynamical ones will automatically yield again (2.4.53), i.e. whatever the values of A and B . Thus, although the actions $S_{A,B}$ are not the same, they do reduce to the same action once the non-dynamical fields are integrated-out. In conclusion, although the action is *not* invariant under the A, B transformation, the physics *is*. It is in this sense that this is a “hidden” gauge symmetry.

2.4.3 de-Sitter Background

As a final display of the power of the harmonic formalism, let us apply it to the case where the background space-time is de-Sitter and see whether it is still a gauge-invariant field which propagates in the d -scalar sector for $\alpha = 0$. This is not guaranteed a priori because on flat space-time we concluded that it was the vDVZ discontinuity which was responsible for this and, as it turns out, there is no discontinuity on a de-Sitter background [4, 5].

It is convenient to work in the following coordinates

$$g_{00} = -1, \quad g_{0i} = 0, \quad g_{ij} = a^2 \delta_{ij}, \quad a \equiv e^{Ht}, \quad (2.4.58)$$

where $a(t)$ is the scale factor and H the (constant) Hubble parameter. We consider directly the case of the linear massive spin-2 field and obtain its action by linearizing the Einstein-Hilbert action with cosmological constant

$$\Lambda \equiv \frac{d(d-1)}{2} H^2, \quad (2.4.59)$$

around the corresponding de-Sitter solution. Also appending a FP mass term and a linear source this gives

$$\begin{aligned} S = \int d^D x \, a^d \left[-\frac{1}{2} \nabla_\mu h_{\nu\rho} \nabla^\mu h^{\nu\rho} + \nabla_\mu h_{\nu\rho} \nabla^\nu h^{\mu\rho} - \nabla_\mu h^{\mu\nu} \nabla_\nu h + \frac{1}{2} \nabla_\mu h \nabla^\mu h \right. \\ \left. + dH^2 \left(h_{\mu\nu} h^{\mu\nu} - \frac{1}{2} h^2 \right) - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) + h_{\mu\nu} T^{\mu\nu} \right], \end{aligned} \quad (2.4.60)$$

where the non-vanishing components of the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ are

$$\Gamma^i_{j0} = H \delta^i_j, \quad \Gamma^0_{ij} = H g_{ij}. \quad (2.4.61)$$

For $m = 0$, we have the gauge symmetry

$$\delta h_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu, \quad (2.4.62)$$

provided the source satisfies the background-covariant conservation equation $\nabla_\mu T^{\mu\nu} = 0$, while for $m^2 = (d-1)H^2$ we have a one-dimensional gauge symmetry

$$\delta h_{\mu\nu} = -\nabla_\mu \nabla_\nu \theta - g_{\mu\nu} H^2 \theta, \quad (2.4.63)$$

provided the source satisfies

$$\nabla_\mu \nabla_\nu T^{\mu\nu} + H^2 T = 0. \quad (2.4.64)$$

The latter case is known as the “partially massless” theory because the gauge symmetry eliminates the d -scalar mode. It is convenient to define the following field strength [12]

$$F_{\mu\nu\rho} \equiv \nabla_\mu h_{\nu\rho} - \nabla_\nu h_{\mu\rho}, \quad F_\mu \equiv g^{\nu\rho} F_{\mu\nu\rho}, \quad (2.4.65)$$

which is invariant under (2.4.63) and in terms of which the action becomes

$$S = \int d^D x a^d \left[-\frac{1}{4} (F_{\mu\nu\rho} F^{\mu\nu\rho} - 2F_\mu F^\mu) - \frac{1}{2} M^2 (h_{\mu\nu} h^{\mu\nu} - h^2) - h_{\mu\nu} T^{\mu\nu} \right], \quad (2.4.66)$$

where

$$M^2 \equiv m^2 - (d-1)H^2, \quad (2.4.67)$$

is precisely zero for the partially massless theory. This representation is quite elegant from the point of view of the partially massless theory $M = 0$ because it exhibits many analogies with electrodynamics: there is a one-dimensional gauge symmetry, the theory can be written as the square of some gauge-invariant field strength and there is a cohomological chain structure between the gauge parameter θ , the field $h_{\mu\nu}$ and the field strength $F_{\mu\nu\rho}$ [12, 13]. Then, M appears as the mass that will break this symmetry and activate the d -scalar mode. In particular, we can already anticipate that for $M^2 < 0$ the theory will be unstable [14, 15].

Here we will focus on the d -scalar sector of the theory only, since this is where the exotic features lie, and we will neglect the source for simplicity. In defining and using harmonic variables we must now pay attention to the fact that the position of the spatial indices matters, i.e. they are displaced using g_{ij} , so for instance

$$\Delta \equiv g^{ij} \partial_i \partial_j = a^{-2} \partial_i \partial_i, \quad (2.4.68)$$

The definitions of the harmonic variables are the same, except for the spatial sectors whose natural generalization is

$$h_{ij} \equiv \frac{1}{d} g_{ij} \phi + \left(\partial_i \partial_j - \frac{1}{d} g_{ij} \Delta \right) \lambda, \quad (2.4.69)$$

$$T_{ij} \equiv g_{ij} p + \left(\partial_i \partial_j - \frac{1}{d} g_{ij} \Delta \right) \sigma. \quad (2.4.70)$$

Note that this changes only the definitions of ϕ and p . The inverse relation now reads

$$\chi = \Delta^{-1} \partial^i h_{0i}, \quad (2.4.71)$$

$$\phi = h_i^i, \quad (2.4.72)$$

$$\lambda = -\frac{1}{d-1} \Delta^{-1} [h_i^i - d \Delta^{-1} \partial^i \partial^j h_{ij}], \quad (2.4.73)$$

where P_i^j is the same as before but one must use g_{ij} to displace its indices now. The gauge transformation (2.4.62) reads

$$\delta\psi = -2\dot{A}, \quad (2.4.74)$$

$$\delta\chi = -A - \dot{B} + 2HB, \quad (2.4.75)$$

$$\delta\phi = -2\Delta B + 2dHA, \quad (2.4.76)$$

$$\delta\lambda = -2B, \quad (2.4.77)$$

so the Bardeen variables are

$$\Psi \equiv \psi - (2\dot{\chi} - \ddot{\lambda} + 2H\dot{\lambda}), \quad (2.4.78)$$

$$\Phi \equiv \Phi_0 - dH(2\chi - \dot{\lambda} + 2H\lambda), \quad (2.4.79)$$

where $\Phi_0 \equiv \Delta\lambda - \phi$ is the Φ of Minkowski space-time. We see that now both combinations include time-derivatives of the original variables so that none of these can be taken as a fundamental field since their initial conditions are undetermined. In particular, this seems to imply that the d -scalar degree of freedom on the FP point $\alpha = 0$ will not be gauge-invariant.

Nevertheless, one must note that the Bardeen variables are the only gauge-invariant combinations (up to combinations among themselves) that are *local in time* in the harmonic variables. This is certainly convenient, although not at all a physical requirement. In fact, as we will see in a moment, if we abandon this property we get access to gauge-invariant combinations that do not suffer from the above initial condition problem.

We can now write down the d -scalar part of the action

$$S_{\text{scal.}} = \frac{d-1}{d} \int d^D x a^d \left[-\frac{1}{2} (\dot{\Phi} - dH\Psi)^2 + \frac{d-2}{2d} \partial_i \Phi \partial^i \Phi + \partial_i \Phi \partial^i \Psi \right. \\ \left. - \frac{1}{2} m^2 (-\Phi_0^2 + 2\Phi_0 \Delta\lambda + \frac{2d}{d-1} (\Delta\lambda - \Phi_0)\psi - \frac{2d}{d-1} \partial_i \chi \partial^i \chi) \right]. \quad (2.4.80)$$

Again, when $m = 0$ we see that only gauge-invariant combinations appear and we retrieve the flat space-time result for $H \rightarrow 0$. For $H \neq 0$, the second and third terms of the first line can be rewritten in a convenient way

$$-\frac{1}{dH} \int d^D x a^d \partial_i \Phi \partial^i (\dot{\Phi} - dH\Psi) = \int d^D x a^d \left[\frac{d-2}{2d} \partial_i \Phi \partial^i \Phi + \partial_i \Phi \partial^i \Psi \right], \quad (2.4.81)$$

where we have used the fact that

$$\partial_i \Phi \partial^i \dot{\Phi} = \frac{1}{2} a^{-2} \partial_t (\partial_i \Phi)^2, \quad (2.4.82)$$

and then integrated by parts the time derivative. Observe also that

$$K \equiv \dot{\Phi} - dH\Psi = \dot{\Phi}_0 - dH\psi, \quad (2.4.83)$$

so that this combination actually only depends on Φ_0 and ψ . We can then consider $\psi, \chi, \Phi_0, \lambda$ as our independent variables and integrate by parts here and there to finally get

$$\begin{aligned} S_{\text{scal.}} = \frac{d-1}{d} \int d^D x a^d \left[-\frac{1}{2} K^2 - \frac{1}{dH} \partial_i (\Phi_0 - 2dH\chi) \partial^i K \right. \\ \left. - \Delta \lambda \left(\dot{K} + dHK + m^2 \Phi_0 + \frac{dm^2}{d-1} \psi \right) \right. \\ \left. + \frac{1}{2} m^2 \left(\Phi_0^2 + \frac{2d}{d-1} \Phi_0 \psi + \frac{2d}{d-1} \partial_i \chi \partial^i \chi \right) \right]. \end{aligned} \quad (2.4.84)$$

We start by integrating-out χ

$$\begin{aligned} S_{\text{scal.}} = \frac{d-1}{d} \int d^D x a^d \left[-\frac{1}{2} K^2 - \frac{d-1}{dm^2} \partial_i K \partial^i K - \frac{1}{dH} \partial_i \Phi_0 \partial^i K \right. \\ \left. - \Delta \lambda \left(\dot{K} + dHK + m^2 \Phi_0 + \frac{dm^2}{d-1} \psi \right) + \frac{1}{2} m^2 \left(\Phi_0^2 + \frac{2d}{d-1} \Phi_0 \psi \right) \right]. \end{aligned}$$

We now rescale our fields

$$\{\psi, \Phi_0, \lambda\} \rightarrow a^{-(d-1)} \{\psi, \Phi_0, \lambda\}, \quad (2.4.85)$$

and trade ψ for the new variable

$$\psi' \equiv \psi + \frac{d-1}{d} \Phi_0, \quad (2.4.86)$$

so that

$$K \rightarrow a^{-(d-1)} (\dot{\Phi}_0 - dH\psi') \equiv a^{-(d-1)} K', \quad (2.4.87)$$

and get

$$\begin{aligned} S_{\text{scal.}} = \frac{d-1}{d} \int d^D x a^{-(d-2)} \left[-\frac{1}{2} K'^2 - \frac{d-1}{dm^2} \partial_i K' \partial^i K' - \frac{1}{dH} \partial_i \Phi_0 \partial^i K' \right. \\ \left. - \Delta \lambda \left(\dot{K}' + HK' + \frac{dm^2}{d-1} \psi' \right) + \frac{1}{2} m^2 \left(-\Phi_0^2 + \frac{2d}{d-1} \Phi_0 \psi' \right) \right]. \end{aligned}$$

We can now trade Φ_0 for the new variable

$$\Omega(t) \equiv \Phi_0(t) - dH \int_{t_i}^t dt' \psi'(t') + \frac{(d-1)H}{M^2} [\dot{\Phi}_0(t_i) - dH\psi'(t_i) + H\Phi_0(t_i)], \quad (2.4.88)$$

where t_i is the time at which the initial conditions are given and we have omitted the $\vec{x}$ dependence for notational simplicity. The choice of the time-independent term will be justified later. As already suggested above, this is a non-local generalization of Φ_0 to de-Sitter space-time, for which the Cauchy problem is well-defined. Indeed, it is gauge invariant, as we will show in a moment, and the initial data $\{\Omega(t_i), \psi'(t_i)\}$ are in bijection with $\{\Phi_0(t_i), \psi'(t_i)\}$

$$\Omega(t_i) = \frac{m^2}{M^2} \Phi_0(t_i) + \frac{(d-1)H}{M^2} [\dot{\Phi}_0(t_i) - dH\psi'(t_i)], \quad (2.4.89)$$

$$\dot{\Omega}(t_i) = \dot{\Phi}_0(t_i) - dH\psi'(t_i), \quad (2.4.90)$$

contrary to the Bardeen variable Φ . We can then invert this to get

$$\Phi_0(t) = \Omega(t) + dH \int_{t_i}^t dt' \psi'(t') - \frac{(d-1)H}{m^2} [\dot{\Omega}_0(t_i) + H\Omega(t_i)], \quad (2.4.91)$$

so, although $K' = \dot{\Omega}$, performing this replacement in the action will yield non-local terms because of the presence of undotted Φ_0 's. However, after integrating-out λ and solving for ψ' , we get that the latter becomes a total time derivative

$$\psi' = -\frac{d-1}{dm^2} (\ddot{\Omega} + H\dot{\Omega}), \quad (2.4.92)$$

so that (2.4.91) becomes local

$$\Phi_0(t) = \frac{M^2}{m^2} \Omega(t) - \frac{(d-1)H}{m^2} \dot{\Omega}(t). \quad (2.4.93)$$

We now understand that the time-independent piece in (2.4.88) was chosen precisely such that it cancels the one arising in the above integral. We are thus left with an action for Ω alone which, after many integrations by parts, gives

$$S_{\text{scal.}} = \frac{d-1}{d} \frac{M^2}{m^2} \int d^D x a^{-(d-2)} \left[-\frac{1}{2} \partial_\mu \Omega \partial^\mu \Omega - \frac{1}{2} M^2 \Omega^2 \right]. \quad (2.4.94)$$

If we rescale back in order to obtain the volume form $\sqrt{-g} = a^d$ for the integration measure

$$\Omega \rightarrow a^{d-1} \Omega, \quad (2.4.95)$$

we get

$$S_{\text{scal.}} = \frac{d-1}{d} \frac{M^2}{m^2} \int d^D x \sqrt{-g} \left[-\frac{1}{2} \partial_\mu \Omega \partial^\mu \Omega - \frac{1}{2} m^2 \Omega^2 \right], \quad (2.4.96)$$

so this has also the effect of replacing M^2 with m^2 in the mass term. The dynamical mode in the d -scalar sector is therefore Ω , which in terms of the original fields reads

$$\begin{aligned} \Omega(t) &\equiv \Phi_0(t) - dH a^{-(d-1)}(t) \int_{t_i}^t dt' a^{d-1}(t') \left(\psi(t') + \frac{d-1}{d} \Phi_0(t') \right) \\ &\quad + \left(\frac{a(t_i)}{a(t)} \right)^{d-1} \frac{(d-1)H}{M^2} [\dot{\Phi}_0(t_i) - dH\psi'(t_i) + H\Phi_0(t_i)] \\ &= a^{-(d-1)}(t) \int_{t_i}^t dt' a^{d-1}(t') K(t') + \left(\frac{a(t_i)}{a(t)} \right)^{d-1} \left[\frac{m^2}{M^2} \Phi_0(t_i) + \frac{(d-1)H}{M^2} K(t_i) \right]. \end{aligned} \quad (2.4.97)$$

Under a gauge transformation (2.4.62) we have that K is gauge-invariant so

$$\delta\Omega(t) \sim \delta\Phi_0(t_i) \sim A(t_i), \quad (2.4.98)$$

and thus Ω is gauge-invariant if we set $A(t_i) = 0$. Note that this restriction is by no means a loss of symmetry since it concerns only a subset of measure zero of the gauge parameters. One can still use such an $A(t)$ to trivialize the time-evolution of a field mode. Thus, after having integrated-out the non-dynamical fields, the d -scalar sector remains gauge-invariant even in de-Sitter space.

A very elegant feature of our result (2.4.96) is that it renders the dependence of the spectrum on M quite transparent. Ω becomes non-dynamical when $M \rightarrow 0$, in which case we reach the partially massless theory with gauge symmetry (2.4.63),¹⁵ while for $M^2 < 0$ that mode becomes a ghost. The stability condition $M^2 \geq 0$ is known as the ‘‘Higuchi bound’’ [15]. We also see that the ‘‘natural variables’’ with respect to the interpretation of M^2 as being the mass of the partially massless theory are the rescaled ones, since it is for these fields that M appears as the mass (2.4.94) and for which Ω involves no a in its definition (2.4.88).

2.5 Propagator

The dynamical content and stability of a linear theory can also be deduced by looking at its propagator. Moreover, since the propagator is an essential building block of perturbative QFT, it is important to be able to ‘‘read from it’’ this important information

¹⁵Since the mapping between Ω and Φ_0 is singular as $M \rightarrow 0$, we should actually check this result by working directly on the $M = 0$ point, in which case integrating out λ to fix ψ' gives $S_{\text{scal.}} = 0$.

of the theory. We will not write explicitly the ϵ prescription here since it depends on whether one is interested in classical or quantum propagation. It will be however useful to use some QFT language, e.g. the number of dynamical fields becomes the number of particle polarizations/states.

2.5.1 Spin 1

Writing the Proca action (2.2.1) in the form

$$S = \int d^D x \left[\frac{1}{2} A_\mu \mathcal{K}^{\mu\nu} A_\nu - A_\mu j^\mu \right], \quad (2.5.1)$$

we can identify the quadratic structure

$$\mathcal{K}^{\mu\nu} = \eta^{\mu\nu} (\square - m^2) - \partial^\mu \partial^\nu. \quad (2.5.2)$$

The propagator is defined by

$$\mathcal{K}^{\mu\rho} D_{\rho\nu} = i \delta_\mu^\nu, \quad (2.5.3)$$

whose solution is

$$D_{\mu\nu}(k) = -\frac{i}{k^2 + m^2} \left(\eta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} \right). \quad (2.5.4)$$

The exchange of a photon between two vertices in the computation of a scattering amplitude will then be controlled by the saturated propagator

$$j^{*\mu}(k) D_{\mu\nu}(k) j'^\nu(k) = j^{*\mu}(k) \left[-\frac{i}{k^2 + m^2} \eta_{\mu\nu} \right] j'^\nu(k), \quad (2.5.5)$$

where here j^μ , j'^μ either represent external on-shell sources, in which case conservation implies $k_\mu j^\mu(k) = k_\mu j'^\mu(k) = 0$, or parts of a Feynman diagram to which the photon is attached, in which case it is the Ward identity¹⁶ which implies these equations. In the classical case, the saturated propagator is what controls the interaction mediated by the electromagnetic field in the perturbative equations of motion of the fields present in the source.

In the massive case we have as many possible inversions of $\mathcal{K}_{\mu\nu}$ as with $\square - m^2$ because of the homogeneous solutions. These are parametrized by all the possi-

¹⁶The Ward identity is usually presented as a direct consequence of gauge symmetry and it can therefore appear as a surprise that it still holds in the massive case. However, note that one can also derive the identity by simply using the operator equation $\partial_\mu \hat{A}^\mu = 0$, which is valid in the massive case, when computing correlation functions with on-shell external momenta $\partial_\mu \langle 0 | T \{ \hat{A}_\mu(x) \dots \} | 0 \rangle = 0$. Thus, the Ward identity still holds in massive electrodynamics, not because $\partial_\mu A^\mu$ contains no propagating degrees of freedom as in the massless case, but because $\partial_\mu A^\mu$ is simply zero on-shell.

ble linear superposition amplitudes $a_i(\vec{k})$ (belonging to some space of integrable functions), that are functions on $\mathbb{R}^d$. Going to the $m = 0$ case enlarges that kernel dramatically because now it also includes all the pure-gauge solutions $A_\mu = \partial_\mu \theta$, parametrized by a function θ on $\mathbb{R}^D$. Thus, on top of the pole contour prescription, which can be translated into a prescription on initial/final conditions, one must also give a prescription for picking a preferred gauge, i.e. one must add a gauge-fixing term. The usual Lorentz-invariant choice is

$$S_{\text{gf}} = -\frac{1}{2\xi} \int d^D x (\partial_\mu A^\mu)^2, \quad (2.5.6)$$

so that

$$\mathcal{K}^{\mu\nu} = \eta^{\mu\nu} \square - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu, \quad (2.5.7)$$

is invertible and¹⁷

$$D_{\mu\nu}(k) = -\frac{i}{k^2} \left(\eta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right). \quad (2.5.8)$$

What matters for the gauge-fixing term to be valid is that the saturated propagator must be independent of it because the physics cannot depend on a choice of gauge. Since $k_\mu j^\mu(k) = 0$, which is also a consequence of gauge symmetry when $m = 0$, we have indeed the ξ -independent result

$$j^{*\mu}(k) D_{\mu\nu}(k) j'^\nu(k) = j^{*\mu}(k) \left[-\frac{i}{k^2} \eta_{\mu\nu} \right] j'^\nu(k). \quad (2.5.9)$$

Comparing with (2.5.5) we note that the interaction between two sources mediated by the photon is continuous in the $m \rightarrow 0$ limit. At the same time however, we know that the massive photon has d polarizations, while the massless one has $d - 1$ polarizations. To understand how a discontinuity in this number can be consistent with a continuous limit at the propagator level, we decompose j^μ and j'^μ harmonically (2.4.2) which in Fourier space gives

$$j_0 = -\rho, \quad j_i(k) = i k_i \sigma(k) + \sigma_i(k), \quad k_i \sigma_i(k) = 0, \quad (2.5.10)$$

and similarly for j'^μ . We then restrict to tree-level diagrams and sources with “mass” $m_s^2 = -k^2$, so that m_s is the “mass” of the virtual photon that is being exchanged. For instance, in the case where the source is made of minimally coupled electrons and positrons we have that $m_s \geq 2m_e$. We also consider the case $m_s > m$ so that we

¹⁷For non-linear theories the gauge-fixing term breaks the unitarity of the S -matrix and one must also include Faddeev–Popov fields to restore it.

do not have to deal with the complications of resonances.¹⁸ We can then write the conservation equation (2.4.36) as

$$\rho = \frac{i\vec{k}^2}{\sqrt{m_s^2 + \vec{k}^2}} \sigma, \quad (2.5.11)$$

and similarly for j'^μ , so that the saturated propagator reads

$$j^{*\mu}(k) D_{\mu\nu}(k) j'^\nu(k) \Big|_{k^2 = -m_s^2} = -\frac{i}{m^2 - m_s^2} \left[\frac{m_s^2 \vec{k}^2}{m_s^2 + \vec{k}^2} \sigma^*(k) \sigma'(k) + \sigma_i^*(k) \sigma'_i(k) \right]. \quad (2.5.12)$$

The first term in the square bracket represents the exchange of the longitudinal photons between the longitudinal modes of the sources, while the second term corresponds to the exchange of a transverse photon between the transverse modes of the sources. We can now focus on the case where $m_s^2 \rightarrow m^2$ from above, so that the photon gets close to being real. It can therefore be considered as an external photon that has been “produced” by j'^μ at $t \rightarrow -\infty$ and then “detected” by its interaction with j^μ at $t \rightarrow \infty$.

This shows how the continuity in the saturated propagator can be reconciled with the discontinuity in the dynamical fields of the photon: the longitudinal information is simply proportional to m^2 for real photons and thus smoothly decouples in the massless limit. It is therefore not enough to look at the unsaturated propagators to deduce the number of dynamical fields in the theory, one must also make use of the conservation equation of the source, which brings in the mass dependence.

Note that the source components that appear are the ones that are being propagated so that counting them gives us a *lower bound* on the number of dynamical fields N_d . In the massive case we have σ_i and σ , that is $N_d \geq d$, while in the massless limit the longitudinal part σ smoothly decouples and becomes unobservable and we are thus left with $N_d \geq d - 1$. Here these inequalities are saturated, as we know. We will see however that this is not always the case for non-local theories in the presence of ghosts.

¹⁸ Demanding heavier sources $m_s > m$ and no loops implies that the virtual photon can never be on-shell, i.e. it is never a “real” photon. Alternatively, if $m_s < m$, then the process in which the photon is on-shell would be kinematically allowed, in which case the propagator would be singular, implying an infinite probability for this process to occur. As in the case of any heavy particle, the resolution of this apparent problem comes by noting that the heavy particle becomes unstable precisely when $m_s < m$, since it can then disintegrate into the source’s particles. By the optical theorem, we then have that the imaginary part of the vacuum polarization diagram becomes non-zero. Since that diagram is responsible for shifting the mass m under radiative corrections in the propagator, we get that the poles of the *renormalized* propagator have a non-vanishing imaginary part. Thus, the case $k^2 = -m_{\text{ren}}^2$, where m_{ren} is the renormalized mass of the photon, is not a singularity of the renormalized propagator but rather the maximum of the so-called “Breit-Wigner” resonance.

Finally, as far as stability is concerned, we have that the saturated propagator (2.5.12) is the one of a healthy scalar times a positive-definite scalar product of j_i and j'_i , so that this theory is stable.

2.5.2 Spin 2

Let us start by identifying the quadratic structure of (2.2.13)

$$\mathcal{K}^{\mu\nu\rho\sigma} \equiv \mathcal{E}^{\mu\nu\rho\sigma} - m^2 (\eta^{\mu(\rho} \eta^{\sigma)\nu} - (1 + \alpha) \eta^{\mu\nu} \eta^{\rho\sigma}), \quad (2.5.13)$$

where $\mathcal{E}$ was defined in (2.2.14). The propagator

$$\mathcal{K}^{\mu\nu\alpha\beta} D_{\alpha\beta\rho\sigma} = i \delta_{(\rho}^{\mu} \delta_{\sigma)}^{\nu}, \quad (2.5.14)$$

is given by

$$\begin{aligned} D_{\mu\nu\rho\sigma}(k) = & -\frac{i}{k^2 + m^2} \left[\frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) - \frac{1}{d} \left(1 - \frac{\alpha}{d} \frac{k^2 + m^2}{\mu^2} \right) \eta_{\mu\nu} \eta_{\rho\sigma} \right. \\ & + \frac{1}{2} \left(\eta_{\mu\rho} \frac{k_\nu k_\sigma}{m^2} + \eta_{\mu\sigma} \frac{k_\nu k_\rho}{m^2} + \eta_{\nu\rho} \frac{k_\mu k_\sigma}{m^2} + \eta_{\nu\sigma} \frac{k_\mu k_\rho}{m^2} \right) \\ & \left. - \frac{1 + 2\alpha}{d} \left(\eta_{\mu\nu} \frac{k_\rho k_\sigma}{\mu^2} + \eta_{\rho\sigma} \frac{k_\mu k_\nu}{\mu^2} \right) + \frac{(1 + 2\alpha)(d - 1)}{d} \frac{k_\mu k_\nu k_\rho k_\sigma}{m^2 \mu^2} \right], \end{aligned}$$

where

$$\mu^2 \equiv m^2 - \alpha \left(\frac{d - 1}{d} k^2 - \frac{d + 1}{d} m^2 \right). \quad (2.5.15)$$

In the saturated propagator with conserved sources the terms with uncontracted k_μ 's drop

$$\begin{aligned} T^{*\mu\nu} D_{\mu\nu\rho\sigma} T'^{\rho\sigma} = & -\frac{i}{k^2 + m^2} \left[T_{\mu\nu}^* T'^{\mu\nu} - \frac{1}{d} \left(1 - \frac{\alpha}{d} \frac{k^2 + m^2}{\mu^2} \right) T^* T' \right] \\ = & -\frac{i}{k^2 + m^2} \left[T_{\mu\nu}^* T'^{\mu\nu} - \frac{1}{d} T^* T' \right] - \frac{\alpha}{d^2} \frac{i}{\mu^2} T^* T'. \end{aligned} \quad (2.5.16)$$

Note that this neatly splits into the Fierz–Pauli propagator $\alpha = 0$ plus an extra scalar propagator which can be written as

$$-\frac{i}{-k^2 + m_{\text{ghost}}^2} \frac{T^* T'}{d(d - 1)}, \quad (2.5.17)$$

with m_{ghost}^2 given precisely by (2.4.52). Indeed, this is the pole corresponding to the ghost since the “kinetic” part $\sim k^2$ in the denominator comes with the wrong sign and the corresponding source is the trace T . In the case $\alpha = -1/2$ we have that $m_{\text{ghost}}^2 = -m^2$, so that this becomes also a tachyon, but then the tensor structure becomes the one of the massless theory

$$T^{*\mu\nu} D_{\mu\nu\rho\sigma} T'^{\rho\sigma} = -\frac{i}{k^2 + m^2} \left[T_{\mu\nu}^* T'^{\mu\nu} - \frac{1}{d-1} T^* T' \right]. \quad (2.5.18)$$

Let us now focus on the case $\alpha = 0$. As we did for the spin-1 case, we can again perform the harmonic decomposition of the sources (2.4.18) and use the conservation equations (2.4.36) with “source mass” $m_s^2 = -k^2$

$$\rho = \frac{i\vec{k}^2}{\sqrt{m_s^2 + \vec{k}^2}} q, \quad q = -\frac{i}{\sqrt{m_s^2 + \vec{k}^2}} \left(p - \frac{d-1}{d} \vec{k}^2 \sigma \right), \quad q_i = \frac{1}{2} \frac{i\vec{k}^2}{\sqrt{m_s^2 + \vec{k}^2}} \sigma_i, \quad (2.5.19)$$

to get

$$T^{*\mu\nu} D_{\mu\nu\rho\sigma} T'^{\rho\sigma} = -\frac{i}{m^2 - m_s^2} \left[x^\dagger M x' + \frac{1}{2} \frac{m_s^2 \vec{k}^2}{m_s^2 + \vec{k}^2} \sigma_i^* \sigma'_i + \sigma_{ij}^* \sigma'_{ij} \right]. \quad (2.5.20)$$

where $x \equiv (p, \frac{d-1}{d} \vec{k}^2 \sigma)$ and M is a 2×2 matrix with eigenvalues

$$\lambda_- = 0, \quad \lambda_+ = \frac{(d^2 - 2d + 2)\vec{k}^4 + 2d\vec{k}^2 m_s^2 + d^2 m_s^4}{d(d-1)(m_s^2 + \vec{k}^2)^2} > 0, \quad (2.5.21)$$

so that there is only one pole corresponding to the d -scalar sector and with the correct sign, as expected. We see that the d -vector and d -tensor sectors are the exact analogues of the d -scalar and d -vector sectors of electrodynamics (2.5.12). Considering the limits $m_s^2 \rightarrow m^2 \rightarrow 0$ we get that the σ_i part smoothly decouples and we are left with only σ_{ij} . In the d -scalar sector however we have the vDVZ discontinuity since

$$\lambda_+ \rightarrow \frac{d^2 - 2d + 2}{d(d-1)} \neq 0, \quad (2.5.22)$$

so this pole remains. We can compare this with the case $m = 0$ where, because of the gauge symmetry, we must add a gauge-fixing term in the action in order to invert the quadratic structure. The usual Lorentz-invariant choice is

$$S_{\text{gf}} = -\frac{1}{\xi} \int d^D x \partial_\mu \bar{h}^{\mu\nu} \partial^\rho \bar{h}_{\rho\nu}, \quad (2.5.23)$$

where $\bar{h}_{\mu\nu}$ has been defined in (2.2.37), in which case one has

$$\begin{aligned} \mathcal{K}^{\mu\nu\rho\sigma} = & \left(\eta^{\mu(\rho} \eta^{\sigma)\nu} - \left(1 - \frac{1}{2\xi} \right) \eta^{\mu\nu} \eta^{\rho\sigma} \right) \square \\ & - \left(1 - \frac{1}{\xi} \right) \left(\eta^{\mu(\rho} \partial^{\sigma)} \partial^{\nu} + \eta^{\nu(\rho} \partial^{\sigma)} \partial^{\mu} - \eta^{\mu\nu} \partial^{\rho} \partial^{\sigma} - \eta^{\rho\sigma} \partial^{\mu} \partial^{\nu} \right), \end{aligned} \quad (2.5.24)$$

with inverse

$$\begin{aligned} D_{\mu\nu\rho\sigma} = & -\frac{i}{k^2} \left[\frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}) - \frac{1}{d-1} \eta_{\mu\nu} \eta_{\rho\sigma} \right. \\ & \left. + \frac{1}{2} (1 - \xi) \left(\eta_{\mu\rho} \frac{k_\nu k_\sigma}{m^2} + \eta_{\mu\sigma} \frac{k_\nu k_\rho}{m^2} + \eta_{\nu\rho} \frac{k_\mu k_\sigma}{m^2} + \eta_{\nu\sigma} \frac{k_\mu k_\rho}{m^2} \right) \right]. \end{aligned} \quad (2.5.25)$$

Comparing the saturated one

$$T^{*\mu\nu} D_{\mu\nu\rho\sigma} T'^{\rho\sigma} = -\frac{i}{k^2 + m^2} \left[T_{\mu\nu}^* T'^{\mu\nu} - \frac{1}{d-1} T^{*T'} \right] = -\frac{i}{m^2 - m_s^2} \sigma_{ij}^* \sigma'_{ij}. \quad (2.5.26)$$

with (2.5.16) for $\alpha = 0$, we see that the discontinuity lies in the factor in front of the $\sim T^* T'$ term which is $1/d$ instead of $1/(d-1)$. This difference is what is precisely needed in order to cancel the d -scalar pole. Finally, here too we can see that the $m \rightarrow 0$ and $\alpha \rightarrow 0$ limits do not commute. Indeed, taking $m \rightarrow 0$ while keeping $\alpha \neq 0$ fixed we get that

$$1 - \frac{\alpha}{d} \frac{k^2 + m^2}{\mu^2} \rightarrow \frac{d}{d-1}, \quad (2.5.27)$$

so (2.5.16) becomes the massless propagator, which is independent of α , and there is thus no vDVZ discontinuity.

2.6 Stückelberg Formalism

In using massive theories so far we have encountered two conceptually disturbing features. First, the gauge symmetry is broken and, second, the number of degrees of freedom is discontinuous in the $m \rightarrow 0$ limit as it suddenly jumps from $2d$ to $2(d-1)$. The Stückelberg trick [4, 5, 16] is an elegant way of killing those two birds with one stone at the level of the action, and with explicit Lorentz covariance. As in the case of propagators, it shows that the degrees of freedom do not change discontinuously as $m \rightarrow 0$, but that some of them simply decouple.

2.6.1 Spin 1

The so-called “Stückelberg trick” amounts to introducing auxiliary fields in a way which is patterned on the gauge transformation itself. In the case of massive electrodynamics we have (2.2.2) so one substitutes

$$A_\mu \rightarrow A_\mu + \frac{1}{m} \partial_\mu \phi, \quad (2.6.1)$$

in (2.2.1), where ϕ is the “Stückelberg field”. Since this technically has the form of a gauge transformation, only the mass term varies and we have that the Proca action becomes

$$S[A] \rightarrow S[A, \phi] = \int d^D x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m A^\mu \partial_\mu \phi + A_\mu j^\mu \right]. \quad (2.6.2)$$

By construction, this action is invariant under the gauge transformation

$$\delta A_\mu = -\partial_\mu \theta, \quad \delta \phi = m \theta, \quad (2.6.3)$$

so ϕ is a redundant (pure-gauge) field. The equations of motion of A_μ and ϕ are, respectively,

$$\partial_\mu F^{\mu\nu} - m^2 A^\nu = -j^\nu + m \partial^\nu \phi, \quad \square \phi = -m \partial_\mu A^\mu, \quad (2.6.4)$$

and we see that the latter is nothing but the divergence of the former. The gauge in which $\phi = 0$ is called the “unitary gauge”, in which case one recovers the equations of Proca theory. However, the advantage of having ϕ around is to keep imposing the gauge condition on the gauge field, and by choosing this condition appropriately, ϕ can then be interpreted as carrying the information of the longitudinal degrees of freedom that are activated in the massive theory. To see this, let us proceed to two different gauge-fixing scenarios.

We first choose to impose the Lorentz gauge $\partial_\mu A^\mu = 0$ so that, along with the equation of motion of A_0 , we can fix the initial conditions of the latter

$$(\Delta - m^2) A_0 = \partial_i \dot{A}_i + m \dot{\phi} - j_0, \quad \dot{A}_0 = \partial_i A_i. \quad (2.6.5)$$

We are then left with the equations

$$(\square - m^2) A_i = -j_i + m \partial_i \phi, \quad \square \phi = 0. \quad (2.6.6)$$

Now we see that, as in the *massless* case in (Sect. 2.2.1), we also have a residual gauge symmetry given by the θ obeying $\square \theta = 0$. However, since the A_i obey a massive Klein–Gordon equation, we cannot use such a θ to kill the homogeneous solution of

$\partial_i A_i$ as in the massless case. Rather, we can use θ to set $\phi = 0$, so that this amounts to choosing the unitary gauge. Thus, with the Lorentz gauge the Stückelberg field cannot represent the longitudinal mode since it obeys a massless Klein–Gordon equation.

Another initial choice of gauge is $\partial_\mu A^\mu = -m\phi$, in which case the conditions on A_0 read

$$(\Delta - m^2)A_0 = \partial_i \dot{A}_i + m\dot{\phi} - j_0, \quad \dot{A}_0 = \partial_i A_i + m\phi, \quad (2.6.7)$$

and the leftover equations are

$$(\square - m^2)A_i = -j_i, \quad (\square - m^2)\phi = 0. \quad (2.6.8)$$

We have again a residual gauge symmetry but it is now parametrized by the θ obeying $(\square - m^2)\theta = 0$. We can thus choose either to set $\phi = 0$ using such a θ , or to eliminate the homogeneous solution of $\partial_i A_i$ as in (Sect. 2.2.1). In the latter case, it is therefore ϕ which survives and represents the degrees of freedom associated with the longitudinal part $\partial_i A_i$, while A_μ contains $2(d-1)$ degrees of freedom as in the massless case. Thus, the interpretation of ϕ depends on the choice of gauge one makes.

Nevertheless, the interpretation in which ϕ represents the 2 degrees of freedom of the longitudinal part is the most appealing because it survives in the $m \rightarrow 0$ limit. Indeed, for $m = 0$ we have that ϕ becomes gauge-invariant and thus an unambiguous degree of freedom. We are then left with massless electrodynamics plus a scalar, totaling $N_f = 2N_d = 2d$. The important feature is that A_μ and ϕ are now decoupled, so if we focus on the dynamics of A_μ then ϕ is unobservable. Just as we saw when studying the propagators, the longitudinal modes do not propagate in the A_μ field anymore.

2.6.2 Spin 2

In the spin-2 case we must pattern the introduction of the Stückelberg field on (2.2.15)

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \frac{1}{m}(\partial_\mu A_\nu + \partial_\nu A_\mu), \quad (2.6.9)$$

in (2.2.13) to get

$$S = \int d^D x \left[\frac{1}{2} h_{\mu\nu} \mathcal{E}^{\mu\nu\rho\sigma} h_{\rho\sigma} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + 2\alpha(\partial_\mu A^\mu)^2 - \frac{1}{2} m^2(h_{\mu\nu} h^{\mu\nu} - (1+\alpha)h^2) - 2m(h^{\mu\nu} \partial_\mu A_\nu - (1+\alpha)h \partial_\mu A^\mu) + h_{\mu\nu} T^{\mu\nu} \right], \quad (2.6.10)$$

where as usual $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, so that the gauge symmetry is restored

$$\delta h_{\mu\nu} = -\partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \quad \delta A_\mu = m\xi_\mu. \quad (2.6.11)$$

Note that for $\alpha = 0$ the equation of motion of A_μ takes the form of the equation of massless electrodynamics with an $h_{\mu\nu}$ -dependent source, so it is invariant under the $U(1)$ transformation (2.2.2). This means that A_μ represents $2(d - 1)$ degrees of freedom, while the difference between Fierz–Pauli theory and the massless theory is $D^2 - D - 2 - (d^2 - d - 2) = 2d$, so if we take the $m \rightarrow 0$ limit now we are still discontinuous in the number of degrees of freedom. We can thus perform a second Stückelberg trick on this field in order to acquire the $U(1)$ symmetry as well. We replace

$$A_\mu \rightarrow A_\mu \rightarrow \frac{1}{m} \partial_\mu \phi, \quad (2.6.12)$$

to get

$$\begin{aligned} S = \int d^D x \left[\frac{1}{2} h_{\mu\nu} \mathcal{E}^{\mu\nu\rho\sigma} h_{\rho\sigma} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + 2\alpha (\partial_\mu A^\mu)^2 + \frac{2\alpha}{m^2} (\Box\phi)^2 \right. \\ \left. - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - (1 + \alpha) h^2) - 2m (h^{\mu\nu} \partial_\mu A_\nu - (1 + \alpha) h \partial_\mu A^\mu) \right. \\ \left. - 2(h^{\mu\nu} \partial_\mu \partial_\nu \phi - (1 + \alpha) h \Box\phi) + \frac{4\alpha}{m} \partial_\mu A^\mu \Box\phi + h_{\mu\nu} T^{\mu\nu} \right], \end{aligned} \quad (2.6.13)$$

which has the gauge symmetry

$$\delta h_{\mu\nu} = 0, \quad \delta A_\mu = -\partial_\mu \theta, \quad \delta \phi = m\theta. \quad (2.6.14)$$

For $\alpha \neq 0$ we see that we have a higher derivative theory for ϕ which means that it carries a healthy and a ghost-like degree of freedom. Indeed, one can integrate-in a second scalar ψ to lower the derivative order by replacing¹⁹

$$\frac{2\alpha}{m^2} (\Box\phi)^2 \rightarrow -2\alpha (\partial_\mu \psi \partial^\mu \phi + \frac{1}{2} m^2 \psi^2), \quad (2.6.15)$$

and then diagonalize the ϕ, ψ kinetic sector to find there is a ghost.²⁰

As for the limit $m \rightarrow 0$, the cases $\alpha = 0$ and $\alpha \neq 0$ must be considered separately as always. In the former case we have that A_μ decouples, while we still have terms $\sim \partial h \partial \phi$. We must thus diagonalize the $h_{\mu\nu}$ and ϕ kinetic sectors by redefining

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{2}{d-1} \eta_{\mu\nu} \phi, \quad (2.6.16)$$

¹⁹The original action is then obtained by integrating-out ψ .

²⁰This is why any other kinetic term than $F_{\mu\nu} F^{\mu\nu}$ for a vector field implies a ghost by the way.

to get

$$S = \int d^D x \left[\frac{1}{2} h'_{\mu\nu} \mathcal{E}^{\mu\nu\rho\sigma} h'_{\rho\sigma} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \frac{2d}{d-1} \partial_\mu \phi \partial^\mu \phi + h'_{\mu\nu} T^{\mu\nu} + \frac{2}{d-1} \phi T \right].$$

We see that although A_μ has totally decoupled, the scalar ϕ remains coupled to the source and is gauge-invariant under (2.6.14). Thus, ϕ still interacts with the system and this is the way the vDVZ discontinuity manifests itself in this formalism. For the $\alpha \neq 0$ case, there is no U(1) gauge symmetry in the equation of A_μ to begin with, so the latter already represents the $2d$ degrees of freedom that are activated by the mass. We therefore do not need to introduce the Stückelberg scalar and can take the $m \rightarrow 0$ limit at the level of the $h_{\mu\nu}$, A_μ action (2.6.10), to get that A_μ decouples, leaving us with the massless theory for $h_{\mu\nu}$.

2.7 Non-local Formulation

Another advantage of the Stückelberg formalism is that it can serve as an intuitive starting point for constructing non-local gauge theories. Here we follow closely the procedure introduced in [17, 18] and also used in our paper [1].

2.7.1 Spin 1

Let us start by solving in a causal way the equation of motion of the Stückelberg field ϕ (2.6.4)

$$\phi = \phi^{\text{hom}} - m \square_{\text{r}}^{-1} \partial_\mu A^\mu, \quad (2.7.1)$$

where ϕ^{hom} is a homogeneous solution $\square \phi^{\text{hom}} = 0$ and $\partial_\mu A^\mu$ must have finite past for this equation to make sense. For notational simplicity, unless specified otherwise, from now on we will only write “ $\square^{-1}$ ” to denote the retarded inversion of $\square$.

Since we know that $\partial_\mu A^\mu$ is not physical, demanding that it has finite past is not too much of a restriction. It would have been way more dramatic if we imposed this condition on all of A_μ , because this would exclude free wave-packet solutions since these extend arbitrarily far into the past. We can now proceed and plug (2.7.1) inside the equation for A_μ to get

$$\partial_\mu F^{\mu\nu} - m^2 \mathcal{P}_\mu^\nu A^\mu = -j'^\nu, \quad (2.7.2)$$

where we have a new conserved source

$$j'^\mu \equiv j^\mu - m \partial^\mu \phi^{\text{hom}}, \quad \partial_\mu j'^\mu = 0, \quad (2.7.3)$$

and we have defined the operator

$$\mathcal{P}_\mu{}^\nu \equiv \delta_\mu^\nu - \partial_\mu \square^{-1} \partial^\nu = \delta_\mu^\nu - \square^{-1} \partial_\mu \partial^\nu, \quad (2.7.4)$$

which has the following nice properties. It is a projector

$$\mathcal{P}_\mu{}^\rho \mathcal{P}_\rho{}^\nu = \delta_\mu^\nu - 2\partial_\mu \square^{-1} \partial^\nu + \partial_\mu \square^{-1} \square \square^{-1} \partial^\nu = \mathcal{P}_\mu{}^\nu, \quad (2.7.5)$$

where we have used the fact that $\square^{-1}$ is a right inverse of $\square$, the projected field $A_\mu^T \equiv \mathcal{P}_\mu{}^\nu A_\nu$ is D -transverse²¹

$$\partial^\mu A_\mu^T = \partial^\mu A_\mu - \square \square^{-1} \partial^\nu A_\nu = 0, \quad (2.7.6)$$

and, under a gauge transformation (2.2.2) where the gauge parameter θ has finite past, varies as

$$\delta A_\mu^T = -\partial_\mu \theta + \partial_\mu \square^{-1} \square \theta = 0. \quad (2.7.7)$$

Indeed, since $\square^{-1}$ acts on $\square \theta$, it only makes sense for $\square \theta$ with finite past, which implies that θ has finite past and also that $\square^{-1} \square = \text{id}$. This condition on the gauge parameter is reminiscent of the condition we encountered on the initial conditions of the gauge parameter on de-Sitter space-time. Again, this does not exclude the possibility of using θ to neutralize a field mode, so it does not diminish the gauge symmetry in any sense. We thus have that A_μ^T is gauge-invariant for all practical purposes.

Going back at (2.7.2) we see that we have reached a gauge invariant description of massive electrodynamics with no extra field, but at the price of non-locality. This may a priori sound a bit surprising because we know that this non-local theory is equivalent to a local one. This means that the physics of (2.7.2) cannot be non-local, i.e. the prediction of the value of some physical observable at x should still only depend on the data in its infinitesimal past light-cone neighbourhood. This is indeed the case because by going to the Lorentz gauge $\partial_\mu A^\mu = 0$ the equations become local. Thus, non-locality is only an artefact of explicit gauge-invariance and actually affects only the pure-gauge modes. The mass term can therefore be understood as the obstruction to having simultaneously both manifest locality and gauge-invariance.

Where Are the Degrees of Freedom?

Let us now try to count the degrees of freedom using (2.7.2). We choose the Lorentz gauge $\partial_\mu A^\mu = 0$ so that we retrieve the equation of motion of Proca theory (2.2.5), but with j'^μ instead of j^μ , i.e. we have the homogeneous solution of ϕ that is still around. This amounts to as many different sources as ϕ has initial data, so we might be worried that our non-local trick might have inserted additional degrees of freedom into the system. Of course there is no miracle, and ϕ^{hom} is eliminated by the residual gauge symmetry one has in the Stückelberg formalism. Indeed, the equations being

²¹This is not a surprise since the right-hand side of (2.7.2) is transverse.

$$(\square - m^2)A_\mu = -j_\mu + m \partial_\mu \phi^{\text{hom}}, \quad \partial_\mu A^\mu = 0, \quad (2.7.8)$$

we can transform with θ such that $\square\theta = 0$ to get

$$(\square - m^2)A_\mu + m^2 \partial_\mu \theta = -j_\mu + m \partial_\mu \phi^{\text{hom}}, \quad \partial_\mu A^\mu = 0. \quad (2.7.9)$$

Since $\square\phi^{\text{hom}} = 0$ as well, we can choose $\theta = m^{-1}\phi^{\text{hom}}$ and retrieve Proca theory exactly. Indeed, remember from Sect. 2.6.1 that in the $\partial_\mu A^\mu = 0$ gauge, ϕ cannot represent the longitudinal mode because it is massless $\square\phi = 0$, so fully gauge-fixing can only result in the unitary gauge $\phi = 0$. This shows us that we could have avoided keeping track of ϕ^{hom} in the above computations since at the end of the day this “freedom” is pure-gauge. In the Stückelberg formalism if we set $\partial_\mu A^\mu = 0$, then we still have a residual gauge-symmetry. In the non-local formalism with $\phi^{\text{hom}} = 0$ if we set $\partial_\mu A^\mu = 0$ we have the Proca equations and thus no residual gauge symmetry.

Nevertheless, we also saw in Sect. 2.6.1 that if we rather choose the gauge $\partial_\mu A^\mu = -m\phi$, then ϕ obeys $(\square - m^2)\phi = 0$, so its homogeneous solution could be interpreted as carrying the longitudinal degrees of freedom of the theory. However, here if ϕ were to carry the plane wave solutions of the longitudinal mode, then the gauge choice $\partial_\mu A^\mu = -m\phi$ would not be admissible because $\partial_\mu A^\mu$ would not have finite past.

We therefore conclude that the Stückelberg fields cannot represent the mode that is activated by the mass in this non-local formulation and thus one can safely set $\phi^{\text{hom}} = 0$. From now on $j'^\mu = j^\mu$ and we will also neglect the homogeneous solutions when integrating-out the Stückelberg fields in the spin-2 case. Indeed, there too the homogeneous solutions of the Stückelberg fields will be massless so that they cannot represent the dynamical fields of the theory. They ultimately correspond to the residual gauge freedom of the Stückelberg formalism.

Filtered Response to Linear Sources

The non-local equation of motion (2.7.2), although quite elegant, can be simplified even more if we restrict to the case where all of A_μ has finite past and thus so does j_μ . This is the case where one is interested in the production of electromagnetic waves by a source with finite past, i.e. when any radiation at future infinity is entirely due to j^μ . Then, one can write

$$A_\mu = \square^{-1} \square A_\mu, \quad (2.7.10)$$

so that (2.7.2) reads

$$\left(1 - \frac{m^2}{\square}\right) \partial_\mu F^{\mu\nu} = -j^\nu. \quad (2.7.11)$$

In this particular case, we have access to a new interpretation of the mass term as a high-pass filter [4, 5, 17, 18]. Indeed, going to “Fourier space”²² and neglecting the pole contour prescription, we have

²²This is actually not really possible for the time coordinate since A_μ will in general not vanish at future infinity because of the waves generated by the source. One should rather use a Laplace transform for t since the support of A_μ is bounded in the past.

$$-\left(1 + \frac{m^2}{k^2}\right) i k_\mu F^{\mu\nu} = -j^\nu, \quad (2.7.12)$$

which can be inverted to give

$$k_\mu F^{\mu\nu} = -\frac{i k^2}{m^2 + k^2} j^\nu. \quad (2.7.13)$$

Now the left-hand side is the kinetic term of ordinary massless electrodynamics, but the source is multiplied by a filter which modulates its intensity. Indeed, for $k^2 \ll m^2$, i.e. for high frequencies and large wave-lengths, the source of $A_\mu(k)$ becomes $\sim k^2$. This is the degravitation analogue for electrodynamics, which “screens” the large scale behaviour of the source [17, 18].

It is important to stress one more time that Eq. (2.7.11) is valid only when studying the response to an external source. More precisely, (2.7.11) only makes sense if $\partial_\mu F^{\mu\nu}$ has finite past, which excludes ingoing radiation at past infinity since that radiation does not obey $\partial_\mu F^{\mu\nu} = 0$ because of the mass. Therefore, (2.7.11) cannot be taken as a classical model covering every feature of massive electrodynamics. For a full description of the theory, with the constraint of past infinity applying only on non-dynamical fields (here $\partial_\mu A^\mu$), one needs to consider (2.7.2).

Propagators Using Projectors

The computation of the propagator in a massive but yet gauge-invariant setting is very instructive, especially in the light of this projector formalism. We can first rewrite (2.7.11) as

$$(\square - m^2) \mathcal{P}_\mu{}^\nu A_\nu = -j_\mu, \quad (2.7.14)$$

so that the operator which must be inverted is

$$\mathcal{K}^{\mu\nu} = (\square - m^2) \mathcal{P}^{\mu\nu}. \quad (2.7.15)$$

As in the massless case, the gauge invariance of the equation is reflected in the fact that $\mathcal{K}$ is proportional to a projector. It gives zero on pure-gauge modes, which means a non-trivial kernel, which means that it is not uniquely invertible. In Sect. 2.5 we have used the standard method for inverting such operators, which is to introduce a gauge-fixing term that will not affect the saturated propagator. In the spirit of the projector formalism developed here, there is actually a natural way of privileging an inverse that is also easily computable. Indeed, we can note that the space in which $\mathcal{K}_{\mu\nu}$ lives is the space of transverse operators and that $\mathcal{P}_\mu{}^\nu$ is the identity element. Thus, as long as we restrict to this subspace, the inversion relation becomes

$$\mathcal{K}^{\mu\rho} D_{\rho\nu} = i \mathcal{P}_\nu{}^\mu, \quad (2.7.16)$$

and admits a unique transverse inverse (up to the homogeneous solution/initial conditions ambiguity)

$$D_{\mu\nu} = -\frac{i}{k^2 + m^2} \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right). \quad (2.7.17)$$

Not surprisingly, in the massless case this corresponds to the Landau gauge $\xi = 0$ in (2.5.8). This is the only choice that cannot be expressed through a gauge fixing term (2.5.6) precisely because it is the only choice which imposes transversality $\partial_\mu A^\mu = 0$, instead of breaking it. In any case, as already noted, since the source is conserved the physically relevant term is the one with no uncontracted k_μ 's. In the spin-2 case however, there will be a whole one-parameter family of transverse operators, so this construction will be very useful.

2.7.2 Spin 2

The equations of motion of (2.6.13) are

$$\mathcal{E}_{\mu\nu\rho\sigma} h^{\rho\sigma} - m^2 (h_{\mu\nu} - (1 + \alpha)\eta_{\mu\nu}h) = -T_{\mu\nu} + 2m(\partial_{(\mu}A_{\nu)} - (1 + \alpha)\eta_{\mu\nu}\partial_\rho A^\rho) + 2(\partial_\mu\partial_\nu\phi - (1 + \alpha)\eta_{\mu\nu}\square\phi), \quad (2.7.18)$$

$$\partial_\mu F^{\mu\nu} - 2\alpha\partial^\nu\partial_\mu A^\mu = -mj^\nu + \frac{2\alpha}{m}\partial^\nu\square\phi, \quad (2.7.19)$$

$$\alpha\square^2\phi = \frac{m^2}{2}\partial_\mu j^\mu - \alpha m\square\partial_\mu A^\mu, \quad (2.7.20)$$

for $h_{\mu\nu}$, A_μ and ϕ , respectively, and we find convenient to define the quantity

$$j^\nu \equiv \partial_\mu h^{\mu\nu} - (1 + \alpha)\partial^\nu h. \quad (2.7.21)$$

Again, note that each one of these equations is the divergence of the previous one. For $\alpha \neq 0$, we can solve for ϕ

$$\phi = \frac{m^2}{2\alpha}\square^{-2}\partial_\mu j^\mu - m\square^{-1}\partial_\mu A^\mu, \quad (2.7.22)$$

where, as anticipated in the spin-1 case, the homogeneous solution $\square^2\phi^{\text{hom}} = 0$ can be safely set to zero since it cannot represent a massive mode and is thus ultimately pure-(residual)gauge. Remember that this expression for ϕ makes sense only if $\partial_\mu j^\mu$ and $\partial_\mu A^\mu$ have finite past. Plugging this inside the equation of A_μ we get

$$\partial_\mu F^{\mu\nu} = -m\mathcal{P}^\nu_\mu j^\mu, \quad (2.7.23)$$

where every term is independently transverse. Now this equation is gauge-invariant so we must fix the gauge in order to solve it. We choose $\partial_\mu A^\mu = 0$, invert $\square$ and

then add a pure-gauge term to get the general solution. This gives, setting again to zero any homogeneous solution,

$$A_\mu = -m\Box^{-1}\mathcal{P}_\mu{}^\nu j_\nu + \partial_\mu\theta. \quad (2.7.24)$$

To perform this inversion we now also need j^μ to have finite past, not just its divergence. This is again ok because j^μ does not represent dynamical fields since it is actually zero in the original formulation (2.2.17). Plugging the solution of A_μ in the one of ϕ we get

$$\phi = \frac{m^2}{2\alpha}\Box^{-2}\partial_\mu j^\mu - m\theta. \quad (2.7.25)$$

where we have used the fact that θ has finite past since $\partial_\mu A^\mu = \Box\theta$ has finite past. Now that both A_μ and ϕ are expressed in terms of $h_{\mu\nu}$ we can plug them in the equation of motion of the latter to get

$$\mathcal{E}_{\mu\nu\rho\sigma}h^{\rho\sigma} - m^2{}_\alpha\mathcal{P}_{\mu\nu\rho\sigma}h^{\rho\sigma} = -T_{\mu\nu}, \quad (2.7.26)$$

where

$$\begin{aligned} {}_\alpha\mathcal{P}_{\mu\nu}{}^{\rho\sigma} &\equiv \delta_{(\mu}^\rho\delta_{\nu)}^\sigma - (1+\alpha)\eta_{\mu\nu}\eta^{\rho\sigma}\left(1 - \frac{1+\alpha}{\alpha}\Box^{-1}\Box\right) - (\delta_{(\mu}^\rho\partial_{\nu)}\Box^{-1}\partial^\sigma + \delta_{(\mu}^\sigma\partial_{\nu)}\Box^{-1}\partial^\rho) \\ &+ (1+\alpha)\eta^{\rho\sigma}\left[2\partial_{(\mu}\Box^{-1}\partial_{\nu)} - \frac{1+2\alpha}{\alpha}\partial_\mu\partial_\nu\Box^{-1}\right] - \frac{1+\alpha}{\alpha}\eta_{\mu\nu}\Box^{-1}\partial_\rho\partial_\sigma \\ &+ \frac{1+2\alpha}{\alpha}\partial_\mu\partial_\nu\Box^{-2}\partial^\rho\partial^\sigma. \end{aligned} \quad (2.7.27)$$

Although we have expressed this such that $\Box^{-1}$ acts separately on $\Box h$ and $\partial_\mu\partial_\nu h^{\mu\nu}$, this requires only that j^μ has finite past to converge. Given the complexity of this structure, here we will directly focus on the case where all of $h_{\mu\nu}$, and thus $T_{\mu\nu}$, has finite past, so that we can commute all these operators at will. The result is then very elegant since it can be expressed in terms of the vector projectors

$${}_\alpha\mathcal{P}_{\mu\nu}{}^{\rho\sigma} = \mathcal{P}_{(\mu}^\rho\mathcal{P}_{\nu)}^\sigma + \frac{1+\alpha}{\alpha}\mathcal{P}_{\mu\nu}\mathcal{P}^{\rho\sigma}, \quad (2.7.28)$$

As anticipated earlier, here we have that ${}_\alpha\mathcal{P}$ is a one-parameter family of operators making the tensor on which they act transverse

$$\partial^\mu{}_\alpha\mathcal{P}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} = 0. \quad (2.7.29)$$

and also gauge-invariant under (2.2.15) for ξ_μ with finite past. It is convenient to switch to another parametrization, namely

$$a = 1 + d(1 + 1/\alpha), \quad \alpha = \frac{d}{a - d - 1}, \quad (2.7.30)$$

and define

$${}_a\mathcal{P}_{\mu\nu}^{\rho\sigma} \equiv {}_0\mathcal{P}_{\mu\nu}^{\rho\sigma} + a {}_s\mathcal{P}_{\mu\nu}^{\rho\sigma} \quad (2.7.31)$$

$$\begin{aligned} &= \delta_{(\mu}^{\rho} \delta_{\nu)}^{\sigma} - \frac{1-a}{d} \eta_{\mu\nu} \eta^{\rho\sigma} - \frac{1}{\square} (\delta_{(\mu}^{\rho} \partial_{\nu)} \partial^{\sigma} + \delta_{(\mu}^{\sigma} \partial_{\nu)} \partial^{\rho}) \\ &+ \frac{1-a}{d\square} (\eta_{\mu\nu} \partial^{\rho} \partial^{\sigma} + \eta^{\rho\sigma} \partial_{\mu} \partial_{\nu}) + \left(1 - \frac{1-a}{d}\right) \frac{1}{\square^2} \partial_{\mu} \partial_{\nu} \partial^{\rho} \partial^{\sigma}, \end{aligned} \quad (2.7.32)$$

where

$${}_0\mathcal{P}_{\mu\nu}^{\rho\sigma} \equiv \mathcal{P}_{(\mu}^{\rho} \mathcal{P}_{\nu)}^{\sigma} - \frac{1}{d} \mathcal{P}_{\mu\nu} \mathcal{P}^{\rho\sigma}, \quad {}_s\mathcal{P}_{\mu\nu}^{\rho\sigma} \equiv \frac{1}{d} \mathcal{P}_{\mu\nu} \mathcal{P}^{\rho\sigma}. \quad (2.7.33)$$

To avoid confusing ${}_s\mathcal{P}$ with ${}_a\mathcal{P}$ where $a = s$, let us stress that the letter “s” will be exclusively used in order to denote the second operator in (2.7.33). Now observe that ${}_0\mathcal{P}$ and ${}_s\mathcal{P}$ are orthogonal projectors

$${}_0\mathcal{P}^2 = {}_0\mathcal{P}, \quad {}_s\mathcal{P}^2 = {}_s\mathcal{P}, \quad {}_0\mathcal{P}{}_s\mathcal{P} = 0, \quad (2.7.34)$$

on the subspaces of transverse-traceless and transverse-pure-trace tensors, respectively. Indeed, ${}_0\mathcal{P}_{\mu\rho\sigma}^{\mu} h^{\rho\sigma} = 0$ so the latter is also invariant under linearized local conformal transformations

$$\delta h_{\mu\nu} = \eta_{\mu\nu} \theta, \quad (2.7.35)$$

for θ with finite past. The obvious advantage of the a parametrization is that now the linear combination and product of two such operators follow the simple rules

$$\alpha {}_a\mathcal{P} + \beta {}_b\mathcal{P} = (\alpha + \beta) \frac{\alpha a + \beta b}{\alpha + \beta} \mathcal{P}, \quad {}_a\mathcal{P} - {}_b\mathcal{P} = (a - b) {}_s\mathcal{P}, \quad {}_a\mathcal{P} {}_b\mathcal{P} = {}_{ab}\mathcal{P}, \quad (2.7.36)$$

so ${}_a\mathcal{P}$ is not a projector unless $a = 0$ or 1 . In the latter case, we have the projector on the subspace of transverse tensors ${}_1\mathcal{P} = {}_0\mathcal{P} + {}_s\mathcal{P}$. Thus, ${}_1\mathcal{P}$, ${}_0\mathcal{P}$ and ${}_s\mathcal{P}$ are the identity elements of the space on which they project.

In terms of α the choice $a = 0$ corresponds to $\alpha = -d/(d+1)$, which is the value for which the mass of the ghost (2.4.52) vanishes. Indeed, since the ghost is the trace h , it is consistent that the mass term in that case is traceless. Interestingly enough, the projector $a = 1$ corresponds to the value $\alpha = -1$. From now on, every time we assign a numerical value to the argument of $\mathcal{P}$ it will be with respect to the “ a ” parametrization (2.7.32).

Now note that the Lichnerowicz operator (2.2.14) takes the form $\mathcal{E} = \square {}_{1-d}\mathcal{P}$, which corresponds to $\alpha = -1/2$. Indeed, this is the only $\mathcal{P}$ that has no $\sim \square^{-2}$ term, so it is the only case where $\square\mathcal{P}$ is a local second-order transverse operator. Therefore, in the case $\alpha = -1/2$, we can rewrite the equation in a compact form analogous to (2.7.11)

$$\left(1 - \frac{m^2}{\square}\right) \mathcal{E}_{\mu\nu\rho\sigma} h^{\rho\sigma} = -T_{\mu\nu}, \quad \alpha = -\frac{1}{2}, \quad (2.7.37)$$

which is the result found in [1, 17, 18].²³ Not surprisingly, for this value of α we also have that, according to (2.4.52),

$$m_{\text{ghost}}^2 = -m^2, \quad (2.7.38)$$

so that the ghost mode is also a tachyon with the same magnitude of mass as the spin-2 modes. To understand why this happens, note that the differential operator corresponding to this equation is

$$\mathcal{K}^{\mu\nu\rho\sigma} = (\square - m^2)_{1-d} \mathcal{P}^{\mu\nu\rho\sigma}. \quad (2.7.39)$$

Since it is transverse but not traceless, the appropriate identity for the inversion is

$$\mathcal{K}^{\mu\nu\alpha\beta} D_{\alpha\beta\rho\sigma} = i \mathcal{P}_{\rho\sigma}^{\mu\nu}, \quad (2.7.40)$$

and thus, using the product rule (2.7.36) the propagator is trivial to compute

$$D_{\mu\nu\rho\sigma} = -\frac{i}{k^2 + m^2} \frac{1}{1-d} \mathcal{P}_{\mu\nu\rho\sigma}. \quad (2.7.41)$$

We see that, because $\mathcal{E} \sim \mathcal{P}$, all the poles are at $k^2 = -m^2$, with the ghost mode having the wrong overall sign, but the same magnitude for the mass. Conversely, this is why the rest of the $\alpha \neq 0$ cases cannot be written as $(\square - m^2)_a \mathcal{P}$ for some a , because the mass of the ghost is not m any more.

To conclude the $\alpha \neq 0$ case (2.7.26), note that in the $m \rightarrow 0$ limit we are left with the massless theory. Thus, as expected, there is no discontinuity. Moreover, as in the spin-1 case, the non-locality is “pure-gauge” since one can fix the gauge

$$\partial_\mu (h^{\mu\nu} - (1 + \alpha)\eta^{\mu\nu} h) = 0, \quad (2.7.42)$$

²³Note that in [17, 18] the authors erroneously concluded that this theory propagates only the d -tensor part of $h_{\mu\nu}$, i.e. it has the same dynamical content as the massless theory, because it has the same tensor structure (adding a gauge-fixing term and inverting one finds that the saturated propagator is indeed (2.5.18)). Their argument is that one has precisely integrated-out the Stückelbergs which correspond to the d -vector and d -scalar modes, so that the latter do not appear in this equation. As we have seen, this is not true because the Stückelbergs do not represent the dynamical fields that are activated by the mass. Moreover, it is not the tensor structure of the propagator alone which determines the dynamical content, otherwise the latter would be the same in massless and massive electrodynamics. As we have also seen, the presence of the mass is important, because it will affect the conservation equation of the source in Fourier space. Indeed, as we pointed out in [1], by expressing the saturated propagator (2.5.18) in terms of the harmonic variables of the conserved sources, we get (2.5.20) with M having both a positive and a negative eigenvalue (the ghost pole). We then have that $M \rightarrow 0$ as $m_s \rightarrow m \rightarrow 0$ so that we have no vDVZ discontinuity, as expected. However, for $m \neq 0$, all the independent components of the source are present and thus so are all the dynamical fields of the local theory.

which remember is possible for $\alpha \neq 0$, to get

$$\mathcal{P}_{\mu\nu\rho\sigma}h^{\rho\sigma} = h_{\mu\nu} - (1 + \alpha)\eta_{\mu\nu}h, \quad (2.7.43)$$

and thus the local equation we started with.

Fierz–Pauli Point

We now pass to the Fierz–Pauli case. We can first observe that the value $\alpha = 0$ corresponds to a diverging a so that the $\mathcal{P}$ operators are not well defined in this limit. However, one should note that now the action is linear in ϕ and its equation of motion (2.7.20) is

$$\partial_\mu j^\mu \equiv \partial_\mu \partial_\nu h^{\mu\nu} - \square h = 0, \quad (2.7.44)$$

to which we will refer as the “scalar equation”. For $h_{\mu\nu}$ with finite past this is equivalent to ${}_s\mathcal{P} \cdot h = 0$, so if the scalar equation holds then ${}_a\mathcal{P} \cdot h = {}_0\mathcal{P} \cdot h$ and we may still use the projectors. Since now $\partial_\mu j^\mu = 0$, the equation of motion of A_μ (2.7.19) has a transverse right-hand side and can be solved as before. The result is then plugged inside (2.7.18) and θ simply redefines ϕ again. In order to determine the latter, we can then take the trace of that equation and isolate ϕ , to get

$$\phi = -\frac{1}{2} \square^{-1} \left[m^2 h + \frac{1}{d} T \right], \quad (2.7.45)$$

where we have used $\partial_\mu j^\mu = 0$ and have put to zero the homogeneous solution since it is massless. Plugging this back inside the equation we get the following system

$$\mathcal{E}_{\mu\nu\rho\sigma}h^{\rho\sigma} - m^2 {}_0\mathcal{P}_{\mu\nu\rho\sigma}h^{\rho\sigma} = -T_{\mu\nu}^{\text{TT}}, \quad (2.7.46)$$

$$\partial_\mu \partial_\nu h^{\mu\nu} - \square h = 0. \quad (2.7.47)$$

where now the source has changed and is actually the traceless-transverse part of $T_{\mu\nu}$

$$T_{\mu\nu}^{\text{TT}} \equiv T_{\mu\nu} - \frac{1}{d} \left(\eta_{\mu\nu} T - \frac{\partial_\mu \partial_\nu}{\square} T \right) \equiv {}_0\mathcal{P}_{\mu\nu}^{\rho\sigma} T_{\rho\sigma}, \quad (2.7.48)$$

thus satisfying

$$\partial^\mu T_{\mu\nu}^{\text{TT}} = 0, \quad T^{\text{TT}} = 0. \quad (2.7.49)$$

Now note that the scalar equation is just the trace of (2.7.46), so that it is not independent and can be dropped. This might appear disturbing because then we are left with the left-hand side of the theory $a = 0$, which is *not* the Fierz–Pauli one $\alpha = 0$, and the corresponding propagator thus has an extra ghost pole. However, when we saturate it with $T_{\mu\nu}^{\text{TT}}$ we retrieve indeed the saturated Fierz–Pauli propagator in terms of $T_{\mu\nu}$. Thus, in this formulation the modification of the source is very relevant. The fact that the Fierz–Pauli theory has one less dynamical field is now reflected in the fact that $h_{\mu\nu}$ “sees”, and thus propagates, one less component of the source. Another

advantage of this formulation is that now the reason for the vDVZ discontinuity at $\alpha = 0$ is obvious, the source remains T^{TT} as $m \rightarrow 0$.

Another option, is to keep the scalar equation and use it to have ${}_a\mathcal{P} \cdot h = {}_0\mathcal{P} \cdot h$ and thus $\mathcal{E} \cdot h = \square {}_0\mathcal{P} \cdot h$, to finally get the following system

$$\left(1 - \frac{m^2}{\square}\right) \mathcal{E}_{\mu\nu\rho\sigma} h^{\rho\sigma} = -T_{\mu\nu}^{\text{TT}}, \quad (2.7.50)$$

$$\partial_\mu \partial_\nu h^{\mu\nu} - \square h = 0. \quad (2.7.51)$$

The first equation is precisely what we have found for the $\alpha = -1/2$ case (2.7.37), but now it is the additional scalar equation which makes the whole difference. It cannot be obtained through a gauge transformation and is responsible for killing the ghost.

Again, since the theory we started with is local, non-locality can only be a pure-gauge effect, although this time this may be a bit less obvious to show because the source term is non-local as well. This is why the source must be part of the gauge-fixing condition

$$\partial_\mu h^{\mu\nu} = -\frac{1}{dm^2} \square \partial^\nu T. \quad (2.7.52)$$

Indeed, with this the scalar equation becomes the equation fixing the trace (2.2.26) and, using this to express the source non-locality in terms of h , we can arrange the terms to get (2.2.24). Equation (2.2.25) is then found by taking the divergence of (2.2.24) and using (2.2.26).

Extra Gauge Symmetry

Using again that all ${}_a\mathcal{P}$ act the same on $h_{\mu\nu}$, yet another interesting formulation of the Fierz–Pauli non-local equations (2.7.51) is

$$(\square - m^2) {}_0\mathcal{P}_{\mu\nu\rho\sigma} h^{\rho\sigma} = -T_{\mu\nu}^{\text{TT}}, \quad (2.7.53)$$

$$\partial_\mu \partial_\nu h^{\mu\nu} - \square h = 0. \quad (2.7.54)$$

The advantage here is that the first equation is invariant under linearized local conformal transformations (2.7.35), and consistently traceless on both sides. However, this is not the case of the scalar equation. We can thus “lift” Fierz–Pauli theory to a non-local gauge theory with one more gauge symmetry

$$(\square - m^2) {}_0\mathcal{P}_{\mu\nu\rho\sigma} h^{\rho\sigma} = -T_{\mu\nu}^{\text{TT}}, \quad (2.7.55)$$

and now interpret the scalar equation as a gauge condition that is reached using (2.7.35) with

$$\theta = -\frac{1}{d} (h - \square^{-1} \partial_\mu \partial_\nu h^{\mu\nu}). \quad (2.7.56)$$

This is a very elegant result because now the ghost mode is also neutralized by a gauge symmetry. Indeed, in the spin-1 case we had $N_d = d$ because there are D fields, one gauge symmetry and no residual symmetry because of the mass. In the spin-2 case we have D^2 fields, D gauge symmetries in general, so that we are left with $N_d = D^2 - D$, except in the $\alpha = 0$ case where an extra gauge symmetry reduces that number by one.

Now the differential operator corresponding to (2.7.55) is

$$\mathcal{K}^{\mu\nu\rho\sigma} = (\square - m^2)_0 \mathcal{P}^{\mu\nu\rho\sigma}. \quad (2.7.57)$$

Since it is both transverse and traceless, the appropriate identity for the inversion is

$$\mathcal{K}^{\mu\nu\alpha\beta} D_{\alpha\beta\rho\sigma} = i \, {}_0\mathcal{P}^{\mu\nu}_{\rho\sigma}, \quad (2.7.58)$$

and thus, using the product rule (2.7.36) the propagator reads

$$D_{\mu\nu\rho\sigma} = -\frac{i}{k^2 + m^2} {}_0\mathcal{P}_{\mu\nu\rho\sigma}. \quad (2.7.59)$$

Saturating it, one finds the Fierz–Pauli result, i.e. (2.5.16) with $\alpha = 0$. This formulation provides us with yet another point of view on the vDVZ discontinuity. Indeed, in the massless theory we saw that the only projector for which $\square\mathcal{P}$ is local is the $a = 1 - d$ one. This gives $\sim \frac{1}{1-d}\mathcal{P}$ for the propagator and the following tensor structure for the saturated one

$$\sim \eta_{\mu(\rho}\eta_{\sigma)\nu} - \frac{1}{d-1} \eta_{\mu\nu}\eta_{\rho\sigma}. \quad (2.7.60)$$

On the other hand, Fierz–Pauli theory, because of the extra gauge symmetry that is needed to kill the ghost in the non-local formulation, must have ${}_0\mathcal{P}$ as its differential operator, and thus the tensor structure for the saturated propagator is

$$\sim \eta_{\mu(\rho}\eta_{\sigma)\nu} - \frac{1}{d} \eta_{\mu\nu}\eta_{\rho\sigma}. \quad (2.7.61)$$

2.7.3 New Non-local Theory

In the case of electrodynamics, the uniqueness of the projector makes the non-local formulation of Proca theory the only stable non-local theory of a massive vector field. In the tensor case, the presence of *two* independent projectors, ${}_0\mathcal{P}$ and ${}_s\mathcal{P}$ defined in (2.7.33), allows us to construct more healthy models than the ones that are obtained from local theories. In particular, as we will see in this thesis, one can construct a novel, genuinely non-local linear theory, that includes the trace scalar but with *no* ghost poles in the propagator. This is possible if we also modify non-locally the

kinetic term, so it will not correspond to simply adding a non-local mass term to linearized GR.

To construct that theory, we take full advantage of the projector formalism developed above to write an equation in which the tensor and scalar modes are diagonalized

$$(\square - m_g^2)_0 \mathcal{P}_{\mu\nu\rho\sigma} h^{\rho\sigma} + (z\square - m_s^2)_s \mathcal{P}_{\mu\nu\rho\sigma} h^{\rho\sigma} = -T_{\mu\nu}, \quad (2.7.62)$$

so that each one of them can have its own mass. The z factor will be useful in tracking ghost-like behaviour. Now since by definition ${}_0\mathcal{P} + z_s\mathcal{P} \equiv {}_z\mathcal{P}$, the only case in which the kinetic part is local, and thus coincides with linearized GR, is

$$z = 1 - d. \quad (2.7.63)$$

To study the stability and particle content of these theories let us compute the corresponding propagator. Because of the scalar sector we have that the differential operator

$$K^{\mu\nu\rho\sigma} \equiv (\square - m_g^2)_0 \mathcal{P}_{\mu\nu\rho\sigma} + (z\square - m_s^2)_s \mathcal{P}_{\mu\nu\rho\sigma}, \quad (2.7.64)$$

is transverse but not traceless, so that the appropriate identity element for the inversion is

$$K^{\mu\nu\alpha\beta} D_{\alpha\beta\rho\sigma} = i {}_1\mathcal{P}^{\mu\nu\rho\sigma}, \quad (2.7.65)$$

and the solution is (using the product rule (2.7.36))

$$D_{\mu\nu\rho\sigma} = -\frac{i}{k^2 + m_g^2} {}_0\mathcal{P}_{\mu\nu\rho\sigma} - \frac{i}{zk^2 + m_s^2} {}_s\mathcal{P}_{\mu\nu\rho\sigma}. \quad (2.7.66)$$

Saturating it with conserved sources we get

$$T^{*\mu\nu} D_{\mu\nu\rho\sigma} T'^{\rho\sigma} = -\frac{i}{k^2 + m_g^2} \left(T_{\mu\nu}^* T'^{\mu\nu} - \frac{1}{d} T^{*T'} \right) - \frac{1}{d} \frac{i}{zk^2 + m_s^2} T^{*T'}, \quad (2.7.67)$$

which is the Fierz–Pauli propagator with mass m_g plus a healthy scalar propagator, for $z > 0$, with mass $m_s/\sqrt{|z|}$. Thus, the first term in (2.7.62) describes the massive $\text{SO}(d)$ -tensor modes, while the second term describes the massive trace mode. This is a remarkable advantage compared to local massive spin-2 theory, where that extra scalar can only be a ghost. In our formalism, instead of having to fight to kill that extra mode allowed by the diffeomorphism symmetry, we have the opportunity to simply let it participate in the dynamics since we can choose z freely. Moreover, its mass is also free, instead of being determined by the one of the tensor modes. Note also that for $m_g \neq 0$ this is not a scalar-tensor theory, nor a bigravity theory in disguise, where the scalar or the second metric would have been integrated-out. Indeed, in scalar-tensor theories the graviton is not massive, while in bigravity theories there is also a massless graviton.

We thus have that stability requires $z > 0$, as it could have been expected from (2.7.62). This means however that, if we want the kinetic term to be the one of GR (2.7.63), then the scalar is a ghost. The exception is when both masses are zero, in which case that mode is neutralized by the residual gauge symmetry of linearized GR. Thus, as in Fierz–Pauli theory, continuity with GR at $m_i \rightarrow 0$ can only be achieved in the presence of a ghost. Conversely, any ghost-free massive theory will have a discontinuity, at the linearized level at least.

This can be easily seen by considering the massless limit $m_g \rightarrow 0$ in the saturated propagator. So let us rewrite the latter as

$$\begin{aligned} T^{*\mu\nu} D_{\mu\nu\rho\sigma} T'^{\rho\sigma} = & -\frac{i}{k^2 + m_g^2} \left(T_{\mu\nu}^* T'^{\mu\nu} - \frac{1}{d-1} T^{*T'} \right) \\ & - \frac{1}{d(d-1)} \frac{i}{k^2 + m_g^2} T^{*T'} - \frac{1}{d} \frac{i}{zk^2 + m_s^2} T^{*T'}, \quad (2.7.68) \end{aligned}$$

so that the first term reduces to the GR result in the $m_g \rightarrow 0$ limit. We see that we are left with the usual vDVZ discontinuity of the Fierz–Pauli propagator, representing the gauge-invariant combination of the two d -scalars in h_{ij} , plus the massive scalar mode. Taking also $m_s \rightarrow 0$, we see that only in the case (2.7.63) does one obtain linearized GR, but then the massive theory has a ghost.

There is however an important difference with FP theory regarding that discontinuity. Here the discontinuity is already visible at the level of the equations of motion (2.7.62), since we do not retrieve the massless local equations in the $m_g, m_s \rightarrow 0$ limit, for $z \neq 1-d$. On the other hand, in FP theory the action tends to the massless one in the $m \rightarrow 0$ limit. The reason for this difference is the presence of projectors, and thus gauge-invariance. Indeed, thanks to the projectors the tensor structure $\mathcal{K}^{\mu\nu\rho\sigma}$ in the equations of motion (2.7.62) is identical²⁴ to the structure of the propagator (2.7.67). Because of this, any discontinuity in the latter must also arise in the former. In FP theory on the other hand, the tensor structure $\mathcal{K}^{\mu\nu\rho\sigma}$ in the action and the one in the propagator $D_{\mu\nu\rho\sigma}$ are not at all the same and one can thus have a discontinuity in the latter that does not show up in the former.

Genuine Non-locality

Let us now try to turn (2.7.62) into a system of local equations by fixing the gauge. The choice which makes the ${}_a\mathcal{P}$ operator local and involves only local operators is

$$\partial_\mu \left(h^{\mu\nu} - \frac{1-a}{D-a} \eta^{\mu\nu} h \right) = 0, \quad (2.7.69)$$

which is accessible since $(1-a)/(D-a) \neq 1$. For generic masses m_g and m_s this gauge does not make the equation local, whatever the choice of a , so the system is genuinely non-local. The only exception is when $m_s^2 = zm_g^2 \equiv zm^2$ because then (2.7.62) can be expressed in terms of a single $\mathcal{P}$ operator

²⁴Up to Klein–Gordon operators.

$$(\square - m^2)_z \mathcal{P}_{\mu\nu\rho\sigma} h^{\rho\sigma} = -T_{\mu\nu}, \quad (2.7.70)$$

and we can fix the (2.7.69) gauge with $a = z$ to get the local system

$$(\square - m^2) \left(h_{\mu\nu} - \frac{1-z}{D-z} \eta_{\mu\nu} h \right) = -T_{\mu\nu}, \quad (2.7.71)$$

$$\partial_\mu \left(h^{\mu\nu} - \frac{1-z}{D-z} \eta^{\mu\nu} h \right) = 0. \quad (2.7.72)$$

This is reminiscent of the situation in local massive spin-2 equations, because (2.7.72) looks like the divergence of (2.7.71). Upon close inspection however, we observe that the analogy does not hold because here the divergence of (2.7.71) implies that $\partial_\mu h^{\mu\nu} - \frac{1-z}{D-z} \partial^\nu h$ is a free dynamical field, not zero. Because of this, these equations do *not* derive from the local action

$$S = \int d^D x \left[\frac{1}{2} h_{\mu\nu} (\square - m^2) \left(h^{\mu\nu} - \frac{1-z}{D-z} \eta^{\mu\nu} h \right) + h_{\mu\nu} T^{\mu\nu} \right], \quad (2.7.73)$$

which describes an obviously unstable theory since it does not have the GR tuning in the kinetic sector. Therefore, even in the case of local gauge-fixed equations, the theory does not derive from a local action and we thus have genuine non-locality.

In the case of local theories, the fact that one could localize the equations by fixing the gauge was a consequence of the fact that the integrated-out fields were pure-gauge. It therefore seems that, if we now wish to localize the above equations by integrating-in some auxiliary fields, the latter will not be pure-gauge, so that these theories cannot be obtained by some Stückelberg-ed local theory. This is not a surprise, since we know Proca and Fierz–Pauli theories to be the only ghost-free local theories of spin-1 and spin-2 dynamics, respectively.

References

1. M. Jaccard, M. Maggiore, E. Mitsou, Phys. Rev. D **88**, 4, 044033 (2013). [arXiv:1305.3034](#)
2. M. Jaccard, M. Maggiore, E. Mitsou, Phys. Rev. D **87**, 4, 044017 (2013). [arXiv:1211.1562](#)
3. A. Nicolis, R. Rattazzi, E. Trincherini, Phys. Rev. D **79**, 064036 (2009). [arXiv:0811.2197](#)
4. K. Hinterbichler, [arXiv:1105.3735](#)
5. K. Hinterbichler, Rev. Mod. Phys. **84**, 671 (2012)
6. D.G. Boulware, S. Deser, Phys. Rev. D **6**, 3368 (1972)
7. S.F. Hassan, R.A. Rosen, Phys. Rev. Lett. **108**, 041101 (2012). [arXiv:1106.3344](#)
8. S.F. Hassan, R.A. Rosen, JHEP **1204**, 123 (2012). [arXiv:1111.2070](#)
9. M. Henneaux, C. Teitelboim, *Quantization of gauge systems* (Princeton University Press, Princeton, 1992)
10. J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980)
11. S. Deser, J. Trubatch, S. Trubatch, Can. J. Phys. **44**, 1715 (1966)
12. S. Deser, A. Waldron, Phys. Rev. D **74**, 084036 (2006). [arXiv:hep-th/0609113](#)
13. K. Hinterbichler, [arXiv:1409.3565](#)

14. S. Deser, A. Waldron, Phys. Lett. B **508**, 347 (2001). [arXiv:hep-th/0103255](#)
15. A. Higuchi, Nucl. Phys. B **282**, 397 (1987)
16. E.C.G. Stueckelberg, Helv. Phys. Acta **30**, 209 (1957)
17. G. Dvali, New J. Phys. **8**, 326 (2006). [arXiv:hep-th/06100134](#)
18. G. Dvali, S. Hofmann, J. Khoury, Phys. Rev. D **76**, 084006 (2007). [arXiv:hep-th/0703027](#)

Infrared Non-local Modifications of General Relativity

Mitsou, E.

2016, XIX, 173 p. 13 illus. in color., Hardcover

ISBN: 978-3-319-31728-1