

Chapter 2

Generic Linear Recurrence Sequences

Let A be a commutative ring with unit and M any A -module. An M -valued linear recurrence sequence (LRS) generalizes the sequence of powers of the roots of a given monic polynomial with A -coefficients. A generic LRS is a linear recurrence sequence with indeterminate coefficients. A main character of this chapter, as of the entire book, is the ring $B_r := \mathbb{Z}[e_1, \dots, e_r]$, thought of as a free polynomial algebra generated by the coefficients of a generic LRS of order $r \geq 1$. The name of the indeterminates, $(e_1, \dots, e_r)$, is reminiscent of the elementary symmetric polynomials in r variables. The ring B_r will later be bestowed the more ambitious task of approximating the bosonic Fock representation of the oscillator Heisenberg algebra.

The present chapter is organized as follows. Section 2.1 is just an opening, to fix notation and to agree to identify M -valued sequences with M -valued formal power series. Section 2.2 gives a closer look at the ring B_r : the notion of partition and Schur polynomials are introduced in 2.2.2 to describe in details its *weight graduation*, defined by assigning degree i to each indeterminate e_i .

Section 2.4 focuses on the universal formula expressing the solution to the Cauchy problem for linear ODEs with constant coefficients and analytic forcing term. The expression is attained by means of a distinguished sequence $(u_i)_{i \in \mathbb{Z}}$ of *universal* B_r -valued generic LRSs. They will then go on to play an important role throughout the text. What ties generic LRSs to generic linear ordinary differential equations (ODEs) is the notion of *formal Laplace*¹ *transform*, introduced in a purely algebraic manner in Section 2.5, following [42]. Here, the Cauchy problem for linear ODEs is discussed, and an example to illustrate the method is fully worked out. Section 2.6 is about *generalized Wronskians* as in [50]. The terminology perhaps is

¹ Pierre-Simon, marquis de Laplace, Beaumont-en-Auge, Normandy, 1749–Paris, 1897. Mathematician and astronomer, he wrote the celebrated *celestial mechanics*. Asked by Napoleon why there was no mention to God in his treatise, as was customary at the time, he answered: je n’avaais pas besoin de cette hypothèse-là: I didn’t need that hypothesis.

not standard, but the same notion has been used by several authors in a variety of different contexts, e.g. [144], and is related with the *higher Wronskians* introduced and used by A. Iarrobino [67, 68] with diverse motivations.

This chapter is elementary in nature. It may be seen as a review of the basic background regarding linear ODEs with constant coefficients. The exposition is however carried out in the perspective of revisiting generalized Wronskians, Schubert calculus for Grassmannians and the boson–fermion correspondence within a unified framework.

2.1 Sequences in Modules over Algebras

2.1.1. Let A be a commutative ring with unit and M be an A -module. For any nonempty subset S of the integers, we denote by M^S the collection of maps $\mathbf{m} : S \rightarrow M$, i.e. the A -module of M -valued sequences defined over S . The image of $s \in S$ is denoted by m_s . We shall be interested in the cases $S = \mathbb{N}$ and $S = \mathbb{Z}$. If $S = \mathbb{N}$, elements of $M^{\mathbb{N}}$ will be represented by the array of their images $(m_0, m_1, \dots)$. Alternatively, let t be any indeterminate over A and call

$$M[[t]] := \left\{ \sum_{j \geq 0} m_j t^j \mid m_j \in M \right\} \quad (2.1)$$

the A -module of *formal power series* with coefficients in M . Because of the natural identification of $M[[t]]$ with $M^{\mathbb{N}}$, we shall also speak of M -valued formal power series. If $M = A$, then $A[[t]]$ is an A -algebra with respect to the usual product

$$\sum_{i \geq 0} a_i t^i \cdot \sum_{j \geq 0} a_j t^j = \sum_{k \geq 0} \left(\sum_{i=0}^k a_i a_{k-i} \right) t^k, \quad a_i \in A. \quad (2.2)$$

An obvious generalization of (2.2) endows $M[[t]]$ with a structure of $A[[t]]$ -module:

$$\sum_{i \geq 0} a_i t^i \cdot \sum_{j \geq 0} m_j t^j = \sum_{k \geq 0} \left(\sum_{i=0}^k a_i m_{k-i} \right) t^k, \quad (a_i, m_j) \in A \times M.$$

2.1.2. If X is another indeterminate over A and $p \in A[X]$ has degree r , we write

$$p(X) = e_0(p)X^r - e_1(p)X^{r-1} + \dots + (-1)^r e_r(p).$$

This means that we denote by $(-1)^i e_i(p)$ the coefficient of X^{r-i} , $0 \leq i \leq r$. The polynomial p is *monic* if $e_0(p) = 1$.

Examples 2.1.3. i) If $p := X^4 - 3X^2 + \sqrt{2}X - 3 \in \mathbb{Z}[\sqrt{2}][X]$, then $e_0(P) = 1$, $e_1(P) = 0$, $e_2(P) = -3$, $e_3(P) = -\sqrt{2}$ and $e_4(P) = -3$;

ii) If $P \in A[X]$ splits as product of r distinct linear factors:

$$p(X) = (X - x_1) \cdots (X - x_r),$$

then $e_1(p), e_2(p), \dots, e_r(p)$ are precisely the elementary symmetric polynomials in the roots $x_1, \dots, x_n$ of p . In this way we follow the convention of [109] for denoting symmetric functions.

Definition 2.1.4. A sequence $\mathbf{m} := (m_0, m_1, \dots) \in M^{\mathbb{N}}$ is called a *linear recurrence sequence* (LRS) of order r if there exists a monic $p \in A[X]$ of degree r such that

$$m_{r+j} - e_1(p)m_{r+j-1} + \cdots + (-1)^r e_r(p)m_j = 0, \quad \forall j \geq 0. \quad (2.3)$$

The polynomial p is the *characteristic polynomial* of the LRS.

Expression (2.3) is also called a *linear difference equation*. All LRSs associated with one polynomial form a submodule of $M^{\mathbb{N}}$ whose elements solve a homogeneous *linear difference equation*.

Examples 2.1.5. i) Let $\alpha \in A$ be a root of $p := X^2 - e_1(p)X + e_2(p) \in A[X]$. Then $(1, \alpha, \alpha^2, \alpha^3, \dots)$ is a LRS of degree 2 having p as characteristic polynomial;
ii) Let $\mathcal{M}$ be a square $n \times n$ matrix with entries in some commutative associative $\mathbb{Z}$ -algebra. By abuse of notation denote by $(-1)^j e_j(\mathcal{M})$ the coefficient of X^{n-j} in the polynomial

$$p_{\mathcal{M}}(X) := \det(X \cdot \mathbb{1}_{n \times n} - \mathcal{M}). \quad (2.4)$$

where $\mathbb{1}_{n \times n}$ denotes the diagonal identity matrix. The *Cayley–Hamilton theorem* (Corollary 4.2.15 to Theorem 4.2.12) says that $p_{\mathcal{M}}(\mathcal{M}) = 0$, namely, $\mathcal{M}$ is a root of (2.4). Hence $(\mathbb{1}_{n \times n}, \mathcal{M}, \mathcal{M}^2, \dots)$ is a LRS of order n , with $p_{\mathcal{M}}(X)$ as *characteristic polynomial*, where $\mathbb{1}_{n \times n}$ is the square identity matrix;

iii) Let $M = A = \mathbb{Z}$ and $p(X) = X^2 - X - 1$. The unique LRS $(f_0, f_1, \dots)$ having characteristic polynomial p and initial conditions $f_0 = f_1 = 1$ is the *Fibonacci sequence* $(1, 1, 2, 3, 5, 8, \dots)$. The equality

$$\sum_{n \geq 0} f_n t^n = \frac{1}{1 - t - t^2}$$

explains that the n th coefficient of the formal Taylor expansion of the right-hand side is precisely f_n .

2.1.6. Let

$$M((t)) := M[t^{-1}, t] = \left\{ \sum_{j \geq -i} m_j t^j \mid m_j \in M, i \in \mathbb{N} \right\}$$

be the A -module of M -valued *formal Laurent series*, i.e. elements of $M((t))$ with only finitely many non-zero negative powers of t . A quick check shows that the kernel of the A -epimorphism $M((t)) \mapsto M[[t]]$ mapping $\sum_{j \geq -i} m_j t^j$ to its ‘holomorphic’ part $\sum_{j \geq 0} m_j t^j$ is the submodule $t^{-1}M[t^{-1}]$. Let

$$\mathbb{J}_M : \frac{M((t))}{t^{-1}M[t^{-1}]} \rightarrow M[[t]] \quad (2.5)$$

be the induced isomorphism. For all $j \geq 0$, define

$$D^j \mathbf{m}(t) := \mathbb{J}_M \left(\frac{\mathbf{m}(t)}{t^j} + t^{-1}M[t^{-1}] \right), \quad (2.6)$$

or more explicitly

$$D^j \sum_{i \geq 0} m_i t^i = \sum_{i \geq 0} m_{i+j} t^i. \quad (2.7)$$

Clearly $D^j \in \text{End}_A(M[[t]])$ is the composite of $D := D^1$ with itself j times. We set $D^0 = \mathbb{1}_M$, the identity endomorphism of M . Via the identification $M[[t]] \cong M^{\mathbb{N}}$, the endomorphism D^j is nothing but the *shift operator* $m_i \mapsto m_{i+j}$.

Example 2.1.7. The unique solution to the equation $\mathbf{D}\mathbf{a}(t) = \mathbf{a}(t)$ with initial condition a_0 is

$$\mathbf{a}(t) = \frac{a_0}{1-t} = a_0(1 + t + t^2 + \cdots).$$

In other words $\frac{a_0}{1-t}$ is related to D as $\exp(a_0 t)$ is related to the formal derivative ∂_t . See Section 2.5.

2.2 Generic Polynomials, Partitions and Schur Determinants

2.2.1. Generic polynomials formalize the idea of polynomials with indeterminate coefficients. There is just one monic generic polynomial in each given degree. For each positive integer r , let $(e_1, \dots, e_r)$ denote a finite sequence of indeterminates over $\mathbb{Z}$, and let B_r be the polynomial ring in $e_1, \dots, e_r$ with integral coefficients

$$B_r := \mathbb{Z}[e_1, \dots, e_r].$$

Henceforth we shall denote by $\mathbf{p}_r(X)$ the *generic monic polynomial of degree r* :

$$\mathbf{p}_r(X) := X^r - e_1 X^{r-1} + \cdots + (-1)^r e_r. \quad (2.8)$$

It specializes to any other monic polynomial of the same degree

$$p(X) := X^r - e_1(p)X^{r-1} + \cdots + (-1)^r e_r(p), \quad e_i(p) \in A,$$

with coefficients in a B_r -algebra A , via the unique ring homomorphism $e_i \mapsto e_i(p)$.

Partitions 2.2.2. Regard the ring B_r as a graded $\mathbb{Z}$ -algebra by declaring i to be the degree of the indeterminate e_i . For $w \geq 0$, let $(B_r)_w$ be the submodule of elements of degree w . For example, $(B_r)_1 = \mathbb{Z}e_1$, $(B_r)_2 = \mathbb{Z}e_1^2 \oplus \mathbb{Z}e_2$ and $(B_r)_3 = \mathbb{Z}e_1^3 \oplus \mathbb{Z}e_1e_2 \oplus \mathbb{Z}e_3$. In general, to describe monomials of degree w of B_r , it is useful to use *partitions of length w* . A *partition* is a monotone non-increasing sequence of non-negative integers

$$\lambda : \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq 0,$$

such that all the *parts* λ_i are zero but finitely many ([33, 109]).

Denote by $\mathcal{P}$ the set of all partitions. If $\lambda := (\lambda_1, \lambda_2, \dots) \in \mathcal{P}$, its *length* $\ell(\lambda) := \#\{i \mid \lambda_i \neq 0\}$ is the number of its non-zero *parts*, and its *weight* is $|\lambda| = \sum_i \lambda_i$. If $\ell(\lambda) = r$, we simply write $\lambda = (\lambda_1, \dots, \lambda_r)$, omitting the infinite sequence of the zero parts. The *null partition*, $(0) := (0, 0, 0, \dots)$, having only null parts, will be simply denoted by 0. The set of partitions of length $k \leq r$ is denoted by $\mathcal{P}_r$: if $\lambda \in \mathcal{P}_r$ has length strictly less than r , we may, if convenient, add a string of $r - k$ zeros to it.

Each $\lambda \in \mathcal{P}$ is a partition of the integer $w := |\lambda|$. The *Young diagram* of $\lambda := (\lambda_1, \dots, \lambda_r) \in \mathcal{P}$ is an array $Y(\lambda)$ of left-justified rows of boxes such that the j th row has λ_j boxes, for $1 \leq j \leq r$. The *conjugate* of a partition λ is the partition λ' whose Young diagram is obtained from $Y(\lambda)$ by exchanging columns with rows: the j th column of $Y(\lambda')$ has as many boxes as the j th row of $Y(\lambda)$. For example, the conjugate to $(3, 2, 2, 1)$ is $(4, 3, 1)$, as shown in Figure 2.1.

Fig. 2.1 The Young diagram of the partition $(3, 2, 2, 1)$ and its conjugate $(4, 3, 1)$



2.2.3. A partition $\lambda \in \mathcal{P}$ can be also expressed as $\lambda := (1^{i_1} 2^{i_2} \cdots n^{i_n})$ to mean that it has i_j parts equal to j , $1 \leq j \leq n$, so that it has length $i_1 + \cdots + i_n$ and weight $i_1 + 2i_2 + \cdots + ni_n$. For example, the partition $\lambda = (3, 2, 2, 1)$ can be equivalently written as $(1^1 2^2 3^1)$. Analogously, $\mu := (2^4 3^2 4^1)$ denotes the partition that in standard notation is $(4, 3, 3, 2, 2, 2, 2)$.

2.2.4. For all integers $n \geq r$, we denote by $\mathcal{P}_{r,n}$ the set of partitions whose Young diagram is contained in an $r \times (n - r)$ rectangle, i.e. in the diagram of the partition $(n - r, \dots, n - r)$. We set $\mathcal{P}_{r,\infty} := \mathcal{P}_r$ (Fig. 2.2).

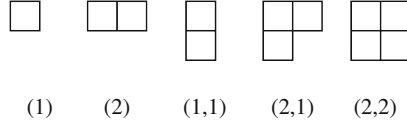


Fig. 2.2 The non-null partitions of $\mathcal{P}_{2,4}$

2.2.5. If $\mathbf{a} := (a_i)_{i \in \mathbb{Z}}$ is a bilateral sequence taking values in an arbitrary commutative associative algebra A and $\lambda \in \mathcal{P}_r$,

$$\Delta_{\lambda}(\mathbf{a}) = \det(a_{\lambda_j + i - j})_{1 \leq i, j \leq r} \in A$$

is the *Schur determinant* associated with the given sequence and partition. In practical terms, one allocates $a_{\lambda_1}, \dots, a_{\lambda_r}$ along the principal diagonal from top-down. Then *above* each a_{λ_i} , in the same column, the index decreases by one unit for each row; *below* a_{λ_i} , in the same column, the index increases by one unit per row. For instance,

$$\Delta_{(211)}(\mathbf{a}) = \begin{vmatrix} a_2 & a_0 & a_{-1} \\ a_3 & a_1 & a_0 \\ a_4 & a_2 & a_1 \end{vmatrix}.$$

2.2.6. As anticipated, partitions provide a useful combinatorial description of the grading of B_r . In fact

$$(B_r)_w = \bigoplus_{\lambda \in \mathcal{P}_r, |\lambda| = w} \mathbb{Z} \cdot \mathbf{e}_{\lambda'}^{\lambda'},$$

where λ' denotes the partition conjugate to λ and $\mathbf{e}^{\lambda'} := e_1^{i_1} \cdot \dots \cdot e_r^{i_r}$, once λ' is written in the form $(1^{i_1} 2^{i_2} \dots)$. Then

$$B_r = \bigoplus_{w \in \mathbb{N}} (B_r)_w \quad (2.9)$$

defines what we shall refer to as the *weight graduation* of B_r .

Example 2.2.7. Let $r \geq 3$. Then $e^{(4,1,1)'} = e_1^3 e_3$, $e^{(2,2,2)'} = e_3^2$, $e^{(321)'} = e_1 e_2 e_3$, $e^{(1,1,1)'} = e_3$ and $e^{(3)'} = e_1^3$.

2.2.8. Besides the monic generic polynomial $\mathfrak{p}_r(X)$, it will be also convenient to use

$$E_r(t) := 1 - e_1 t + \dots + (-1)^r e_r t^r \in B_r[t]. \quad (2.10)$$

Considering the rings $B_r((t)) = B_r[t^{-1}, t]$ and $B_r((X)) = B_r[X^{-1}, X]$ of *formal Laurent series* in t and X , respectively, the relationship between $E_r(t)$ and $\mathfrak{p}_r(X)$ is the obvious one:

$$E_r(t) = t^r \mathfrak{p}_r\left(\frac{1}{t}\right) \quad \text{and} \quad \mathfrak{p}_r(X) = X^r E_r\left(\frac{1}{X}\right).$$

Let $H_r(t) := \sum_{n \in \mathbb{Z}} h_n t^n \in B_r[[t^{-1}, t]]$ be given by

$$H_r(t) := \frac{1}{E_r(t)} = 1 + \sum_{n \geq 1} (1 - E_r(t))^n. \quad (2.11)$$

Because of the definition $h_j = 0$ if $j < 0$, $h_0 = 1$, while for every $i > 0$, the term h_i is an explicit polynomial in $e_1, \dots, e_r$, which is homogeneous of degree i with respect to the grading (2.9) of B_r . For example,

$$h_1 = e_1, \quad h_2 = e_1^2 - e_2, \quad h_3 = e_1^3 - 2e_1e_2 + e_3.$$

In general, for $n \geq 0$, h_n can be computed recursively using the equation $H_r(t)E_r(t) = 1$, equivalent to (2.11). Indeed

$$E_r(t) \sum_{n \geq 0} h_n t^n = 1 \quad (2.12)$$

which holds if and only if $h_0 = 1$ and the coefficient of t^n , in the left-hand side of (2.12), vanishes for all $n \geq 1$, i.e. if and only if

$$h_n - e_1 h_{n-1} + \dots + (-1)^n e_n h_0 = 0, \quad (2.13)$$

with the usual convention that $e_n = 0$ if $n > r$. The Schur polynomial $\Delta_\lambda(H_r)$ (cf. 2.2.5) associated with the sequence $H_r := (h_j)_{j \in \mathbb{Z}}$ of the coefficients of $H_r(t)$ and to the partition $\lambda \in \mathcal{P}_r$ is an element of $(B_r)_{|\lambda|}$. For example, for the partition $\lambda = (2, 2, 1)$ of the integer 5,

$$\Delta_{(2,2,1)}(H_r) = \begin{vmatrix} h_2 & h_1 & 0 \\ h_3 & h_2 & 1 \\ h_4 & h_3 & h_1 \end{vmatrix} = h_1 h_2^2 - h_1^2 h_3 - h_2 h_3 + h_1 h_4 \in (B_r)_5.$$

Using (2.13) one can easily check (see, e.g. [109]) that

$$e_j = \Delta_{(1^j)}(H_r) \quad \text{and} \quad h_n = \Delta_{(1^n)}(E_r) := \det(e_{j-i+1})_{0 \leq i, j \leq n}.$$

where we recall from Section 2.2.6 that (1^j) is the partition with j parts equal to 1.

Remark 2.2.9. For the coefficients of the formal power series $u_0 = H_r(t)$, the more careful notation $h_{r,n}$ should be preferred, in place of just h_n , to keep track of their dependence on r . To make the notation less heavy, we decided however to drop the subscript r , hoping for the context being sufficient to avoid confusions.

2.2.10. The sequence $H_r := (h_j)_{j \in \mathbb{Z}}$ of coefficients of the formal power series $H_r(t)$ may be alternatively defined as

$$\sum_{n \geq 0} \frac{h_n}{X^{r+n}} := \frac{1}{\mathfrak{p}_r(X)}. \quad (2.14)$$

More explicitly

$$h_n = \text{Res}_X \frac{X^{r+n-1}}{\mathfrak{p}_r(X)},$$

where Res_X denotes the *residue* of a Laurent series in the indeterminate X , which by definition is the coefficient of X^{-1} . See also [96, 97] and Section 5.8.

Note that the ring B_r can be thought of as a quotient of a polynomial ring with infinitely many indeterminates, by considering the unique $\mathbb{Z}$ -module homomorphism

$$\pi : \mathbb{Z}[X_1, X_2, \dots] \longrightarrow B_r,$$

mapping $X_i \mapsto h_i$. This map is clearly a ring epimorphism, because each element of B_r is a polynomial in $e_1, \dots, e_r$ and each e_i is a weighted homogeneous polynomial of degree i in $h_1, \dots, h_i$. The quotient

$$\mathbb{Z}[h_1, h_2, \dots] := \frac{\mathbb{Z}[X_1, X_2, \dots]}{\ker(\pi)} = B_r$$

will be denoted by $\mathbb{Z}[H_r]$.

2.3 Generic Linear Recurrence Sequences

The Fundamental Sequence 2.3.1. A special role in the theory is played by the *fundamental sequence* $(u_i)_{i \in \mathbb{Z}}$ of $B_r[[t]]$, which will return in Chapter 7, defined for $i \geq 0$ by

$$u_i := D^i(H_r(t)) \quad \text{and} \quad u_{-i} := t^i H_r(t), \quad (2.15)$$

or more explicitly

$$u_i = t^i + \sum_{n \geq 0} h_{n-i} t^n \quad \text{and} \quad u_{-i} = h_i + \sum_{n \geq 1} h_{n+i} t^n. \quad (2.16)$$

Equations 2.16 show that the terms of the sequence (u_i) are linearly independent over the integers, because the powers $(t^n)_{n>0}$ and the coefficients $(h_n)_{n\geq 0}$ are. Notice, in particular, that $u_0 = H_r(t)$ and that

$$D^i u_j = u_{i+j}, \quad \forall (i, j) \in \mathbb{N} \times \mathbb{Z} \quad (2.17)$$

by the very definition of the endomorphism D of $A[[t]]$ (cf. (2.6) for $M = A$).

The same remark as in 2.2.9 applies to the sequence $(u_i)_{i \in \mathbb{Z}}$ defined in (2.15), which in turn would deserve to be denoted $(u_{r,i})$

2.3.2. Let now M be a module over a B_r -algebra A , fixed once and for all, and define linear maps $\mathbb{U}_i : M[[t]] \rightarrow M$ via

$$\sum_{i \geq 0} \mathbb{U}_i(\mathbf{m}(t)) t^i := E_r(t) \mathbf{m}(t), \quad (\mathbf{m}(t) \in M[[t]]). \quad (2.18)$$

Comparing the coefficients of t^i on both sides of (2.18), we obtain for all $i \geq 0$

$$\mathbb{U}_i(\mathbf{m}(t)) = m_i + \sum_{j=1}^r (-1)^j e_j m_{i-j}, \quad (2.19)$$

with the convention that $m_j = 0$ if $j < 0$. So, for example,

$$\begin{aligned} \mathbb{U}_0(\mathbf{m}(t)) &= m_0, & \mathbb{U}_1(\mathbf{m}(t)) &= m_1 - e_1 m_0, \\ \mathbb{U}_2(\mathbf{m}(t)) &= m_2 - e_1 m_1 + e_2 m_0, & \mathbb{U}_3(\mathbf{m}(t)) &= m_3 - e_1 m_2 + e_2 m_1 - e_3 m_0. \end{aligned}$$

Proposition 2.3.3. Each $\mathbf{m}(t) \in M[[t]]$ admits the unique expansion

$$\mathbf{m}(t) = \sum_{j \geq 0} \mathbb{U}_j(\mathbf{m}(t)) u_{-j}. \quad (2.20)$$

In particular, if $M = A$,

$$\mathbb{U}_j(u_{-i}) = \delta_{ji}. \quad (2.21)$$

Proof. Rewrite equality (2.18) by inverting $E_r(t)$ in $B_r[[t]]$:

$$\mathbf{m}(t) = \sum_{j \geq 0} \mathbb{U}_j(\mathbf{m}(t)) \frac{t^j}{E_r(t)} = \sum_{j \geq 0} \mathbb{U}_j(\mathbf{m}(t)) t^j H_r(t) = \sum_{j \geq 0} \mathbb{U}_j(\mathbf{m}(t)) u_{-j}.$$

In particular $t^j = E_r(t) H_r(t) t^j = E_r(t) u_{-j} = \sum_{i \geq 0} \mathbb{U}_i(u_{-j}) t^i$ if $M = A$ and (2.21) follows. $\square$

If $M := A[X]$, consider the formal power series

$$\frac{1}{1 - Xt} := \sum_{n \geq 0} X^n t^n \in A[X][[t]]. \quad (2.22)$$

Proposition 2.3.4. For $j \geq 0$,

$$\mathbb{U}_j \left(\frac{1}{1 - Xt} \right) = \mathfrak{p}_j(X) := X^j - e_1 X^{j-1} + \cdots + (-1)^r e_r X^{j-r}$$

with the agreement that $X^i = 0$ if $i < 0$.

Proof. Since

$$\begin{aligned} \sum_{j \geq 0} \mathbb{U}_j \left(\frac{1}{1 - Xt} \right) t^j &= \frac{E_r(t)}{1 - Xt} = \left(1 + \sum_{j=1}^r (-1)^j e_j t^j \right) \sum_{n \geq 0} X^n t^n \\ &= \sum_{j \geq 0} (X^j - e_1 X^{j-1} + \cdots + (-1)^r e_r X^{j-r}) t^j, \end{aligned}$$

the claim follows. $\square$

Definition 2.3.5. The sequence $(m_0, m_1, \dots) \in M^{\mathbb{N}}$ is a *generic LRS* of order r if

$$\mathbb{U}_{r+j}(\mathbf{m}(t)) = m_{r+j} - e_1 m_{r+j-1} + \cdots + (-1)^r e_r m_j = 0$$

for all $j \geq 0$.

The r -tuple $(m_0, m_1, \dots, m_{r-1})$ is said to be the *initial data* of the generic LRS. The characteristic polynomial of the generic LRS is clearly $\mathfrak{p}_r(X)$, the generic polynomial of degree r . Adopting the language of M -valued formal power series, we can say that $\mathbf{m}(t) = \sum_{j \geq 0} m_j t^j$ is a generic LRS if and only if $\mathbb{U}_{r+j}(\mathbf{m}(t)) = 0$ for all $j \geq 0$.

Proposition 2.3.6. A formal power series $\mathbf{m}(t) \in M[[t]]$ is a generic LRS of order r if and only if $\mathfrak{p}_r(D)\mathbf{m}(t) = 0$, where $\mathfrak{p}_r(D)$ denotes the endomorphism of $M[[t]]$ obtained by evaluating $\mathfrak{p}_r(X)$ at D .

Proof. Note that

$$\begin{aligned} \mathfrak{p}_r(D)\mathbf{m}(t) &= (D^r - e_1 D^{r-1} + \cdots + (-1)^r e_r) \sum_{j \geq 0} m_j t^j \\ &= \sum_{j \geq 0} (m_{r+j} - e_1 m_{r+j-1} + \cdots + (-1)^r e_r m_j) t^j = \sum_{j \geq 0} \mathbb{U}_{r+j}(\mathbf{m}(t)) t^j. \end{aligned}$$

Therefore $\mathfrak{p}_r(D)\mathbf{m}(t) = 0$ if and only if $\mathbb{U}_{r+j}(\mathbf{m}(t)) = 0$ for all $j \geq 0$. $\square$

2.3.7. Let K_r be the B_r -submodule $\ker \mathfrak{p}_r(D)$ of $B_r[[t]]$:

$$K_r := \{u \in B_r[[t]] \mid \mathfrak{p}_r(D)u = 0\} \quad (2.23)$$

and define in addition $K_r(A) := K_r \otimes_{\mathbb{Z}} A$ and $K_r(M) := K_r(A) \otimes_A M$.

Proposition 2.3.8. *Each $\mathbf{m}(t) \in K_r(M)$ can be uniquely written as*

$$\mathbf{m}(t) = u_0 \mathbb{U}_0(\mathbf{m}(t)) + u_{-1} \mathbb{U}_1(\mathbf{m}(t)) + \cdots + u_{-r+1} \mathbb{U}_{-r+1}(\mathbf{m}(t)). \quad (2.24)$$

Proof. If $\mathbf{m}(t) \in K_r(M)$, then $\mathbb{U}_{r+j}(\mathbf{m}(t)) = 0$ for all $j \geq 0$, and then (2.24) follows from (2.20). $\square$

Corollary 2.3.9. *The r -tuple $(u_0, u_{-1}, \dots, u_{-r+1})$ is an A -basis of $K_r(A)$.*

Proof. Each $\mathbf{a}(t) \in K_r(A)$ can be uniquely written as $\sum_{j=0}^{r-1} \mathbb{U}_j(\mathbf{a}(t))u_{-j}$ by virtue of Proposition 2.3.8. In addition $\mathbb{U}_{r+j}(u_{-i}) = \delta_{i,r+j}$ is 0 in the range $0 \leq i \leq r-1$, i.e. $u_{-i} \in K_r(A)$. $\square$

Remark 2.3.10. What about u_j if the index j does not satisfy $-r+1 \leq j \leq 0$? First of all, if $j > 0$, then $u_j \in K_r(A)$ for any B_r -algebras A . In fact

$$\mathfrak{p}_r(D)u_j = \mathfrak{p}_r(D)D^j u_0 = D^j \mathfrak{p}_r(D)u_0 = 0,$$

where in the first equality, we used (2.17). In particular

$$u_j = \mathbb{U}_0(u_j)u_0 + \mathbb{U}_1(u_j)u_{-1} + \cdots + \mathbb{U}_{-r+1}(u_j)u_{-r+1}, \quad (2.25)$$

where

$$\mathbb{U}_i(u_j) = h_{i+j} - e_1 h_{i+j-1} + \cdots + (-1)^{i-1} e_{i-1} h_j. \quad (2.26)$$

If $j < -r+1$, then $j = -r-i$ for some $i \geq 0$ and

$$\mathfrak{p}_r(D)u_{-r-i} = t^i. \quad (2.27)$$

In fact

$$\mathfrak{p}_r(D)u_{-r} = \mathfrak{p}_r\left(\frac{1}{t}\right)t^r u_0 = E_r(t)u_0 = E_r(t)H_r(t) = 1,$$

and consequently

$$\mathfrak{p}_r(D)u_{-r-i} = \mathfrak{p}_r(D)t^i u_{-r} = t^i \mathfrak{p}_r(D)u_{-r} = t^i,$$

which proves (2.27).

Remark 2.3.11. Observe that $\mathbf{m}(t)$ is a generic LRS if and only if (cf. 2.1.6)

$$\mathbb{J}_M\left(\mathfrak{p}_r\left(\frac{1}{t}\right)\frac{\mathbf{m}(t)}{t} + t^{-1}M[t^{-1}]\right) = 0$$

in $M((t))$. In this case

$$\begin{aligned}\mathbf{m}(t) &= \mathcal{U}_0(\mathbf{m}(t))u_0 + \mathcal{U}_1(\mathbf{m}(t))tu_0 + \cdots + \mathcal{U}_{r-1}(\mathbf{m}(t))t^{r-1}u_0 \\ &= \mathcal{U}_0(\mathbf{m}(t))D^{r-1}u_{-r+1} + \mathcal{U}_1(\mathbf{m}(t))D^{r-2}u_{-r+1} + \cdots + \mathcal{U}_{r-1}(\mathbf{m}(t))u_{-r+1},\end{aligned}$$

and
$$\mathcal{U}_j(\mathbf{m}(t)) = \text{Res}_t \left(\mathfrak{p}_j \left(\frac{1}{t} \right) \frac{\mathbf{m}(t)}{t} \right), \quad (0 \leq j \leq r-1).$$

The situation is analogous to that of M -valued formal distributions in the module

$$M[[z^{\pm 1}, w^{\pm 1}]] := M[[z^{-1}, z, w^{-1}, w]]$$

that belong to the kernel of the multiplication by a power of $z - w$. In fact, if $\mathbf{m}(z, w) \in M[[z^{\pm 1}, w^{\pm 1}]]$, then the equality $(z - w)^r \mathbf{m}(z, w) = 0$ holds if and only if

$$\mathbf{m}(z, w) = c^0(w)\delta(z - w) + c^{(1)}(w)\partial_w\delta(z - w) + \cdots + c^{(r-1)}(w)\partial_w^{(r-1)}\delta(z - w),$$

where $\delta(z - w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n$ is the formal Dirac δ -function,

$$\partial_w^{(j)}\delta(z - w) := \frac{1}{j!} \frac{\partial^j \delta(z - w)}{\partial w^j}$$

and

$$c^{(j)}(w) = \text{Res}_z (z - w)^j \mathbf{m}(z, w)$$

See [73, p. 17] or [74, Theorem 1.6].

2.4 Cauchy Problems for Linear Recurrence Sequences

Definition 2.4.1. Two M -valued formal power series $\mathbf{m}(t)$ and $\tilde{\mathbf{m}}(t)$ share the same *initial conditions* modulo (t^r) , if $\mathbf{m}(t) - \tilde{\mathbf{m}}(t) \in t^r M[[t]]$.

The requirement is equivalent to $m_i = \tilde{m}_i$ for $0 \leq i \leq r-1$. An easy induction argument shows that $\mathbf{m}(t)$ and $\tilde{\mathbf{m}}(t)$ have the same initial conditions modulo (t^r) if and only if $\mathcal{U}_i(\mathbf{m}(t)) = \mathcal{U}_i(\tilde{\mathbf{m}}(t))$ for $0 \leq i \leq r-1$.

Lemma 2.4.2. If $\mathbf{m}(t) \in K_r(M)$ and $\mathcal{U}_i(\mathbf{m}(t)) = 0$ for $0 \leq i \leq r-1$, then $\mathbf{m}(t) = 0$.

Proof. Just note that by hypothesis, all coefficients of expansion (2.20) of $\mathbf{m}(t)$ vanish. $\square$

If $\mathbf{f} := (f_0, f_1, \dots)$ is another sequence of indeterminates over A , set $A[\mathbf{f}] := A[f_0, f_1, \dots]$, $M[\mathbf{f}] := M \otimes_A A[\mathbf{f}]$, and consider $\mathbf{f}(t) = \sum_{j \geq 0} f_j t^j$. Extend $\mathfrak{p}_r(D)$ to an endomorphism of $M[\mathbf{f}][[t]]$ in the obvious way.

Proposition 2.4.3 (Cauchy theorem for a generic LRS). *Let $\tilde{\mathbf{m}}(t) \in M[[t]] \subseteq M[\mathbf{f}][[t]]$. Then*

$$\mathbf{m}(t) := \mathfrak{U}_0(\tilde{\mathbf{m}}(t))u_0 + \cdots + \mathfrak{U}_{-r+1}(\tilde{\mathbf{m}}(t))u_{-r+1} + \sum_{j \geq 0} u_{-r-j} f_j \quad (2.28)$$

is the unique element of $\mathfrak{p}_r(D)^{-1}(\mathbf{f}(t))$ sharing with $\tilde{\mathbf{m}}(t)$ the same initial conditions modulo (t^r) .

Proof. Apply $\mathfrak{p}_r(D)$ to both sides of (2.28), exploiting its ‘ M -linearity’:

$$\mathfrak{p}_r(D)\mathbf{m}(t) = \sum_{j=0}^{r-1} \mathfrak{U}_j(\tilde{\mathbf{m}}(t))\mathfrak{p}_r(D)u_{-j} + \sum_{j \geq 0} f_j \mathfrak{p}_r(D)u_{-r-j}.$$

The equality $\mathfrak{p}_r(D)u_{-j} = 0$ for $0 \leq j \leq r-1$ together with (2.27) implies $\mathfrak{p}_r(D)\mathbf{m}(t) = \sum_{j \geq 0} f_j t^j$ as required. It is clear that (2.28) shares the same initial condition as $\tilde{\mathbf{m}}(t)$. If $\mathbf{m}'(t)$ were another element of $\mathfrak{p}_r(D)^{-1}(\mathbf{f}(t))$ with the same initial conditions as $\tilde{\mathbf{m}}(t)$, one would obtain $\mathbf{m}(t) - \mathbf{m}'(t) \in K_r(M)$, with the same initial conditions as the null series $0 \in M[[t]]$. Hence $\mathbf{m}(t) = \mathbf{m}'(t)$ because of Lemma 2.4.2. $\square$

Proposition 2.4.3, saying that the map $\mathfrak{p}_r(D) : M[[t]] \rightarrow M[[t]]$ is surjective, together with the definition of $K_r(M)$, implies:

Corollary 2.4.4. The following Cauchy sequence

$$0 \longrightarrow K_r(M) \hookrightarrow M[[t]] \xrightarrow{\mathfrak{p}_r(D)} M[[t]] \longrightarrow 0 \quad (2.29)$$

is exact.

Universality 2.4.5. Formula (2.28) is universal in the following sense. Let A be any $\mathbb{Z}$ -algebra, M any A -module and $P \in A[X]$ any monic polynomial of degree r . Given $\mathbf{m}(t)$ and $\mathbf{g}(t) := \sum_{j \geq 0} g_j t^j$ two M -valued formal power series, the unique element of $P(D)^{-1}(\mathbf{g}(t))$ sharing the same initial conditions as $\mathbf{m}(t)$ is

$$\sum_{j=0}^{r-1} \mathfrak{U}_j(\mathbf{m}(t))u_{-j} + \sum_{j \geq 0} g_j u_{-r-j},$$

where A is regarded as a $B_r[\mathbf{f}]$ -module under the unique $\mathbb{Z}$ -algebra homomorphism defined by $e_i \mapsto e_i(P)$ and $f_j \mapsto g_j$.

2.5 Formal Laplace Transform and Linear ODEs

Definition 2.5.1. Let M be a module over a $\mathbb{Z}$ -algebra A . We call formal Laplace transform the A -linear map $\mathcal{L} : M[[t]] \rightarrow M[[t]]$ given by

$$\mathcal{L}\left(\sum_{j \geq 0} m_j t^j\right) = \sum_{j \geq 0} j! m_j t^j. \quad (2.30)$$

It is easy to verify that $\mathcal{L}$ is invertible if and only if A contains the rational numbers. In this case

$$\mathcal{L}^{-1}\left(\sum_{j \geq 0} m_j t^j\right) = \sum_{j \geq 0} m_j \frac{t^j}{j!}.$$

Define the endomorphism $\partial_t : M[[t]] \rightarrow M[[t]]$ as

$$\partial_t \left(\sum_{j \geq 0} m_j t^j \right) = \sum_{j \geq 0} (j+1) m_{j+1} t^j.$$

Proposition 2.5.2. The following diagram commutes

$$\begin{array}{ccc} M[[t]] & \xrightarrow{\mathcal{L}} & M[[t]] \\ \partial_t \downarrow & & \downarrow D \\ M[[t]] & \xrightarrow{\mathcal{L}} & M[[t]] \end{array}$$

In particular, $\mathcal{L}$ maps $\ker \mathfrak{p}_r(\partial_t)$ to $K_r(M)$.

Proof. Just a matter of routine calculation,

$$\mathcal{L} \partial_t \sum_{j \geq 0} m_j t^j = \sum_{j \geq 0} (j+1)! m_{j+1} t^j = D \sum_{j \geq 0} j! m_j t^j = D \mathcal{L} \sum_{j \geq 0} m_j t^j.$$

An easy induction shows that, $\mathcal{L} \circ \partial_t^i = D^i \circ \mathcal{L}$ and so $\mathcal{L}$ maps $\ker \mathfrak{p}_r(\partial_t)$ to $K_r(M)$. $\square$

If $\mathcal{L}$ is invertible, then $\partial_t = \mathcal{L}^{-1} D \mathcal{L}$, and, by induction, $\partial_t^i = \mathcal{L}^{-1} D^i \mathcal{L}$, which implies that $\ker \mathfrak{p}_r(\partial_t)$ is isomorphic to $K_r(M)$. In other words, to determine $\ker \mathfrak{p}_r(\partial_t)$, it suffices to determine $K_r(M)$. From now on we assume $M = A$. Then ∂_t is a derivation of $A[[t]]$, i.e.

$$\partial_t(\mathbf{a}(t)\mathbf{b}(t)) = \partial_t \mathbf{a}(t) \cdot \mathbf{b}(t) + \mathbf{a}(t) \cdot \partial_t \mathbf{b}(t)$$

a well-known property left as an exercise.

Lemma 2.5.3. If L is invertible, any $\mathbf{f}(t) \in A[[t]]$ admits an expansion of the type

$$\mathbf{f}(t) = \sum_{j \geq 0} \mathbb{U}_j(\mathcal{L}(\mathbf{f}(t)))\mathcal{L}^{-1}(u_{-j}). \quad (2.31)$$

Proof. Since $\mathcal{L}(\mathbf{f}(t)) = \sum_{j \geq 0} \mathbb{U}_j(\mathcal{L}(\mathbf{f}(t)))u_{-j}$, taking the inverse transform of both sides and using A -linearity of $\mathcal{L}$ give formula (2.31). $\square$

2.5.4. Given $\mathbf{f}(t) \in A[[t]]$, the preimage $\mathfrak{p}_r(\partial_t)^{-1}(\mathbf{f}(t))$ is, by definition, the set of the solutions to the generic linear *ordinary differential equation* (ODE)

$$y^{(r)} - e_1 y^{(r-1)} + \cdots + (-1)^r e_r y = \mathbf{f}(t), \quad (2.32)$$

where $y^{(i)} := \partial_t^i y$. Proposition 2.4.3 has the following important consequence.

Corollary 2.5.5. The unique element of $\mathfrak{p}_r(\partial_t)^{-1}(\mathbf{f}(t))$ sharing the same initial conditions of a given $\varphi \in A[[t]]$ modulo (t^r) is

$$y = \sum_{i=0}^{r-1} \mathbb{U}_i(\mathcal{L}(\varphi))\mathcal{L}^{-1}(u_{-i}) + \sum_{j \geq 0} j! f_j \mathcal{L}^{-1}(u_{-r-j}). \quad (2.33)$$

Proof. Just apply $\mathfrak{p}_r(\partial_t) = \mathcal{L}^{-1} \mathfrak{p}_r(D) \mathcal{L}$ to both sides of (2.33):

$$\mathfrak{p}_r(\partial_t)y = \mathcal{L}^{-1} \mathfrak{p}_r(D) \mathcal{L} \left(\sum_{i=0}^{r-1} \mathbb{U}_i(\mathcal{L}(\varphi))\mathcal{L}^{-1}(u_{-i}) \right) \quad (2.34)$$

$$+ \mathcal{L}^{-1} \mathfrak{p}_r(D) \mathcal{L} \left(\sum_{j \geq 0} j! f_j \mathcal{L}^{-1}(u_{-r-j}) \right). \quad (2.35)$$

Using the A -linearity of $\mathcal{L}$ and the fact that $\mathfrak{p}_r(D)u_{-i} = 0$ for $0 \leq i \leq r-1$, the first summand of (2.35) vanishes, and the second one is

$$\mathcal{L}^{-1} \mathfrak{p}_r(D) \left(\sum_{j \geq 0} j! f_j u_{-r-j} \right) = \mathcal{L}^{-1} \left(\sum_{j \geq 0} j! f_j t^j \right) = \mathcal{L}^{-1}(\mathcal{L}(\mathbf{f}(t))) = \mathbf{f}(t).$$

The uniqueness is obvious. $\square$

Example 2.5.6. It is worth to discuss a couple of examples. Suppose one wants to solve the linear ODE

$$y'' + \omega^2 y = t^n. \quad (2.36)$$

Write $n + 2$ in the form $4p + q$ where $p \in \mathbb{N}$ and $q \in \{0, 1, 2, 3\}$, so that (2.36) reads

$$y'' + \omega^2 y = t^{4p+q-2}, \quad (4p + q \geq 2).$$

Solving (2.36) then amounts to solving

$$(D^2 + \omega^2)\mathcal{L}(y) = (4p + q - 2)!t^{4p+q-2}.$$

There is a unique morphism $B_2 \rightarrow \mathbb{Q}$ sending $e_1 \mapsto 0$ and $e_2 \mapsto \omega^2$. Take $\phi = a_0 + a_1 t \in \mathbb{Q}[t]$ and look for the unique solution such that $y - \phi = 0 \pmod{(t^2)}$. Then

$$u_0 = H_2(t) = \frac{1}{1 + \omega^2 t^2} = 1 + \sum_{j \geq 1} (-1)^j \omega^{2j} t^{2j}$$

and

$$u_{-1} = tu_0 = \frac{1}{\omega} \left(\omega t + \sum_{j \geq 1} (-1)^j \omega^{2j+1} t^{2j+1} \right).$$

The solution to the given Cauchy problem is then

$$\mathcal{L}(y) = a_0 u_0 + a_1 u_{-1} + (4p + q - 2)! u_{-4p-q}$$

Now

$$\begin{aligned} u_{-4p-q} &= t^{4p+q} u_0 = \frac{1}{\omega^{4p+q}} (\omega t)^{4p+q} u_0 \\ &= \frac{1}{3} (2q-1)(q-1)(q-3) u_0 + \frac{1}{3\omega} q(q-2)(q-4) \omega u_{-1} \\ &\quad + \frac{1}{6} q(q-1)(q-2) \omega t - \frac{1}{2} q(q-1)(q-3) - \sum_{a=0}^p (-1)^a \omega^{2a+q} t^{2a+q}, \end{aligned}$$

a formula that the reader can check as exercise. Since

$$\mathcal{L}^{-1}(u_0) = \cos(\omega t) \quad \text{and} \quad \mathcal{L}^{-1}(u_{-1}) = \frac{1}{\omega} \sin(\omega t)$$

one finally obtains

$$y = a_0 \cos \omega t + \frac{a_1}{\omega} \sin \omega t + (4p + q - 2)! \mathcal{L}^{-1}(u_{-4p-q}),$$

where

$$\begin{aligned}\mathcal{L}^{-1}(u_{-4p-q}) &= \mathcal{L}^{-1}(t^{4p+q}u_0) = \frac{1}{\omega^{4p+q}}\mathcal{L}^{-1}((\omega t)^{4p+q}u_0) \\ &= \frac{1}{\omega^{4p+q}}\left(\frac{(2q-1)(q-1)(q-3)}{3}\cos\omega t + \frac{q(q-2)(q-4)}{3}\sin\omega t \right. \\ &\quad \left. - \sum_{a=0}^p (-1)^{2a+q}\omega^{2a+q}\frac{t^{2a+q}}{(2a+q)!} + \frac{1}{6}q(q-1)(q-2)\omega t - \frac{1}{2}q(q-1)(q-3)\right)\end{aligned}$$

If, for instance, we put $p = 1$ and $q = 3$,

$$\begin{aligned}y &= a_0 \cos \omega t + \frac{a_1}{\omega} \sin \omega t + \frac{5!}{\omega^7} \left(\omega t - \frac{\omega^3 t^3}{3!} + \frac{\omega^5 t^5}{5!} - \sin \omega t \right) \\ &= a_0 \cos \omega t + \left(\frac{a_1}{\omega} - \frac{5!}{\omega^7} \right) \sin \omega t + \frac{5!}{\omega^6} \left(t - \frac{\omega^2 t^3}{3!} + \frac{\omega^4 t^5}{5!} \right).\end{aligned}$$

is the general solution to

$$y'' + \omega^2 y = t^5.$$

2.6 Generalized Wronskians

2.6.1. Recall that if M, N are A -modules, a multilinear map $\varphi : M^r \rightarrow N$ is said to be *alternating* if $\varphi(m_1, \dots, m_r) = 0$ whenever $m_i = m_j$ for some $1 \leq i < j \leq r$. Let $C_1, \dots, C_r$ be r columns of A^r , in such a way that $(C_1, \dots, C_r)$ is an $r \times r$ square matrix. As in the undergraduate courses, by the determinant of $(C_1, \dots, C_r)$, we understand the unique multilinear alternating form taking the value 1 on the canonical basis of A^n .

Denote by

$$\mathbf{u}_r := (u_0, u_{-1}, \dots, u_{-r+1})$$

the canonical basis of the A -module $\ker(\mathfrak{p}_r(D) \cdot) \subseteq A[[t]]$.

Definition 2.6.2. Given an ordered r -tuple $\mathcal{F} := (f_1, f_2, \dots, f_r)$ of arbitrary polynomials in $A[X]$, the $\mathcal{F}$ -Wronskian of $\mathbf{u}_r$ is the determinant:

$$W_{\mathcal{F}}(\mathbf{u}_r) := \begin{vmatrix} f_r(D)u_0 & f_r(D)u_{-1} & \dots & f_r(D)u_{-r+1} \\ f_{r-1}(D)u_0 & f_{r-1}(D)u_{-1} & \dots & f_{r-1}(D)u_{-r+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_1(D)u_0 & f_1(D)u_{-1} & \dots & f_1(D)u_{-r+1} \end{vmatrix} \in A[[t]]$$

where $f_i(D)u_{-j}$ means applying the operator $f_i(D)$ to u_{-j} .

Remark 2.6.3. The Wronskian is A -multilinear and alternating in $f_1, \dots, f_r$. Since $\mathfrak{p}_r(D)u_{-i} = 0$ for $-r + 1 \leq i \leq 0$, we have that if $f_i \equiv g_i \pmod{\mathfrak{p}_r(X)}$, then

$$W_{\mathcal{F}}(\mathbf{u}_r) = W_{\mathcal{G}}(\mathbf{u}_r),$$

where we set $\mathcal{G} = (g_1, \dots, g_r) \in A[X]^r$.

2.6.4. For $\lambda \in \mathcal{P}_r$, define

$$\mathbf{X}^\lambda := (X^{\lambda_r}, X^{1+\lambda_{r-1}}, \dots, X^{r-1+\lambda_1}) \in A[X]^r$$

and $\mathbf{X} := \mathbf{X}^{(0)}$, where (0) denotes the null partition $(0, 0, \dots, 0)$. Then

$$W_{\mathbf{X}^\lambda}(\mathbf{u}_r) = \det(D^{i-1+\lambda_{r-i+1}}u_{1-j})_{1 \leq i, j \leq r} = \begin{vmatrix} u_{\lambda_r} & u_{-1+\lambda_r} & \dots & u_{-r+1+\lambda_r} \\ u_{1+\lambda_{r-1}} & u_{\lambda_{r-1}} & \dots & u_{-r+2+\lambda_{r-1}} \\ \vdots & \vdots & \ddots & \vdots \\ u_{r-1+\lambda_1} & u_{r-2+\lambda_1} & \dots & u_{\lambda_1} \end{vmatrix}.$$

As in 2.2.10, the *residue* $\text{Res}_X(g)$ of a Laurent series $g = \sum_{i \in \mathbb{Z}} g_i X^i$ ($n \in \mathbb{Z}$) is the coefficient g_{-1} of X^{-1} . To ease notation, in this section we shall omit the subscript X to indicate the residue.

The following definition is due to Laksov and Thorup [96, 97]:

Definition 2.6.5. The *residue* of an ordered r -tuple

$$g_i := \sum_{j \leq n_i} g_{ij} X^j, \quad 0 \leq i \leq r-1.$$

of Laurent series with B_r -coefficients is

$$\begin{aligned} & \text{Res}(g_0, g_1, \dots, g_{r-1}) := \\ &= \begin{vmatrix} \text{Res}(g_0) & \text{Res}(g_1) & \dots & \text{Res}(g_{r-1}) \\ \text{Res}(Xg_0) & \text{Res}(Xg_1) & \dots & \text{Res}(Xg_{r-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Res}(X^{r-1}g_0) & \text{Res}(X^{r-1}g_1) & \dots & \text{Res}(X^{r-1}g_{r-1}) \end{vmatrix}. \end{aligned} \quad (2.37)$$

Clearly $\text{Res}(g_0, g_1, \dots, g_{r-1})$ is B_r -multilinear and alternating:

$$\text{Res}(g_{\mathfrak{s}(0)}, g_{\mathfrak{s}(1)}, \dots, g_{\mathfrak{s}(r-1)}) = \text{sgn}(\mathfrak{s}) \text{Res}(g_0, g_1, \dots, g_{r-1}),$$

where $\mathfrak{s} \in S_r$ (the permutations on r elements) and $\text{sgn}(\mathfrak{s})$ denotes its parity ± 1 . If at least one of the g_j is a polynomial, the j th row of the determinant (2.37) vanishes, which causes

$$\text{Res}(g_0, g_1, \dots, g_{r-1}) = 0.$$

Lemma 2.6.6. *Let $f(X) = a_0X^\lambda + a_1X^{\lambda-1} + \cdots + a_\lambda$ be a polynomial of degree $\leq \lambda$ with coefficients in a B_r -algebra A . Then, for all $1 \leq i \leq r$,*

$$\text{Res} \left(\frac{X^{i-1}f(X)}{\mathfrak{p}_r(X)} \right) = \sum_{j=0}^{\lambda} a_j h_{i-r+\lambda-j}. \quad (2.38)$$

Proof. Equality (2.14) gives

$$h_{i-r+\lambda-j} = \text{Res} \left(\frac{X^{i-1+\lambda-j}}{\mathfrak{p}_r(X)} \right) = \text{Res} \left(\frac{X^{i-1}X^{\lambda-j}}{\mathfrak{p}_r(X)} \right)$$

Since taking the residue is A -linear, we eventually get formula (2.38). $\square$

Theorem 2.6.7 (Cf. [98, Theorem 2.7(2)] and [49, p. 286]). *Let $\mathcal{F} := (f_1, \dots, f_r) \in A[X]^r$. Then*

$$\text{i)} \quad W_{\mathbf{X}}(\mathbf{u}_r) \neq 0;$$

$$\text{ii)} \quad W_{\mathcal{F}}(\mathbf{u}_r) = \text{Res} \left(\frac{f_1}{\mathfrak{p}_r(X)}, \dots, \frac{f_r}{\mathfrak{p}_r(X)} \right) W_{\mathbf{X}}(\mathbf{u}_r); \quad (2.39)$$

$$\text{iii)} \quad W_{\mathbf{X}^\lambda}(\mathbf{u}_r) = \Delta_\lambda(H_r)W_{\mathbf{X}}(\mathbf{u}_r). \quad (2.40)$$

In particular, if the specialization morphism $B_r \rightarrow A$ is injective, the generalized Wronskians $\{W_{\mathbf{X}^\lambda}(\mathbf{u}_r) \mid \lambda \in \mathcal{P}_r\}$ are linearly independent over the integers.

Proof. Given a formal power series, $\mathbf{a} \in A[[t]]$, denote by $\mathbf{a}(0) := \mathbf{a} \bmod (t)$ its constant term. Notice that if $\mathbf{a}, \mathbf{b} \in A[[t]]$, then $(\mathbf{a} \cdot \mathbf{b})(0) = \mathbf{a}(0) \cdot \mathbf{b}(0)$, i.e. the ‘evaluation at $t = 0$ ’ is a ring homomorphism $A[[t]] \rightarrow A$. Thus (i) follows from the fact that

$$\begin{aligned} W_{\mathbf{X}}(\mathbf{u}_r)(0) &= \begin{vmatrix} u_0 & u_{-1} & \cdots & u_{-r+1} \\ u_1 & u_0 & \cdots & u_{-r+2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{r-1} & u_{r-2} & \cdots & u_0 \end{vmatrix} (0) = \begin{vmatrix} u_0(0) & u_{-1}(0) & \cdots & u_{-r+1}(0) \\ u_1(0) & u_0(0) & \cdots & u_{-r+2}(0) \\ \vdots & \vdots & \ddots & \vdots \\ u_{r-1}(0) & u_{r-2}(0) & \cdots & u_0(0) \end{vmatrix} = \\ &= \begin{vmatrix} 1 & 0 & \cdots & 0 \\ h_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{r-1} & h_{r-1} & \cdots & 1 \end{vmatrix} = 1, \end{aligned}$$

which proves that $W_{\mathbf{X}}(\mathbf{u}_r)$ is a unit in $A[[t]]$ and hence invertible.

(ii) For $i = 1, \dots, n$, write

$$f_i(X) = q_i(X)\mathfrak{p}_r(X) + \rho_i(X),$$

with $\deg(\rho_i) < r$. According to Remark 2.6.3, we have $W_{\mathcal{F}}(\mathbf{u}_r) = W_{(\rho_1, \dots, \rho_r)}(\mathbf{u}_r)$ and

$$\text{Res}\left(\frac{f_1}{p_r(X)}, \dots, \frac{f_r}{p_r(X)}\right) = \text{Res}\left(\frac{\rho_1}{p_r(X)}, \dots, \frac{\rho_r}{p_r(X)}\right).$$

We may consequently assume that $\deg(f_i) < r$ for $i = 1, \dots, r$. Since

$$W_{\mathcal{F}}(\mathbf{u}_r) \quad \text{and} \quad \text{Res}\left(\frac{f_1}{p_r(X)}, \dots, \frac{f_r}{p_r(X)}\right),$$

are both linear in $(f_1, \dots, f_r)$, we may assume that $f_i(X) = X^{m_i}$ with $0 \leq m_i < r$ for $i = 1, \dots, r$. However both sides of (2.39) are alternating in $f_1, \dots, f_r$, so we may assume that $0 \leq m_r < \dots < m_2 < m_1 < r$. Hence $m_i = r - i$ for $i = 1, \dots, r$. In this case the theorem follows from

$$\text{Res}\left(\frac{X^{r-1}}{p_r(X)}, \dots, \frac{1}{p_r(X)}\right) = 1.$$

(iii) To prove formula (2.40), it suffices to recall that

$$\text{Res}\left(\frac{X^{\lambda_i + r - i}}{p_r(X)}\right) = h_{\lambda_i - i + 1}.$$

We have

$$\text{Res}\left(\frac{X^{\lambda_1 + r - 1}}{p_r(X)}, \dots, \frac{X^{\lambda_r}}{p_r(X)}\right) = \Delta_{\lambda}(H_r) := \begin{vmatrix} h_{\lambda_1} & \dots & h_{\lambda_1 + r - 1} \\ \vdots & \ddots & \vdots \\ h_{\lambda_r - r + 1} & \dots & h_{\lambda_r} \end{vmatrix}, \quad (2.41)$$

and then (2.40) follows from item (ii) above. Since the Schur polynomials $\Delta_{\lambda}(H_r)$ are well known to form a $\mathbb{Z}$ -basis of B_r (e.g. [109, Section I.3]), as we shall also see in Chapter 5, it follows that the generalized Wronskians $\{W_{\mathbf{X}^{\lambda}}(\mathbf{u}_r) \mid \lambda \in \mathcal{P}_r\}$ are linearly independent over the integers as well, by item (i) and (2.40). $\square$

2.6.8. A quick check shows that $DW_{\mathbf{X}}(\mathbf{u}_r) = W_{\mathbf{X}^{(1)}}(\mathbf{u}_r)$. Equation (2.40) reads in this case as

$$W_{\mathbf{X}^{(1)}}(\mathbf{u}_r) = h_1 W_{\mathbf{X}}(\mathbf{u}_r) = e_1 W_{\mathbf{X}}(\mathbf{u}_r), \quad (2.42)$$

Formula (2.42) is the purely algebraic version of the celebrated theorem by Abel and Liouville ([6, p. 195]), stating that the derivative of the Wronskian of a fundamental system of solutions to a linear ODE is proportional to the Wronskian itself (see also [39]). It implies that if the Wronskian does not vanish at a point, it vanishes nowhere. Formula 2.42 is our prototype of the boson–fermion correspondence, in the sense of Chapter 7.

2.6.9. If A contains the rationals, the formal Laplace transform $\mathcal{L}$ yields the isomorphism $\mathcal{L}^{-1} : A[[t]] \rightarrow A[[t]]$ described in Proposition 2.5.2. Then, given $\mathcal{F} := (f_1, \dots, f_r) \in A[X]^r$ and

$$\mathcal{L}^{-1}(\mathbf{u}_r) = (\mathcal{L}^{-1}(u_0), \dots, \mathcal{L}^{-1}(u_{r-1}))$$

one can define $W_{\mathcal{F}}(\mathcal{L}^{-1}(\mathbf{u}_r))$, as in 2.6.2, just replacing D by ∂_t and u_i by $v_i := \mathcal{L}^{-1}(u_i)$. Notice that

$$\partial_t v_i = \mathcal{L}^{-1} \circ \mathcal{L} \partial_t \circ \mathcal{L}^{-1}(u_i) = \mathcal{L}^{-1}(Du_i) = \mathcal{L}^{-1}(u_{i+1}) = v_{i+1}.$$

In particular $\mathbf{v}_r := (v_0, v_{-1}, \dots, v_{-r+1})$ is a basis of the solutions to the linear ODE $\mathfrak{p}_r(\partial_t)y = 0$, and formula (2.39) can be rewritten in the form

$$W_{\mathcal{F}}(\mathbf{v}_r) := W_{\mathcal{F}}(\mathcal{L}^{-1}(\mathbf{u}_r)) = \text{Res} \left(\frac{f_1}{\mathfrak{p}_r(X)}, \dots, \frac{f_r}{\mathfrak{p}_r(X)} \right) W_{\mathbf{X}}(\mathbf{v}_r),$$

i.e. the ‘ratio’

$$\frac{W_{\mathcal{F}}(\mathbf{v}_r)}{W_{\mathbf{X}}(\mathbf{v}_r)}$$

is the same one would get from formula (2.40). Notice that

$$W_{\mathbf{X}}(\mathbf{v}_r) = \begin{vmatrix} v_0 & v_{-1} & \cdots & v_{-r+1} \\ v_1 & v_0 & \cdots & v_{-r+2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{r-1} & v_{r-2} & \cdots & v_0 \end{vmatrix} = \begin{vmatrix} v_0 & v_{-1} & \cdots & v_{-r+1} \\ \partial_t v_0 & \partial_t v_{-1} & \cdots & \partial_t v_{-r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_t^{r-1} v_0 & \partial_t^{r-1} v_{-1} & \cdots & \partial_t^{r-1} v_{-r+1} \end{vmatrix}$$

is the usual Wronskian we learn from the undergraduate calculus.

2.6.10. In particular (2.42) can be phrased, in the hypothesis of Section 2.6.9 by saying that the derivative of the Wronskian is proportional to the Wronskian itself, a fact which is equivalent to the fact that the determinant of the exponential of a square matrix is the exponential of its trace. Cf. Exercise 4.5.3.

Generalized Wronskians are related with derivatives of Wronskians. Let us write $W_{\lambda} := W_{\mathbf{X}^{\lambda}}$ for short, so that, e.g.

$$\partial_t W_0(\mathbf{v}_r) = W_{(1)}(\mathbf{v}_r). \quad (2.43)$$

In general, the j th derivative of a Wronskian is a linear combination of generalized Wronskians:

$$\partial_t^j W_0(\mathbf{v}_r) = \sum_{|\lambda|=j} g_{\lambda} W_{\lambda}(\mathbf{v}_r). \quad (2.44)$$

This is a consequence of (2.43) and an easy induction based on the following *Pieri rule for generalized Wronskians*:

$$\partial_t W_{\lambda}(\mathbf{v}_r) = \sum_{\mathbf{a}} W_{(\mathbf{a}(1)+\lambda_r, \mathbf{a}(2)+\lambda_{r-1}, \dots, \mathbf{a}(r)+\lambda_1)}(\mathbf{v}_r), \quad (2.45)$$

where the summation is taken over all the sequences $\mathbf{a} : \{0, 1, \dots, r-1\} \rightarrow \{0, 1\}$ such that $\sum_{j=0}^{r-1} \mathbf{a}(j) = 1$ but cancelling all the terms for which $\mathbf{a} + \lambda$ is not a partition.

Recall now from [33] that each box of a Young diagram $Y(\lambda)$ of a partition λ determines a *hook*, consisting of that box and of all the boxes in its row to the right of the box and its column below the box. The *hook length* of the box is the number of boxes in its hook. So, for instance, filling the Young diagram of the partition $(3, 2, 1, 1)$ with the hook length of each box gives the following *Young tableau* (Fig. 2.3):

Fig. 2.3 The hook length filling of $(4, 3, 1, 1)$

7	4	3	1
6	2	1	
1			
1			

Then an amazing fact occurs, observed by I. Scherbak:

Theorem 2.6.11. (see [49, 50]). The coefficient g_{λ} in (2.44) can be computed via the *hook length* formula:

$$g_{\lambda} = \frac{|\lambda|!}{k_1 \cdots k_j} = \frac{j!}{k_1 \cdots k_j}, \quad (2.46)$$

where k_i is the hook length of each box of the Young diagram of the partition λ .

A consequence of 2.6.11 is the following remarkable fact, already observed in [38]:

Corollary 2.6.12. The coefficient $g_{(n-r)^r}$ multiplying $W_{(n-r)^r}(\mathbf{v}_r)$ in the expansion of $\partial_t^{r(n-r)} W_0(\mathbf{v}_r)$ is precisely the Plücker degree (see Section 5.3) of the Grassmannian $G_{r,n}$, parametrizing r -dimensional linear subspaces of $\mathbb{C}^n$, i.e.

$$g_{(d-r)^{r+1}} = \frac{1!2! \cdots r! \cdot r(n-r)!}{(n-r)!(d-r+1)! \cdots d!}.$$

Example 2.6.13. Let $\mathbf{v}_2 = (v_0, v_1)$. Then

$$\begin{aligned} \partial_t^4 W_0(\mathbf{v}_2) &= \partial_t^3 \circ \partial_t W_0(\mathbf{v}_2) = \partial_t^3 W_{(1)}(\mathbf{v}_2) = \partial_t^2 (W_{(2)}(\mathbf{v}_2) + W_{(1,1)}(\mathbf{v}_2)) \\ &= \partial_t (2W_{(2,1)}(\mathbf{v}_2) + W_{(3)}(\mathbf{v}_2)) \\ &= 2W_{(2,2)}(\mathbf{v}_2) + 2W_{(3,1)}(\mathbf{v}_2) + W_{(4)}(\mathbf{v}_2) \end{aligned}$$

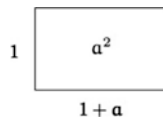
and the coefficient 2 multiplying $W_{(2,2)}(\mathbf{v}_2)$ is the Plücker degree of the Grassmannian $G_1(\mathbb{P}^3) := G_{2,4}(\mathbb{C})$ in its Plücker embedding, i.e. the *number of lines of $\mathbb{P}^3_{\mathbb{C}}$ meeting four others in general position*. All the other coefficients have enumerative interpretation in either the classical or the quantum cohomology of the Grassmannian (see, e.g. [38]).

2.7 Notes and References

- The Fibonacci² sequence $1, 1, 2, 3, 5, 8, \dots$, whose n th term ($n \geq 3$) is the sum of the preceding two, was born to model the growth of a colony of breeding rabbits and is perhaps the most popular example of LRS. For instance, the Italian artist Mario Merz³ realized many neon-light installations of the Fibonacci sequence, one of which, *Il volo dei numeri*⁴, was draped on the spire of the Mole Antonelliana, the symbol of the city of Torino, which nowadays hosts the National Cinema Museum⁵. The Fibonacci numbers obey to the same recursive law enjoyed by the powers $(1, a, a^2, \dots)$ of one of the two roots $(1 \pm \sqrt{5})/2$ of the polynomial $X^2 - X - 1$. The equality $a^2 = 1(1 + a)$ proves the link with the famous *golden ratio*, which the ancient Greek architects used to design the planimetry of their temples (Fig. 2.4).
- There is an enormous deal of literature concerning LRSs, for example, [57, 92, 113, 115, 147, 153, 154] just to cite a few. Their applications are countless, starting from algebra and algebraic geometry (see [115]) to industrial mathematics. General excellent expository references are [2, 17, 69], while [113] and [153, 154] describe applications to other sciences.
- Linear ODEs with constant coefficients and analytic source term are entirely governed by the algebra of LRSs: see [49, 50], but also [42] where a generalized formal Laplace transform is associated with any sequence of invertible elements in a ring. For more applications look at the paper [151]. The inverse of the formal Laplace transform of the formal series u_{-r+1} in Corollary 2.3.9 is precisely the *impulse response solution* of the generic linear ODE of order r . See Camporesi [13] for a truly enlightening exposition on that.

Fig. 2.4 A golden rectangle

with $a = \frac{1}{2} + \frac{\sqrt{5}}{2}$



²Leonardo Pisano, known as Fibonacci (Pisa, 1170–1240), wrote the famous *Liber Abaci* in 1202 where the name ‘zero’ comes from ‘Zephyrus’, a wind blowing from the West.

³Mario Merz (Milano, 1925–2003) was a painter and sculptor who used poor materials for his creations (<http://fondazionemerz.org/en/mario-merz/>).

⁴The flight of numbers

⁵<http://www.museocinema.it/>

- The notions of generic LRS and generic linear ODE were studied in [49] with the purpose to prove a Giambelli (or Jacobi–Trudi) formula for generalized Wronskians, associated with a fundamental system of linear ODEs. That formula generalizes the well-known result whereby the derivative of a Wronskian is proportional to the Wronskian itself; see also [39]. The construction of the universal basis of solutions to the Cauchy problem [49] essentially amounts to the explicit construction of the $\mathcal{D}$ -module associated with a generic linear differential operator of order r (see, e.g. [16, Chapter 6] and [65, Example 1.2.4]). This is completely transparent in the algebraic presentation of [42].
- To deal with LRSs, we used the language of formal power series. As we wanted to keep that to a minimum, we did not mention the beautiful calculus of formal distributions as presented in [73, Chapter 2] and [30, 103]. Formal calculus is essential to develop the theory of vertex algebras in a purely algebraic way. One of its most powerful tools, called ‘operator product expansion’, allows to expand formal Laurent series in two (or more) variables as linear combinations of *formal Dirac deltas* and their derivatives.
- The fact that generalized Wronskians (see the survey [50]) are parametrized by partitions is a consequence of Schubert calculus, not conversely. This is a good point to recall that the generating function of $p_r(w)$ (the number of partitions of length at most r of the integer w) is given by

$$\sum_{k \geq 0} p_r(k) q^k = \frac{1}{\prod_{i=1}^r (1 - q^i)},$$

a formula essentially due to Euler [78, p. 15]. In turn, this equals the q -rank of B_r :

$$\text{rank}_q B_r := \sum_{w \geq 0} \text{rank}(B_r)_w q^w = \sum_{w \geq 0} p_r(w) q^w = \frac{1}{\prod_{i=1}^r (1 - q^i)}.$$

For instance, $\text{rank}(B_5)_3 = p_5(3) = 3$ and $\text{rank}(B_5)_8 = p_5(8) = 18$. Standard references for the combinatorial, representation theoretical and algebro-geometrical aspects of partitions are [33, 109].

2.8 Exercises

2.8.1. Show that a sequence $(a_0, a_i, \dots)$ of complex numbers is a LRS if and only if its associated formal power series $\sum_{j \geq 0} a_j z^j$ is the formal Taylor expansion of a rational function, i.e. the ratio of two polynomials $f, g \in \mathbb{C}[z]$.

2.8.2. Let A be a $\mathbb{Q}$ -algebra, $D : A[[t]] \rightarrow A[[t]]$ as in (2.7) and $\partial_t := \mathcal{L} \circ D \circ \mathcal{L}^{-1}$. Prove that ∂_t satisfies the Leibniz rule:

$$\partial_t(\mathbf{a}(t)\mathbf{b}(t)) = \partial_t \mathbf{a}(t) \cdot \mathbf{b}(t) + \mathbf{a}(t) \cdot \partial_t \mathbf{b}(t)$$

although D does not.

2.8.3. Let A be a ring with unit and $A[[t]]$ the algebra of formal power series in the formal variable t . Prove that $\mathbf{a}(t) = \sum_{i \geq 0} a_i t^i$ is invertible if and only if a_0 is. If in addition A is commutative, then

$$\frac{1}{\mathbf{a}(t)} = a_0^{-1} \sum_{i \geq 0} (-1)^i a_0^{-i} \mathbf{a}(t)^i.$$

2.8.4. Let $A := \mathbb{Z}[\mathbf{t}] := \mathbb{Z}[t_1, \dots, t_r]$ and define

$$1 - e_1(\mathbf{t})z + \dots + (-1)^r e_r(\mathbf{t})z^r := \prod_{i=1}^r (1 - t_i z).$$

The coefficient $e_i(\mathbf{t})$ of z^i is called i th elementary symmetric polynomial of degree i in the indeterminates $t_1, \dots, t_r$. Let $h_j(\mathbf{t})$ denote the sum of all monomials of degree j in $\mathbf{t} := (t_1, \dots, t_r)$. Show that

$$\sum_{j \geq 0} h_j t^j = \frac{1}{\prod_{i=1}^r (1 - t_i z)}.$$

The term $h_j(\mathbf{t})$ is called the *complete symmetric polynomial* of degree j .

2.8.5. Write $\mathbf{x} := (x_1, x_2)$, and let $H_2(\mathbf{x})$ be the sequence $(h_1(\mathbf{x}), h_2(\mathbf{x}), \dots)$ of complete symmetric polynomials in (x_1, x_2) . Consider the *generalized Wronskian* associated with $\mathbf{f} := (\exp(x_1 t), \exp(x_2 t))$:

$$W_{(\lambda_1, \lambda_2)}(\mathbf{f}) := \begin{vmatrix} \partial_t^{\lambda_2} \exp(x_1 t) & \partial_t^{\lambda_2} \exp(x_2 t) \\ \partial_t^{1+\lambda_1} \exp(x_1 t) & \partial_t^{1+\lambda_1} \exp(x_2 t) \end{vmatrix}$$

- i) Show that $W_{(\lambda_1, \lambda_2)}(\mathbf{f}) := \Delta_{(\lambda_1, \lambda_2)}(H_2(\mathbf{x}))(x_1 - x_2) \exp((x_1 + x_2)t)$. In particular $W_{(0,0)}(\mathbf{f}) = (x_1 - x_2) \exp((x_1 - x_2)t)$. Thus *Jacobi–Trudy* formula for generalized Wronskians holds:

$$W_{(\lambda_1, \lambda_2)}(\mathbf{f}) = \Delta_{(\lambda_1, \lambda_2)}(H_2) W_{(0,0)}(\mathbf{f}).$$

- ii) Show that $\partial_t W_{(0,0)}(\mathbf{f}) = e_1(\mathbf{x}) W_{(0,0)}(\mathbf{f})$.
 iii) Compute $\partial_t^4 W_{(0,0)}(\mathbf{f})$ as a $\mathbb{Z}$ -linear combinations of generalized Wronskians. What is the coefficient of $W_{(2,2)}(\mathbf{f})$?

2.8.6. Prove the Pieri rule (2.45) for generalized Wronskians (hint: write the generalized Wronskian determinant $W_\lambda(\mathbf{v}_r)$ in the form

$$\partial_t^{\lambda-r} \mathbf{v}_r \wedge \partial_t^{1+\lambda_r-1} \mathbf{v}_r \wedge \dots \wedge \partial_t^{r-1+\lambda_1} \mathbf{v}_r$$

where $\partial_t^{i+\lambda_r-i} \mathbf{v}_r$ represents the i th row of the Wronskian. Taking the derivative of $W_\lambda(\mathbf{v}_r)$ then amounts to apply Leibniz rule of ∂_t with respect to the rows).

2.8.7. Show that if $1 \leq r \leq 2$, then $\Delta_{(2,2,1)}(H_r) = 0$.

2.8.8. Let

$$\exp(t) := \sum_{n \geq 0} \frac{t^n}{n!} \in \mathbb{Q}[[t]].$$

It solves the linear ODE $\ddot{y} - 3\dot{y} + 2y = 0$, where $\dot{y}$ denotes the derivative with respect to the indeterminate. Find the expression of $\exp(t)$ as a linear combination of $\mathcal{L}^{-1}(u_0)$ and $\mathcal{L}^{-1}(u_{-1})$, where $\mathcal{L}$ denotes the formal Laplace transform as in 2.5.

2.8.9. Find the set of solutions of $\ddot{y} - 3\dot{y} + 2y = t^2$ and $\ddot{y} - 3\dot{y} + 2y = \exp(t)$. Notice that $\exp(t)$ solves the associated homogeneous equation, and so we are in the case of *resonance*.

2.8.10. Consider the linear equation of order two $\partial_t^2 y - (x_1 + x_2)\partial_t y + x_1 x_2 \cdot y = 0$, with coefficients in $\widetilde{B}_2 := \mathbb{Q}[x_1, x_2]$. Then $\exp(t_1 z), \exp(t_2 z) \in \widetilde{B}_2[[z]]$ are $\mathbb{Z}$ -linearly independent, but they do not form a basis over the integers. Find the expression of $\exp(x_i t)$ as linear combination of a basis of solutions over $\widetilde{B}_2$. Show that $(\exp(x_1 t), \exp(x_2 t))$ form a basis of solutions in the localized ring $\widetilde{B}_2[[z]]_{(x_1 - x_2)}$.

2.8.11. Generalize Exercise 2.8.10. Let $e_i(\mathbf{x}) := e_i(x_1, \dots, x_r)$ be the elementary symmetric polynomials in $(x_1, \dots, x_r)$ of degree i . Then $\exp(x_i t)$, $1 \leq i \leq r$, are linearly independent solutions of the linear ODE:

$$\partial_t^r y - e_1(\mathbf{x}_r)\partial_t^{r-1} y + \dots + (-1)^r e_r(\mathbf{x}_r)y = 0$$

but they are not a basis over $\mathbb{Q}[x_1, \dots, x_r]$. Show they are a basis up to inverting the *Vandermonde determinant*:

$$V(x_1, \dots, x_r) = \prod_{1 \leq i < j \leq r} (x_i - x_j)$$

2.8.12. Consider the homogeneous linear equation with $B_2 \otimes_{\mathbb{Z}} \mathbb{Q}$ -coefficients:

$$\ddot{y} - e_1 \dot{y} + e_2 y = 0. \quad (2.47)$$

Let $B_2[\xi] := \frac{B_2[X]}{(X^2 - e_1 X + e_2)}$, where $\xi := X + (X^2 - e_1 X + e_2)$. Then the polynomial $X^2 - e_1 X + e_2$ has a root in $B[\xi]$. Show that $\exp(\xi \cdot t)$ is a solution of (2.47), and express it as a $B[\xi]$ -linear combination of u_0 and u_{-1} . Deduce the Euler formulas

$$\exp(\pm \sqrt{-1}t) = \cos t \pm \sqrt{-1} \sin t$$

2.8.13. Check that

$$\mathcal{M} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Q}[a, b, c, d]^{2 \times 2}$$

is a root of the polynomial $T^2 - (a + d)T + ad - bc = 0$ and, using this fact, that

$$\exp(\mathcal{M}t) = \sum_{n \geq 0} \frac{\mathcal{M}^n t^n}{n!}$$

is a solution of $\ddot{y} - (a + d)\dot{y} + (ad - bc)y = 0$. Deduce an expression for the exponential of $\mathcal{M} \cdot t$ by writing it as a linear combination of $v_0 := \mathcal{L}^{-1}(u_0)$ and $v_1 := \mathcal{L}^{-1}(u_{-1})$, with matrix coefficients.

2.8.14. Let $\mathfrak{p}_r(\partial_t) := \partial_t^r - e_1 \partial_t^{r-1} + \dots + (-1)^r e_r$ be the generic linear ODO of order r , so that

$$\{y_1 \in B_r[[t]] \mid \mathfrak{p}_r(\partial_t)y_1 = 0\} \quad (2.48)$$

is a generic linear ODE of order r . For all $2 \leq i \leq r$, define $y_i := \partial_t y_{i-1}$.

i) Show that (2.48) is equivalent to a system of linear ODE of the first order:

$$\partial_t \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} = C(e_1, \dots, e_r) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix}$$

where $C(e_1, \dots, e_r) \in B_r^{r \times r}$ is said to be *companion matrix*.

ii) Determine the expression of the companion matrix $C(e_1, \dots, e_r)$.

iii) For all $n \geq 0$, determine explicit expression for $C(e_1, e_2)^n$ and $C(e_1, e_2, e_3)^n$.

(Companion matrices are very important. They are associated with cyclic endomorphisms of a vector space and cyclic vectors of an endomorphism generate invariant subspaces also known as *Krylov subspaces*. See Tomei [143, Section 2.2].)

2.8.15. Write $H_r(z) = \sum_{n \geq 0} h_n z^n = \exp\left(\sum_{i \geq 1} x_i z^i\right)$, and compute h_n as an explicit polynomial of $(x_1, x_2, \dots)$ for the first few values of n . Express x_{r+1} as a B_r -linear combination of $(x_1, \dots, x_r)$.

2.8.16. Let M be a module over a $\mathbb{Z}$ -algebra A . Prove that the formal Laplace transform defined in 2.5 is invertible if and only if A contains the rational numbers.

2.8.17. Recall the usual definition of *Laplace transform* of an analytic function f :

$$L(f)(s) = \int_0^\infty e^{-st} f(t) dt,$$

with suitable restrictions on the domain of s depending on the choice of the function. How is L related to the formal Laplace transform defined in Section 2.5?

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