Chapter 2
Nonlinear Eigenvalue Problems in Dunford–Pettis Spaces

In this chapter, we present some variants of Leray–Schauder type fixed point theorems and eigenvalue results for decomposable single-valued nonlinear weakly compact operators in Dunford–Pettis spaces.

2.1 Introduction

In this section we discuss operator equations of the form

$$GTx = \lambda x, \tag{2.1}$$

in appropriate spaces of functions, where by GT we mean the composition $G \circ T$ of single-valued mappings and λ is a scalar. Recently, several authors [37, 38, 135] have taken advantage of the representation $F = GT$ and established fixed point theorems for F-self mapping on closed convex subset of Banach spaces. In applications to construct a set Ω of a space E such that F takes Ω back into Ω is very difficult and sometimes impossible. As a result, it makes sense to discuss maps $F : \Omega \longrightarrow X$. To do this, one of the most important tools in nonlinear analysis is the Leray–Schauder principle. Due to the lack of compactness for many problems posed in L^1-spaces, we also give alternatives of Leray–Schauder type for some nonlinear weakly compact composite operators $F = GT$ in Dunford–Pettis spaces, where G and T verify some sequential conditions (($\mathcal{H}_1$) and ($\mathcal{H}_2$)). Let S be a nonlinear operator from a Banach space X into itself. We will use the following two conditions.

$$(\mathcal{H}_1) \qquad \begin{cases} \textit{If } \{x_n\}_{n\in\mathbb{N}} \text{ is a weakly convergent sequence in } E, \textit{ then} \\ \{Sx_n\}_{n\in\mathbb{N}} \text{ has a strongly convergent subsequence in } E. \end{cases}$$

A.B. Amar, D. O'Regan, *Topological Fixed Point Theory for Singlevalued and Multivalued Mappings and Applications*, DOI 10.1007/978-3-319-31948-3_2

$$(\mathcal{H}_2) \qquad \begin{cases} \textit{If } \{x_n\}_{n\in\mathbb{N}} \text{ is a weakly convergent sequence in } E, \textit{ then} \\ \{Sx_n\}_{n\in\mathbb{N}} \text{ has a weakly convergent subsequence in } E. \end{cases}$$

In the literature a continuous map satisfying the condition $(\mathcal{H}_1)$ is called (ws)-compact [100].

The following fixed point result will be used throughout this section. The proof follows from the Schauder fixed point theorem.

Theorem 2.1. *Let Ω be a nonempty closed convex subset of a Banach space X. Assume that $F : \Omega \longrightarrow \Omega$ is a continuous map which verifies $(\mathcal{H}_1)$. If $F(\Omega)$ is weakly relatively compact, then there exists $x \in \Omega$ such that $Fx = x$.*

2.2 Nonlinear Eigenvalue Problems

We use Theorem 2.1 to obtain a nonlinear alternative of Leray–Schauder type for decomposable nonlinear weakly compact operators in Dunford–Pettis spaces.

Theorem 2.2. *Let X be a Dunford–Pettis space, Ω a nonempty closed convex subset of X, U a relatively open subset of Ω and $z \in U$. If $G : X \to X$ and $T : \overline{U} \to X$ are operators satisfying:*

1. *G is a bounded linear weakly compact operator.*
2. *T is a nonlinear continuous operator satisfying $(\mathcal{H}_2)$.*
3. *$T(\overline{U})$ is bounded and $G(T(\overline{U})) \subset \Omega$.*

Then, either

$(\mathcal{A}_1)$ *GT has a fixed point in $\overline{U}$, or*

$(\mathcal{A}_2)$ *there is a point $x \in \partial_\Omega U$ (the boundary of U in Ω) and $\lambda \in (0,1)$ with $x = (1-\lambda)z + \lambda GTx$.*

Remark 2.1. (a) Since Ω is closed, the closure in Ω of U and closure are the same, for $U \subset \Omega$.

(b) For $U \subset \Omega$, we have $\partial_\Omega U = \overline{U} \cap \overline{\Omega\backslash U}$.

Proof. Consider $GT : \overline{U} \to \Omega$. Suppose $(\mathcal{A}_2)$ does not hold. Also without loss of generality, assume that the operator GT has no fixed point in $\partial_\Omega U$ (otherwise we are finished, i.e., $(\mathcal{A}_1)$ occurs). Let D be the set defined by

$$D = \left\{ x \in \overline{U} \ : \ x = (1-\lambda)z + \lambda GTx, \text{ for some } \lambda \in [0,1] \right\}.$$

Now $D \neq \emptyset$ since $z \in D$. Also D is closed. To see this, let (x_n) be a sequence in D such that $x_n \to x \in \overline{U}$. For all $n \in \mathbb{N}$, there exists a $\lambda_n \in [0,1]$ such that $x_n = (1-\lambda)z + \lambda_n GTx_n$. Now $\lambda_n \in [0,1]$, so we can extract a subsequence $\{\lambda_{n_j}\}_j$ such that $\lambda_{n_j} \longrightarrow \lambda \in [0,1]$. So, by the continuity of the operators G and T we obtain that $(1-\lambda_{n_j})z+\lambda_{n_j}GTx_{n_j}\longrightarrow(1-\lambda)z+\lambda GTx$. Hence $x=(1-\lambda)z+\lambda GTx$ and $x\in D$. Next, we shall prove that the set D is sequentially compact. To see this, let $\{x_n\}_n$ be any sequence in D. Since $GT(D)$ is weakly relatively compact, we obtain by the Eberlein–Šmulian theorem (Theorem 1.8) that there exists a subsequence $\{x_{n_j}\}_j$ of $\{x_n\}_n$ with $GTx_{nj}\rightharpoonup y$ for some $y\in\Omega$. We have $x_{n_j}=(1-\lambda_{n_j})z+\lambda_{n_j}GTx_{n_j}$ for some $\lambda_{n_j}\in[0,1]$. Passing eventually to a new subsequence, we may assume that $\lambda_{n_j}\to\lambda$ for $\lambda\in[0,1]$. So, $\{x_{n_j}\}_j\rightharpoonup(1-\lambda)z+\lambda y$. Next, since T verifies $(\mathcal{H}_2)$, then $\{Tx_{n_j}\}_j$ has a weakly convergent subsequence, say $\{Tx_{n_{j_k}}\}_k$. Using the fact that the linear operator G is weakly compact together with Proposition 1.18, we infer that the sequence $\{GTx_{n_{j_k}}\}_k$ is strongly convergent. Hence $\{x_{n_{j_k}}\}_k$ is strongly convergent as well. Hence D is compact. Because E is a Hausdorff locally convex space, we have that E is completely regular (see Remark 1.11). Since $D\cap(\Omega\setminus U)=\emptyset$, then by Proposition 1.1, there is a continuous function $\varphi\ :\ \Omega\longrightarrow[0,1]$, such that $\varphi(x)=1$ for $x\in D$ and $\varphi(x)=0$ for $x\in\Omega\setminus U$. Since Ω is convex, $z\in\Omega$, we can define the operator $\widehat{GT}:\Omega\to\Omega$ by

$$\widehat{GT}x=\begin{cases}(1-\varphi(x))z+\varphi(x)GTx, & \text{if } x\in U,\\ z, & \text{if } x\in\Omega\setminus U.\end{cases}$$

We first check that $\widehat{GT}$ is continuous and satisfies $(\mathcal{H}_1)$. Indeed, we have $\partial_\Omega U=\partial_\Omega\overline{U}$ and the operators φ and GT are continuous, so $\widehat{GT}$ is continuous. By an argument similar to the one used above, it is easy to show that GT satisfies $(\mathcal{H}_1)$. Since $[0,1]$ is compact, it follows that $\widehat{GT}$ satisfies $(\mathcal{H}_1)$. Now, the set $GT(\overline{U})$ is weakly relatively compact. Applying the Krein -Šmulian theorem (Theorem 1.10), we have that the set $D_*=\overline{\mathrm{conv}}(GT(\overline{U})\cup\{z\})$ is convex and weakly compact. Moreover, $\widehat{GT}(D_*)\subset D_*$. Thus, all the assumptions of Theorem 2.1 are satisfied for the operator $\widehat{GT}$. Therefore, there exists $x_0\in\Omega$ with $\widehat{GT}x_0=x_0$. From the definition of $\widehat{GT}$, x_0 must be an element of U. Then $x_0=(1-\varphi(x_0))z+\varphi(x_0)GTx_0$, which implies that $x_0\in D$ and so $\varphi(x_0)=1$. Accordingly, $GTx_0=x_0$ and the proof is complete. ■

Corollary 2.1. *Let E be a Dunford–Pettis space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω and $z\in U$. Suppose $G:E\to E$ is a bounded linear weakly compact operator and $T:\overline{U}\to E$ is a nonlinear continuous operator satisfying $(\mathcal{H}_2)$, $T(\overline{U})$ is bounded and $G(T(\overline{U}))\subset\Omega$. Also, assume that GT satisfies the Leray–Schauder boundary condition*

$$x\neq(1-\lambda)z+\lambda GTx,\ \ \lambda\in(0,1),\ x\in\partial_\Omega U,$$

then the set of fixed point of GT in $\overline{U}$ is nonempty and compact.

Proof. By Theorem 2.2, the operator GT has a fixed point in $\overline{U}$. Let $\mathcal{S}=\Big\{x\in\overline{U}\ :\ GTx=x,\Big\}$ be the fixed point set of GT. Since the operators G and T

are continuous, $\mathcal{S}$ is obviously a closed subset of $\overline{U}$ such that $GT(\mathcal{S}) = \mathcal{S}$. Following an argument similar to that in Theorem 2.2, we obtain that $\mathcal{S}$ is sequentially compact and, hence, it is compact. ■

As a special case, we obtain a fixed point theorem of Rothe type [175] for decomposable nonlinear weakly compact operators.

Corollary 2.2. *Let E be a Dunford–Pettis space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω and $z \in U$. Suppose $G : E \to E$ is a bounded linear weakly compact operator and $T : \overline{U} \to E$ is a nonlinear continuous operator satisfying $(\mathcal{H}_2)$ and $T(\overline{U})$ is bounded. In addition, assume that $\overline{U}$ is starshaped with respect to z and $G(T(\partial_\Omega U)) \subseteq \overline{U}$. Then the set of fixed points of F in $\overline{U}$ is nonempty and compact.*

Proof. Because $\overline{U}$ is starshaped with respect to θ and $GT(\partial_\Omega U) \subseteq \overline{U}$, then $x \neq (1-\lambda)z + \lambda GTx, \lambda \in (0,1), x \in \partial_\Omega U$. Applying Corollary 2.1, the set of fixed points of GT in $\overline{U}$ is nonempty and compact. ■

Corollary 2.3. *Let E be a Dunford–Pettis space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω and $\theta \in U$. Suppose $G : E \to E$ is a bounded linear weakly compact operator and $T : \overline{U} \to E$ is a nonlinear continuous operator satisfying $(\mathcal{H}_2)$, $T(\overline{U})$ is bounded and $G(T(\overline{U})) \subset \Omega$. In addition, suppose GT has no fixed point in $\overline{U}$. Then, there exist an $x \in \partial_\Omega U$ and $\lambda \in (0,1)$ such that $x = \lambda GTx$.*

Corollary 2.4. *Let E be a Dunford–Pettis space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω with $\theta \in U$ and $\alpha \geq 1$. Suppose $G : E \to E$ is a bounded linear weakly compact operator and $T : \overline{U} \to E$ is a nonlinear continuous operator satisfying $(\mathcal{H}_2)$, $T(\overline{U})$ is bounded and $G(T(\overline{U})) \subset \Omega$. In addition, assume that there is a real number $k > \alpha$ such that*

$$G(T(\overline{U})) \cap (k.U) = \emptyset. \tag{2.2}$$

Then there exist an $x \in \partial_\Omega U$ and $\lambda \geq k$ such that $GTx = \lambda x$.

Remark 2.2. θ is the zero vector of E.

Proof. Consider $F = GT : \overline{U} \to \Omega$. We suppose that for all $x \in \partial_\Omega U$ and $\lambda \geq k, F(x) \neq \lambda x$. Let $F_1 = \frac{1}{k} F$ and

$$D = \left\{ x \in \overline{U} \ : \ x = \lambda F_1 x, \text{ for some } \lambda \in [0,1] \right\}.$$

The set D is nonempty because $\theta \in D$. Following an argument similar to that in Theorem 2.2, we obtain that D is compact. Now we show that $D \cap (\Omega \setminus U) = \emptyset$. If this is not the case, there exists an $x \in \Omega \setminus U$ and $\beta \in [0,1]$ such that $\beta F_1 x = x$.

If $\beta = 0$, then $x = \theta$, which contradicts $\theta \in U$. If $\beta \neq 0$, then $Fx = \frac{k}{\beta}x$ ($\frac{k}{\beta} \geq k$), which contradicts (3.15). Thus, $D \cap (\Omega \setminus U) = \emptyset$. Let $F_1^* : \Omega \to \Omega$ the operator defined by

$$F_1^* x = \begin{cases} \varphi(x)F_1 x, & \text{if } x \in U, \\ z, & \text{if } x \in \Omega \setminus U. \end{cases}$$

Following arguments similar to those used in the proof of Theorem 2.2, we prove that F_1^* has a fixed point $y \in \Omega$. If $y \notin U$, $\varphi(y) = 0$ and $y = \theta$, which contradicts the hypothesis $\theta \in U$. Then $y \in U$ and $y = \varphi(y)Fy$, which implies that $y \in D$, and so $\varphi(y) = 1$ and $Fy = ky$. Hence, $F(\overline{U}) \cap (k.U) \neq \emptyset$, another contradiction. Accordingly, there exist an $x \in \partial_\Omega U$ and $\lambda \geq k$ such that $Fx = GTx = \lambda x$. ■

In the rest of this section we shall discuss nonlinear Leray–Schauder alternatives for decomposable nonlinear positive operators. Let E_1 and E_2 be two Banach lattice spaces, with positive cones E_1^+ and E_2^+, respectively. An operator F from E_1 into E_2 is said to be positive if it carries the positive cône E_1^+ into E_2^+ (i.e., $F(E_1^+) \subset E_2^+$).

Theorem 2.3. *Let Ω be a nonempty closed convex subset of a Banach lattice E such that $\Omega^+ := \Omega \cap E^+ \neq \emptyset$. Assume $F : \Omega \longrightarrow \Omega$ is a positive continuous operator satisfying $(\mathcal{H}_1)$. If $F(\Omega)$ is weakly relatively compact, then F has at least a positive fixed point in Ω.*

Proof. Clearly, the set Ω^+ is a closed convex subset of E^+ and $F(\Omega^+) \subseteq \Omega^+$. Also, $F(\Omega^+) \subseteq F(\Omega)$, so $F(\Omega^+)$ is weakly relatively compact. Now, it suffices to apply Theorem 2.1 to prove that F has a fixed point in $\Omega^+ \subseteq \Omega$. ■

Theorem 2.4. *Let E be a Dunford–Pettis lattice space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω and $z \in U \cap E^+$. Suppose $G : E \to E$ is a positive bounded linear weakly compact operator and $T : \overline{U} \to E$ is a positive nonlinear continuous operator satisfying $(\mathcal{H}_2)$, $T(\overline{U})$ is bounded and $G(T(\overline{U})) \subset \Omega$. Then, either*

- $(\mathcal{A}_1)$ *GT has a positive fixed point in $\overline{U}$, or*
- $(\mathcal{A}_2)$ *there is a point $x \in \partial_\Omega U \cap E^+$ (the positive boundary of U in Ω) and $\lambda \in (0,1)$ with $x = (1-\lambda)z + \lambda GTx$.*

Proof. Consider $GT : \overline{U} \to \Omega$. Suppose $(\mathcal{A}_2)$ does not hold. Also without loss of generality, assume that the operator GT has no positive fixed point in $\partial_\Omega U$ (otherwise we are finished, i.e., $(\mathcal{A}_1)$ occurs). Let D be the set defined by

$$D = \left\{ x \in \overline{U} \cap E^+ : x = (1-\lambda)z + \lambda GTx, \text{ for some ł} \in [0,1] \right\}.$$

Now $D \neq \emptyset$ since $z \in D$. Because E is a normed lattice, E^+ is closed, and so, $\overline{U} \cap E^+$ is a closed subset of Ω. Following arguments similar to those used in the proof of Theorem 2.2, we prove that D is compact. Because E is a Hausdorff locally convex space, we have that E is completely regular (see Remark 1.11). Since $D \cap (\Omega \setminus U) = \emptyset$, then by Proposition 1.1, there is a continuous function $\varphi : \Omega \longrightarrow [0, 1]$, such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in \Omega \setminus U$. Since Ω is convex, $z \in \Omega$, we can define the operator $\widehat{GT} : \Omega \to \Omega$ by

$$\widehat{GT}x = \begin{cases} (1 - \varphi(x))z + \varphi(x)GTx, & \text{if } x \in U, \\ z, & \text{if } x \in \Omega \setminus U. \end{cases}$$

Clearly, the operator $\widehat{GT}$ is positive. By an argument similar to that in Theorem 2.2 and using Theorem 2.3, we prove that there exists a positive element $x_0 \in \Omega$ with $\widehat{GT}x_0 = x_0$. If $x_0 \notin U$, $\varphi(x_0) = 0$ and $x_0 = z$, which contradicts the hypothesis $z \in U$. Then, $x_0 \in U$ and $x_0 = (1-\varphi(x_0))z + \varphi(x_0)GTx_0$, which implies that $x_0 \in D$ and so $\varphi(x_0) = 1$. Accordingly, $GTx_0 = x_0$ and x_0 is a positive fixed point of GT which completes the proof. ■

Corollary 2.5. *Let E be a Dunford–Pettis lattice space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω and $z \in U \cap E^+$. Suppose $G : E \to E$ is a positive bounded linear weakly compact operator and $T : \overline{U} \to E$ is a positive nonlinear continuous operator satisfying $(\mathcal{H}_2)$, $T(\overline{U})$ is bounded and $G(T(\overline{U})) \subset \Omega$. Also, assume that GT satisfies the Leray–Schauder boundary condition*

$$x \neq (1 - \lambda)z + \lambda GTx, \ \lambda \in (0, 1), \ x \in \partial_\Omega U \cap E^+,$$

then the set of positive fixed points of GT in $\overline{U}$ is nonempty and compact.

http://www.springer.com/978-3-319-31947-6

Topological Fixed Point Theory for Singlevalued and Multivalued Mappings and Applications
Ben Amar, A.; O' Regan, D.
2016, X, 194 p., Hardcover
ISBN: 978-3-319-31947-6