Chapter 2
Kinematic Analysis of Mechanisms. Relative Velocity and Acceleration. Instant Centers of Rotation

Abstract Kinematic analysis of a mechanism consists of calculating position, velocity and acceleration of any of its points or links. To carry out such an analysis, we have to know linkage dimensions as well as position, velocity and acceleration of as many points or links as degrees of freedom the linkage has. We will point out two different methods to calculate velocity of a point or link in a mechanism: the relative velocity method and the instant center of rotation method.

2.1 Velocity in Mechanisms

We will point out two different methods to calculate velocity of a point or link in a mechanism: the relative velocity method and the instant center of rotation method. However, before getting into the explanation of these methods, we will introduce the basic concepts for their development.

2.1.1 Position, Displacement and Velocity of a Point

To analyze motion in a system, we have to define its position and displacement previously. The movement of a point is a series of displacements in time, along successive positions.

2.1.1.1 Position of a Point

The position of a point is defined according to a reference frame. The coordinate system in a plane can be Cartesian or polar (Fig. 2.1).

A. Simón Mata et al., *Fundamentals of Machine Theory and Mechanisms*, Mechanisms and Machine Science 40, DOI 10.1007/978-3-319-31970-4_2

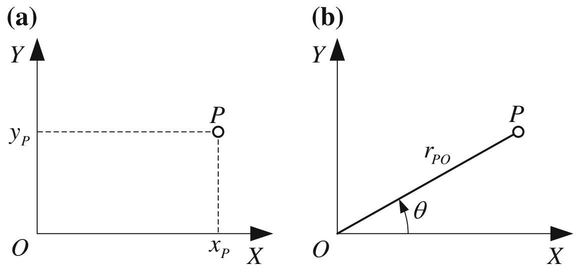


Fig. 2.1 **a** Cartesian and polar coordinates of point P in a plane. **b** Polar coordinates of the same point

In any coordinate system, we have to define the following:

- Origin of coordinates: starting point from where measurements start.
- Axis of coordinates: established directions to measure distances and angles.
- Unit system: units to quantify distances.

If a polar coordinate system is used, the position of a point is defined by a vector called $\mathbf{r}_{PO}$ connecting the origin of coordinates O with the mentioned point. If O is a point on the frame this vector gives us the absolute position of point P and we will call it $\mathbf{r}_P$.

In most practical situations, an absolute reference system, considered stationary, is used. The stationary system coordinates do not depend on time. The absolute position of a point is defined as its position seen from this absolute reference system. If the reference system moves with respect to a stationary system, the position of the point is considered a relative position.

Anyway, this choice is not fundamental in kinematics as the movements to be studied will be relative. Take, for example, the suspension of a car where movements might refer to the car body, without considering whether the car is moving or not. Movements in the suspension system can be regarded as absolute motion with respect to the car body.

2.1.1.2 Displacement of a Point

When a point changes its position, a displacement takes place. If at instant t the point is at position P and at instant $t + \Delta t$, the point is at P', displacement during Δt is defined as the vector that measures the change in position (Eq. 2.1):

$$\Delta \mathbf{r} = \mathbf{r}_{P'} - \mathbf{r}_P \tag{2.1}$$

Displacement is a vector that connects point P at instant t with point P' at instant $t + \Delta t$ and does not depend on the path followed by the point but on the initial and final positions (Fig. 2.2).

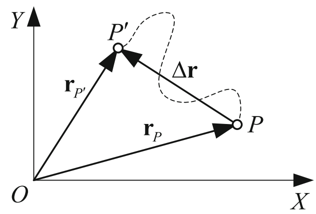


Fig. 2.2 Displacement of point P in a plane during instant Δt

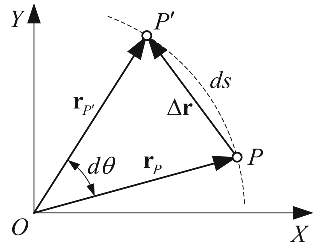


Fig. 2.3 Displacement of point P in a plane during instant dt close to zero

2.1.1.3 Velocity of a Point

The ratio between point displacement and time spent carrying it out is referred to as average velocity of that point. Therefore, average velocity is a vector of magnitude $\Delta\mathbf{r}/\Delta t$ and has the same direction as displacement vector $\Delta\mathbf{r}$. If the time during which displacement takes place is close to zero, the velocity of the point is called instant velocity, or simply velocity (Eq. 2.2):

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \tag{2.2}$$

The instant velocity vector magnitude is dr/dt. In an infinitesimal position change, the direction of the displacement vector coincides with the trajectory. When O is the instantaneous center of the trajectory of point P, we can express the instant velocity magnitude as Eq. (2.3):

$$v_P = \frac{dr}{dt} = \frac{ds}{dt} = \frac{d\theta}{dt} \cdot r_P = \omega \cdot r_P \tag{2.3}$$

The direction of this velocity is the same as $d\mathbf{r}$ which, at the same time, is tangent to the motion trajectory of point P (Fig. 2.3).

2.1.2 Position, Displacement and Angular Velocity of a Rigid Body

Any movement of a rigid body can be considered a combination of two motions: the displacement of a point in the rigid body and its rotation with respect to the point.

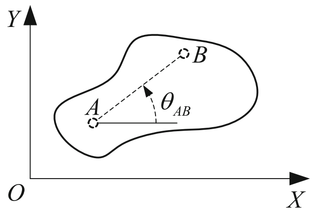


Fig. 2.4 Angular position of a rigid body θ_{AB}

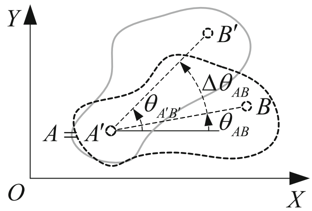


Fig. 2.5 Angular displacement of a rigid body $\Delta\theta_{AB}$

In the last section, we defined the displacement of a point, so the next subject to be studied is the rotation of a rigid body.

2.1.2.1 Angular Position of a Rigid Body

To define the angular position of a rigid body, we just need to know the angle formed by the axis of the coordinate system and reference line *AB* (Fig. 2.4).

2.1.2.2 Angular Displacement of a Rigid Body

When a rigid body changes its angular position from θ_{AB} to $\theta_{A'B'}$, angular displacement $\Delta\theta_{AB}$ takes place (Fig. 2.5).

$$\theta_{A'B'} = \theta_{AB} + \Delta\theta_{AB} \tag{2.4}$$

The angular displacement of a rigid body, $\Delta\theta_{AB}$, does not depend on the trajectory followed but on the initial and final angular position (Eq. 2.4).

2.1.2.3 Angular Velocity of a Rigid Body

We define the angular velocity of a rigid body as the ratio between angular displacement and its duration. If this time is, *dt* close to zero, this velocity is called instant angular velocity or simply angular velocity (Eq. 2.5).

$$\omega_{AB} = \frac{d\theta_{AB}}{dt} \tag{2.5}$$

2.1.3 Relative Velocity Method

In this section we will develop the relative velocity method that will allow calculating linear and angular velocities of points and links in a mechanism.

2.1.3.1 Relative Velocity Between Two Points

Let A be a point that travels from position A to position A' during time interval Δt and let B be a point that moves from position B to position B' in the same time interval (Fig. 2.6).

Absolute displacements of points A and B are given by vectors $\Delta\mathbf{r}_A$ and $\Delta\mathbf{r}_B$. Relative displacement of point B with respect to A is given by vector $\Delta\mathbf{r}_{BA}$, so it verifies (Eq. 2.6):

$$\Delta\mathbf{r}_B = \Delta\mathbf{r}_A + \Delta\mathbf{r}_{BA} \tag{2.6}$$

In other words, we can consider that point B moves to position B' with displacement equal to the one for point A to reach point B'' followed by another displacement, from point B'' to point B'. The latter coincides with vector $\Delta\mathbf{r}_{BA}$ for relative displacement. We can assert the same for the displacement of point A (Eq. 2.7), hence:

$$\Delta\mathbf{r}_A = \Delta\mathbf{r}_B + \Delta\mathbf{r}_{AB} \tag{2.7}$$

Evidently $\Delta\mathbf{r}_{BA}$ and $\Delta\mathbf{r}_{AB}$ are two vectors with the same magnitude but opposite directions.

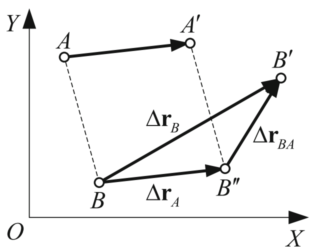


Fig. 2.6 Absolute displacements of points A and B, $\Delta\mathbf{r}_A$ and $\Delta\mathbf{r}_B$, and relative displacement of point B with respect to A, $\Delta\mathbf{r}_{BA}$

If we regard these as infinitesimal displacements and relate them to time dt, the time it took them to take place, we obtain the value of the relative velocities by Eq. (2.8):

$$\frac{d\mathbf{r}_B}{dt} = \frac{d\mathbf{r}_A}{dt} + \frac{d\mathbf{r}_{BA}}{dt} \Rightarrow \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \tag{2.8}$$

Therefore, the velocity of a point can be determined by the velocity of another point and their relative velocity.

As we have mentioned before, relative displacements $\Delta\mathbf{r}_{BA}$ and $\Delta\mathbf{r}_{AB}$ have opposite directions. Therefore relative velocities $\mathbf{v}_{BA}$ and $\mathbf{v}_{AB}$ will be two vectors with the same magnitude that also have opposite directions (Eq. 2.9).

$$\mathbf{v}_{BA} = -\mathbf{v}_{AB} \tag{2.9}$$

2.1.3.2 Relative Velocity Between Two Points of the Same Link

Let AB be a reference line on a body that changes its position to $A'B'$ during time interval Δt.

As studied in the previous section, the vector equation for the displacement of point B is Eq. (2.6) (Fig. 2.7a). In the case of A and B belonging to the same body, distance $\overline{AB}$ does not change, so the only possible relative movement between A and B is a rotation of radius $\overline{AB}$. This way, relative displacement will always be a rotation of point B about point A (Fig. 2.7b).

If we divide these displacements by the time interval in which they happened, we obtain Eq. (2.11):

$$\frac{\Delta\mathbf{r}_B}{\Delta t} = \frac{\Delta\mathbf{r}_A}{\Delta t} + \frac{\Delta\mathbf{r}_{BA}}{\Delta t} \Rightarrow \mathbf{V}_B = \mathbf{V}_A + \mathbf{V}_{BA} \tag{2.10}$$

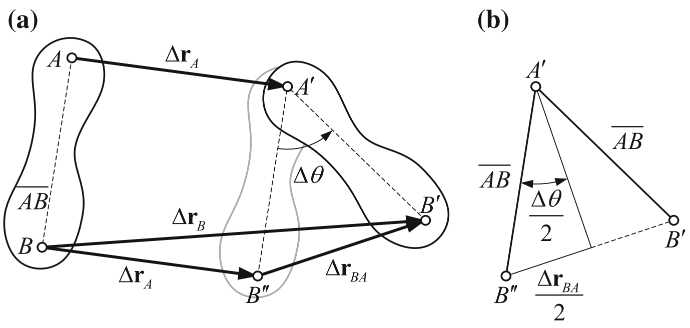


Fig. 2.7 **a** Relative displacement of point B with respect to point A (both being part of the same link). **b** Rotation of point B about point A

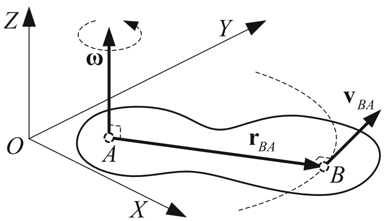


Fig. 2.8 Relative velocity of point B with respect to point A (both being part of the same link)

The value of $\mathbf{V}_{BA}$ (average relative velocity of point B with respect to point A) can be determined by using Eq. (2.11):

$$V_{BA} = \frac{\Delta r_{BA}}{\Delta t} = \frac{2\sin(\Delta\theta/2)}{\Delta t} \tag{2.11}$$

where $\Delta\theta$ is the angular displacement of body AB (Fig. 2.7b). If all displacements take place during an infinitesimal period of time, dt, then average velocities become instant velocities (Eq. 2.12):

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \tag{2.12}$$

This way, the velocity of point B can be obtained by adding relative instant velocity $\mathbf{v}_{BA}$ to the velocity of point A.

To obtain the magnitude of relative instant velocity $\mathbf{v}_{BA}$ in Eq. (2.13), we have to consider that the time during which displacement takes place is close to zero in Eq. (2.11)

$$v_{BA} = \lim_{\Delta t \to 0} \frac{\Delta r_{BA}}{\Delta t} = \frac{2\sin(d\theta/2)}{dt}\overline{AB} \simeq \frac{d\theta}{dt}\overline{AB} = \omega\overline{AB} \tag{2.13}$$

Therefore, any point on a rigid body, such as B, moves relatively to any other point on the same body, such as A, with velocity $\mathbf{v}_{BA}$, which can be expressed as a vector of magnitude equal to the product of the angular velocity of the body multiplied by the distance between both points (Eq. 2.14). Its direction is given by the angular velocity of the body, perpendicular to the straight line connecting both points (Fig. 2.8).

$$\mathbf{v}_{BA} = \boldsymbol{\omega} \wedge \mathbf{r}_{BA} \tag{2.14}$$

2.1.3.3 Application of the Relative Velocity Method to One Link

Equation (2.13) is the basis for the relative velocity method. It is a vector equation that allows us to calculate two algebraic unknowns such as one magnitude and one direction, two magnitudes or two directions.

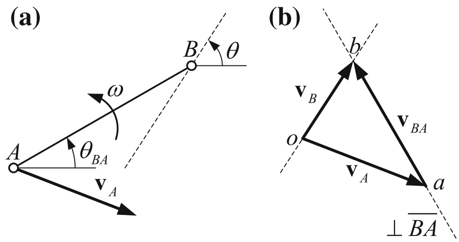


Fig. 2.9 **a** Calculation of the point B velocity magnitude knowing its direction and vector $\mathbf{v}_A$. **b** Velocity diagram

Figure 2.9a shows points A and B of a link moving at unknown angular velocity ω. Suppose that we know the velocity of point A, $\mathbf{v}_A$, and the velocity direction of point B. To calculate the velocity magnitude of point B, we use Eq. (2.13). Studying every parameter in the equation:

- $\mathbf{v}_A$ is a vector defined as $\mathbf{v}_A = v_{A_x}\hat{\mathbf{i}} + v_{A_y}\hat{\mathbf{j}}$ with known magnitude and direction.
- $\mathbf{v}_B$ is a vector with known direction and unknown magnitude. Assuming it is moving upward to the left (Fig. 2.9a), the direction of this vector will be given by angle θ and it will be defined as Eq. (2.15):

$$\mathbf{v}_B = v_B \cos\theta\hat{\mathbf{i}} + v_B \sin\theta\hat{\mathbf{j}} \tag{2.15}$$

- $\mathbf{v}_{BA}$ is a vector of unknown magnitude due to the fact that we do not know the angular velocity value, ω, of the rigid body. Its direction is perpendicular to segment line $\overline{AB}$ (Fig. 2.9b). Therefore, it can be obtained in Eq. (2.16):

$$\begin{aligned}\mathbf{v}_{BA} = \boldsymbol{\omega} \wedge \mathbf{r}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega \\ r_{BA_x} & r_{BA_y} & 0 \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega \\ \overline{BA}\cos\theta_{BA} & \overline{BA}\sin\theta_{BA} & 0 \end{vmatrix} \\ = -r_{BA_y}\omega\hat{\mathbf{i}} + r_{BA_x}\omega\hat{\mathbf{j}} = \overline{BA}\omega(-\sin\theta_{BA}\hat{\mathbf{i}} + \cos\theta_{BA}\hat{\mathbf{j}})\end{aligned} \tag{2.16}$$

where $r_{BA_x} = \overline{AB}\cos\theta_{AB}$ and $r_{BA_y} = \overline{AB}\sin\theta_{AB}$.

If we plug the velocity vectors into Eq. (2.13), we obtain Eq. (2.17):

$$\begin{aligned}v_B\cos\theta\hat{\mathbf{i}} + v_B\sin\theta\hat{\mathbf{j}} &= v_{A_x}\hat{\mathbf{i}} + v_{A_y}\hat{\mathbf{j}} - r_{BA_x}\omega\hat{\mathbf{i}} + r_{BA_y}\omega\hat{\mathbf{j}} \\ &= v_{A_x}\hat{\mathbf{i}} + v_{A_y}\hat{\mathbf{j}} - \overline{BA}\omega\sin\theta_{BA}\hat{\mathbf{i}} + \overline{BA}\omega\cos\theta_{BA}\hat{\mathbf{j}}\end{aligned} \tag{2.17}$$

If we break the velocity vectors in the equation into their components, two algebraic equations are obtained (Eq. 2.18):

$$\left.\begin{aligned} v_B \cos\theta &= v_{A_x} - \overline{BA}\omega \sin\theta_{BA} \\ v_B \sin\theta &= v_{A_y} + \overline{BA}\omega \cos\theta_{BA} \end{aligned}\right\} \tag{2.18}$$

We get two equations where the $\mathbf{v}_B$ magnitude and angular velocity ω are the unknowns, so the problem is completely defined. Once the magnitudes have been calculated by solving the system of Eq. (2.18), we obtain the rotation direction of ω and the direction of $\mathbf{v}_B$ depending on the + or − magnitude sign. In the example in Fig. 2.9a, the values obtained from Eq. (2.18) are positive for angular velocity ω as well as for the velocity magnitude of point B, v_B. This means that both have same directions from the ones that were assumed to write the equations. Therefore, point B moves upward right and the body rotates counterclockwise.

Equation (2.13) can also be solved graphically using a velocity diagram (Fig. 2.9b). Starting from point o (velocity pole), a straight line equal to the value of known velocity $\mathbf{v}_A$ is drawn using a scale factor. The velocity polygon is closed drawing the known direction of $\mathbf{v}_B$ from the pole and velocity direction $\mathbf{v}_{BA}$ (perpendicular to $\overline{AB}$) from the end point of $\mathbf{v}_A$. The intersection of these two directions defines the end points of vectors $\mathbf{v}_{BA}$ and $\mathbf{v}_B$. Measuring their length and using the scale factor, we obtain their magnitudes.

2.1.3.4 Calculation of Velocities in a Four-Bar Mechanism

Figure 2.10 represents a four-bar linkage in which we know the dimensions of all the links: $\overline{O_2A}$, $\overline{AB}$, $\overline{O_4B}$ and $\overline{O_2O_4}$. This mechanism has one degree of freedom, which means that the position and velocity of any point on any link can be determined from the position and velocity of one link. Assume that we know θ_2 and ω_2 and that we want to find the values of θ_3, θ_4, ω_3 and ω_4. To calculate θ_3 and θ_4, we can simply draw a scale diagram of the linkage at position θ_2 (Fig. 2.10) or solve the necessary trigonometric equations (Appendix A).

Once the link positions are obtained, we can start determining the velocities. First, velocity $\mathbf{v}_A$ will be calculated:

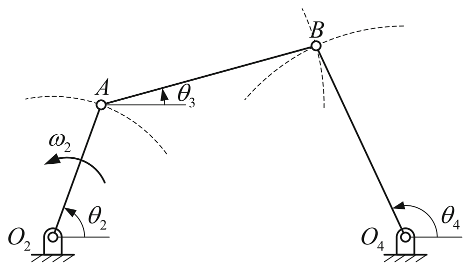


Fig. 2.10 Four-bar linkage where all the link dimensions are know as well as the position and velocity of link 2

- The horizontal and vertical components of $\mathbf{v}_A$ are given by the expression (Eq. 2.19):

$$\mathbf{v}_A = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{AO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_2 \\ \overline{AO_2}\cos\theta_2 & \overline{AO_2}\sin\theta_2 & 0 \end{vmatrix} \tag{2.19}$$

As the direction of ω_2 is counterclockwise (Fig. 2.10), its value in the previous equation will be negative. Point A describes a rotational motion about O_2 with a radius of $r_2 = \overline{O_2A}$ and an angular velocity of ω_2, so the direction of $\mathbf{v}_A$ will be perpendicular to $\overline{O_2A}$ to the left according to the rotation of link 2 (Fig. 2.11).

- Once $\mathbf{v}_A$ is known, we can obtain $\mathbf{v}_B$ with the following expression (Eq. 2.20):

$$\mathbf{v}_B = \boldsymbol{\omega}_4 \wedge \mathbf{r}_{BO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_4 \\ \overline{BO_4}\cos\theta_4 & \overline{BO_4}\sin\theta_4 & 0 \end{vmatrix} \tag{2.20}$$

Since point B rotates about steady point O_4 with a radius of $\overline{BO_4}$ and an angular velocity of ω_4, we cannot calculate the magnitude of $\mathbf{v}_B$ due to the fact that ω_4 is unknown. The direction of the linear velocity of point B has to be perpendicular to turning radius $\overline{BO_4}$ (Eq. 2.21). We can use the relative velocity method to find the magnitude of velocity $\mathbf{v}_B$:

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \tag{2.21}$$

Vector $\mathbf{v}_A$ as well as the direction of vector $\mathbf{v}_B$ are known in vector equation (2.21). We will now study vector $\mathbf{v}_{BA}$.

- The horizontal and vertical components of the point B relative velocity considering its rotation about point A are (Eq. 2.22):

$$\mathbf{v}_{BA} = \boldsymbol{\omega}_3 \wedge \mathbf{r}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_3 \\ \overline{BA}\cos\theta_3 & \overline{BA}\sin\theta_3 & 0 \end{vmatrix} \tag{2.22}$$

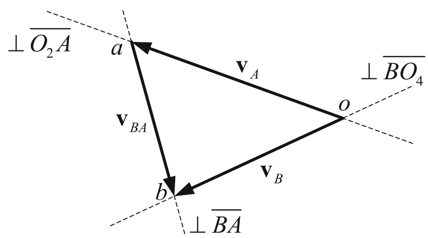


Fig. 2.11 Velocity diagram for the given position and velocity of link 2

Since the angular velocity of link 3 is unknown, we cannot calculate the magnitude of $\mathbf{v}_{BA}$. The direction of $\mathbf{v}_{BA}$ is known since the relative velocity of a point that rotates about another is always perpendicular to the radius joining them. In this case, the direction will be perpendicular to $\overline{BA}$.

This way we confirm that Eq. (2.21) has two unknowns. In (Fig. 2.11) this equation is solved graphically to calculate the $\mathbf{v}_{BA}$ and $\mathbf{v}_B$ magnitudes the same way as in Fig. 2.9.

If we want to solve Eq. (2.21) mathematically, the unknowns are the ω_3 and ω_4 magnitudes. To obtain these values, we have to solve the vector equation (Eq. 2.23):

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_4 \\ \overline{BO_4}\cos\theta_4 & \overline{BO_4}\sin\theta_4 & 0 \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_2 \\ \overline{AO_2}\cos\theta_2 & \overline{AO_2}\sin\theta_2 & 0 \end{vmatrix} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_3 \\ \overline{BA}\cos\theta_3 & \overline{BA}\sin\theta_3 & 0 \end{vmatrix} \tag{2.23}$$

By developing and separating components, we obtain two algebraic equations (Eq. 2.24) where we can clear ω_3 and ω_4.

$$\left.\begin{array}{l} \overline{BO_4}\omega_4\sin\theta_4 = \overline{AO_2}\omega_2\sin\theta_2 + \overline{BA}\omega_3\sin\theta_3 \\ \overline{BO_4}\omega_4\cos\theta_4 = \overline{AO_2}\omega_2\cos\theta_2 + \overline{BA}\omega_3\cos\theta_3 \end{array}\right\} \tag{2.24}$$

Once the angular velocities are obtained, we can represent velocities $\mathbf{v}_A$, $\mathbf{v}_B$ and $\mathbf{v}_{BA}$ according to their components (Fig. 2.11).

Assume that we add point C to link 3 in the previous mechanism as shown in Fig. 2.12a and that we want to calculate its velocity. In this case, the value of angle θ'_3 is already known since angle β is a given value of the problem. Hence, $\theta'_3 = 360° - (\beta - \theta_3)$.

To obtain the velocity of point C once ω_3 has been determined, we make use of vector equation $\mathbf{v}_C = \mathbf{v}_A + \mathbf{v}_{CA}$, $\mathbf{v}_{CA}$ where is perpendicular to $\overline{CA}$ and its value is:

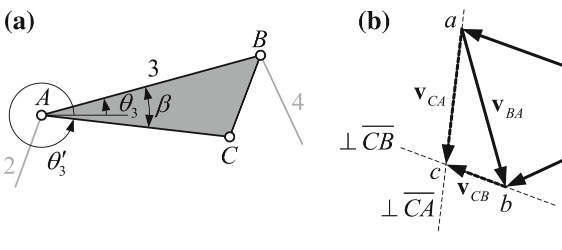


Fig. 2.12 **a** Four-bar linkage with new point C on link 3. **b** Velocity diagram

$$\mathbf{v}_{CA} = \boldsymbol{\omega}_3 \wedge \mathbf{r}_{CA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_3 \\ \overline{CA}\cos\theta_3' & \overline{CA}\sin\theta_3' & 0 \end{vmatrix} \tag{2.25}$$

Vector $\mathbf{v}_{CA}$ is obtained directly from Eq. (2.25) since angular velocity ω_3 is already known. $\mathbf{v}_C$ can be calculated adding the two known vectors, $\mathbf{v}_A$ and $\mathbf{v}_{CA}$. The velocity of point C can also be calculated based on the velocity of point B by using Eq. (2.26)

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{CB} \tag{2.26}$$

Figure 2.12b shows the calculation of $\mathbf{v}_C$ graphically.

2.1.3.5 Velocity Calculation in a Crankshaft Linkage

To calculate link velocities in a crankshaft linkage such as the one in Fig. 2.13, we start by calculating the positions of links 3 and 4. We consider that dimensions $\overline{O_2A}$ and $\overline{AB}$ are already known as well as the direction of the piston trajectory line and its distance r_{B_y} to O_2. If we draw a scale diagram of the linkage, the positions of links 3 and 4 are determined for a given position of link 2. We can also obtain their position by solving the following trigonometric equations (Eq. 2.27) (Appendix A):

$$\left.\begin{array}{l} \mu = \arcsin\dfrac{\overline{\mathrm{O_2A}}\sin\theta_2 + r_{B_y}}{\overline{AB}} \\ r_{B_x} = \overline{\mathrm{O_2A}}\cos\theta_2 + \overline{AB}\cos\mu \\ \theta_3 = 360^\circ - \mu \end{array}\right\} \tag{2.27}$$

From this point, the calculation of the velocity of point A is the same as the one previously done for the four-bar linkage (Eq. 2.28).

$$\mathbf{v}_A = \boldsymbol{\omega}_2 \wedge \mathbf{r}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_2 \\ r_2\cos\theta_2 & r_2\sin\theta_2 & 0 \end{vmatrix} \tag{2.28}$$

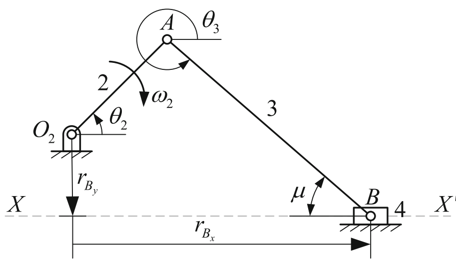


Fig. 2.13 Crankshaft linkage: positions of links 3 and 4 are determined for a given position of link 2

Since point A is rotating with respect to steady point O_2, the direction of velocity $\mathbf{v}_A$ is perpendicular to $\overline{O_2A}$ and it points in the same direction as the angular velocity of link 2, that is, ω_2.

We will now study velocity $\mathbf{v}_B$:

- The magnitude of velocity $\mathbf{v}_B$ is unknown. As the trajectory of point B moves along a straight line, its turning radius is infinite and its angular velocity is zero. Therefore, we cannot determine its velocity magnitude in terms of its angular velocity and turning radius.
- The direction of $\mathbf{v}_B$ is the same as the trajectory of the piston, XX'. Consequently, velocity $\mathbf{v}_B$ can be written as Eq. (2.29):

$$\mathbf{v}_B = v_B\hat{\mathbf{i}} \tag{2.29}$$

To calculate $\mathbf{v}_B$ we need to make use of the relative velocity method (Eq. 2.30):

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \tag{2.30}$$

The magnitude and direction of vector $\mathbf{v}_{BA}$ are given by Eq. (2.31):

$$\mathbf{v}_{BA} = \boldsymbol{\omega}_3 \wedge \mathbf{r}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_3 \\ \overline{BA}\cos\theta_3 & \overline{BA}\sin\theta_3 & 0 \end{vmatrix} \tag{2.31}$$

Plugging the results into velocity equation (2.30), we obtain Eq. (2.32):

$$v_B\hat{\mathbf{i}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_2 \\ \overline{AO_2}\cos\theta_2 & \overline{AO_2}\sin\theta_2 & 0 \end{vmatrix} + \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_3 \\ \overline{BA}\cos\theta_3 & \overline{BA}\sin\theta_3 & 0 \end{vmatrix} \tag{2.32}$$

Breaking it into its components, we define the following equation system (Eq. 2.33):

$$\left.\begin{array}{l} v_B = \overline{-AO_2}\omega_2\sin\theta_2 - \overline{BA}\omega_3\sin\theta_3 \\ 0 = \overline{AO_2}\omega_2\cos\theta_2 + \overline{BA}\omega_3\cos\theta_3 \end{array}\right\} \tag{2.33}$$

From which the magnitude of velocity $\mathbf{v}_B$ and angular velocity ω_3 are obtained. Once these velocities are known, we can represent them as shown in Fig. 2.14.

2.1.3.6 Velocity Analysis in a Slider Linkage

To analyze the slider linkage in Fig. 2.15a, we will start by calculating the position of links 3 and 4. As in previous examples, we know the length and position of link

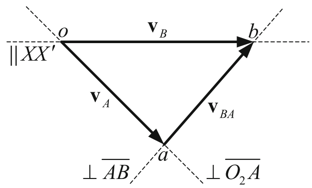


Fig. 2.14 Calculation of velocities in a crankshaft linkage

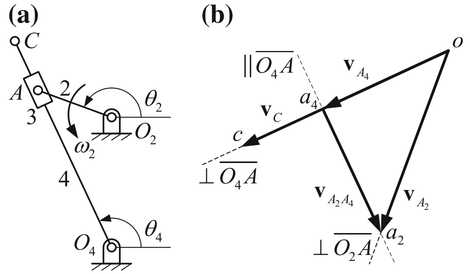


Fig. 2.15 Slider linkage: **a** positions of links 3 and 4 are determined for a given position of link 2, **b** calculation of velocities in a slider linkage

2 as well as the distance between both steady supports $\overline{O_2O_4}$. The position of link 4 is graphically determined by the line that joins O_4 and A. If we use trigonometry for our analysis, the equations needed are (Eq. 2.34) (Appendix A):

$$\left.\begin{array}{l}\overline{O_4A} = \sqrt{\overline{O_2O_4}^2 + \overline{O_2A}^2 - 2\,\overline{O_2O_4}\,\overline{O_2A}\cos(270^\circ - \theta_2)} \\ \theta_4 = \arccos\frac{\overline{O_2A}\cos\theta_2}{\overline{O_4A}}\end{array}\right\} \tag{2.34}$$

In the diagram, let A be a point that belongs to links 2 and 3 as in previous examples for the four-bar and crank-shaft linkages. It is not necessary to distinguish A_2 and A_3 as they are actually the same point. However, there is another point, A_4 in link 4, which coincides with A_2 at the instant represented in Fig. 2.15a. Nonetheless, point A_4 rotates about steady point O_4 while A_2 rotates about O_2. Due to this, they follow different trajectories at different velocities.

The velocity of point A_2 is perpendicular to $\overline{O_2A}$ and its magnitude and direction are represented by Eq. (2.35):

$$\mathbf{v}_{A_2} = \boldsymbol{\omega}_2 \wedge \mathbf{r}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_2 \\ r_2\cos\theta_2 & r_2\sin\theta_2 & 0 \end{vmatrix} \tag{2.35}$$

The velocity of A_4 is perpendicular to $\overline{O_4A}$ and its magnitude is unknown because it depends on the angular velocity of link 4. Since point A_4 belongs to link 4 and it is rotating about steady point O_4, its velocity is represented by Eq. (2.36):

$$\mathbf{v}_{A_4} = \boldsymbol{\omega}_4 \wedge \mathbf{r}_{A_4O_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_4 \\ \overline{A_4O_4}\cos\theta_4 & \overline{A_4O_4}\sin\theta_4 & 0 \end{vmatrix} \tag{2.36}$$

To calculate $\mathbf{v}_{A_4}$, we will make use of the relative velocity method (Eq. 2.37):

$$\mathbf{v}_{A_2} = \mathbf{v}_{A_4} + \mathbf{v}_{A_2A_4} \tag{2.37}$$

To calculate the velocities and solve this vector equation, we have to study vector $\mathbf{v}_{A_2A_4}$ first:

- The magnitude of $\mathbf{v}_{A_2A_4}$ is unknown and represents the velocity at which link 3 slides over link 4.
- The direction of $\mathbf{v}_{A_2A_4}$ coincides with direction $\overline{O_4A}$. Therefore, this velocity is represented by Eq. (2.38):

$$\mathbf{v}_{A_2A_4} = v_{A_2A_4}\cos\theta_4\hat{\mathbf{i}} + v_{A_2A_4}\sin\theta_4\hat{\mathbf{j}} \tag{2.38}$$

Using Eq. (2.37), we obtain Eq. (2.39) with two algebraic unknowns (angular velocity ω_4 and the magnitude of velocity $\mathbf{v}_{A_2A_4}$):

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_2 \\ \overline{AO_2}\cos\theta_2 & \overline{AO_2}\sin\theta_2 & 0 \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_4 \\ \overline{O_4A}\cos\theta_4 & \overline{O_4A}\sin\theta_4 & 0 \end{vmatrix} + v_{A_2A_4}\cos\theta_4\hat{\mathbf{i}} + v_{A_2A_4}\sin\theta_4\hat{\mathbf{j}} \tag{2.39}$$

This can be solved by breaking the equation into its components (Eq. 2.40):

$$\left.\begin{aligned} -\overline{AO_2}\omega_2\sin\theta_2 &= -\overline{O_4A}\omega_4\sin\theta_4 + v_{A_2A_4}\cos\theta_4 \\ \overline{AO_2}\omega_2\cos\theta_2 &= \overline{O_4A}\omega_4\cos\theta_4 + v_{A_2A_4}\sin\theta_4 \end{aligned}\right\} \tag{2.40}$$

Once the velocities have been obtained, we can represent them in the polygon shown on Fig. 2.15b.

2.1.3.7 Velocity Images

In the velocity polygon shown in Fig. 2.16b, the sides of triangle $\triangle abc$ are perpendicular to those of triangle $\triangle ABC$ of the linkage in Fig. 2.16a. The reason for this is that relative velocities are always perpendicular to their radius and, consequently, triangles $\triangle abc$ and $\triangle ABC$ are similar, with a scale ratio that depends on ω_3. The velocity image of link 3 is a triangle similar to the link, rotated 90° in the direction of ω_3.

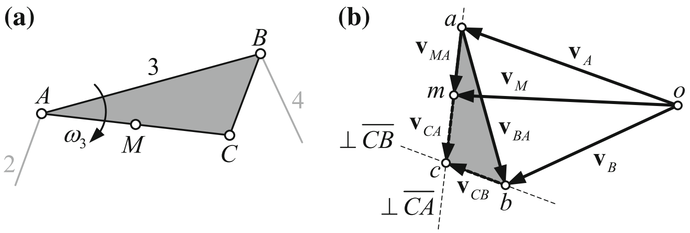


Fig. 2.16 **a** Four bar linkage with coupler point C. **b** Triangle $\triangle abc$ in the velocity diagram in grey represents the velocity image of link 3

Every side or link has its image in the velocity polygon. This way $\overline{ab}$, $\overline{bc}$ and $\overline{ac}$ are the images of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively. Vector $\overrightarrow{oa} = \mathbf{v}_A$ starting at pole o is the image of $\overline{O_2A}$ and vector $\overrightarrow{ob} = \mathbf{v}_B$ is the image of $\overline{O_4B}$. Moreover, the image of the frame link is pole o with null velocity. We can verify that velocities departing from o are always absolute velocities while velocities departing from any other point are relative ones.

If we add point M to link 3 of the linkage in Fig. 2.16a, we can obtain its velocity in the velocity polygon by looking for its image. We can verify that distance $\overline{am}$ in the velocity diagram is given by Eq. (2.41):

$$\frac{\overline{ab}}{\overline{AB}} = \frac{\overline{am}}{\overline{AM}} \Longrightarrow \overline{am} = \overline{AM}\,\frac{\overline{ab}}{\overline{AB}} \tag{2.41}$$

In conclusion, once the image of the velocity of a link has been obtained, it is very simple to calculate the velocity of any point in it. Finding the image of the point in the velocity polygon is enough. The vector that joins pole o with the image of a point represents its absolute velocity.

2.1.3.8 Application to Superior Pairs

This method can be applied to cams or geared teeth. In Fig. 2.17a, let link 2 be the driving element and link 3 the follower. Angular velocity ω_2 of the driving link is known.

In the considered instant, link 2 transmits movement to link 3 in point A. However, we have to distinguish between point A of link 2 (A_2) and point A of link 3 (A_3). These two points have different velocities and, consequently, there will be a relative velocity $\mathbf{v}_{A_3A_2}$ between them. We know that the vector sum in Eq. (2.42) has to be met:

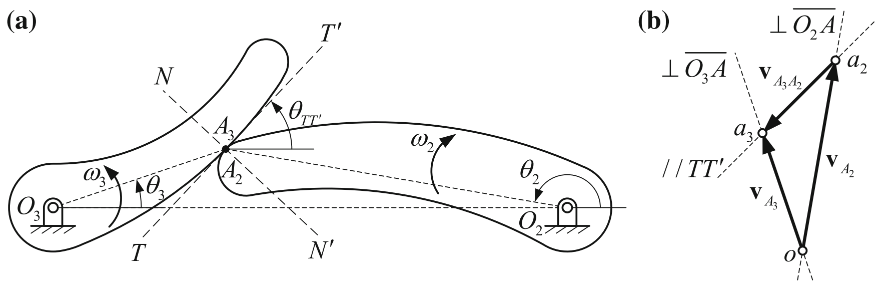


Fig. 2.17 **a** Superior pair linkage. **b** Calculation of velocities in a superior pair

$$\mathbf{v}_{A_3} = \mathbf{v}_{A_2} + \mathbf{v}_{A_3A_2} \tag{2.42}$$

The velocity of point A_2 is perpendicular to its turning radius $\overline{O_2A}$ while the velocity of point A_3 is perpendicular to $\overline{O_3A}$. To calculate these two velocities, we can use Eq. (2.43) in which ω_3 is unknown:

$$\begin{aligned} \mathbf{v}_{A_2} &= \boldsymbol{\omega}_2 \wedge \mathbf{r}_{AO_2} \\ \mathbf{v}_{A_3} &= \boldsymbol{\omega}_3 \wedge \mathbf{r}_{AO_3} \end{aligned} \tag{2.43}$$

Relative velocity $\mathbf{v}_{A_3A_2}$ of point A_3 relative to A_2 has an unknown magnitude. To find it, we need to determine the direction of vector $\mathbf{v}_{A_3A_2}$. Since the links are rigid, there is no relative motion in direction $\overline{NN'}$ due to physical constraints. Hence, relative motion happens at point A along the tangential line to the surface. This way, the direction of $\mathbf{v}_{A_3A_2}$ will coincide with tangential line TT' (Eq. 2.44):

$$\mathbf{v}_{A_3A_2} = v_{A_3A_2} \cos\theta_{TT'}\hat{\mathbf{i}} + v_{A_3A_2} \sin\theta_{TT'}\hat{\mathbf{j}} \tag{2.44}$$

Angular velocity ω_3 and linear velocity $\mathbf{v}_{A_3A_2}$ can be determined by rewriting Eq. (2.42) using the two velocity components of each vector (Eq. 2.45):

$$\left.\begin{aligned} -\overline{O_3A}\omega_3 \sin\theta_3 &= -\overline{O_2A}\omega_2 \sin\theta_2 + v_{A_3A_2} \cos\theta_{TT'} \\ \overline{O_3A}\omega_3 \cos\theta_3 &= \overline{O_2A}\omega_2 \cos\theta_2 + v_{A_3A_2} \sin\theta_{TT'} \end{aligned}\right\} \tag{2.45}$$

Example 1 Determine velocities $\mathbf{v}_B$ and $\mathbf{v}_C$ of the four-bar mechanism in Fig. 2.18a. Its dimensions are: $\overline{O_2O_4} = 15\,\text{cm}$, $\overline{O_2A} = 6\,\text{cm}$, $\overline{AB} = 11\,\text{cm}$, $\overline{O_4B} = 9\,\text{cm}$, $\overline{AC} = 8\,\text{cm}$ and $\widehat{BAC} = 30°$. The input angle is $\theta_2 = 60°$ and the angular velocity of the driving link is $\omega_2 = -20\,\text{rad/s}$ (clockwise direction).

Angles θ_3, θ_4 and θ_3' can be obtained by applying the trigonometric method (Eqs. 2.46–2.51) developed in Appendix A where angles β, ϕ and δ are represented in Fig. 2.18b.

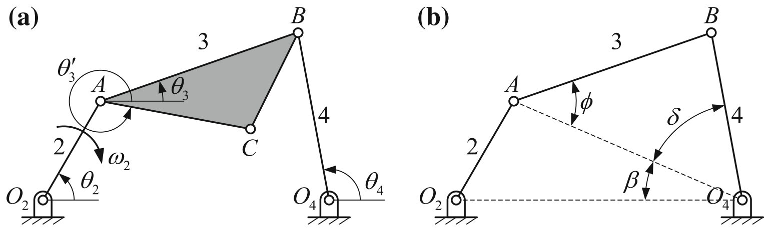


Fig. 2.18 **a** Four bar linkage. **b** Position calculation of links 3 and 4 in a four-bar linkage using the trigonometric method

$$\overline{O_4A} = \sqrt{15^2 + 6^2 - 2 \cdot 15 \cdot 6 \cdot \cos 60^\circ} = 13.08\,\text{cm} \tag{2.46}$$

$$\beta = \arcsin\left(\frac{6}{13.08}\sin 60^\circ\right) = 23.41^\circ \tag{2.47}$$

$$\phi = \arccos\left(\frac{11^2 + 13.08^2 - 9^2}{2 \cdot 11 \cdot 13.08}\right) = 42.81^\circ \tag{2.48}$$

$$\delta = \arcsin\left(\frac{11}{9}\sin 42.81^\circ\right) = 56.17^\circ \tag{2.49}$$

$$\begin{aligned} \theta_3 &= \phi - \beta = 19.4^\circ \\ \theta_4 &= 180^\circ - (\beta + \delta) = 100.42^\circ \end{aligned} \tag{2.50}$$

$$\theta_3' = 360^\circ - \left(\widehat{BAC} - \theta_3\right) = 349.4^\circ \tag{2.51}$$

To calculate the velocity of point B, we will apply the relative velocity method. We start by analyzing velocities $\mathbf{v}_A$, $\mathbf{v}_B$ and $\mathbf{v}_{BA}$ (Eqs. 2.52–2.54).

$$\mathbf{v}_A = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{AO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -20 \\ 6\cos 60^\circ & 6\sin 60^\circ & 0 \end{vmatrix} = 103.9\hat{\mathbf{i}} - 60\hat{\mathbf{j}} \tag{2.52}$$

Operating with these components, we calculate its magnitude and direction:

$$\mathbf{v}_A = 120\,\text{cm/s}\,\angle 330^\circ$$

$$\mathbf{v}_{BA} = \boldsymbol{\omega}_3 \wedge \mathbf{r}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_3 \\ 11\cos 19.4^\circ & 11\sin 19.4^\circ & 0 \end{vmatrix} = -3.56\omega_3\mathbf{i} + 10.38\omega_3\hat{\mathbf{j}} \tag{2.53}$$

$$\mathbf{v}_B = \boldsymbol{\omega}_4 \wedge \mathbf{r}_{BO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_4 \\ 9\cos 100.4^\circ & 9\sin 100.4^\circ & 0 \end{vmatrix} = -8.85\omega_4\mathbf{i} + -1.62\omega_4\hat{\mathbf{j}} \tag{2.54}$$

To calculate ω_3 and ω_4 we use the relative velocity (Eq. 2.55):

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \tag{2.55}$$

Clearing the components, we obtain Eq. (2.56):

$$\left.\begin{array}{l} -8.85\omega_4 = 103.9 - 3.65\omega_3 \\ -1.62\omega_4 = -60 + 10.38\omega_3 \end{array}\right\} \tag{2.56}$$

From which the following values for angular velocity $\omega_3 = 7.16$ rad/s clockwise and $\omega_4 = -8.78$ rad/s counterclockwise can be worked out. Operating with these values in Eqs. (2.53) and (2.54), we obtain velocities $\mathbf{v}_B$ and $\mathbf{v}_{BA}$:

$$\mathbf{v}_B = 77.75\hat{\mathbf{i}} + 14.28\hat{\mathbf{j}} = 79.1\,\text{cm/s}\,\angle 10.4^\circ$$

$$\mathbf{v}_{BA} = -26.13\hat{\mathbf{i}} + 74.32\hat{\mathbf{j}} = 78.78\,\text{cm/s}\,\angle 109.4^\circ$$

To calculate the velocity of point C, we apply the relative velocity equation, $\mathbf{v}_C = \mathbf{v}_A + \mathbf{v}_{CA}$, where $\mathbf{v}_A$ is already known and $\mathbf{v}_{CA}$ is given by Eq. (2.57):

$$\mathbf{v}_{CA} = \boldsymbol{\omega}_3 \wedge \mathbf{r}_{CA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 7.16 \\ 8\cos 349.4^\circ & 8\sin 349.4^\circ & 0 \end{vmatrix} = 10.53\hat{\mathbf{i}} + 56.3\hat{\mathbf{j}} \tag{2.57}$$

$$\mathbf{v}_{CA} = 57.28\,\text{cm/s}\,\angle 79.4^\circ$$

Using these values in the relative velocity equation, we obtain:

$$\mathbf{v}_C = 114.4\hat{\mathbf{i}} - 3.7\hat{\mathbf{j}} = 114.46\,\text{cm/s}\,\angle 358.1^\circ$$

Example 2 Calculate velocity $\mathbf{v}_B$ in the crank-shaft linkage shown in Fig. 2.19. Consider the dimensions to be as follows: $\overline{O_2A} = 3$ cm, $\overline{AB} = 7$ cm and $y = 1.5$ cm. The trajectory followed by the piston is horizontal. The input angle is $\theta_2 = 60^\circ$ and link 2 moves with angular velocity $\omega_2 = -10$ rad/s (clockwise).

We start by solving the position problem (Eqs. 2.58–2.60) using the trigonometric method (Fig. 2.19):

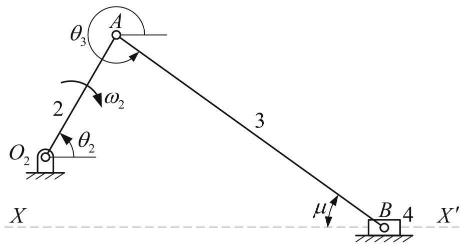


Fig. 2.19 Calculation of the position of the links for a given input angle in a crank-shaft linkage

$$\mu = \arcsin\frac{3\sin 60^\circ + 1.5}{7} = 35.8^\circ \tag{2.58}$$

$$x_B = 3\cos 60^\circ + 7\cos 35.8^\circ = 7.1\,\text{cm} \tag{2.59}$$

$$\theta_3 = 360^\circ - 35.8 = 324.2^\circ \tag{2.60}$$

The velocity of point B is obtained from relative velocity (Eq. 2.61):

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \tag{2.61}$$

where $\mathbf{v}_A$, $\mathbf{v}_B$ and $\mathbf{v}_{BA}$ are given by Eqs. (2.62)–(2.64):

$$\mathbf{v}_A = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{AO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -10 \\ 3\cos 60^\circ & 3\sin 60^\circ & 0 \end{vmatrix} = 25.98\hat{\mathbf{i}} - 15\hat{\mathbf{j}} \tag{2.62}$$

$$\mathbf{v}_A = 30\,\text{cm/s}\,\angle 330^\circ$$

$$\mathbf{v}_{BA} = \boldsymbol{\omega}_3 \wedge \mathbf{r}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_3 \\ 7\cos 324.2^\circ & 7\sin 324.2^\circ & 0 \end{vmatrix} = 4.09\omega_3\hat{\mathbf{i}} + 5.68\omega_3\hat{\mathbf{j}} \tag{2.63}$$

$$\mathbf{v}_B = v_B\hat{\mathbf{i}} \tag{2.64}$$

Using these values in the relative velocity (Eq. 2.61) we obtain Eq. (2.65):

$$\left.\begin{aligned} v_B &= 25.98 + 4.09\omega_3 \\ 0 &= -15 + 5.68\omega_3 \end{aligned}\right\} \tag{2.65}$$

Ultimately, resulting in the following values for angular and linear velocities $\omega_3 = 2.64\,\text{rad/s}$ counterclockwise and $v_B = 36.78\,\text{cm/s}$. Thus, the velocities will be:

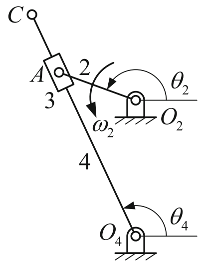


Fig. 2.20 Position and velocity calculation of the links in a slider linkage. The unknowns are θ_4, $\overline{O_4A}$, ω_4 and $v_{A_2A_4}$

$$\mathbf{v}_B = 36.78\hat{\mathbf{i}} = 36.78\,\text{cm/s}\,\angle 0^\circ$$

$$\mathbf{v}_{BA} = 10.79\hat{\mathbf{i}} + 15\hat{\mathbf{j}} = 18.48\,\text{cm/s}\,\angle 54.24^\circ$$

Example 3 Calculate velocity $\mathbf{v}_C$ of the slider linkage in Fig. 2.20 when the dimensions of the links are $\overline{O_2A} = 3$ cm, $\overline{O_2O_4} = 5$ cm, $\overline{O_4C} = 9$ cm and the input angle is $\theta_2 = 160^\circ$. The input link moves counterclockwise with angular velocity $\omega_2 = 10$ rad/s.

The position problem can easily be solved using the trigonometric method (Eqs. 2.66 and 2.67):

$$\overline{O_4A} = \sqrt{5^2 + 3^2 - 2\cdot 5\cdot 3\cos(270^\circ - 160^\circ)} = 6.65\,\text{cm} \tag{2.66}$$

$$\theta_4 = \arccos\frac{3\cos 160^\circ}{6.65} = 115.08^\circ \tag{2.67}$$

In order to calculate the velocity of point C, we first have to calculate the velocity of point A_4 which temporarily coincides with A_2 at the time instant considered while being part of link 4. We can relate $\mathbf{v}_{A_2}$ and $\mathbf{v}_{A_4}$ with relative velocity (Eq. 2.68):

$$\mathbf{v}_{A_2} = \mathbf{v}_{A_4} + \mathbf{v}_{A_2A_4} \tag{2.68}$$

where $\mathbf{v}_{A_2}$, $\mathbf{v}_{A_4}$ and $\mathbf{v}_{A_2A_4}$ are given by Eqs. (2.69)–(2.71).

$$\mathbf{v}_{A_2} = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{AO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 10 \\ 3\cos 160^\circ & 3\sin 160^\circ & 0 \end{vmatrix} = -10.26\hat{\mathbf{i}} - 28.19\hat{\mathbf{j}} \tag{2.69}$$

$$\mathbf{v}_{A_2} = 30\,\text{cm/s}\,\angle 250^\circ$$

$$\mathbf{v}_{A_4} = \boldsymbol{\omega}_4 \wedge \mathbf{r}_{AO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_4 \\ 6.65\cos 115.08^{\circ} & 6.65\sin 115.08^{\circ} & 0 \end{vmatrix} = -6.02\omega_4\hat{\mathbf{i}} - 2.82\omega_4\hat{\mathbf{j}} \tag{2.70}$$

$$\mathbf{v}_{A_2A_4} = v_{A_2A_4}\cos 115.08^{\circ}\hat{\mathbf{i}} + v_{A_2A_4}\sin 115.08^{\circ}\hat{\mathbf{j}} = -0.42v_{A_2A_4}\hat{\mathbf{i}} + 0.91v_{A_2A_4}\hat{\mathbf{j}} \tag{2.71}$$

Using these values in Eq. (2.68) and clearing the components, we obtain Eq. (2.72):

$$\left.\begin{array}{l} -10.26 = -6.02\omega_4 - 0.42v_{A_2A_4} \\ -28.19 = -2.82\omega_4 + 0.9v_{A_2A_4} \end{array}\right\} \tag{2.72}$$

We calculate the values of angular velocity $\omega_4 = -3.19$ rad/s clockwise and the magnitude of $v_{A_2A_4} = -21.32$ cm/s. The negative sign indicates that the angle of $\mathbf{v}_{A_2A_4}$ is not θ_4 but $\theta_4 + 180°$. Consequently, the velocity values are Eqs. (2.73) and (2.74):

$$\mathbf{v}_{A_4} = -19.2\hat{\mathbf{i}} - 9\hat{\mathbf{j}} = 21.2\,\text{cm/s}\,\angle 205.1^{\circ} \tag{2.73}$$

$$\mathbf{v}_{A_2A_4} = 8.95\hat{\mathbf{i}} - 19.19\hat{\mathbf{j}} = 21.32\,\text{cm/s}\,\angle 295^{\circ} \tag{2.74}$$

To calculate the velocity of point C, we make use of Eq. (2.75):

$$\mathbf{v}_C = \boldsymbol{\omega}_4 \wedge \mathbf{r}_{CO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 3.19 \\ 9\cos 115.08^{\circ} & 9\sin 115.08^{\circ} & 0 \end{vmatrix} = -26\hat{\mathbf{i}} - 12.17\hat{\mathbf{j}} \tag{2.75}$$

$$\mathbf{v}_C = 28.7\,\text{cm/s}\,\angle 205.1^{\circ}$$

Example 4 In the mixing machine in Fig. 2.21a, calculate the velocity of extreme point C of the spatula knowing that the motor of the mixer moves counterclockwise with an angular velocity of 95.5 rpm and $\theta_2 = 0°$. The dimensions in the drawing are in centimeters.

The kinematic skeleton of the mixing machine is shown in Fig. 2.21b. To determine the position of the linkage, we have to calculate the value of angle θ_3 and distance $\overline{O_4A}$. To do so, we apply Eq. (2.76):

$$\left.\begin{array}{l} \mu = \arctan\dfrac{r_2}{r_1} = \arctan\dfrac{7}{11} = 32.47^{\circ} \\ \overline{O_4A} = \sqrt{r_1^2 + r_2^2} = \sqrt{11^2 + 7^2} = 13.04\,\text{cm} \\ \theta_3 = 90^{\circ} + \mu = 122.47^{\circ} \end{array}\right\} \tag{2.76}$$

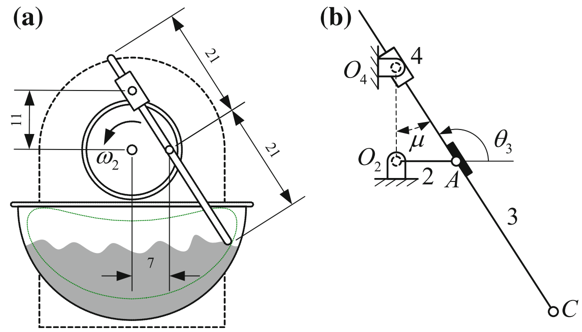


Fig. 2.21 **a** Mixing machine. **b** Kinematic skeleton

Before starting the calculation of velocities, we have to convert the given input velocity from rpm into rad/s (Eq. 2.77):

$$\omega_2 = 95.5 \ \text{rpm}\frac{2\pi\,\text{rad}}{60\,\text{s}} = 10\,\text{rad/s} \tag{2.77}$$

To calculate the velocity of point C, we first have to solve the velocity of point O_3, which coincides with the position of point O_4 at the instant considered while still being part of link 3. Since points O_3 and A belong to link 3, we can use the relative velocity method to relate their velocities (Eq. 2.78):

$$\mathbf{v}_{O_3} = \mathbf{v}_A + \mathbf{v}_{O_3A} \tag{2.78}$$

where $\mathbf{v}_A$ and $\mathbf{v}_{O_3A}$ are given by Eqs. (2.79) and (2.80):

$$\mathbf{v}_A = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{AO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 10 \\ 7\cos 0^\circ & 7\sin 0^\circ & 0 \end{vmatrix} = 70\hat{\mathbf{j}} \tag{2.79}$$

$$\mathbf{v}_A = 70\,\text{cm/s}\,\angle 90^\circ$$

$$\mathbf{v}_{O_3A} = \boldsymbol{\omega}_3 \wedge \mathbf{r}_{AO_3} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_3 \\ \overline{AO_3}\cos 122.47^\circ & \overline{AO_3}\sin 122.47^\circ & 0 \end{vmatrix} = -11\omega_3\hat{\mathbf{i}} - 7\omega_3\hat{\mathbf{j}} \tag{2.80}$$

Using these values in Eq. (2.78), we obtain Eq. (2.81):

$$\mathbf{v}_{O_3} = (70\hat{\mathbf{j}}) + (-11\omega_3\hat{\mathbf{i}} - 7\omega_3\hat{\mathbf{j}}) \tag{2.81}$$

However, in Eq. (2.81) the direction as well as the magnitude of velocity $\mathbf{v}_{O_3}$ remain unknown. To obtain information on this velocity, we will relate the velocity of point O_3 with the velocity of point O_4 by using Eq. (2.82):

$$\mathbf{v}_{O_3} = \mathbf{v}_{O_4} + \mathbf{v}_{O_3O_4} \tag{2.82}$$

In this equation, the velocity of point O_4 is zero, $\mathbf{v}_{O_4} = 0$, since it is a fixed point. Therefore, the velocity of point O_3 has the same magnitude and direction as the relative velocity between points O_3 and O_4. The direction of this velocity is given by link 3. Hence, the velocity of point O_3 is defined as Eq. (2.83):

$$\mathbf{v}_{O_3} = v_{O_3} \cos 122.47\hat{\mathbf{i}} + v_{O_3} \sin 122.47\hat{\mathbf{j}} \tag{2.83}$$

Evening out Eqs. (2.81) and (2.83), we obtain Eq. (2.84):

$$v_{O_3} \cos 122.47\hat{\mathbf{i}} + v_{O_3} \sin 122.47\hat{\mathbf{j}} = 70\hat{\mathbf{j}} + (-11\omega_3\hat{\mathbf{i}} - 7\omega_3\hat{\mathbf{j}}) \tag{2.84}$$

By separating the components, we obtain Eq. (2.85):

$$\left.\begin{aligned} v_{O_3} \cos 122.47 &= -11\omega_3 \\ v_{O_3} \sin 122.47 &= 70 + 7\omega_3 \end{aligned}\right\} \tag{2.85}$$

Solving Eq. (2.85), we obtain the values for angular velocity $\omega_3 = 2.88$ rad/s clockwise and the velocity magnitude of point O_3, $v_{O_3} = 58.71$ cm/s. This way, the vector velocity of point O_3 is defined by Eq. (2.86):

$$\mathbf{v}_{O_3} = -31.52\hat{\mathbf{i}} + 49.53\hat{\mathbf{j}} = 58.71\,\text{cm/s}\,\angle 122.47^\circ \tag{2.86}$$

Eventually, in order to calculate the velocity of point C, we apply velocity (Eq. 2.87):

$$\mathbf{v}_C = \mathbf{v}_A + \mathbf{v}_{CA} \tag{2.87}$$

where relative velocity between points C and A is Eq. (2.88):

$$\mathbf{v}_{CA} = \boldsymbol{\omega}_3 \wedge \mathbf{r}_{CA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 2.88 \\ 21\cos 302.47^\circ & 21\sin 302.47^\circ & 0 \end{vmatrix} = 51.03\hat{\mathbf{i}} + 32.45\hat{\mathbf{j}} \tag{2.88}$$

Operating with the known values in Eq. (2.87), we obtain the vector velocity of point C:

$$\mathbf{v}_C = 51.03\hat{\mathbf{i}} + 102.45\hat{\mathbf{j}} = 114.4\,\text{cm/s}\,\angle 63.52^\circ$$

Once all the velocities are defined, we can represent them in the velocity polygon (Fig. 2.21c).

2.1.4 Instant Center of Rotation Method

Any planar displacement of a rigid body can be considered a rotation about a point. This point is called instantaneous center or instant center of rotation (I.C.R.).

2.1.4.1 Instant Center of Rotation of a Rigid Body

Let a rigid body move from position AB to position $A'B'$ (Fig. 2.22). Position change could be due to a pure rotation of triangle $\triangle OAB$ about O, intersection point of the bisectors of segments AA' and BB'. We can obtain the displacement of points A and B by using their distance to center O and the angular displacement of the body, $\Delta\theta$ (Eq. 2.89).

$$\left.\begin{array}{l}\Delta r_A = \overline{AA'} = 2\overline{OA}\sin\frac{\Delta\theta}{2}\\ \Delta r_B = \overline{BB'} = 2\overline{OB}\sin\frac{\Delta\theta}{2}\end{array}\right\} \tag{2.89}$$

Considering the time to be infinitesimal, we can consider the body to be rotating about O, the instant rotation center. Displacements will be Eq. (2.90):

$$\left.\begin{array}{l}dr_A = 2\overline{OA}\sin\frac{d\theta}{2} = \overline{OA}d\theta\\ dr_B = 2\overline{OB}\sin\frac{d\theta}{2} = \overline{OB}d\theta\end{array}\right\} \tag{2.90}$$

Dividing both displacements by the time spent, dt, we find the instant velocities of points A and B. Their directions are perpendicular to radius $\overline{OA}$ and $\overline{OB}$ respectively and their magnitudes are Eq. (2.91):

$$\left.\begin{array}{l}v_A = \overline{OA}\frac{d\theta}{dt} = \overline{OA}\omega\\ v_B = \overline{OB}\frac{d\theta}{dt} = \overline{OB}\omega\end{array}\right\} \tag{2.91}$$

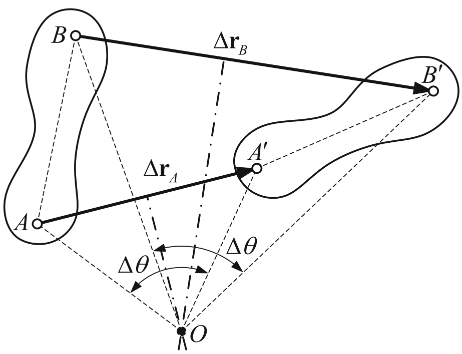


Fig. 2.22 A planar movement of the rigid body AB can be considered a rotation about point O

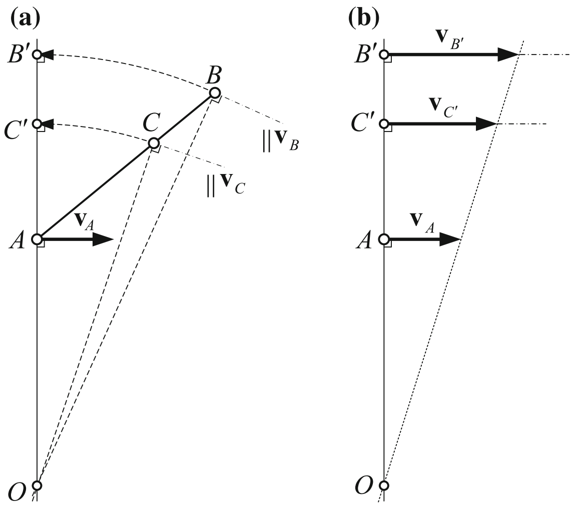


Fig. 2.23 Graphical calculation of direction (**a**) and magnitude (**b**) of the velocities of points B and C knowing $\mathbf{v}_A$ and the direction of $\mathbf{v}_B$

This way, it is verified that, at a certain instant of time, point O is the rotation center of points A and B. The magnitude of the velocity of any point in the body will be Eq. (2.92):

$$v = R\omega \tag{2.92}$$

where:

- R is the instant rotation radius of the point (distance from the point to O).
- ω is the angular velocity of the body measured in radians per second.

Velocity of every point in a link will have direction perpendicular to its instant rotation radius. Thus, if we know the direction of the velocities of two points of a link, we can find the ICR of the link on the intersection of two perpendicular lines to both velocities.

Consider that in the link in Fig. 2.23a, we know the magnitude and direction of point A velocity and the direction of point B velocity. The ICR of the link has to be on the intersection of the perpendicular lines to $\mathbf{v}_A$ and $\mathbf{v}_B$; even though the latter magnitude is unknown, we do know its direction. Once the ICR of the link is determined, we can calculate its angular velocity (Eq. 2.91) and so $\omega = v_A / \overline{OA}$. Ultimately, once the ICR and angular velocity of the link are known, we can calculate the velocity of any point C in the link. The magnitude of the velocity of point C is $v_C = \overline{OC}\omega$ and its direction is perpendicular to $\overline{OC}$.

In many cases, it is simpler to calculate velocity magnitudes with graphical methods. Figure 2.23b shows how velocities $\mathbf{v}_B$ and $\mathbf{v}_C$ can be calculated by means of a graphic method once the ICR of a rigid body and the velocity of one of its

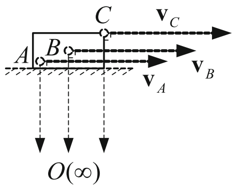


Fig. 2.24 The ICR of a body moving on a plane with pure translation is placed at the infinite

points (in this case $\mathbf{v}_A$) are known. If we fold up points B and C over line OA, it must be verified that the triangles with their sides formed by each velocity and the rotation radius of each point are similar (Eq. 2.93), since:

$$\frac{v_A}{\overline{OA}} = \frac{v_B}{\overline{OB'}} = \frac{v_C}{\overline{OC'}} = \omega \tag{2.93}$$

In the case of a body moving on a plane with no angular velocity (pure translation), its ICR is placed at the infinite since all points of the body have the same velocity and the perpendicular lines to such velocities intersect at the infinite (Fig. 2.24).

2.1.4.2 Instant Center of Rotation of a Pair of Links

So far, we have looked at the ICR of a link relative to a stationary reference system. However, we can define the ICR of a pair of links, not taking into account if one of them is fixed or not. This ICR between the two links is the point one link rotates about with respect to the other.

In Fig. 2.25, point I_{23} is the ICR of link 2 relative to link 3. In other words, link 2 rotates about this point relative to link 3. There is one point of each link that coincides in position with this ICR. If we consider that link 3 is moving, these two points move at the same absolute velocity, that is, null relative velocity. This is the only couple of points - one of each link - that has zero relative velocity at the instant studied.

To help us understand the ICR concept of a pair of links, we are going to calculate the ones corresponding to a four-bar linkage. Notice that in the linkage in Fig. 2.26, there is one ICR for every two links. To know the number of ICRs in a linkage, we have to establish all possible combinations of the number of links

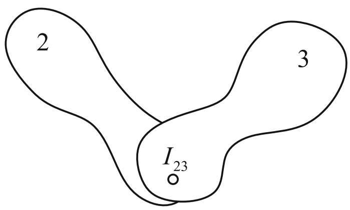


Fig. 2.25 The ICR between links 2 and 3 is the point link 2 rotates about with respect to link 3 or vice versa

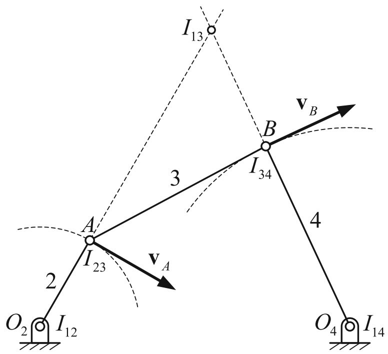


Fig. 2.26 ICR I_{13} is on the intersection point of two lines perpendicular to the velocity vectors of points A and B of link 3 with respect to link 1

taking two at a time since I_{ij} is the same IRC as I_{ji}. Therefore, the number of IRCs is given by Eq. (2.94):

$$NICRs = \frac{N(N-1)}{2} = 6 \tag{2.94}$$

where:

- *NICRs* is the number of ICRs
- N is the number of links.

The obvious ones are I_{12}, I_{23}, I_{34} and I_{14} since every couple of links is joined by a hinge, which is the rotating point of one link relative to another. Remember that the velocity of any point in the link has to be perpendicular to its instant rotation radius. In consequence, considering that points A and B are part of link 3, I_{13} is on the intersection of two lines perpendicular to the velocity vectors of points A and B (Fig. 2.26).

ICR I_{24} is obtained the same way but considering the inversion shown in Fig. 2.27. As in kinematic inversions, relative motion between links is maintained.

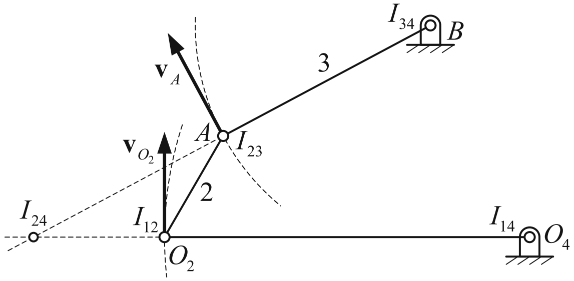


Fig. 2.27 ICR I_{24} is on the intersection of two lines perpendicular to the velocity vectors of points A and O_2 of link 2 with respect to link 4

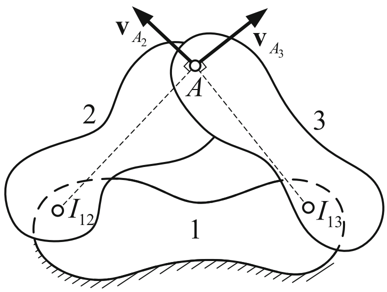


Fig. 2.28 The velocity vector of ICR I_{23} has to be perpendicular to ICR I_{12} and to ICR I_{13}. Therefore, it has to be located on the straight line defined by ICR I_{12} and ICR I_{13}

2.1.4.3 Kennedy's Theorem

Also known as the Three Centers Theorem, it is used to find the ICR of a linkage without having to look into its kinematic inversions as we did in the last example. Kennedy's Theorem states that all three ICRs of three links with planar motion have to be aligned on a straight line.

In order to demonstrate this theorem, first note Fig. 2.28 representing a set of three links (1, 2, 3) that have relative motion. Links 2 and 3 are joined to link 1 making two rotating pairs. Therefore, ICRs I_{12} and I_{13} are easy to locate.

Links 2 and 3 are not physically joined. However, as previously studied in this chapter, there is a point link 2 rotates about, relative to link 3, at a given instant. This point is ICR I_{23}. Initially, we do not know where to locate it, so we are going to assume it coincides with point A.

In this case, point A would act as a hinge that joins links 2 and 3. In other words, we could consider it as a point that is part of links 2 and 3 at the same time. If we consider it to be a point of link 2, its velocity with respect to link 1 has to be perpendicular to the rotating radius $\overline{I_{12}A}$ (Fig. 2.28). However, if we consider it to be part of link 3, it has to rotate about I_{13} with a radius of $\overline{I_{13}A}$.

This gives us different directions for the velocity vectors of points A_2 and A_3, which means that there is a relative velocity between them. Therefore, point A cannot be ICR I_{23}. If the velocity of ICR I_{23} has to have the same direction when calculated as a point of link 2 and a point of link 3, ICR I_{23} has to be located on the straight line defined by I_{12} and I_{13}.

This rule is known as Kennedy's Theorem, which says that the three relative ICRs of any three links have to be located on a straight line. This law is valid for any set of three links that has relative planar motion, even if none of them is the ground link (frame).

2.1.4.4 Locating the ICRs of a Linkage

To locate the ICRs of the links in a linkage, we will apply the following rules:

1. Identify the ones corresponding to rotating kinematic pairs. The ICR is the point that identifies the axis of the pair (hinge).

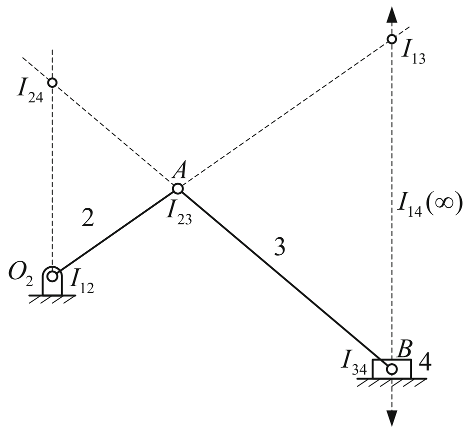


Fig. 2.29 Instant Centers of Rotation of links 1, 2, 3 and 4 in a slider-crank linkage

2. In sliding pairs, the ICR is on the curvature center of the path followed by the slide.
3. The rest of ICRs can be obtained by means of the application of Kennedy's Theorem to sets of three links in the linkage.

Example 5 Find the Instant Centers of Rotation of all the links in the slider-crank linkage in Fig. 2.29.

First, we identify ICRs I_{12}, I_{23}, I_{34}, and I_{14} that correspond to the four kinematic pairs in the linkage. Notice that ICR I_{14} is located at the infinite as the slider path is a straight line.

Next, we apply Kennedy's Theorem to links 1, 2 and 3. According to this theorem ICRs I_{13}, I_{23} and I_{14} have to be aligned. The same way, if we take links 1, 3 and 4, ICRs I_{13}, I_{34} and I_{14} also have to be aligned. By drawing the two straight lines, we find the position of ICR I_{13}.

To find ICR I_{24}, we proceed the same way applying Kennedy's Theorem to links 1, 2, 4 on one side and 2, 3, 4 on the other.

Example 6 Find the Instant Centers of Rotation of the links of the four-bar linkage in Fig. 2.30a.

To help us to locate all ICRs we are going to make use of a polygon formed by as many sides as there are links in the linkage to analyze. In this case, we use a four-sided polygon. Next, we number the vertex from 1 to 4 (Fig. 2.30b). Every side or diagonal of the polygon represents an ICR. In this case, the sides represent I_{12}, I_{23}, I_{34} and I_{14}. Both diagonals represent ICRs I_{13} and I_{24}. We will trace those sides or diagonals representing known ICRs with a solid line and the unknown ones with a dotted line.

In the example in Fig. 2.30a, ICRs I_{12}, I_{23}, I_{34} and I_{14} are known while ICRs I_{24} and I_{13} are unknown. In order to find ICR I_{24} we apply Kennedy's Theorem making use of the polygon. To find the two ICRs that are aligned with ICR I_{24}, we define a

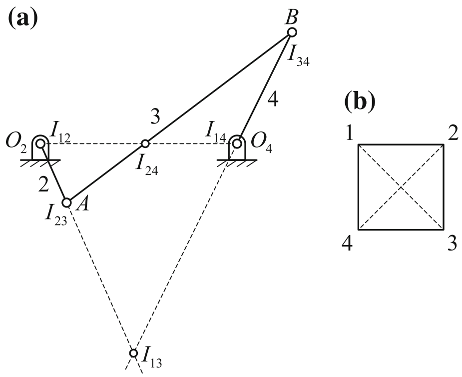


Fig. 2.30 **a** Instant Centers of Rotation of links 1, 2, 3 and 4 in a four-bar linkage. **b** Polygon to analyze all ICRs

triangle in the polygon, where two sides represent already known ICRs (for instance, I_{23} and I_{34}) and a third side representing the unknown ICR (in this case I_{24}). We will repeat the operation with ICRs I_{12}, I_{14} and I_{24}. We find ICR I_{24} on the intersection of lines $I_{23}I_{24}$ and $I_{12}I_{14}$. To find ICR I_{13}, we define triangles I_{12}, I_{23}, I_{13} and I_{14}, I_{34}, I_{13}. ICR I_{13} is on the intersection of lines $I_{12}I_{23}$ and $I_{14}I_{34}$.

Example 7 Find the Instant Centers of Rotation of the links in the slider linkage in Fig. 2.31.

In the example in Fig. 2.31a, ICRs I_{12}, I_{23}, I_{34} and I_{14} are known while ICRs I_{24} and I_{13} are unknown. In order to find these ICRs we apply Kennedy's Theorem making use of the polygon (Fig. 2.31b) the same way we did in the last example.

Example 8 Find the all the Instant Centers of Rotation in the mechanism in Fig. 2.32a. Link 2 is an eccentric wheel that rotates about O_2 transmitting a rolling motion without slipping to link 3, which is a roller joined at point A to link 4 in straight motion inside a vertical guide.

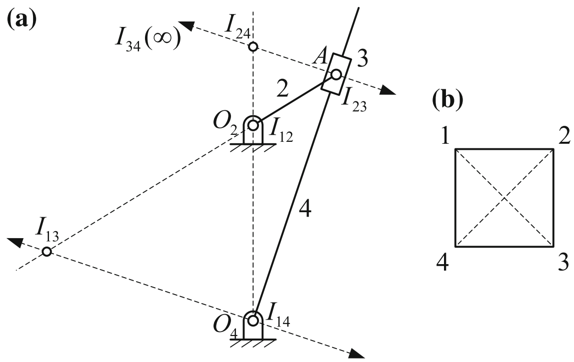


Fig. 2.31 **a** Instant Centers of Rotation of links 1, 2, 3 and 4 in a slider linkage. **b** Polygon to analyze all ICRs

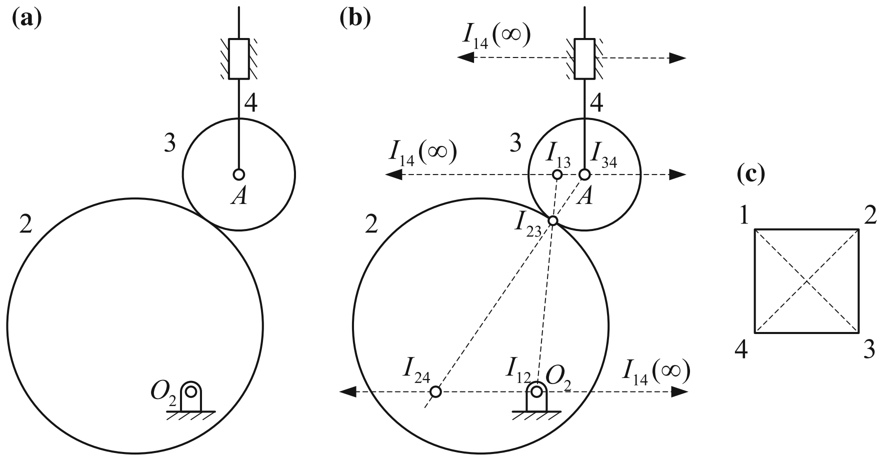


Fig. 2.32 **a** Instant Centers of Rotation of links 1, 2, 3 and 4 in a mechanism with two wheels and a slider. **b** ICRs. **c** Polygon to analyze all ICRs

The known ICRs are I_{12}, I_{23}, I_{34} and I_{14}. ICR I_{13} is on the intersection of $I_{12}I_{23}$ and $I_{14}I_{34}$ and ICR I_{24} is on the intersection of lines $I_{23}I_{34}$ and $I_{12}I_{14}$.

Example 9 Find the ICRs of the links in the five-bar linkage shown in Fig. 2.33a. Link 2 rolls and slips over link 3.

The known ICRs are I_{12}, I_{13}, I_{15}, I_{34}, I_{35} and I_{45}. ICR I_{23} is on the intersection of $I_{12}I_{13}$ and a line perpendicular to the contours of links 2 and 3 at the contact point (Fig. 2.33b). The rest of the ICRs can easily be found by applying Kennedy's Theorem making use of the polygon the same way we did in the previous examples (Fig. 2.33c).

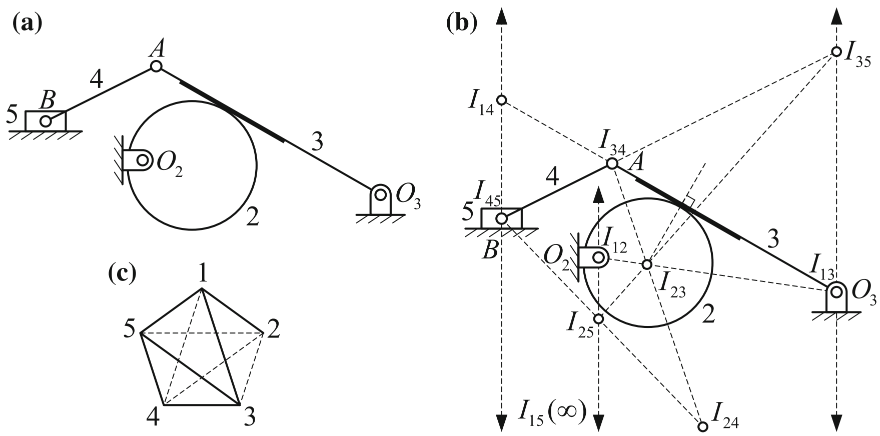


Fig. 2.33 **a** Mechanism with 5 links, **b** ICRs of all links in the linkage, **c** polygon helping to apply Kennedy's Theorem

2.1.4.5 Calculating Velocities with ICRs

We have already studied the relative velocity method for the calculation of point velocity in a linkage. Although it is a simple method to apply, it has one inconvenience. In order to calculate the velocity of one link, we need to calculate the velocities of all the links that connect it to the input link.

Calculating velocity by using instantaneous centers of rotation allows us to directly calculate the velocity of any point in a linkage without having to first calculate the velocities of other points.

Figure 2.34 shows a six-bar linkage in which the velocity of point A is already known. To calculate the velocity of point *D* by means of the relative velocity method, we first have to calculate the velocities of points *B* and *C*.

With the ICR method, it is not necessary to calculate the velocity of a point that physically joins the links. By calculating the relative ICR of two links, we can consider that we know the velocity of a point that is equally part of both links.

It is important to stress that the ICR behaves as if it were part of both links simultaneously and, consequently, its velocity is the same, no matter which link we look at to find it.

The process to calculate velocity is as follows:

4. We identify the following links:
   - The link the point with known velocity belongs to (in this example point *A*).
   - The link to which the point with unknown velocity belongs (point *D*).
   - The frame link.

   In the example of Fig. 2.34, the link with known velocity is link 2, the one with unknown velocity is link 6 and link 1 is the frame.
5. We identify all three relative ICRs of the mentioned links (I_{12}, I_{16} and I_{26} in the example) which are aligned according to Kennedy's Theorem.
6. We calculate the velocity of the ICR between the two non-fixed links $\mathbf{v}_{26}$, considering that the ICR is a point that belongs to the link with known velocity. In this case, I_{26} will be considered part of link 2 and it will revolve about I_{12}.
7. We consider ICR I_{26} a point in the link with unknown velocity (link 6 in this example). Knowing the velocity of a point in this link, $\mathbf{v}_{26}$, and its center of rotation, I_{16}, the velocity of any other point in the same link can easily be calculated.

This problem is solved in Example 13 of this chapter.

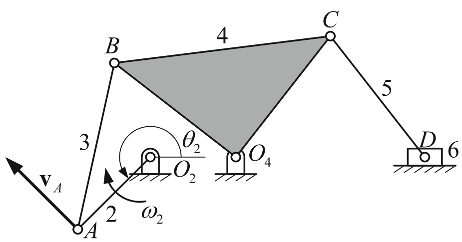


Fig. 2.34 Six-bar linkage with known velocity of point *A*

2.1.4.6 Application of ICRs to a Four-Bar Linkage

Figure 2.35 shows a four-bar linkage in which the velocity vector of point A, $\mathbf{v}_A$, is known and the velocity of point B, $\mathbf{v}_B$, is the one to be calculated. The steps to be followed are:

8. We identify the link the point of known velocity belongs to (in this example link 2). We also have to identify the link the point with unknown velocity belongs to (link 4), and the frame (link 1).
9. We locate the three ICRs between these three links: I_{12}, I_{14} and I_{24}. The straight line they form will be used as a folding line for points A and B.
1. We obtain velocity magnitude v_{24} as if I_{24} was part of link 2. Figure 2.35 shows the graphic calculation of this velocity making use of $\mathbf{v}_A$. See the analytical calculation in Eqs. (2.95)–(2.97).

$$v_{I_{24}} = \overline{I_{12}I_{24}}\omega_2 \tag{2.95}$$

$$v_A = \overline{I_{12}I_{23}}\omega_2 \tag{2.96}$$

Dividing and clearing $v_{I_{24}}$:

$$v_{I_{24}} = \frac{\overline{I_{12}I_{24}}}{\overline{I_{12}I_{23}}} v_A \tag{2.97}$$

2. ICR I_{24} is now considered a point on link 4. The velocity of point B is graphically obtained by drawing two similar triangles: the first one defined by sides $\overline{I_{12}I_{24}}$ and and the second one by sides $\overline{I_{14}B}$ (Fig. 2.35). It can also be obtained analytically in Eqs. (2.98)–(2.100):

$$v_{I_{24}} = \overline{I_{14}I_{24}}\omega_4 \tag{2.98}$$

$$v_B = \overline{I_{14}I_{34}}\omega_4 \tag{2.99}$$

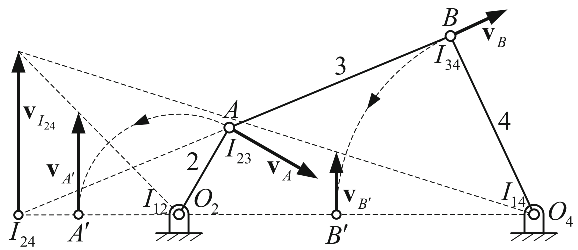


Fig. 2.35 Calculation of the velocity of point B in a four-bar linkage with the ICR method

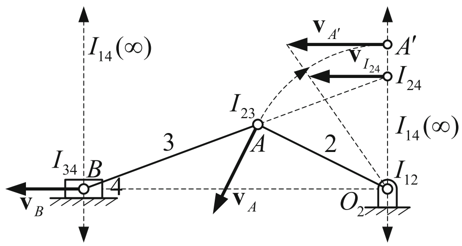


Fig. 2.36 Velocity calculation of point B in a crank-shaft linkage using the ICR method

Dividing and clearing

$$v_B = \frac{\overline{I_{14}I_{34}}}{\overline{I_{14}I_{24}}} v_{I_{24}} \tag{2.100}$$

If the angular velocity of link 4 is required, it can easily be calculated using the ω_2 value in Eqs. (2.101)–(2.103):

$$\omega_2 = \frac{v_{I_{24}}}{\overline{I_{12}I_{24}}} \tag{2.101}$$

$$\omega_4 = \frac{v_{I_{24}}}{\overline{I_{14}I_{24}}} \tag{2.102}$$

$$\omega_4 = \frac{\overline{I_{12}I_{24}}}{\overline{I_{14}I_{24}}} \omega_2 \tag{2.103}$$

2.1.4.7 Application of the ICR Method to a Crank-shaft Linkage

We assume velocity vector $\mathbf{v}_A$ of point A to be known and we want to calculate $\mathbf{v}_B$ for point B (Fig. 2.36).

10. The link with known velocity is link 2. We want to find the velocity of link 4, while link 1 is fixed.
11. We locate the three ICRs related to these links: I_{12}, I_{24} and I_{14}.
12. We calculate velocity of ICR I_{24}, regarded as a point of link 2.
13. We consider ICR I_{24} as part of link 4. Note that all the points in link 4 have the same velocity. Consequently, if we know velocity $v_{I_{24}}$, we already know the velocity of point B: $\mathbf{v}_B = \mathbf{v}_{I_{24}}$.

2.2 Accelerations in Mechanisms

In this section we will start by defining the components of the linear acceleration of a point. Then we will develop the relative acceleration method that will allow us to calculate the linear and angular accelerations of all points and links in a mechanism.

These accelerations will be needed in order to continue with the dynamic analysis in future chapters.

2.2.1 Acceleration of a Point

The acceleration of a point is the relationship between the change of its velocity vector and time.

Point A moves from position A to A' along a curve during time Δt and changes its velocity vector from $\mathbf{v}_A$ to $\mathbf{v}_{A'}$ (Fig. 2.37a). Vector $\Delta\mathbf{v}$ measures this velocity change (Fig. 2.37b).

The $\Delta\mathbf{v}/\Delta t$ ratio, that is to say, the variation of velocity divided by the time it takes for that change to happen, is the average acceleration. When the time considered is infinitesimal, then, $\Delta\mathbf{v}/\Delta t$ becomes $d\mathbf{v}/dt$ and this is called instantaneous acceleration or just acceleration.

From Fig. 2.37b we deduce that $\Delta\mathbf{v} = \Delta\mathbf{v}_1 + \Delta\mathbf{v}_2$, where, since the magnitude of vector $\mathbf{v}_A$ is equal $\overline{om} = \overline{on}$, we can assert that:

- $\Delta\mathbf{v}_1$ represents the change in direction of velocity $\mathbf{v}_A$, thus $\mathbf{v}_A + \Delta\mathbf{v}_1 = \mathbf{v}_{A'} - \Delta\mathbf{v}_2$ is a vector with the same direction as $\mathbf{v}_{A'}$ and the magnitude of $\mathbf{v}_A$.
- $\Delta\mathbf{v}_2$ represents the change in magnitude (magnitude change) of the velocity of point A when it switches from one position to another. Its magnitude is the difference between the magnitudes of vectors $\mathbf{v}_A$ and $\mathbf{v}_{A'}$.

Relating these changes in velocity and the time it took for them to happen, we obtain average acceleration vector $\mathbf{A}$ of point A (Eq. 2.104) when it moves from point A to A'.

$$\mathbf{A}_A = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\Delta\mathbf{v}_1}{\Delta t} + \frac{\Delta\mathbf{v}_2}{\Delta t} \tag{2.104}$$

This average acceleration has two components. One is only responsible for the change in direction ($\Delta\mathbf{v}_1/\Delta t$), and the other is responsible for the change in velocity

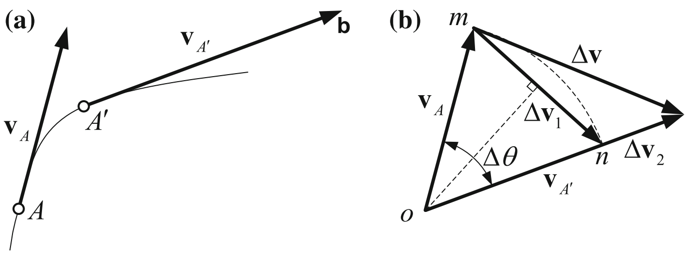


Fig. 2.37 **a** Change of point A velocity while changing its position from A to A' following a curve in Δt time. **b** Velocity change vector

magnitude ($\Delta\mathbf{v}_2/\Delta t$). In Fig. 2.37b we can calculate the magnitudes of $\Delta\mathbf{v}_1$ and $\Delta\mathbf{v}_2$ (Eqs. 2.108 and 2.109):

$$\Delta v_1 = 2v_A \sin\frac{\Delta\theta}{2} \tag{2.105}$$

$$\Delta v_2 = v_2 - v_1 \tag{2.106}$$

The directions of $\Delta\mathbf{v}_1$ and $\Delta\mathbf{v}_2$ in the limit as Δt approaches zero are respectively perpendicular and parallel to velocity vector $\mathbf{v}_A$, that is, normal and tangential to the trajectory at point A. These vectors are called normal and tangential accelerations, $\mathbf{a}_A^n$ and $\mathbf{a}_A^t$. The acceleration vector can be obtained by adding these two components (Eq. 2.107):

$$\mathbf{a}_A = \mathbf{a}_A^n + \mathbf{a}_A^t \tag{2.107}$$

The magnitudes of these components can be calculated as follows in Eqs. (2.108) and (2.109):

$$a_A^n = \lim_{\Delta t\to 0}\left(2v_A\frac{\sin\Delta\theta/2}{\Delta t}\right) \simeq v_A\frac{d\theta}{dt} = v_A\omega = R\omega^2 = \frac{v_A^2}{R} \tag{2.108}$$

$$a_A^t = \lim_{\Delta t\to 0}\frac{v_{A'} - v_A}{\Delta t} = \frac{dv_A}{dt} = R\frac{d\omega}{dt} + \omega\frac{dR}{dt} = R\alpha + \omega\frac{dR}{dt} \tag{2.109}$$

where:

- v is the velocity of point A.
- R is the trajectory radius at point A.
- ω is the angular velocity of the radius.
- α is the angular acceleration of the radius.
- dR/dt is the radius variation with respect to time.

To sum up, acceleration of a point A can be broken into two components:

- The first one is called normal acceleration, $\mathbf{a}_A^n$. Its direction is normal to the trajectory followed by point A and it points towards the trajectory center (Fig. 2.38). This component is responsible for the change in velocity direction and its magnitude is Eq. (2.110):

$$a_A^n = R\omega^2 = \frac{v_A^2}{R} \tag{2.110}$$

- The second component, known as tangential acceleration, $\mathbf{a}_A^t$, has a direction tangential to the trajectory, that is, the same as the velocity vector of point A. It can point towards the same side as the velocity or towards the opposite one;

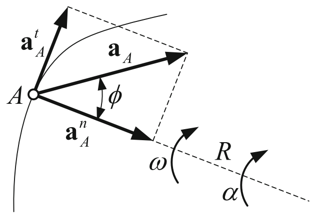


Fig. 2.38 The acceleration of a point has a normal component that points towards the center of the trajectory and a tangential component whose direction is tangential to the trajectory

it depends on whether the velocity magnitude increases or decreases. Tangential acceleration is responsible for the change in magnitude of the velocity vector and its value is Eq. (2.111):

$$a_A^t = R\alpha + \omega \frac{dR}{dt} \tag{2.111}$$

If the trajectory radius is constant, dR/dt is zero and the value of the tangential acceleration is $a_A^t = R\alpha$.

The magnitude of the acceleration can be determined by the magnitudes of its normal and tangential components. Equation (2.112) will be applied:

$$a_A = \sqrt{(a_A^n)^2 + (a_A^t)^2} \tag{2.112}$$

Finally, the angle formed by the acceleration vector and the normal direction to the trajectory is defined by Eq. (2.113):

$$\phi = \arctan \frac{a_A^t}{a_A^n} = \arctan \frac{\alpha}{\omega^2} \tag{2.113}$$

Equation (2.113) is only valid when the radius is constant.

2.2.2 *Relative Acceleration of Two Points*

The relative acceleration of point A with respect to point B is the ratio between the change in their relative velocity vector and time.

Let us assume that point A moves from position A to A' in the same period of time it takes B to reach position B'. The velocities of points A and B are $\mathbf{v}_A$ and $\mathbf{v}_B$ and their change is given by vectors $\Delta\mathbf{v}_A$ and $\Delta\mathbf{v}_B$ (Fig. 2.39). This way, the new velocities will be Eqs. (2.114) and (2.115):

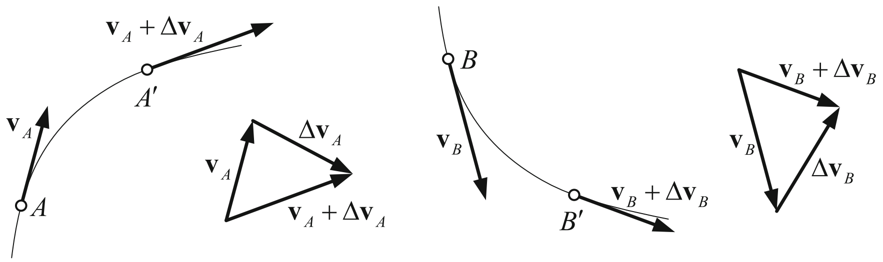


Fig. 2.39 Velocity change vectors $\Delta\mathbf{v}_A$ and $\Delta\mathbf{v}_B$ of points A and B when moving to new positions A' and B' respectively

$$\mathbf{v}_{A'} = \mathbf{v}_A + \Delta\mathbf{v}_A \tag{2.114}$$

$$\mathbf{v}_{B'} = \mathbf{v}_B + \Delta\mathbf{v}_B \tag{2.115}$$

On the other side, Eq. (2.116) that gives us the value of relative velocity $\mathbf{v}_{BA}$ between A and B is (Fig. 2.40a):

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A \tag{2.116}$$

And between A' and B' it is Eq. (2.117) (Fig. 2.40b):

$$\mathbf{v}_{BA} + \Delta\mathbf{v}_{BA} = (\mathbf{v}_B + \Delta\mathbf{v}_B) - (\mathbf{v}_A + \Delta\mathbf{v}_A) \tag{2.117}$$

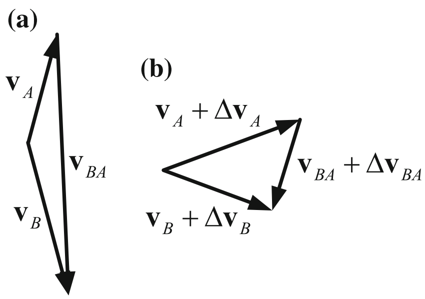
If we plug the value of relative velocity $\mathbf{v}_{BA}$ from Eq. (2.116) in Eq. (2.117), we obtain Eq. (2.118):

$$(\mathbf{v}_B - \mathbf{v}_A) + \Delta\mathbf{v}_{BA} = (\mathbf{v}_B + \Delta\mathbf{v}_B) - (\mathbf{v}_A + \Delta\mathbf{v}_A) \tag{2.118}$$

By simplifying the previous equation, we get Eq. (2.119):

$$\Delta\mathbf{v}_{BA} = \Delta\mathbf{v}_B - \Delta\mathbf{v}_A \tag{2.119}$$

Fig. 2.40 Relative velocity between **a** points A and B, **b** points A' and B'



After rearranging Eq. (2.120):

$$\Delta\mathbf{v}_B = \Delta\mathbf{v}_A + \Delta\mathbf{v}_{BA} \tag{2.120}$$

This way, if we divide Eq. (2.120) by the period of time, Δt, we obtain Eq. (2.121):

$$\frac{\Delta\mathbf{v}_B}{\Delta t} = \frac{\Delta\mathbf{v}_A}{\Delta t} + \frac{\Delta\mathbf{v}_{BA}}{\Delta t} \tag{2.121}$$

Each one of the terms in equation (Eq. 2.121) is an average acceleration (Eq. 2.122):

$$\mathbf{A}_B = \mathbf{A}_A + \mathbf{A}_{BA} \tag{2.122}$$

When Δt approaches zero (dt), the average accelerations become instantaneous accelerations (Eq. 2.123):

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} \tag{2.123}$$

Therefore, the acceleration vector of point B equals the sum of the acceleration vector of point A plus the relative acceleration vector of point B with respect to point A. The latter has a normal as well as a tangential component (Eq. 2.124):

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}^n_{BA} + \mathbf{a}^t_{BA} \tag{2.124}$$

2.2.3 *Relative Acceleration of Two Points in the Same Rigid Body*

As the distance between two points of a rigid body cannot change, relative motion between them is a rotation of one point about the other. In the example shown in Fig. 2.41, point B rotates about point A, both being part of a link that moves with angular velocity ω and angular acceleration α. The relative acceleration vector of point B with respect to point A can be broken into two components:

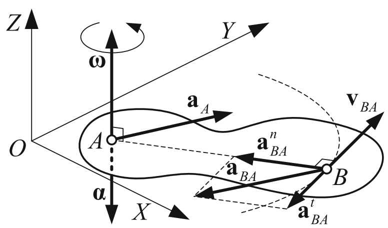


Fig. 2.41 Relative acceleration of point B with respect to point A both being in the same link

- The normal component, $\mathbf{a}_{BA}^{n}$, is always perpendicular to the relative velocity vector and it points towards the center of curvature of the trajectory. In this case, it points towards point A.
- The tangential component, $\mathbf{a}_{BA}^{t}$, has the same direction as the relative velocity vector. In the example shown in Fig. 2.41, as the direction of angular acceleration α opposes the direction of angular velocity ω, the tangential component points in the opposite direction to relative velocity $\mathbf{v}_{BA}$, which means that the magnitude of this velocity is decreasing.

These normal and tangential components of the relative acceleration of point B with respect to point A can be obtained with Eqs. (2.125) and (2.126):

$$\mathbf{a}_{BA}^{n} = \boldsymbol{\omega} \wedge \mathbf{v}_{BA} = \boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \mathbf{r}_{BA} \tag{2.125}$$

$$\mathbf{a}_{BA}^{t} = \boldsymbol{\alpha} \wedge \mathbf{r}_{BA} \tag{2.126}$$

The angle formed by the relative acceleration vector and the normal direction to the trajectory is Eq. (2.127):

$$\phi = \arctan\frac{\mathbf{a}_{BA}^{t}}{\mathbf{a}_{BA}^{n}} = \arctan\frac{\alpha}{\omega^2} \tag{2.127}$$

In other words, angle ϕ is independent from distance $\overline{AB}$. It only depends on the acceleration α and the angular velocity ω.

With these components, we can calculate the acceleration of point B based on the one of point A (Eq. 2.128):

$$\mathbf{a}_B = a_{A_x}\hat{\mathbf{i}} + a_{A_y}\hat{\mathbf{j}} + \boldsymbol{\omega} \wedge \mathbf{v}_{BA} + \boldsymbol{\alpha} \wedge \mathbf{r}_{BA} \tag{2.128}$$

In the case of a rigid body that revolves about steady point O (Fig. 2.42) the absolute acceleration vector of point P (a generic point of the body) is the relative acceleration with respect to point O (Eq. 2.129):

$$\mathbf{a}_P = \mathbf{a}_O + \mathbf{a}_{PO}^{n} + \mathbf{a}_{PO}^{t} = \mathbf{a}_{PO}^{n} + \mathbf{a}_{PO}^{t} \tag{2.129}$$

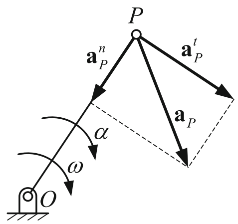


Fig. 2.42 Normal and tangential components of the acceleration of point P on a link that revolves about steady point O

Where the normal and tangential components are Eqs. (2.130) and (2.131):

$$\mathbf{a}^n_{PO} = \boldsymbol{\omega} \wedge \mathbf{v}_{PO} = \boldsymbol{\omega} \wedge \boldsymbol{\omega} \wedge \mathbf{r}_{PO} \tag{2.130}$$

$$\mathbf{a}^t_{PO} = \boldsymbol{\alpha} \wedge \mathbf{r}_{PO} \tag{2.131}$$

2.2.4 Computing Acceleration in a Four-Bar Linkage

To apply the relative acceleration method, we will calculate the acceleration of points B and C in the linkage in Example 1 of this chapter (Fig. 2.43). We know angular velocity $\omega_2 = -20\ \mathrm{rad/s}$ clockwise and angular acceleration $\alpha_2 = 150\ \mathrm{rad/s^2}$ counterclockwise of the motor link as well as the geometrical data of the linkage. We will also make use of the following results obtained from the position and velocity analysis in Example 1:

$$\begin{aligned} \theta_3 &= 19.4^\circ \\ \theta'_3 &= 349.4^\circ \\ \theta_4 &= 100.42^\circ \end{aligned} \qquad \begin{aligned} \omega_3 &= 7.16\ \mathrm{rad/s} \\ \omega_4 &= -8.78\ \mathrm{rad/s} \end{aligned} \qquad \begin{aligned} \mathbf{v}_A &= 103.9\hat{\mathbf{i}} - 60\hat{\mathbf{j}} \\ \mathbf{v}_B &= 77.75\hat{\mathbf{i}} + 14.28\hat{\mathbf{j}} \\ \mathbf{v}_{BA} &= -26.13\hat{\mathbf{i}} + 74.32\hat{\mathbf{j}} \\ \mathbf{v}_C &= 114.4\hat{\mathbf{i}} - 3.7\hat{\mathbf{j}} \\ \mathbf{v}_{CA} &= 10.53\hat{\mathbf{i}} + 56.3\hat{\mathbf{j}} \end{aligned}$$

To solve the problem we will apply the vector equation that relates the accelerations of points B and A (Eq. 2.132):

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} = (\mathbf{a}^n_B + \mathbf{a}^t_B) = (\mathbf{a}^n_A + \mathbf{a}^t_A) + (\mathbf{a}^n_{BA} + \mathbf{a}^t_{BA}) \tag{2.132}$$

In general, normal components will be known, since they depend on velocity, while tangential components will be unknown as they depend on angular acceleration, α. We will start calculating the acceleration of point A. Then we will analyze the acceleration of point B with respect to point A. Next, we will study the acceleration of point B and, finally, we will obtain the acceleration of point C.

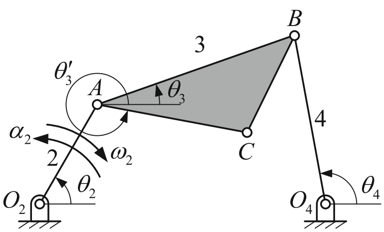


Fig. 2.43 Four-bar linkage with known angular velocity and acceleration of the input link

Acceleration Vector of Point A:

Point A rotates about steady point O_2, so the normal acceleration component is given by Eq. (2.133):

$$\mathbf{a}_A^n = \boldsymbol{\omega}_2 \wedge \mathbf{v}_A = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -20 \\ 103.9 & -60 & 0 \end{vmatrix} = -1200\hat{\mathbf{i}} - 2078\hat{\mathbf{j}} \tag{2.133}$$

$$\mathbf{a}_A^n = 2400\,\mathrm{cm/s^2}\,\angle 240°$$

We can verify that this vector has a perpendicular direction to $\mathbf{v}_A$ and it points towards the trajectory curvature center of point A, that is to say, towards O_2.

The tangential component of the acceleration vector of point A is given by Eq. (2.134):

$$\mathbf{a}_A^t = \boldsymbol{\alpha}_2 \wedge \mathbf{r}_{AO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 150 \\ 6\cos 60° & 6\sin 60° & 0 \end{vmatrix} = -779.42\hat{\mathbf{i}} + 450\hat{\mathbf{j}} \tag{2.134}$$

$$\mathbf{a}_A^t = 900\,\mathrm{cm/s^2}\,\angle 150°$$

Acceleration vector $\mathbf{a}_A^t$ is parallel to velocity vector $\mathbf{v}_A$ but in the opposite direction, as the direction of angular acceleration α_2 is opposite to the direction of angular velocity ω_2.

We can start drawing the acceleration polygon (Fig. 2.44) by tracing vectors $\mathbf{a}_A^n$ and $\mathbf{a}_A^t$. We define point a in the polygon at the end of vector acceleration $\mathbf{a}_A$. The acceleration image of link $\overline{O_2A}$ is $\overline{oa}$.

Relative Acceleration of Point B with Respect to Point A:

Since relative motion is a revolution of point B about point A, the normal component of the relative acceleration of B with respect to A is Eq. (2.135):

$$\mathbf{a}_{BA}^n = \boldsymbol{\omega}_3 \wedge \mathbf{v}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 7.16 \\ -26.13 & 74.32 & 0 \end{vmatrix} = -532.13\hat{\mathbf{i}} - 187.1\hat{\mathbf{j}} \tag{2.135}$$

$$\mathbf{a}_{BA}^n = 564.06\,\mathrm{cm/s^2}\,\angle 199.4°$$

The direction of this vector is perpendicular to velocity $\mathbf{v}_{BA}$ and it heads towards the trajectory center of point B, In other words, the direction is from B to A.

The tangential component of the relative acceleration vector of B with respect to A is expressed as Eq. (2.136):

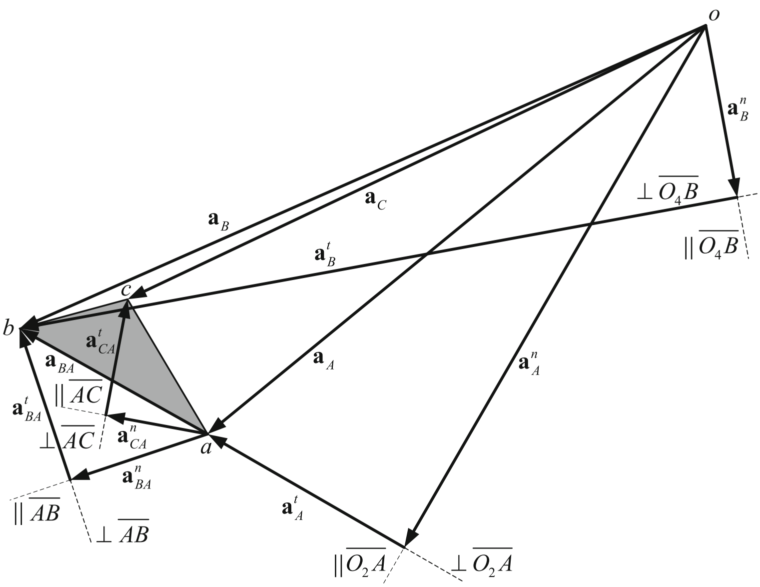


Fig. 2.44 Acceleration polygon of the four-bar linkage in Fig. 2.43

$$\mathbf{a}_{BA}^t = \boldsymbol{\alpha}_3 \wedge \mathbf{r}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_3 \\ 11\cos 19.4^\circ & 11\sin 19.4^\circ & 0 \end{vmatrix} = -3.65\alpha_3\hat{\mathbf{i}} + 10.38\alpha_3\hat{\mathbf{j}} \tag{2.136}$$

To calculate the value of $\mathbf{a}_{BA}^t$, we need to know α_3, which will be obtained in the next step.

Acceleration of Point B:

This point rotates about O_4, so the normal component of its acceleration (Eq. 2.137) will be:

$$\mathbf{a}_B^n = \boldsymbol{\omega}_4 \wedge \mathbf{v}_B = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -8.78 \\ 77.75 & 14.28 & 0 \end{vmatrix} = 125.37\hat{\mathbf{i}} - 682.64\hat{\mathbf{j}} \tag{2.137}$$

$$\mathbf{a}_B^n = 694.06\,\text{cm/s}^2 \angle 280.4^\circ$$

This vector is perpendicular to $\mathbf{v}_B$ and heads towards the trajectory curvature center of point B. In other words, from B to O_4.

The tangential acceleration is defined by Eq. (2.138):

$$\mathbf{a}_B^t = \boldsymbol{\alpha}_4 \wedge \mathbf{r}_{BO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_4 \\ 9\cos 100.4^\circ & 9\sin 100.4^\circ & 0 \end{vmatrix} = -8.85\alpha_4\hat{\mathbf{i}} - 1.62\alpha_4\hat{\mathbf{j}} \tag{2.138}$$

This component depends on α_4, which is another unknown that we need to find. In order to determine it, we need to plug the obtained values in the acceleration vector (Eqs. 2.139 and 2.140):

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} = (\mathbf{a}_A^n + \mathbf{a}_A^t) + (\mathbf{a}_{BA}^n + \mathbf{a}_{BA}^t) \tag{2.139}$$

$$\begin{aligned} &(125.37\hat{\mathbf{i}} - 682.64\hat{\mathbf{j}}) + (-8.85\alpha_4\hat{\mathbf{i}} - 1.62\alpha_4\hat{\mathbf{j}}) \\ &\quad = (-1200\hat{\mathbf{i}} - 2078\hat{\mathbf{j}}) + (-779.42\hat{\mathbf{i}} + 450\hat{\mathbf{j}}) \\ &\qquad + (-532.13\hat{\mathbf{i}} - 187.1\hat{\mathbf{j}}) + (-3.65\alpha_3\hat{\mathbf{i}} + 10.38\alpha_3\hat{\mathbf{j}}) \end{aligned} \tag{2.140}$$

Breaking Eq. (2.140) into its components, we obtain Eq. (2.141):

$$\left.\begin{array}{l} 125.37 - 8.85\alpha_4 = -1200 - 779.42 - 532.13 - 3.65\alpha_3 \\ -682.64 - 1.62\alpha_4 = -2078 + 450 - 187.1 + 10.38\alpha_3 \end{array}\right\} \tag{2.141}$$

By solving the system, the angular accelerations are obtained: $\alpha_3 = 58.81\ \text{rad/s}^2$ and $\alpha_4 = 322.21\ \text{rad/s}^2$. They can be used in Eqs. (2.136)–(2.138) to calculate the values of the tangential components.

$$\mathbf{a}_{BA}^t = -214.65\hat{\mathbf{i}} + 610.48\hat{\mathbf{j}} = 647.08\,\text{cm/s}^2\ \angle 109.37^\circ$$

$$\mathbf{a}_B^t = -2851.55\hat{\mathbf{i}} - 522\hat{\mathbf{j}} = 2898.9\,\text{cm/s}^2\ \angle 190.37^\circ$$

Acceleration of Point C:

Finally, we can find the acceleration of point C (Eq. 2.142) by using the following vector equation:

$$\mathbf{a}_C = \mathbf{a}_A + \mathbf{a}_{CA} = \mathbf{a}_A + \mathbf{a}_{CA}^n + \mathbf{a}_{CA}^t \tag{2.142}$$

We already know acceleration $\mathbf{a}_A$ and we can calculate the components of the relative acceleration of point C with respect to point A (Eqs. 2.143 and 2.144):

$$\mathbf{a}_{CA}^n = \boldsymbol{\omega}_3 \wedge \mathbf{v}_{CA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 7.16 \\ 10.53 & 56.3 & 0 \end{vmatrix} = -403.11\hat{\mathbf{i}} + 75.39\hat{\mathbf{j}} \tag{2.143}$$

$$\mathbf{a}_{CA}^{n} = 410.1\,\mathrm{cm/s^2}\ \angle 169.4^\circ$$

$$\mathbf{a}_{CA}^{t} = \boldsymbol{\alpha}_3 \wedge \mathbf{r}_{CA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 58.81 \\ 8\cos 349.4^\circ & 8\sin 349.4^\circ & 0 \end{vmatrix} = 86.54\hat{\mathbf{i}} + 462.45\hat{\mathbf{j}} \tag{2.144}$$

$$\mathbf{a}_{CA}^{t} = 470.48\,\mathrm{cm/s^2}\ \angle 79.4^\circ$$

Hence, the acceleration of point C is Eq. (2.145):

$$\begin{aligned} \mathbf{a}_C &= (-1200\hat{\mathbf{i}} - 2078\hat{\mathbf{j}}) + (-779.42\hat{\mathbf{i}} + 450\hat{\mathbf{j}}) \\ &\quad + (-403.11\hat{\mathbf{i}} + 75.39\hat{\mathbf{j}}) + (86.54\hat{\mathbf{i}} + 462.45\hat{\mathbf{j}}) \\ &= -2295.99\hat{\mathbf{i}} - 1090.16\hat{\mathbf{j}} \end{aligned} \tag{2.145}$$

$$\mathbf{a}_C = 2541.66\,\mathrm{cm/s^2}\ \angle 205.4^\circ$$

Once all the accelerations have been obtained, they can be represented in an acceleration polygon like the one shown in Fig. 2.44.

In the acceleration polygon in Fig. 2.44, triangle $\triangle abc$ is defined by the end points of the absolute acceleration vectors of points A, B and C. The same as in velocity analysis, triangle $\triangle abc$ in the polygon is similar to triangle $\triangle ABC$ in the mechanism (Eq. 2.146).

$$\frac{\overline{ab}}{\overline{AB}} = \frac{\overline{ac}}{\overline{AC}} = \frac{\overline{bc}}{\overline{BC}} \tag{2.146}$$

2.2.4.1 Accelerations in a Slider-crank Linkage

Figure 2.45 shows the slider-crank mechanism whose velocity was calculated in Example 2 of Sect. 2.1.3.8. The crank rotates with constant angular velocity of -10 rad/s clockwise. We want to find the acceleration of point B.

We will use the results obtained in Example 2:

$$\theta_3 = 324.26 \quad \omega_3 = 2.64\ \mathrm{rad/s} \quad \begin{aligned} \mathbf{v}_A &= 25.98\hat{\mathbf{i}} - 15\hat{\mathbf{j}} \\ \mathbf{v}_{BA} &= 10.79\hat{\mathbf{i}} + 15\hat{\mathbf{j}} \\ \mathbf{v}_B &= 36.78\hat{\mathbf{i}} \end{aligned}$$

To calculate the acceleration of point B (Eq. 2.147), we apply the following vector equation:

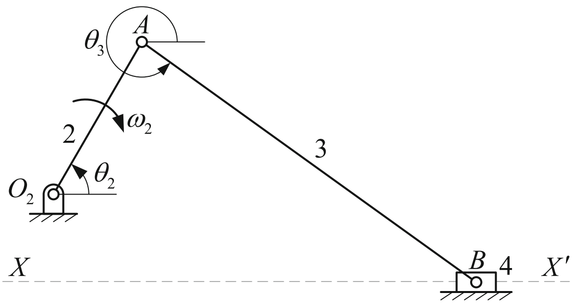


Fig. 2.45 Slider-crank linkage with constant angular velocity in link 2

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{BA} = (\mathbf{a}_A^n + \mathbf{a}_A^t) + (\mathbf{a}_{BA}^n + \mathbf{a}_{BA}^t) \tag{2.147}$$

Acceleration Vector of Point A:

The normal component of acceleration $\mathbf{a}_A^n$ is given by Eq. (2.148):

$$\mathbf{a}_A^n = \boldsymbol{\omega}_2 \wedge \mathbf{v}_A = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -10 \\ 25.98 & -15 & 0 \end{vmatrix} = -150\hat{\mathbf{i}} - 259.8\hat{\mathbf{j}} \tag{2.148}$$

$$\mathbf{a}_A^n = 300\,\text{cm/s}^2\,\angle 240^\circ$$

This vector is perpendicular to velocity $\mathbf{v}_A$ and it heads towards the trajectory curvature center of point A, that is to say, from A to O_2.

The tangential component of the vector is zero since the angular velocity of link 2 is constant, that is, $\alpha_2 = 0$.

Relative Acceleration of Point B with Respect to Point A:

The normal component of the relative acceleration vector of point B with respect to point A is given by Eq. (2.149):

$$\mathbf{a}_{BA}^n = \boldsymbol{\omega}_3 \wedge \mathbf{v}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 2.64 \\ 10.79 & 15 & 0 \end{vmatrix} = -39.6\hat{\mathbf{i}} + 28.49\hat{\mathbf{j}} \tag{2.149}$$

$$\mathbf{a}_{BA}^n = 48.78\,\text{cm/s}^2\,\angle 144.27^\circ$$

This vector is perpendicular to velocity $\mathbf{v}_{BA}$ and heads from B towards A.

The tangential component of the relative acceleration vector of point B with respect to point A is given by Eq. (2.150):

$$\mathbf{a}_{BA}^t = \boldsymbol{\alpha}_3 \wedge \mathbf{r}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_3 \\ 7\cos 324.26^\circ & 7\sin 324.26^\circ & 0 \end{vmatrix} = 4.09\alpha_3\hat{\mathbf{i}} + 5.68\alpha_3\hat{\mathbf{j}} \tag{2.150}$$

Prior to calculating its value, we have to find α_3.

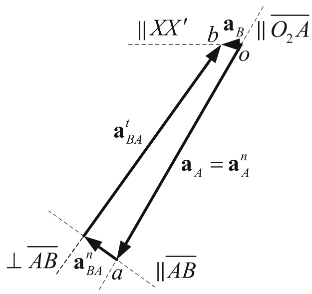


Fig. 2.46 Acceleration polygon of the slider-crank linkage in Fig. (2.45)

Acceleration of Point B:

We have to take into account that link 4 has a pure translational motion following the trajectory defined by line XX' (Fig. 2.45). The direction of the acceleration of point B coincides with the trajectory direction (Eq. 2.151):

$$\mathbf{a}_B = \mathbf{a}^t_B = a_B \hat{\mathbf{i}} \tag{2.151}$$

By substituting Eqs. (2.148)–(2.150) in Eq. (2.147) we obtain Eq. (2.152):

$$a_B \hat{\mathbf{i}} = (-150\hat{\mathbf{i}} - 259.8\hat{\mathbf{j}}) + (-39.6\hat{\mathbf{i}} + 28.49\hat{\mathbf{j}}) + (4.09\alpha_3 \hat{\mathbf{i}} + 5.68\alpha_3 \hat{\mathbf{j}}) \tag{2.152}$$

The following algebraic components are obtained if we break this acceleration vector into its components (Eq. 2.153):

$$\left.\begin{aligned} a_B &= -150 - 39.6 + 4.09\alpha_3 \\ 0 &= -259.8 + 28.49 + 5.68\alpha_3 \end{aligned}\right\} \tag{2.153}$$

Based on these equations, we find angular acceleration, $\alpha_3 = 40.72 \text{ rad/s}^2$, and the magnitude of the linear acceleration at point B, $a_B = -23.04 \text{ cm/s}^2$. This way, the remaining acceleration value is Eq. (2.154):

$$\mathbf{a}^t_{BA} = 166.54\hat{\mathbf{i}} + 231.29\hat{\mathbf{j}} = 285 \text{ cm/s}^2 \angle 54.24° \tag{2.154}$$

$$\mathbf{a}_B = -23.04\hat{\mathbf{i}} = 23.04 \text{ cm/s}^2 \angle 180°$$

Once all the accelerations have been determined, we can represent them in an acceleration polygon as the one shown in Fig. 2.46.

2.2.5 *The Coriolis Component of Acceleration*

When a body moves along a trajectory defined over a rotating body, the acceleration of any point on the first body relative to a coinciding point on the second body will have, in addition to the normal and tangential components, a new one named the Coriolis acceleration. To demonstrate its value, we will use a simple example. Although a demonstration in a particular situation is not valid to demonstrate a generic situation, we will use this example because of its simplicity.

In Fig. 2.47, link 3 represents a slide that moves over straight line. Point P_3 of link 3 is above point P_2 of link 2. Therefore, the position of both points coincides at the instant represented in the figure.

The acceleration of point P_3 can be computed as Eq. (2.155):

$$\mathbf{a}_{P_3} = \mathbf{a}_{P_2} + \mathbf{a}_{P_3P_2} \tag{2.155}$$

Relative acceleration $\mathbf{a}_{P_3P_2}$ has, in addition to normal and tangential components studied so far, a new component called the Coriolis acceleration (Eq. 2.156):

$$\mathbf{a}_{P_3P_2} = \mathbf{a}^n_{P_3P_2} + \mathbf{a}^t_{P_3P_2} + \mathbf{a}^c_{P_3P_2} \tag{2.156}$$

To demonstrate the existence of this new component, see Fig. 2.48. It represents slider 3 moving with constant relative velocity $\mathbf{v}_{P_3P_2} = \mathbf{v}$ over link 2, which, at the same time, rotates with constant angular velocity ω_2. P_3 is a point of slider 3 that moves along trajectory O_2F on body 2. P_2 is a point of link 2 that coincides with P_3 at the instant represented.

In a dt time interval, line O_2F rotates about O_2, angle $d\theta$, and moves to position O_2F'. In the same period of time point P_2 moves to P_2' and P_3 moves to P_3'. This last displacement can be regarded as the sum of displacements $\Delta P_2P_2'$, $\Delta P_2'B$ and $\Delta BP_3'$.

Displacement $\Delta P_2P_2'$ takes place at constant velocity since $\overline{O_2P_2}$ and ω_2 are constant. Movement $\Delta P_2'B$ also takes place at constant velocity as $\mathbf{v}_{P_3P_2}$ is constant.

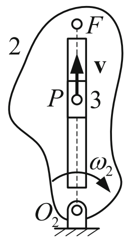


Fig. 2.47 Link 3 moves along a trajectory defined over link 2 which rotates with angular velocity ω_2

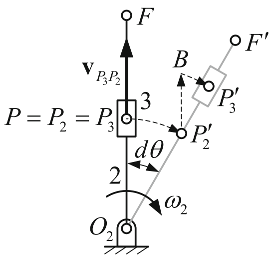


Fig. 2.48 Link 3 moves with constant relative velocity along a straight trajectory defined over link 2 which, at the same time, rotates with constant angular velocity ω_2

However, displacement $\Delta BP'_3$ is triggered by an acceleration. To obtain this acceleration, we start by calculating the length of arc $\overset{\frown}{BP'_3}$ (Eq. 2.157):

$$\overset{\frown}{BP'_3} = \overline{P'_2B}d\theta \tag{2.157}$$

But $\overline{P'_2B} = v_{P_3P_2}dt$ and $d\theta = \omega_2 dt$, which yields (Eq. 2.158):

$$\overset{\frown}{BP'_3} = \omega_2 v_{P_3P_2} dt^2 \tag{2.158}$$

The velocity of point P_3 is perpendicular to line O_2F and its magnitude is $\omega_2\overline{O_2P_3}$. Since ω_2 is constant and $\overline{O_2P_3}$ increases its value with a constant ratio, the magnitude of the velocity of point P_3, perpendicular to line O_2F, changes uniformly, that is, with constant acceleration.

In general, a displacement (ds) with constant acceleration (a) is defined by Eq. (2.159):

$$ds = \frac{1}{2}adt^2 \tag{2.159}$$

Then BP'_3 is expressed as Eq. (2.160):

$$\overset{\frown}{BP'_3} = \frac{1}{2}adt^2 \tag{2.160}$$

Evening out the two equations for arc $\overset{\frown}{BP'_3}$, we obtain Eq. (2.161):

$$\omega_2 v_{P_3P_2} dt^2 = \frac{1}{2}adt^2 \tag{2.161}$$

Finally, we clear the acceleration value Eq. (2.162):

$$a = 2\omega_2 v_{P_3P_2} \tag{2.162}$$

where a is known as the Coriolis component of the acceleration of point P_3 in honor of the great French mathematician of the XIX century. The Coriolis acceleration component is a vector perpendicular to the relative velocity vector. Its direction can be determined by rotating vector $\mathbf{v}_{P_3P_2}$ 90° in the direction of ω_2.

The Coriolis acceleration component vector can be obtained mathematically with Eq. (2.163):

$$\mathbf{a}^c_{P_3P_2} = 2\boldsymbol{\omega}_2 \wedge \mathbf{v}_{P_3P_2} \tag{2.163}$$

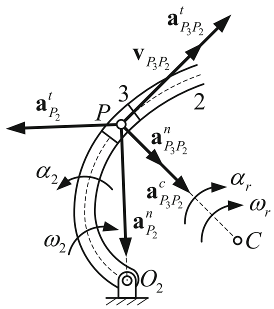


Fig. 2.49 Link 3 moves along a curved trajectory over link 2 while the latter is rotating with angular velocity ω_2

A general case of relative motion on a plane between two rigid bodies is shown in Fig. 2.49. P_2 is a point on body 2 and P_3 is a point on link 3, which moves along a curved trajectory over body 2 with its center in point C.

The absolute acceleration of point P_3 is Eq. (2.164):

$$\mathbf{a}_{P_3} = \mathbf{a}_{P_2} + \mathbf{a}_{P_3P_2} \tag{2.164}$$

Or, expressed in terms of their intrinsic components (Eq. 2.165):

$$\mathbf{a}^n_{P_3} + \mathbf{a}^t_{P_3} = \mathbf{a}^n_{P_2} + \mathbf{a}^t_{P_2} + \mathbf{a}^n_{P_3P_2} + \mathbf{a}^t_{P_3P_2} + \mathbf{a}^c_{P_3P_2} \tag{2.165}$$

where the Coriolis component is part of the relative acceleration of point P_3 with respect to P_2 and its value is given by Eq. (2.163).

The radius of the trajectory followed by P_3 over link 2 at the instant shown in Fig. 2.49 is $\overline{CP}$. Consequently, the normal and tangential acceleration vectors of P_3 with respect P_2 (Eqs. 2.166 and 2.167) are normal and tangential to the trajectory at the instant considered and their values are:

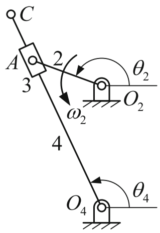
$$\mathbf{a}^n_{P_3P_2} = \boldsymbol{\omega}_r \wedge \mathbf{v}_{P_3P_2} \tag{2.166}$$

$$\mathbf{a}^t_{P_3P_2} = \boldsymbol{\alpha}_r \wedge \mathbf{r}_{PC} \tag{2.167}$$

2.2.5.1 Accelerations in a Quick-return Mechanism

In the mechanism in Fig. 2.50, link 2 is the motor link. We want to calculate the velocity and acceleration of link 4. The information on the length of the links as well as angular velocity ω_2 are the same as in Example 3. Angular acceleration of the motor link is zero, that is, $\alpha_2 = 0$.

Fig. 2.50 Quick-return linkage with constant angular velocity



Based on the results obtained in Example 3:

$$\theta_4 = 115.08^\circ \quad \omega_4 = 3.19\ \mathrm{rad/s^2} \quad \begin{aligned} \mathbf{v}_{A_2} &= -10.26\hat{\mathbf{i}} - 28.19\hat{\mathbf{j}} \\ \mathbf{v}_{A_4} &= -19.2\hat{\mathbf{i}} - 9\hat{\mathbf{j}} \\ \mathbf{v}_{A_2A_4} &= 8.95\hat{\mathbf{i}} - 19.19\hat{\mathbf{j}} \\ \mathbf{v}_C &= -26\hat{\mathbf{i}} - 12.17\hat{\mathbf{j}} \end{aligned}$$

To calculate the accelerations, we use Eq. (2.168):

$$\mathbf{a}^n_{A_2} + \mathbf{a}^t_{A_2} = \mathbf{a}^n_{A_4} + \mathbf{a}^t_{A_4} + \mathbf{a}^n_{A_2A_4} + \mathbf{a}^t_{A_2A_4} + \mathbf{a}^c_{A_2A_4} \tag{2.168}$$

We study the value of each component, starting with the acceleration of point A_2 (Eqs. 2.169 and 2.170):

$$\mathbf{a}^n_{A_2} = \boldsymbol{\omega}_2 \wedge \mathbf{v}_{A_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 10 \\ -10.26 & -28.19 & 0 \end{vmatrix} = 281.9\hat{\mathbf{i}} - 102.6\hat{\mathbf{j}} \tag{2.169}$$

$$\mathbf{a}^n_{A_2} = 300\,\mathrm{cm/s^2}\,\angle 340^\circ$$

$$\mathbf{a}^t_{A_2} = \boldsymbol{\alpha}_2 \wedge \mathbf{r}_{AO_2} = 0 \tag{2.170}$$

We continue with the acceleration of point A_4 (Eqs. 2.171 and 2.172):

$$\mathbf{a}^n_{A_4} = \boldsymbol{\omega}_4 \wedge \mathbf{v}_{A_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 3.19 \\ -19.2 & -9 & 0 \end{vmatrix} = 28.71\hat{\mathbf{i}} - 61.23\hat{\mathbf{j}} \tag{2.171}$$

$$\mathbf{a}^n_{A_4} = 67.63\,\mathrm{cm/s^2}\,\angle 295.08^\circ$$

$$\begin{aligned} \mathbf{a}^t_{A_4} = \boldsymbol{\alpha}_4 \wedge \mathbf{r}_{AO_4} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_4 \\ \overline{O_4A}\cos 115.08^\circ & \overline{O_4A}\sin 115.08^\circ & 0 \end{vmatrix} \\ &= -6.02\alpha_4\hat{\mathbf{i}} - 2.82\alpha_4\hat{\mathbf{j}} \end{aligned} \tag{2.172}$$

We will now study the value of the acceleration of point A_2 relative to point A_4. Because the relative motion of point A_2 with respect to point A_4 follows a straight trajectory, the normal component is zero, $\mathbf{a}^t_{A_2A_4} = 0$. The tangential component has the direction of link 4 (Eq. 2.173), consequently:

$$\mathbf{a}^t_{A_2A_4} = a^t_{A_2A_4} \cos 115.08° \hat{\mathbf{i}} + a^t_{A_2A_4} \sin 115.08° \hat{\mathbf{j}} \tag{2.173}$$

Finally, we calculate the Coriolis component of the acceleration (Eq. 2.174):

$$\mathbf{a}^c_{A_2A_4} = 2\boldsymbol{\omega}_4 \wedge \mathbf{v}_{A_2A_4} = 2\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 3.19 \\ 8.95 & -19.19 & 0 \end{vmatrix} = 122.43\hat{\mathbf{i}} + 57.1\hat{\mathbf{j}} \tag{2.174}$$

In Eq. (2.175) we plug the calculated values in the vector equation (Eq. 2.168):

$$\begin{aligned}(281.9\hat{\mathbf{i}} - 102.6\hat{\mathbf{j}}) = {} & (28.71\hat{\mathbf{i}} - 61.23\hat{\mathbf{j}}) + (-6.02\alpha_4\hat{\mathbf{i}} - 2.82\alpha_4\hat{\mathbf{j}}) \\ & + (a^t_{A_2A_4} \cos 115.08° \hat{\mathbf{i}} + a^t_{A_2A_4} \sin 115.08° \hat{\mathbf{j}}) \\ & + (122.43\hat{\mathbf{i}} + 57.1\hat{\mathbf{j}})\end{aligned} \tag{2.175}$$

By separating the components, Eq. (2.176) is obtained:

$$\left.\begin{aligned} 281.9 = 28.71 - 6.02\alpha_4 + a^t_{A_2A_4} \cos 115.08° + 122.43 \\ -102.6 = -61.23 - 2.82\alpha_4 + a^t_{A_2A_4} \sin 115.08° + 57.1 \end{aligned}\right\} \tag{2.176}$$

This way, we find the value of angular acceleration $\alpha_4 = -11.56\ \text{rad/s}^2$ and the magnitude of tangential relative acceleration $a^t_{A_2A_4} = -145.63\ \text{cm/s}^2$. The accelerations then remain as follows:

$$\mathbf{a}^t_{A_4} = 69.59\hat{\mathbf{i}} + 32.6\hat{\mathbf{j}} = 76.85\ \text{cm/s}^2 \angle 25°$$

$$\mathbf{a}^t_{A_2A_4} = 61.73\hat{\mathbf{i}} - 131.9\hat{\mathbf{j}} = 145.63\ \text{cm/s}^2 \angle 295.08°$$

We proceed to calculating the acceleration of point C (Eqs. 2.177 and 2.178):

$$\mathbf{a}^n_C = \boldsymbol{\omega}_4 \wedge \mathbf{v}_C = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 3.19 \\ -26 & -12.17 & 0 \end{vmatrix} = 38.82\hat{\mathbf{i}} - 82.94\hat{\mathbf{j}} \tag{2.177}$$

$$\mathbf{a}^n_C = 91.57\ \text{cm/s}^2 \angle 295.08°$$

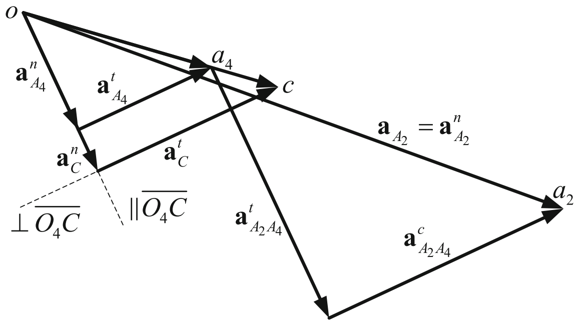


Fig. 2.51 Acceleration polygon of the quick-return linkage in Fig. 2.50

$$\mathbf{a}^t_C = \boldsymbol{\alpha}_4 \wedge \mathbf{r}_{CO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -11.56 \\ \overline{O_4C}\cos 115.08^\circ & \overline{O_4C}\sin 115.08^\circ & 0 \end{vmatrix} = 94.23\hat{\mathbf{i}} + 44.1\hat{\mathbf{j}} \tag{2.178}$$

$$\mathbf{a}^t_C = 104.04\,\mathrm{cm/s^2}\,\angle 25.08^\circ$$

Once all the accelerations have been determined, they can be represented in an acceleration polygon as in previous examples (Fig. 2.51).

2.3 Exercises with Their Solutions

In this section we will carry out the kinematic analysis of different mechanisms by applying the methods developed in this chapter up to now.

Example 10 In the mixing machine in Example 4 (Fig. 2.52a), calculate the velocity of point C (Fig. 2.52b) by means of the ICR method once the velocity of point A is known. Use the relative acceleration method to calculate the acceleration of point C knowing that the motor (link 2) moves at constant angular velocity $\omega_2 = 10$ rad/s counterclockwise, in other words, $\alpha_2 = 0$.

We start by calculating the velocity of point C by using the ICR method. Since the point of known velocity, A, is part of link 3 and point C also belongs to link 3, we only need to find the ICR of links 1 and 3. That is to say, I_{13}. This center is shown in Fig. 2.53.

For velocity calculation purposes, we first have to find the velocity of point A (Eq. 2.179) considered part of link 2. We know that $\omega_2 = 10$ rad/s and distance $\overline{I_{12}A} = 7$ cm:

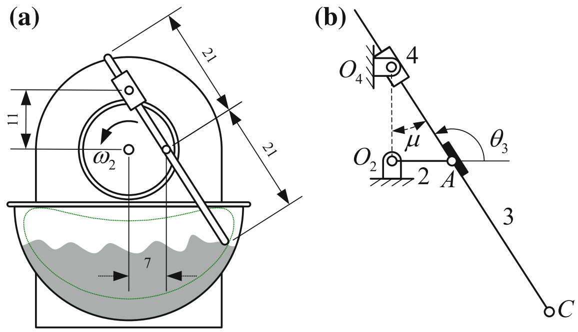


Fig. 2.52 **a** Mixing machine. **b** Kinematic skeleton

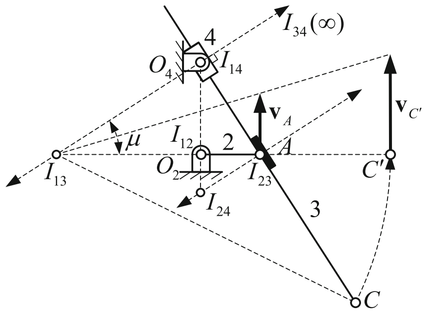


Fig. 2.53 Calculation of the point C velocity using the ICR method

$$v_A = \omega_2 \overline{I_{12}A} = 70\,\text{cm/s} \tag{2.179}$$

This same velocity associated to link 3 (Eq. 2.180) is:

$$v_A = \omega_3 \overline{I_{13}I_{23}} \Rightarrow \omega_3 = \frac{v_A}{\overline{I_{13}I_{23}}} \tag{2.180}$$

To determine the angular velocity of link 3, we need to calculate distance $\overline{I_{13}I_{23}}$ (Eq. 2.181). We will make use of values $\mu = 32.47^\circ$ and $\overline{O_4A} = 13.04\,\text{cm}$ obtained in Example 4:

$$\overline{I_{13}I_{23}} = \frac{\overline{O_4A}}{\sin\mu} = 24.289\,\text{cm} \tag{2.181}$$

$$\omega_3 = \frac{v_A}{\overline{I_{13}I_{23}}} = 2.88\ \text{rad/s} \tag{2.182}$$

Finally, we can calculate the velocity of point C (Eq. 2.183) knowing that $\theta_3 = 122.47^\circ$ and $\overline{AC} = 21\,\text{cm}$:

$$v_C = \omega_3 \overline{I_{13}C} = \omega_3 \sqrt{\overline{I_{13}I_{23}}^2 + \overline{AC}^2 - 2 \cdot \overline{I_{13}I_{23}} \cos\theta_3} \tag{2.183}$$

$$v_C = 114.4\,\text{cm/s}$$

To calculate accelerations, we use the position and velocity results obtained in Example 4:

$$\theta_3 = 122.47^\circ \quad \omega_3 = 288\text{ rad/s} \qquad \begin{aligned} \mathbf{v}_A &= 70\hat{\mathbf{j}} \\ \mathbf{v}_{O_3} &= \mathbf{v}_{O_3O_4} = -31.52\hat{\mathbf{i}} - 20.16\hat{\mathbf{j}} \\ \mathbf{v}_C &= 51.03\hat{\mathbf{i}} + 102.45\hat{\mathbf{j}} \\ \mathbf{v}_{O_3A} &= -31.68\hat{\mathbf{i}} - 20.16\hat{\mathbf{j}} \\ \mathbf{v}_{CA} &= 51.03\hat{\mathbf{i}} + 32.45\hat{\mathbf{j}} \end{aligned}$$

We start by calculating the acceleration of point O_3 on link 3 (Eqs. 2.184 and 2.185), which coincides with O_4 at the instant being studied.

$$\mathbf{a}_{O_3} = \mathbf{a}_A^n + \mathbf{a}_A^t + \mathbf{a}_{O_3A}^n + \mathbf{a}_{O_3A}^t \tag{2.184}$$

$$\mathbf{a}_{O_3} = \mathbf{a}_{O_4}^n + \mathbf{a}_{O_4}^t + \mathbf{a}_{O_3O_4}^n + \mathbf{a}_{O_3O_4}^t + \mathbf{a}_{O_3O_4}^c \tag{2.185}$$

In these equations, $\mathbf{a}_A^t = 0$ due to the fact that the angular acceleration of link 2 is zero. Acceleration $\mathbf{a}_{O_4} = 0$ because point O_4 is on the frame. Finally, acceleration $\mathbf{a}_{O_3O_4}^n = 0$ since the relative motion of point O_3 with respect to link 4 follows a straight path.

The rest of the acceleration components (Eqs. 2.186–2.190) have the following values:

$$\mathbf{a}_A^n = \boldsymbol{\omega}_2 \wedge \mathbf{v}_A = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 10 \\ 0 & 70 & 0 \end{vmatrix} = -700\hat{\mathbf{i}} \tag{2.186}$$

$$\mathbf{a}_A^n = 700\,\text{cm/s}^2\,\angle 180^\circ$$

$$\mathbf{a}_{O_3A}^n = \boldsymbol{\omega}_3 \wedge \mathbf{v}_{O_3A} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 2.88 \\ -31.68 & -20.16 & 0 \end{vmatrix} = 58.06\hat{\mathbf{i}} - 91.24\hat{\mathbf{j}} \tag{2.187}$$

$$\mathbf{a}_{O_3A}^n = 108.15\,\text{cm/s}^2\,\angle 302.47^\circ$$

$$\mathbf{a}^t_{O_3A} = \boldsymbol{\alpha}_3 \wedge \mathbf{r}_{O_3A} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_4 \\ \overline{AO_3}\cos 122.47^\circ & \overline{AO_3}\sin 122.47^\circ & 0 \end{vmatrix} = -11\alpha_3\hat{\mathbf{i}} - 7\alpha_3\hat{\mathbf{j}} \tag{2.188}$$

The relative acceleration component of will have the following values:

$$\mathbf{a}^t_{O_3O_4} = a^t_{O_3O_4}\cos 122.47^\circ\hat{\mathbf{i}} + a^t_{O_3O_4}\sin 122.47^\circ\hat{\mathbf{j}} \tag{2.189}$$

$$\mathbf{a}^c_{O_3O_4} = 2\boldsymbol{\omega}_3 \wedge \mathbf{v}_{O_3O_4} = 2\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 2.88 \\ -31.52 & 49.53 & 0 \end{vmatrix} = -285.29\hat{\mathbf{i}} - 181.56\hat{\mathbf{j}} \tag{2.190}$$

$$\mathbf{a}^c_{O_3O_4} = 338.16\,\mathrm{cm/s^2}\ \angle 212.47^\circ$$

Evening out the two vector equations that define the value of the acceleration of point O_3 and introducing the calculated values, we obtain Eq. (2.191):

$$\begin{aligned}&(-700\hat{\mathbf{i}}) + (58.06\hat{\mathbf{i}} - 91.24\hat{\mathbf{j}}) + (-11\alpha_3\hat{\mathbf{i}} - 7\alpha_3\hat{\mathbf{j}}) \\ &\quad = a^t_{O_3O_4}(-0.54\hat{\mathbf{i}} + 0.843\hat{\mathbf{j}}) + (-285.29\hat{\mathbf{i}} - 181.56\hat{\mathbf{j}})\end{aligned} \tag{2.191}$$

By breaking up the components, we obtain Eq. (2.192):

$$\left.\begin{aligned}-700 + 58.06 - 11\alpha_3 &= 0.54a^t_{O_3O_4} - 285.29 \\ -91.24 - 7\alpha_3 &= 0.843a^t_{O_3O_4} - 181.56\end{aligned}\right\} \tag{2.192}$$

This way, the angular acceleration of link 3 can be found, $\alpha_3 = -19.3\ \mathrm{rad/s^2}$, as well as the magnitude of the relative tangential acceleration, $a^t_{O_3O_4} = 267.37\,\mathrm{cm/s^2}$.

Ultimately, we apply Eq. (2.193) to calculate the acceleration of point C:

$$\mathbf{a}_C = \mathbf{a}^n_A + \mathbf{a}^t_A + \mathbf{a}^n_{CA} + \mathbf{a}^t_{CA} \tag{2.193}$$

Relative accelerations (Eqs. 2.194 and 2.195) can be worked out as follows:

$$\mathbf{a}^n_{CA} = \boldsymbol{\omega}_3 \wedge \mathbf{v}_{CA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 2.88 \\ 51.03 & 32.45 & 0 \end{vmatrix} = -93.46\hat{\mathbf{i}} + 146.97\hat{\mathbf{j}} \tag{2.194}$$

$$\mathbf{a}^t_{CA} = \boldsymbol{\alpha}_3 \wedge \mathbf{r}_{CA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -19.3 \\ 11.27 & -17.71 & 0 \end{vmatrix} = -341.94\hat{\mathbf{i}} - 217.59\hat{\mathbf{j}} \tag{2.195}$$

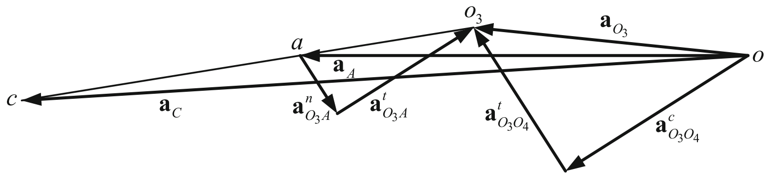


Fig. 2.54 Acceleration polygon of the mixing machine shown in Fig. (2.52)

Therefore, the acceleration of point C is Eq. (2.196):

$$\mathbf{a}_C = (-700\hat{\mathbf{i}}) + (-93.46\hat{\mathbf{i}} + 146.97\hat{\mathbf{j}}) + (-341.94\hat{\mathbf{i}} - 217.59\hat{\mathbf{j}}) \tag{2.196}$$

$$\mathbf{a}_C = -1135.4\hat{\mathbf{i}} - 70.62\hat{\mathbf{j}} = 1137.6\,\text{cm/s}^2\,\angle 183.56^\circ$$

Once the acceleration problem has been solved, we can represent the vectors obtained and draw the acceleration polygon in Fig. 2.54:

Example 11 Figure 2.55a represents a mechanism that is part of a calculating machine that carries out multiplications and divisions. At the instant shown, knowing that $\overline{O_2A} = 1$ cm, $\overline{O_2B} = 0.5$ cm, input $x = 0.25$ cm and link 2 moves with constant angular velocity of $\omega_2 = 1$ rad/s counterclockwise, calculate:

1. Which constant value, k, the input has to be multiplied by to obtain output, y. That is to say, the value of constant k in equation $y = kx$.
2. The velocity vector of points A, B, C and D using the relative velocity method.
3. The acceleration vector of point C.
4. The velocity vector of point C using the ICR method. Use the velocity of point A calculated in question 2.

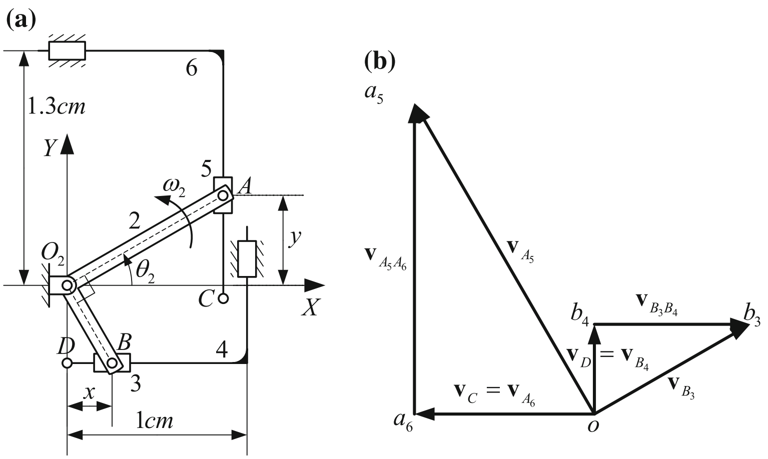


Fig. 2.55 Calculating machine (**a**) and its velocity polygon (**b**) for x = 0.25 cm and $\omega_2 = 1$ rad/s

1. We start by calculating x and y (Eq. 2.197) in terms of θ_2.

$$x = \overline{O_2A}\sin\theta_2$$
$$y = \overline{O_2B}\sin\theta_2 \quad (2.197)$$

Constant k will be given by Eq. (2.198):

$$k = \frac{y}{x} = \frac{\overline{O_2A}}{\overline{O_2B}} = 2 \quad (2.198)$$

2. Angle θ_2 needs to be determined to calculate the velocity of point A:

$$y = kx = 2 \cdot 0.25\,\text{cm} = 0.5\,\text{cm} \Rightarrow \theta_2 = \arcsin\frac{0.5}{1} = 30^\circ$$

We now define the relationship between the velocity of point A_5 and the velocity of point A_6 (Eq. 2.199), which coincides with it at that certain instant but belongs to link 6:

$$\mathbf{v}_{A_5} = \mathbf{v}_{A_6} + \mathbf{v}_{A_5A_6} \quad (2.199)$$

This yields the following velocity values (Eqs. 2.200 and 2.201):

$$\mathbf{v}_{A_5} = \mathbf{v}_{A_2} = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{AO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 1 \\ 1\cos 30^\circ & 1\sin 30^\circ & 0 \end{vmatrix} = -0.5\hat{\mathbf{i}} + 0.866\hat{\mathbf{j}} \quad (2.200)$$

$$\mathbf{v}_{A_5} = 1\,\text{cm/s}\,\angle 120^\circ$$

A_6 is a point on link 6 with straight horizontal motion. Relative motion of points A_5 and A_6 is also a straight movement but with vertical direction.

$$\mathbf{v}_{A_6} = v_{A_6}\hat{\mathbf{i}}$$
$$\mathbf{v}_{A_5A_6} = v_{A_5A_6}\hat{\mathbf{j}} \quad (2.201)$$

We introduce these values in Eq. (2.199) and we obtain Eq. (2.202):

$$0.5\hat{\mathbf{i}} + 0.866\hat{\mathbf{j}} = v_{A_6}\hat{\mathbf{i}} + v_{A_5A_6}\hat{\mathbf{j}} \quad (2.202)$$

The unknowns can easily be determined by separating the components.

$$\mathbf{v}_{A_6} = -0.5\hat{\mathbf{i}} = 0.5\,\text{cm/s}\,\angle 180^\circ$$
$$\mathbf{v}_{A_5A_6} = 0.866\hat{\mathbf{j}} = 0.866\,\text{cm/s}\,\angle 90^\circ$$

Link 6 moves with straight translational motion. Therefore, all its points have the same velocity. Since point C belongs to this link, its velocity is:

$$\mathbf{v}_C = \mathbf{v}_{A_6} = -0.5\hat{\mathbf{i}} = 0.5\,\text{cm/s}\,\angle 180^\circ$$

To calculate the velocity of point D, we proceed in a similar way. We have to calculate the velocity of point B_4:

$$\mathbf{v}_{B_3} = \mathbf{v}_{B_4} + \mathbf{v}_{B_3B_4} \tag{2.203}$$

In Eq. (2.203), the velocities are given by Eqs. (2.204) and (2.205):

$$\mathbf{v}_{B_3} = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{BO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 1 \\ 0.5\cos 300^\circ & 0.5\sin 300^\circ & 0 \end{vmatrix} = 0.43\hat{\mathbf{i}} + 0.25\hat{\mathbf{j}} \tag{2.204}$$

$$\mathbf{v}_{B_3} = 0.5\,\text{cm/s}\,\angle 30^\circ$$

$$\begin{aligned} \mathbf{v}_{B_4} &= v_{B_4}\hat{\mathbf{j}} \\ \mathbf{v}_{B_3B_4} &= v_{B_3B_4}\hat{\mathbf{i}} \end{aligned} \tag{2.205}$$

Plugging these values into Eq. (2.203), we obtain Eq. (2.206):

$$0.43\hat{\mathbf{i}} + 0.25\hat{\mathbf{j}} = v_{B_4}\hat{\mathbf{j}} + v_{B_3B_4}\hat{\mathbf{i}} \tag{2.206}$$

From which we can calculate the following values:

$$\mathbf{v}_{B_4} = 0.25\hat{\mathbf{j}} = 0.25\,\text{cm/s}\,\angle 90^\circ$$
$$\mathbf{v}_{B_3B_4} = 0.43\hat{\mathbf{i}} = 0.43\,\text{cm/s}\,\angle 0^\circ$$

Since point D belongs to link 4, which moves along a straight line, its velocity will be the same as the point B_4 velocity:

$$\mathbf{v}_D = \mathbf{v}_{B_4} = 0.25\hat{\mathbf{j}} = 0.25\,\text{cm/s}\,\angle 90^\circ$$

Once all the velocities have been determined, they can be represented in a velocity polygon (Fig. 2.55b).

3. To calculate the acceleration of point C, we start by defining Eq. (2.207), which relates the accelerations of point A_5 on link 5 with A_6 on link 6.

$$\mathbf{a}_{A_5}^n + \mathbf{a}_{A_5}^t = \mathbf{a}_{A_6}^n + \mathbf{a}_{A_6}^t + \mathbf{a}_{A_5A_6}^n + \mathbf{a}_{A_5A_6}^t + \mathbf{a}_{A_5A_6}^c \tag{2.207}$$

The values of the acceleration components Eq. (2.208) in Eq. (2.207) are:

$$\mathbf{a}_{A_5}^n = \mathbf{a}_{A_2}^n = \boldsymbol{\omega}_2 \wedge \mathbf{v}_{A_5} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 1 \\ -0.5 & 0.866 & 0 \end{vmatrix} = -0.866\hat{\mathbf{i}} - 0.5\hat{\mathbf{j}} \tag{2.208}$$

$$\begin{aligned} \mathbf{a}_{A_5}^t &= \mathbf{a}_{A_2}^t = 0 \\ \mathbf{a}_{A_6}^n &= 0 \\ \mathbf{a}_{A_6}^t &= a_{A_6}^t \hat{\mathbf{i}} \\ \mathbf{a}_{A_5A_6}^n &= 0 \\ \mathbf{a}_{A_5A_6}^t &= a_{A_5A_6}^t \hat{\mathbf{j}} \\ \mathbf{a}_{A_5A_6}^c &= 0 \end{aligned}$$

Introducing these values in Eq. (2.207), we obtain Eq. (2.209):

$$-0.866\hat{\mathbf{i}} - 0.5\hat{\mathbf{j}} = a_{A_6}^t \hat{\mathbf{i}} + a_{A_5A_6}^t \hat{\mathbf{j}} \tag{2.209}$$

Clearing the components, we obtain the values of the accelerations:

$$\begin{aligned} \mathbf{a}_{A_6}^t &= -0.866\hat{\mathbf{i}} = 0.866\,\text{cm/s}^2\ \angle 180^\circ \\ \mathbf{a}_{A_5A_6}^t &= -0.5\hat{\mathbf{i}} = 0.5\,\text{cm/s}^2\ \angle 270^\circ \end{aligned}$$

Once more, as point C belongs to link 6, which moves with straight translational motion, the acceleration of point C is the same as the acceleration of point A_6:

$$\mathbf{a}_C = \mathbf{a}_{A_6} = -0.866\hat{\mathbf{i}} = 0.866\,\text{cm/s}^2\ \angle 180^\circ$$

Once the accelerations have been determined, they can be represented in an acceleration polygon (Fig. 2.56).

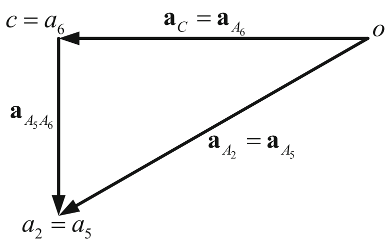


Fig. 2.56 Acceleration polygon of points A_2, A_5, A_6 and C in the mechanism shown in Fig. (2.55a)

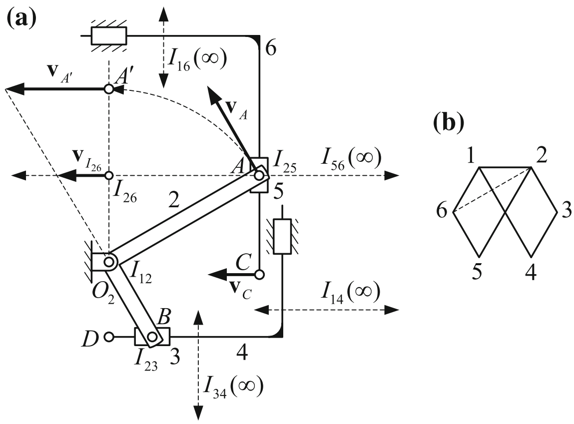


Fig. 2.57 **a** Calculation of the velocity of point C with the ICR method using velocity $\mathbf{v}_{A_2}$. **b** Polygon to analyze ICRs positions

4. To calculate the velocity of point C by means of the ICR method, we start by calculating the relative instantaneous centers of rotation of link 1 (frame), link 2 (the link point A belongs to) and link 6 (the link point C belongs to).

Once ICRs I_{12}, I_{16} and I_{26} have been obtained (Fig. 2.57a), we can calculate the velocity of I_{26}. At the considered instant, I_{26} is part of links 2 and 6 simultaneously. As a point on link 2, it rotates about O_2 and its velocity is Eq. (2.210):

$$v_A = \omega_2 \overline{I_{12}A}$$

$$v_{I_{26}} = \frac{v_A}{\overline{I_{12}A}} \overline{I_{12}I_{26}} = \frac{1}{1} 0.5 = 0.5\,\mathrm{cm/s} \tag{2.210}$$

Moreover, I_{26} also belongs to link 6, which moves with linear translational motion and all its points have the same velocity. Hence, $\mathbf{v}_C = \mathbf{v}_{I_{26}}$ (Fig. 2.57a).

Example 12 Figure 2.58 represents a Scotch Yoke mechanism used in the assembly line of a production chain. The lengths of the links are $\overline{O_2A} = 1$ m and $\overline{O_2B} = 1.2$ m. At the instant considered, angle $\theta_2 = 60°$ and angular velocity $\omega_2 = -1$ rad/s (clockwise). Knowing that link 2 moves with constant velocity, calculate:

1. The acceleration of point C.
2. The velocity of point C by using the ICR method when the velocity of point B is known.

1. First of all, we make use of the vector (Eq. 2.211) to calculate the velocities:

$$\mathbf{v}_{A_3} = \mathbf{v}_{A_4} + \mathbf{v}_{A_3A_4} \tag{2.211}$$

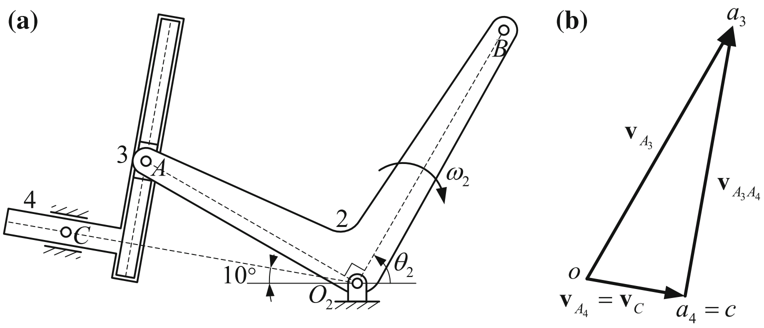


Fig. 2.58 **a** Scotch Yoke mechanism. **b** Velocity polygon

Next, we obtain the expressions of these components Eqs. (2.212)–(2.214). We start by calculating $\mathbf{v}_{A_3}$ (Eq. 2.212) knowing that $\overline{O_2A}$ forms an angle of $60° + 90° = 150°$ with the X-axis (Fig. 2.58a) and that $\omega_2 = -1$ rad/s clockwise.

$$\mathbf{v}_{A_3} = \mathbf{v}_{A_2} = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{AO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -1 \\ 1\cos 150° & 1\sin 150° & 0 \end{vmatrix} = 0.5\hat{\mathbf{i}} + 0.866\hat{\mathbf{j}} \tag{2.212}$$

$$\mathbf{v}_{A_3} = \mathbf{v}_{A_2} = 1\,\text{m/s}\,\angle 60°$$

The trajectory of link 4 forms an angle of $180° - 10° = 170°$ with the X-axis (Fig. 2.58a), thus:

$$\mathbf{v}_{A_4} = v_{A_4}\cos 170°\hat{\mathbf{i}} + v_{A_4}\sin 170°\hat{\mathbf{j}} = -0.985v_{A_4}\hat{\mathbf{i}} + 0.174v_{A_4}\hat{\mathbf{j}} \tag{2.213}$$

The motion of A_3 with respect to A_4 follows a straight line that forms an angle of $80°$ with the X-axis, consequently:

$$\mathbf{v}_{A_3A_4} = v_{A_3A_4}\cos 80°\hat{\mathbf{i}} + v_{A_3A_4}\sin 80°\hat{\mathbf{j}} = 0.174v_{A_3A_4}\hat{\mathbf{i}} + 0.985v_{A_3A_4}\hat{\mathbf{j}} \tag{2.214}$$

We plug these values into relative velocity (Eq. 2.211) and we obtain Eq. (2.215):

$$(0.5\hat{\mathbf{i}} + 0.866\hat{\mathbf{j}}) = (-0.985v_{A_4}\hat{\mathbf{i}} + 0.174v_{A_4}\hat{\mathbf{j}}) + (0.174v_{A_3A_4}\hat{\mathbf{i}} + 0.985v_{A_3A_4}\hat{\mathbf{j}}) \tag{2.215}$$

By separating the vector components, we find the system of algebraic equations (Eq. 2.216):

$$\left.\begin{array}{l} 0.5 = -0.985 v_{A_4} + 0.174 v_{A_3A_4} \\ 0.866 = 0.174 v_{A_4} + 0.985 v_{A_3A_4} \end{array}\right\} \tag{2.216}$$

In this system we can find the magnitude of the point A_4 velocity, $v_{A_4} = -0.34\,\mathrm{m/s}$, and the magnitude of the relative velocity, $v_{A_3A_4} = 0.94\,\mathrm{m/s}$. With these values, we can calculate the velocity vectors:

$$\mathbf{v}_{A_4} = 0.335\hat{\mathbf{i}} - 0.059\hat{\mathbf{j}} = 0.34\,\mathrm{m/s}\angle 350^\circ$$
$$\mathbf{v}_{A_3A_4} = 0.163\hat{\mathbf{i}} + 0.925\hat{\mathbf{j}} = 0.94\,\mathrm{m/s}\angle 80^\circ$$

Since all points on link 4 follow a straight path, the velocity of point C is the same as the one of point A_4:

$$\mathbf{v}_C = \mathbf{v}_{A_4} = 0.335\hat{\mathbf{i}} - 0.059\hat{\mathbf{j}} = 0.34\,\mathrm{m/s}\angle 350^\circ$$

Figure 2.58b shows the velocity polygon corresponding to these results.

To calculate the acceleration, we apply Eq. (2.217):

$$\mathbf{a}_{A_3} = \mathbf{a}_{A_4} + \mathbf{a}_{A_3A_4} \tag{2.217}$$

where the components of the accelerations (Eqs. 2.218–2.220) can be expressed as:

$$\mathbf{a}^n_{A_3} = \mathbf{a}^n_{A_2} = \boldsymbol{\omega}_2 \wedge \mathbf{v}_{A_3} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -1 \\ 0.5 & 0.866 & 0 \end{vmatrix} = 0.866\hat{\mathbf{i}} - 0.5\hat{\mathbf{j}} \tag{2.218}$$

$$\mathbf{a}^t_{A_3} = 0$$
$$\mathbf{a}^n_{A_4} = 0$$

$$\mathbf{a}_{A_4} = \mathbf{a}^t_{A_4} = a_{A_4}\cos 170^\circ\hat{\mathbf{i}} + a_{A_4}\sin 170^\circ\hat{\mathbf{j}} \tag{2.219}$$

$$\mathbf{a}_{A_3A_4} = \mathbf{a}^t_{A_3A_4} = a_{A_3A_4}\cos 80^\circ\hat{\mathbf{i}} + a_{A_3A_4}\sin 80^\circ\hat{\mathbf{j}} \tag{2.220}$$

We obtain Eq. (2.221) plugging these values into relative acceleration (Eq. 2.217):

$$(0.866\hat{\mathbf{i}} - 0.5\hat{\mathbf{j}}) = (-0.985 a_{A_4}\hat{\mathbf{i}} + 0.174 a_{A_4}\hat{\mathbf{j}}) + (0.174 a_{A_3A_4}\hat{\mathbf{i}} + 0.985 a_{A_3A_4}\hat{\mathbf{j}}) \tag{2.221}$$

Separating the vector components (Eq. 2.222):

$$\left.\begin{array}{l} 0.866 = -0.985 a_{A_4} + 0.174 a_{A_3A_4} \\ -0.5 = 0.174 a_{A_4} + 0.985 a_{A_3A_4} \end{array}\right\} \tag{2.222}$$

Finally, solving the algebraic equation system, we obtain:

$$\mathbf{a}_{A_4} = 0.926\hat{\mathbf{i}} - 0.163\hat{\mathbf{j}} = 0.94\,\mathrm{m/s^2}\,\angle 350^\circ$$

$$\mathbf{a}_{A_3A_4} = -0.059\hat{\mathbf{i}} - 0.335\hat{\mathbf{j}} = 0.34\,\mathrm{m/s^2}\,\angle 260^\circ$$

Since link 4 moves with translational motion without rotation, the acceleration of point C is the same as the one of point A_4:

$$\mathbf{a}_C = \mathbf{a}_{A_4} = 0.926\hat{\mathbf{i}} - 0.163\hat{\mathbf{j}} = 0.94\,\mathrm{m/s^2}\,\angle 350^\circ$$

We can see these results represented in the acceleration polygon in Fig. 2.59.

1. We now calculate the velocity of point C using the ICR method. Since the known velocity belongs to a point in link 2 (point B) and the one we want to find corresponds to a point of link 4 (point C), the relative ICRs of links 1, 2 and 4 need to be determined. In other words, we need to determine ICRs I_{12}, I_{14} and I_{24}. Once the centers have been obtained (Fig. 2.60), we calculate the velocity of I_{24}. Point B rotates about point O_2 with the following velocity:

$$v_B = \omega_2 \overline{I_{12}B} = 1.2\,\mathrm{m/s}$$
$$v_{I_{24}} = \frac{v_B}{\overline{I_{12}B}}\overline{I_{12}I_{24}} = \frac{1.2}{1.2}\overline{O_2A}\sin 20^\circ = 0.34\,\mathrm{m/s}$$

Since all points in link 4 follow a straight path, instantaneous center I_{24} yields the velocity of any point of link 4. The velocity of point C is:

$$v_C = v_{I_{24}} = 0.34\,\mathrm{m/s}$$

Its direction is as shown in Fig. 2.60.

Example 13 Figure 2.61 represents a quick return mechanism that has the following dimensions: $\overline{O_2O_4} = 25\,\mathrm{mm}$, $\overline{O_2A} = 30\,\mathrm{mm}$, $\overline{AB} = 50\,\mathrm{mm}$, $\overline{O_4B} = \overline{O_4C} = 45\,\mathrm{mm}$ and $\overline{CD} = 50\,\mathrm{mm}$. The angle between sides $\overline{O_4B}$ and $\overline{O_4C}$ of the triangle

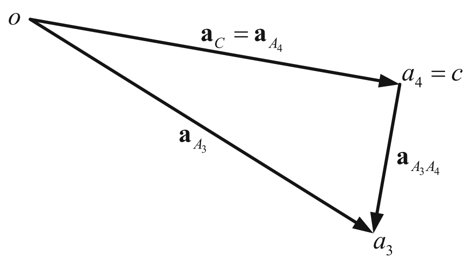


Fig. 2.59 Acceleration polygon of points A_3, A_4 and C of the Scotch Yoke mechanism in Fig. (2.58a)

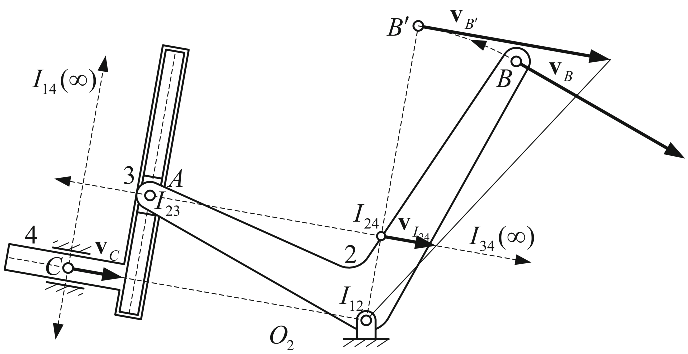
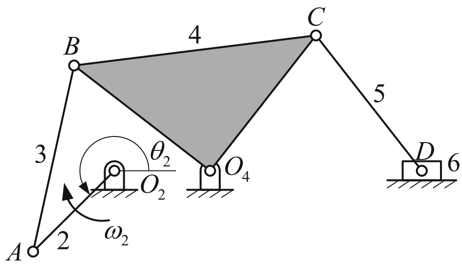


Fig. 2.60 Calculation of the velocity of point C with the ICR method using velocity $\mathbf{v}_B$

Fig. 2.61 Quick-return mechanism



$\triangle BO_4C$ is 90°. Calculate the velocity of point D of the slider when the position and velocity of the input link are $\theta_2 = 225°$ and $\omega_2 = 80\,\text{rpm}$ clockwise. Use the ICR method.

Consider that point D, of unknown velocity, is part of link 6 (it could also be considered part of link 5) and that point A, of known velocity, is part of link 2 (it is also part of link 3). We have to find the relative ICRs of links 1, 2 and 6, that is, centers I_{12}, I_{16} and I_{26}.

Figure 2.62 shows the ICRs and the auxiliary lines used to find them. Before starting the calculation, we have to convert the velocity of the input link from rpm into rad/s ($80\ \text{rpm} = 8.38\,\text{rad/s}$).

In order to obtain the velocity of point D graphically, we need to know the velocity of point A (Eq. 2.223):

$$v_A = \omega_2 \overline{I_{12}A} \tag{2.223}$$

Figure 2.62 shows how is $\mathbf{v}_{I_{26}}$ calculated graphically using $\mathbf{v}_A$. However, this velocity (Eq. 2.224) can also be obtained mathematically:

$$v_{26} = \omega_2 \overline{I_{12}I_{26}} \tag{2.224}$$

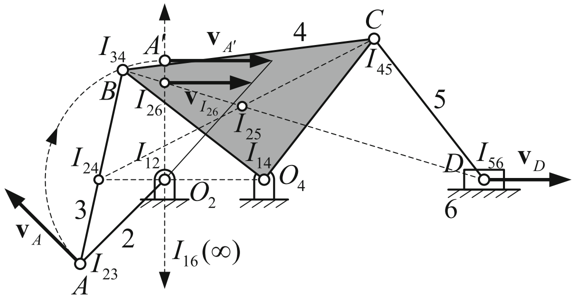


Fig. 2.62 Calculation of the velocity of point D with the ICR method knowing $\mathbf{v}_A$

Since link 6 is moving with non-angular velocity, all the points in this link have the same velocity:

$$v_D = v_{I_{26}} = \omega_2 \overline{I_{12}I_{26}} = 211.8\,\text{mm/s} \tag{2.225}$$

Its direction is shown in Fig. 2.62.

Example 14 The mechanism shown in Fig. 2.63 represents the second inversion of a slider-crank mechanism where the slider follows a circular arc trajectory with radius $R = 24.4\,\text{cm}$. The mechanism has the following dimensions: $\overline{O_2P} = 10\,\text{cm}$ and $\overline{O_2O_4} = 14.85\,\text{cm}$. Consider that link 2 rotates clockwise with a constant velocity of $\omega_2 = 12$ rad/s clockwise and that $\theta_2 = 180°$ at the instant considered. Find the velocity and acceleration of link 4. Point Q of link 4 and point P of link 2 are superposed.

We start by solving the position problem using trigonometry (Eqs. 2.226–2.229). We apply the law of cosines to triangle $\triangle O_4O_2P$:

$$\overline{O_4P} = \sqrt{\overline{O_2O_4}^2 + \overline{O_2P}^2 - 2\,\overline{O_2O_4}\,\overline{O_2P}\cos 90°} = 17.9\,\text{cm} \tag{2.226}$$

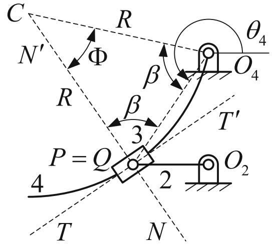


Fig. 2.63 Inverted slider-crank mechanism with the slider following a curved path. Point P on link 2 is coincident with point Q on link 4 at the studied instant

We also know that:

$$\begin{aligned} \overline{O_4P}\cos(\theta_4 - 180^\circ) &= \overline{O_2P} \\ \theta_4 = 180^\circ + \arccos\frac{10}{17.9} &= 236.06^\circ \end{aligned} \tag{2.227}$$

Applying the law of cosines to triangle $\triangle O_4PC$:

$$\begin{aligned} \overline{O_4P}^2 &= R^2 + R^2 - 2R^2\cos\Phi \\ \Phi &= \arccos\frac{2R^2 - \overline{O_4P}^2}{2R^2} = 43.04^\circ \end{aligned} \tag{2.228}$$

$$2\beta + \Phi = 180^\circ \Rightarrow \beta = \frac{180^\circ - \Phi}{2} = 68.48^\circ \tag{2.229}$$

In Fig. 2.63 we can observe that the normal direction to link 4 on point Q is NN'. Its angle is defined by $\theta_{NN'} = \theta_4 - 180^\circ + \beta = 124.54^\circ$. The tangential direction to link 4 on point Q is defined by TT' and its angle is $\theta_{TT'} = \theta_4 - 180^\circ + \beta - 90^\circ = 34.54^\circ$.

Applying the relative velocity method to points P and Q (Eq. 2.230):

$$\mathbf{v}_P = \mathbf{v}_Q + \mathbf{v}_{PQ} \tag{2.230}$$

The absolute velocity of point P (Eq. 2.231), the extreme point of link 2, is:

$$\mathbf{v}_P = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{PO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -12 \\ 10\cos 180^\circ & 10\cos 180^\circ & 0 \end{vmatrix} = 120\hat{\mathbf{j}} \tag{2.231}$$

$$\mathbf{v}_P = 120\,\text{cm/s}\,\angle 90^\circ$$

The direction of velocity $\mathbf{v}_{PQ}$ is tangential to the trajectory followed by link 3 when it slides inside the guide rail of link 4 (Eq. 2.232). Nevertheless, its magnitude will be one of the unknowns of the problem.

$$\mathbf{v}_{PQ} = v_{PQ}\cos\theta_{TT'}\hat{\mathbf{i}} + v_{PQ}\sin\theta_{TT'}\hat{\mathbf{j}} \tag{2.232}$$

This relative velocity (Eq. 2.233) can also be expressed as the vector product of the angular velocity of radius R associated to slider movement and the vector that goes from the center of curvature C to point P:

$$\mathbf{v}_{PQ} = \boldsymbol{\omega}_R \wedge \mathbf{r}_{PC} \tag{2.233}$$

The velocity of point Q (Eq. 2.234) is:

$$\mathbf{v}_Q = \boldsymbol{\omega}_4 \wedge \mathbf{r}_{QO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_4 \\ 17.9\cos 236.06^{\circ} & 17.9\sin 236.06^{\circ} & 0 \end{vmatrix} = 14.85\omega_4\hat{\mathbf{i}} - 10\omega_4\hat{\mathbf{j}} \tag{2.234}$$

Substituting each vector in the relative velocity expression and separating its components, we obtain the following system of two equations and two unknowns (Eq. 2.235), ω_4 and v_{PQ}:

$$\left.\begin{array}{r} 0 = 14.85\omega_4 + 0.823v_{PQ} \\ 120 = -10\omega_4 + 0.567v_{PQ} \end{array}\right\} \tag{2.235}$$

$$\omega_4 = -5.94 \text{ rad/s}$$
$$v_{PQ} = 106.96\,\text{cm/s}$$

Hence, velocities $\mathbf{v}_Q$ and $\mathbf{v}_{PQ}$ are:

$$\mathbf{v}_Q = -88.21\hat{\mathbf{i}} + 59.4\hat{\mathbf{j}} = 106.33\,\text{cm/s}\,\angle 146.06^{\circ}$$

$$\mathbf{v}_{PQ} = 88.1\hat{\mathbf{i}} + 60.64\hat{\mathbf{j}} = 106.96\,\text{cm/s}\,\angle 34.54^{\circ}$$

Once the velocities have been obtained, the velocity polygon can be drawn (Fig. 2.64).

Once vector $\mathbf{v}_{PQ}$ is defined, we have to calculate ω_R in Eq. (2.236). This value will be needed to solve the acceleration problem.

$$\mathbf{v}_{PQ} = \boldsymbol{\omega}_R \wedge \mathbf{r}_{PC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_R \\ R\cos(\theta_{NN'} + 180^{\circ}) & R\sin(\theta_{NN'} + 180^{\circ}) & 0 \end{vmatrix} \tag{2.236}$$

$$88.1\hat{\mathbf{i}} + 60.64\hat{\mathbf{j}} = -R\omega_R \sin 304.54^{\circ}\hat{\mathbf{i}} + R\omega_R \cos 304.54^{\circ}\hat{\mathbf{j}}$$

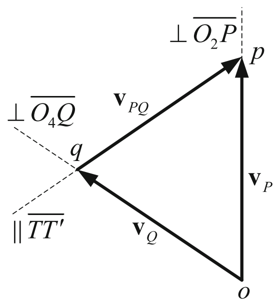


Fig. 2.64 Velocity polygon of the mechanism shown in Fig. (2.63)

Clearing, we obtain:

$$\omega_R = \frac{-88.1}{R \sin 304.54^\circ} = 4.38 \text{ rad/s}$$

Finally, we can study the acceleration problem. The angular velocity of link 2 is constant. Therefore, its angular acceleration (α_2) is null. Defining the vector expression of the relative acceleration (Eq. 2.237) between P and Q:

$$\mathbf{a}_P = \mathbf{a}_Q + \mathbf{a}_{PQ} \tag{2.237}$$

$$\mathbf{a}_P^n + \mathbf{a}_P^t = \mathbf{a}_Q^n + \mathbf{a}_Q^t + \mathbf{a}_{PQ}^n + \mathbf{a}_{PQ}^t + \mathbf{a}_{PQ}^c \tag{2.238}$$

And analyzing each vector in Eq. (2.238), we obtain Eqs. (2.239)–(2.245):

$$\mathbf{a}_P^n = \boldsymbol{\omega}_2 \wedge \mathbf{v}_P = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -12 \\ 0 & 120 & 0 \end{vmatrix} = 1440\hat{\mathbf{i}} \tag{2.239}$$

$$\mathbf{a}_P^t = \boldsymbol{\alpha}_2 \wedge \mathbf{r}_{PO_2} = 0 \tag{2.240}$$

$$\mathbf{a}_P = 1440\hat{\mathbf{i}} = 1440 \,\text{cm/s}^2 \angle 0^\circ$$

The normal acceleration of point Q (Eq. 2.241) is:

$$\mathbf{a}_Q^n = \boldsymbol{\omega}_4 \wedge \mathbf{v}_Q = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 5.94 \\ -88.21 & 59.4 & 0 \end{vmatrix} = 352.62\hat{\mathbf{i}} + 523.97\hat{\mathbf{j}} \tag{2.241}$$

$$\mathbf{a}_Q^n = 631.6 \,\text{cm/s}^2 \angle 56.06^\circ$$

The tangential acceleration of point Q (Eq. 2.242) is:

$$\mathbf{a}_Q^t = \boldsymbol{\alpha}_4 \wedge \mathbf{r}_{QO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_4 \\ 17.9 \cos 236.06^\circ & 17.9 \sin 236.06^\circ & 0 \end{vmatrix} = 14.85\alpha_4\hat{\mathbf{i}} - 10\alpha_4\hat{\mathbf{j}} \tag{2.242}$$

The normal component of the acceleration of point P relative to point Q is Eq. (2.243):

$$\mathbf{a}_{PQ}^n = \boldsymbol{\omega}_R \wedge \mathbf{v}_{PQ} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 4.38 \\ 88.1 & 60.64 & 0 \end{vmatrix} = -265.62\hat{\mathbf{i}} + 385.9\hat{\mathbf{j}} \tag{2.243}$$

$$\mathbf{a}_{PQ}^{n} = 468.1\,\text{cm/s}^2\ \angle 124.54^\circ$$

This vector is perpendicular to velocity $\mathbf{v}_{PQ}$ and it points towards the trajectory curvature center followed by link 3 when it slides along the link 4, that is to say, from P to C. The tangential component of the acceleration of P relative to Q (Eq. 2.244) is:

$$\begin{aligned}\mathbf{a}_{PQ}^{t} = \boldsymbol{\alpha}_R \wedge \mathbf{r}_{PC} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_R \\ 24.4\cos 304.54^\circ & 24.4\cos 304.54^\circ & 0 \end{vmatrix} \\ &= 20.1\alpha_R\hat{\mathbf{i}} + 13.83\alpha_R\hat{\mathbf{j}}\end{aligned} \tag{2.244}$$

And the Coriolis component of the acceleration of P relative to Q (Eq. 2.245) is:

$$\mathbf{a}_{PQ}^{c} = 2\boldsymbol{\omega}_4 \wedge \mathbf{v}_{PQ} = 2\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -5.94 \\ 88.1 & 60.64 & 0 \end{vmatrix} = 720.45\hat{\mathbf{i}} - 1046.7\hat{\mathbf{j}} \tag{2.245}$$

$$\mathbf{a}_{PQ}^{c} = 1270.7\,\text{cm/s}^2\ \angle 304.54^\circ$$

Plugging these values into Eq. (2.238) and separating its components, we obtain the system of two equations and two unknowns, α_R and α_4, Eq. (2.246).

$$\begin{aligned}(1440\hat{\mathbf{i}}) = {} & (352.62\hat{\mathbf{i}} + 523.97\hat{\mathbf{j}}) + (14.85\alpha_4\hat{\mathbf{i}} - 10\alpha_4\hat{\mathbf{j}}) \\ & + (-265.62\hat{\mathbf{i}} + 385.9\hat{\mathbf{j}}) + (20.1\alpha_R\hat{\mathbf{i}} + 13.83\alpha_R\hat{\mathbf{j}}) \\ & + (720.45\hat{\mathbf{i}} - 1046.7\hat{\mathbf{j}})\end{aligned}$$

$$\left.\begin{aligned}1440 &= 352.62 + 14.85\alpha_4 - 265.62 + 20.1\alpha_R + 720.45 \\ 0 &= 523.97 - 10\alpha_4 + 385.9 + 13.83\alpha_R - 1046.7\end{aligned}\right\} \tag{2.246}$$

where:

$$\begin{aligned}\alpha_4 &= 14.77\ \text{rad/s}^2 \\ \alpha_R &= 20.56\ \text{rad/s}^2\end{aligned}$$

With these values we can calculate the tangential components of the acceleration of point Q relative to O_4 and to P.

$$\begin{aligned}\mathbf{a}_{Q}^{t} &= 219.33\hat{\mathbf{i}} - 147.7\hat{\mathbf{j}} = 264.38\,\text{cm/s}^2\ \angle 326.06^\circ \\ \mathbf{a}_{PQ}^{t} &= 413.26\hat{\mathbf{i}} + 284.34\hat{\mathbf{j}} = 501.7\,\text{cm/s}^2\ \angle 34.54^\circ\end{aligned}$$

With these values, the acceleration polygon can be drawn (Fig. 2.65).

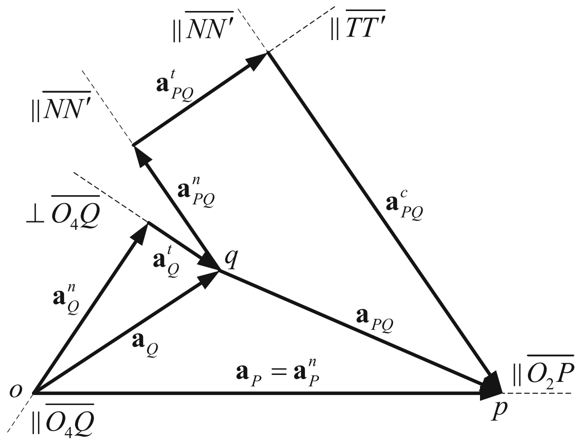


Fig. 2.65 Acceleration polygon of the mechanism shown in Fig. (2.63)

Example 15 Make a complete kinematic analysis of the shaft press mechanism in Fig. 2.66, provided that: $\theta_2 = 241^\circ$ and $\omega_2 = -10\,\mathrm{rad/s}$ clockwise constant. Piston 8 follows a trajectory along a vertical line that passes through O_6. The dimensions of the mechanism are: $\overline{O_2A} = 1.4\,\mathrm{cm}$, $\overline{AB} = 3.5\,\mathrm{cm}$, $\overline{O_4B} = 1.4\,\mathrm{cm}$, $\mathbf{r}_{O_2} = (0, 0)\,\mathrm{cm}$, $\mathbf{r}_{O_4} = (-3.1, -1.6)\,\mathrm{cm}$, $\overline{BC} = 3.7\,\mathrm{cm}$, $\overline{CD} = 5\,\mathrm{cm}$, $\overline{O_6D} = 1.2\,\mathrm{cm}$, $\mathbf{r}_{O_6} = (0, -2.8)\,\mathrm{cm}$, $\overline{O_6E} = 1.5\,\mathrm{cm}$, $\psi = 67.5^\circ$ and $\overline{EF} = 2.3\,\mathrm{cm}$. Use the relative velocity and acceleration methods.

Before we start to calculate velocities, it is necessary to solve the position problem. We will use the trigonometric method (Eqs. 2.247–2.266) developed in Appendix A for this purpose.

We start by studying the position of links 3 and 4 (Fig. 2.67).

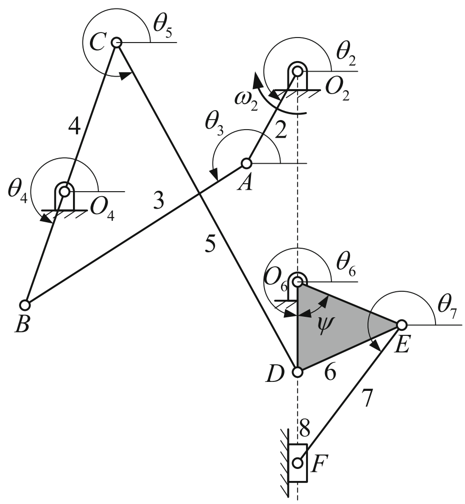
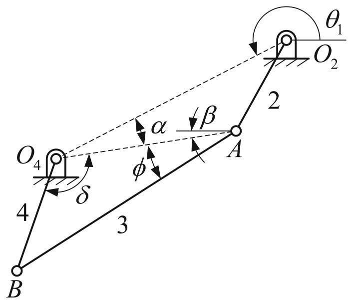


Fig. 2.66 Shaft press mechanism

Fig. 2.67 Calculation of the position of links 3 and 4 with the trigonometric method



$$\overline{O_2O_4} = \sqrt{x_{O_4}^2 + y_{O_4}^2} = \sqrt{3.1^2 + 1.6^2} = 3.49\,\text{cm} \tag{2.247}$$

$$\theta_1 = 180^\circ + \arctan\frac{y_{O_4}}{x_{O_4}} = 207.3^\circ \tag{2.248}$$

The application of the law of cosines to triangle $\triangle O_2O_4A$ yields:

$$\overline{O_4A} = \sqrt{\overline{O_2O_4}^2 + \overline{O_2A}^2 - 2\,\overline{O_2O_4}\,\overline{O_2A}\cos(\theta_2 - \theta_1)} = 2.54\,\text{cm} \tag{2.249}$$

We also apply the law of sines to the same triangle:

$$\overline{O_2A}\sin(\theta_2 - \theta_1) = \overline{O_4A}\sin\alpha \tag{2.250}$$

$$\alpha = \arcsin\frac{1.4\sin 33.7^\circ}{2.45} = 18.47^\circ$$

Hence:

$$\beta = \theta_1 - 180^\circ - \alpha = 27.3^\circ - 18.47^\circ = 8.83^\circ \tag{2.251}$$

Next, we apply the law of cosines to triangle $\triangle O_4AB$:

$$\overline{O_4B}^2 = \sqrt{\overline{O_4A}^2 + \overline{AB}^2 - 2\,\overline{O_4A}\,\overline{AB}\cos\phi} \tag{2.252}$$

$$\phi = \arccos\frac{2.45^2 + 3.5^2 - 1.6^2}{2\cdot 2.45\cdot 3.5} = 23.81^\circ$$

Thus:

$$\theta_3 = 180^\circ + \beta + \phi = 180^\circ + 8.83^\circ + 23.81^\circ = 212.64^\circ \tag{2.253}$$

Finally, we apply the law of sines to triangle $\triangle O_4AB$:

$$\overline{O_4B}\sin(180^\circ - \delta) = \overline{AB}\sin\phi \tag{2.254}$$

$$\delta = 180^\circ - \arcsin\frac{3.2\sin 23.81^\circ}{1.6} = 117.98^\circ$$

This yields:

$$\theta_4 = 360^\circ + \beta - \delta = 360^\circ + 8.83^\circ - 117.98 = 250.85^\circ \tag{2.255}$$

Next, we study links 5 and 6 (Fig. 2.68).

$$\overline{O_4O_6} = \sqrt{\left(x_{O_4} - x_{O_6}\right)^2 + \left(y_{O_4} - y_{O_6}\right)^2} = \sqrt{3.1^2 + 1.2^2} = 3.32\,\text{cm} \tag{2.256}$$

$$\theta_1' = \arctan\frac{y_{O_4} - y_{O_6}}{x_{O_4} - x_{O_6}} = \arctan\frac{-1.2}{3.1} = 338.84^\circ \tag{2.257}$$

We apply the law of cosines to triangle $\triangle O_4O_6C$:

$$\overline{O_6C} = \sqrt{\overline{O_4C}^2 + \overline{O_4O_6}^2 - 2\,\overline{O_4O_6}\;\overline{O_4C}\cos(\theta_4 + 180^\circ - \theta_{1'})} = 3.99\,\text{cm} \tag{2.258}$$

We also apply the law of sines to the same triangle:

$$\overline{O_4C}\sin(\theta_4 + 180^\circ - \theta_{1'}) = \overline{O_6C}\sin\alpha \tag{2.259}$$

$$\alpha = \arcsin\frac{2.1\sin 92.01^\circ}{3.99} = 31.735^\circ$$

$$\beta = \alpha + 360^\circ - \theta_{1'} = 31.735^\circ + 360^\circ - 338.84^\circ = 52.895^\circ \tag{2.260}$$

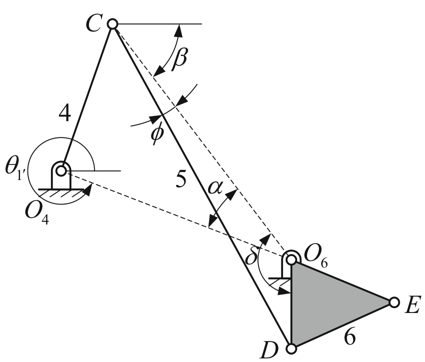


Fig. 2.68 Variables defined to calculate the position of links 5 and 6 with the trigonometric method

Next, we apply the law of cosines to triangle $\triangle O_6CD$:

$$\overline{O_6D}^2 = \overline{O_6C}^2 + \overline{CD}^2 - 2\,\overline{O_6C}\;\overline{CD}\cos\phi \tag{2.261}$$

$$\phi = \arccos\frac{3.99^2 + 5^2 - 1.2^2}{2\cdot 3.99\cdot 5} = 8.32^\circ$$

Thus:

$$\theta_5 = 360^\circ - \beta - \phi = 360^\circ - 52.895^\circ - 8.32^\circ = 298.8^\circ \tag{2.262}$$

The application of the law of sines on triangle $\triangle O_6CD$ yields:

$$\overline{O_6D}\sin(180^\circ - \delta) = \overline{CD}\sin\phi \tag{2.263}$$

$$\delta = 180^\circ - \arcsin\frac{5\sin 8.32^\circ}{1.2} = 142.92^\circ$$

Thus:

$$\theta_6 = 180^\circ - \beta + \delta = 180^\circ - 52.895^\circ + 142.92^\circ = 270^\circ$$

Finally, we will solve the position problem for links 7 and 8 (Fig. 2.69).

$$\overline{O_6E}\sin\psi = \overline{EF}\sin\mu \tag{2.264}$$

$$\mu = \arcsin\frac{1.5\sin 67.5^\circ}{2.3} = 37.05^\circ$$

Hence:

$$\theta_7 = 270^\circ - \mu = 232.95^\circ \tag{2.265}$$

Projecting $\overline{O_6E}$ and $\overline{EF}$ on the vertical axis, we obtain:

$$\overline{O_6E}\cos\psi + \overline{EF}\cos\mu = \overline{FO_{6y}} \tag{2.266}$$

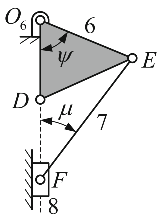


Fig. 2.69 Variables defined to calculate the position of links 7 and 8 using the trigonometric method

$$\overline{FO_6}_y = 1.5\cos 67.5^\circ + 2.3\cos 37.05^\circ = 2.41\,\text{cm}$$

The position of point F with respect to O_2 will be given by Eq. (2.267):

$$\overline{FO_2}_y = \overline{FO_6}_y + \overline{O_6O_2}_y = -2.41 - 2.8 = -5.21\,\text{cm} \tag{2.267}$$

Once the position of each link has been obtained, we can solve the velocity problem. We start by analyzing the velocities of points A and B. The velocity of point A (Eq. 2.268), which is the end of the crank (link 2), is:

$$\mathbf{v}_A = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{AO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_2 \\ \overline{O_2A}\cos\theta_2 & \overline{O_2A}\sin\theta_2 & 0 \end{vmatrix} = 12.24\hat{\mathbf{i}} - 6.79\hat{\mathbf{j}} \tag{2.268}$$

$$\mathbf{v}_A = 14\,\text{cm/s}\,\angle 331^\circ$$

Taking into account that point A is also part of link 3, we can calculate the velocity of any other point (point B) on the same link by means of the relative velocity (Eq. 2.269):

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{BA} \tag{2.269}$$

The velocity of point B of link 4 (Eq. 2.270) is:

$$\mathbf{v}_B = \boldsymbol{\omega}_4 \wedge \mathbf{r}_{BO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_4 \\ \overline{O_4B}\cos\theta_4 & \overline{O_4B}\sin\theta_4 & 0 \end{vmatrix} = 1.511\omega_4\hat{\mathbf{i}} - 0.525\omega_4\hat{\mathbf{j}} \tag{2.270}$$

And the velocity of point B relative to point A (Eq. 2.271) is:

$$\mathbf{v}_{BA} = \boldsymbol{\omega}_3 \wedge \mathbf{r}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_3 \\ \overline{AB}\cos\theta_3 & \overline{AB}\sin\theta_3 & 0 \end{vmatrix} = 1.888\omega_3\hat{\mathbf{i}} - 2.947\omega_3\hat{\mathbf{j}} \tag{2.271}$$

Substituting these values in the relative velocity (Eq. 2.269) and separating this equation into its components, we obtain Eq. (2.272):

$$(1.511\omega_4\hat{\mathbf{i}} - 0.525\omega_4\hat{\mathbf{j}}) = (12.24\hat{\mathbf{i}} - 6.79\hat{\mathbf{j}}) + (1.888\omega_3\hat{\mathbf{i}} - 2.947\omega_3\hat{\mathbf{j}})$$

$$\left.\begin{aligned} 1.511\omega_4 &= 12.24 + 1.888\omega_3 \\ -0.525\omega_4 &= -6.79 - 2.947\omega_3 \end{aligned}\right\} \tag{2.272}$$

The solution of Eq. (2.272) is:

$$\omega_3 = -1.11 \text{ rad/s}$$
$$\omega_4 = 6.71 \text{ rad/s}$$

Using the angular velocities of links 3 and 4 we can now calculate vectors $\mathbf{v}_B$, $\mathbf{v}_{BA}$ and $\mathbf{v}_C$ (Eq. 2.273):

$$\mathbf{v}_B = 10.17\hat{\mathbf{i}} - 3.55\hat{\mathbf{j}} = 10.77\,\text{cm/s}\,\angle 340.85^\circ$$
$$\mathbf{v}_{BA} = -2.1\hat{\mathbf{i}} + 3.27\hat{\mathbf{j}} = 3.885\,\text{cm/s}\,\angle 122.6^\circ$$

$$\begin{aligned}\mathbf{v}_C = \boldsymbol{\omega}_4 \wedge \mathbf{r}_{CO_4} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_4 \\ \overline{O_4C}\cos(\theta_4 + 180^\circ) & \overline{O_4C}\sin(\theta_4 + 180^\circ) & 0 \end{vmatrix} \\ &= -13.31\hat{\mathbf{i}} + 4.62\hat{\mathbf{j}} \end{aligned} \tag{2.273}$$

$$\mathbf{v}_C = 14.1\,\text{cm/s}\,\angle 160.85^\circ$$

We will continue the velocity analysis with points D and E. Assuming now that point C belongs to link 5, we can calculate the velocity of point D of the same link by means of the relative velocity (Eq. 2.274):

$$\mathbf{v}_D = \mathbf{v}_C + \mathbf{v}_{DC} \tag{2.274}$$

Hence, the velocity of point D of link 6 (Eq. 2.275) is:

$$\mathbf{v}_D = \boldsymbol{\omega}_6 \wedge \mathbf{r}_{DO_6} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_6 \\ \overline{O_6D}\cos\theta_6 & \overline{O_6D}\sin\theta_6 & 0 \end{vmatrix} = -1.2\omega_6\hat{\mathbf{i}} \tag{2.275}$$

The relative velocity of point D relative to point C (Eq. 2.276) can be expressed as:

$$\mathbf{v}_{DC} = \boldsymbol{\omega}_5 \wedge \mathbf{r}_{DC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_5 \\ \overline{CD}\cos\theta_5 & \overline{CD}\sin\theta_5 & 0 \end{vmatrix} = -4.382\omega_5\hat{\mathbf{i}} - 2.409\omega_5\hat{\mathbf{j}} \tag{2.276}$$

Plugging these values into the relative velocity (Eq. 2.274) and separating the components, we obtain Eq. (2.277):

$$(-1.2\omega_6\hat{\mathbf{i}}) = (-13.31\hat{\mathbf{i}} + 4.62\hat{\mathbf{j}}) + (-4.382\omega_5\hat{\mathbf{i}} - 2.409\omega_5\hat{\mathbf{j}})$$

$$\left.\begin{array}{l} -1.2\omega_6 = -13.31 - 4.382\omega_5 \\ 0 = 4.62 - 2.409\omega_5 \end{array}\right\} \tag{2.277}$$

where the values of the angular velocities can be obtained:

$$\omega_5 = -1.92 \text{ rad/s}$$
$$\omega_6 = -18.09 \text{ rad/s}$$

Therefore, the velocities of points D and E (Eq. 2.278) are:

$$\mathbf{v}_D = -21.7\hat{\mathbf{i}} = 21.71 \, \text{cm/s} \angle 180^\circ$$
$$\mathbf{v}_{DC} = -8.41\hat{\mathbf{i}} - 4.625\hat{\mathbf{j}} = 9.6 \, \text{cm/s} \angle 208.8^\circ$$

$$\begin{aligned} \mathbf{v}_E = \boldsymbol{\omega}_6 \wedge \mathbf{r}_{EO_6} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_6 \\ \overline{O_6E}\cos(\theta_6 + \psi) & \overline{O_6E}\sin(\theta_6 + \psi) & 0 \end{vmatrix} \\ &= -10.38\hat{\mathbf{i}} - 25.07\hat{\mathbf{j}} \end{aligned} \tag{2.278}$$

$$\mathbf{v}_E = 27.13 \, \text{cm/s} \angle 247.5^\circ$$

Finally, to calculate the velocity of point F we use Eq. (2.279):

$$\mathbf{v}_F = \mathbf{v}_E + \mathbf{v}_{FE} \tag{2.279}$$

The velocity of point F of link 8 is a vector that has the direction of the Y-axis since the displacement of the piston follows a vertical trajectory (Eq. 2.280). Therefore, the velocity of point F can be expressed as:

$$\mathbf{v}_F = v_F\hat{\mathbf{j}} \tag{2.280}$$

The velocity of F relative to E is given by Eq. (2.281):

$$\mathbf{v}_{FE} = \boldsymbol{\omega}_7 \wedge \mathbf{r}_{FE} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_7 \\ \overline{EF}\cos\theta_7 & \overline{EF}\sin\theta_7 & 0 \end{vmatrix} = 1.835\omega_7\hat{\mathbf{i}} - 1.385\omega_7\hat{\mathbf{j}} \tag{2.281}$$

Plugging these values into Eq. (2.279) and separating its components, we obtain Eq. (2.282):

$$(\mathrm{v}_F\hat{\mathbf{j}}) = (-10.38\hat{\mathbf{i}} - 25.07\hat{\mathbf{j}}) + (1.835\omega_7\hat{\mathbf{i}} - 1.385\omega_7\hat{\mathbf{j}})$$

$$\left.\begin{aligned} 0 &= -10.38 + 1.835\omega_7 \\ \mathrm{v}_F &= -25.07 - 1.385\omega_7 \end{aligned}\right\} \tag{2.282}$$

where the unknowns can easily be calculated.

$$\begin{aligned} \omega_7 &= 5.65\ \mathrm{rad/s} \\ v_F &= -32.91\ \mathrm{cm/s} \end{aligned}$$

With these values vectors $\mathbf{v}_F$ and $\mathbf{v}_{FE}$ can be completely defined:

$$\begin{aligned} \mathbf{v}_F &= -32.91\hat{\mathbf{j}} = 32.91\ \mathrm{cm/s}\ \angle 270^\circ \\ \mathbf{v}_{FE} &= 10.37\hat{\mathbf{i}} - 7.83\hat{\mathbf{j}} = 12.995\ \mathrm{cm/s}\ \angle 322.9^\circ \end{aligned}$$

Figure 2.70 shows the velocity polygon, which was constructed by drawing the velocity vectors to scale. All the absolute velocities were drawn starting from the same point, o, called the pole of velocities.

After solving the position and velocity problems, we can calculate the accelerations of the links of the mechanism. We will start by analyzing the acceleration of points A and B.

The acceleration of point A (Eq. 2.283) is:

$$\mathbf{a}_A^n = \boldsymbol{\omega}_2 \wedge \mathbf{v}_A = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 10 \\ 12.24 & -6.79 & 0 \end{vmatrix} = 67.9\hat{\mathbf{i}} + 122.4\hat{\mathbf{j}} \tag{2.283}$$

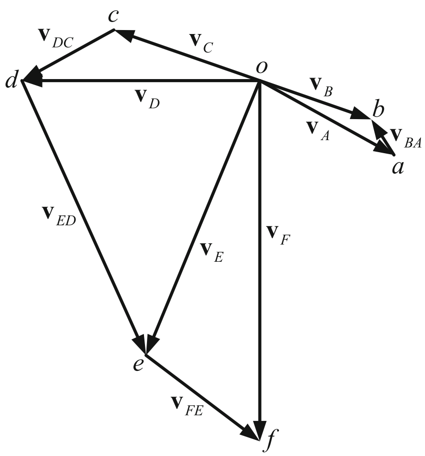


Fig. 2.70 Velocity polygon of the mechanism shown in Fig. 2.66

$$\mathbf{a}_A^t = 0$$

$$\mathbf{a}_A = \mathbf{a}_A^n = 67.9\hat{\mathbf{i}} + 122.4\hat{\mathbf{j}} = 140\,\text{cm/s}^2\ \angle 61^\circ$$

The relationship between the accelerations of points A and B (Eq. 2.284) is:

$$\mathbf{a}_B^n + \mathbf{a}_B^t = \mathbf{a}_A^n + \mathbf{a}_A^t + \mathbf{a}_{BA}^n + \mathbf{a}_{BA}^t \tag{2.284}$$

We continue by analyzing the rest of the vectors in Eq. (2.284). The normal component of the acceleration of point B (Eq. 2.285) is:

$$\mathbf{a}_B^n = \boldsymbol{\omega}_4 \wedge \mathbf{v}_B = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_4 \\ 10.17 & -3.55 & 0 \end{vmatrix} = 23.63\hat{\mathbf{i}} + 68.05\hat{\mathbf{j}} \tag{2.285}$$

$$\mathbf{a}_B^n = 72.04\,\text{cm/s}^2\ \angle 70.85^\circ$$

The tangential component of the acceleration of point B (Eq. 2.286) is:

$$\mathbf{a}_B^t = \boldsymbol{\alpha}_4 \wedge \mathbf{r}_{BO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_4 \\ \overline{BO_4}\cos\theta_4 & \overline{BO_4}\sin\theta_4 & 0 \end{vmatrix} = 1.511\alpha_4\hat{\mathbf{i}} - 0.5249\alpha_4\hat{\mathbf{j}} \tag{2.286}$$

The normal component of the acceleration of point B relative to point A (Eq. 2.287) is:

$$\mathbf{a}_{BA}^n = \boldsymbol{\omega}_3 \wedge \mathbf{v}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_3 \\ -2.1 & 3.27 & 0 \end{vmatrix} = 3.63\hat{\mathbf{i}} + 2.326\hat{\mathbf{j}} \tag{2.287}$$

$$\mathbf{a}_{BA}^n = 4.31\,\text{cm/s}^2\ \angle 32.64^\circ$$

Finally, the tangential component of the acceleration of B relative to A (Eq. 2.288) is:

$$\mathbf{a}_{BA}^t = \boldsymbol{\alpha}_3 \wedge \mathbf{r}_{BA} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_3 \\ \overline{AB}\cos\theta_3 & \overline{AB}\sin\theta_3 & 0 \end{vmatrix} = 1.888\alpha_3\hat{\mathbf{i}} - 2.947\alpha_3\hat{\mathbf{j}} \tag{2.288}$$

Plugging each vector into Eq. (2.284) and projecting its components onto the X-axis and Y-axis, we obtain a system with two unknowns, α_3 and α_4, Eq. (2.289).

$$
\begin{aligned}
&(23.63\hat{\mathbf{i}} + 68.05\hat{\mathbf{j}}) + (1.511\alpha_4\hat{\mathbf{i}} - 0.5249\alpha_4\hat{\mathbf{j}}) \\
&= (67.9\hat{\mathbf{i}} + 122.4\hat{\mathbf{j}}) + (3.63\hat{\mathbf{i}} + 2.326\hat{\mathbf{j}}) \\
&\quad + (1.888\alpha_3\hat{\mathbf{i}} - 2.947\alpha_3\hat{\mathbf{j}})
\end{aligned}
$$

$$
\left.\begin{aligned}
23.63 + 1.511\alpha_4 &= 67.9 + 3.63 + 1.888\alpha_3 \\
68.05 - 0.5249\alpha_4 &= 122.4 + 2.326 - 2.947\alpha_3
\end{aligned}\right\} \tag{2.289}
$$

Solving this system, we obtain Eq. (2.290):

$$
\left.\begin{aligned}
\alpha_3 &= 31.9 \text{ rad/s}^2 \\
\alpha_4 &= 71.48 \text{ rad/s}^2
\end{aligned}\right\} \tag{2.290}
$$

With these values we can calculate the absolute acceleration vectors of points B and C and the acceleration vector of point B with respect to point A (Eq. 2.291):

$$
\begin{aligned}
\mathbf{a}_B &= 131.67\hat{\mathbf{i}} + 30.53\hat{\mathbf{j}} = 135.16\,\text{cm/s}^2\,\angle 13.1^\circ \\
\mathbf{a}_{BA} &= 63.85\hat{\mathbf{i}} - 91.69\hat{\mathbf{j}} = 111.73\,\text{cm/s}^2\,\angle 304.85^\circ \\
\mathbf{a}_C^n &= \boldsymbol{\omega}_4 \wedge \mathbf{v}_C = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 6.71 \\ -13.31 & 4.62 & 0 \end{vmatrix} = -31\hat{\mathbf{i}} - 89.31\hat{\mathbf{j}} \\
\mathbf{a}_C^t &= \boldsymbol{\alpha}_4 \wedge \mathbf{r}_{CO_4} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 71.48 \\ 2.3\cos 70.85^\circ & 2.3\sin 70.85^\circ & 0 \end{vmatrix} = -15.53\hat{\mathbf{i}} + 53.61\hat{\mathbf{j}}
\end{aligned}
$$

$$
\mathbf{a}_C = \mathbf{a}_C^n + \mathbf{a}_C^t = -186.3\hat{\mathbf{i}} - 35.7\hat{\mathbf{j}} = 189.68\,\text{cm/s}^2\,\angle 183.33^\circ \tag{2.291}
$$

We will continue by studying the accelerations of points D and E. The relationship between the accelerations of points D and C is given by Eq. (2.292):

$$
\mathbf{a}_D = \mathbf{a}_D^n + \mathbf{a}_D^t = \mathbf{a}_C^n + \mathbf{a}_C^t + \mathbf{a}_{DC}^n + \mathbf{a}_{DC}^t \tag{2.292}
$$

The remaining vectors (Eqs. 2.293–2.295) in Eq. (2.292) are:

$$
\mathbf{a}_D^n = \boldsymbol{\omega}_6 \wedge \mathbf{v}_D = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -18.09 \\ -21.7 & 0 & 0 \end{vmatrix} = 392.7\hat{\mathbf{j}}
$$

$$
\mathbf{a}_D^t = \boldsymbol{\alpha}_6 \wedge \mathbf{r}_{DO_6} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_6 \\ 1.2\cos 270^\circ & 1.2\sin 270^\circ & 0 \end{vmatrix} = 1.2\alpha_6\hat{\mathbf{i}}
$$

$$\mathbf{a}_D = 1.2\alpha_6\hat{\mathbf{i}} + 392.7\hat{\mathbf{j}} \tag{2.293}$$

$$\mathbf{a}^t_{DC} = \boldsymbol{\alpha}_5 \wedge \mathbf{r}_{DC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_5 \\ 5\cos 298.8^\circ & 5\sin 298.8^\circ & 0 \end{vmatrix} = 4.38\alpha_5\hat{\mathbf{i}} + 2.41\alpha_5\hat{\mathbf{j}} \tag{2.294}$$

$$\mathbf{a}^n_{DC} = \boldsymbol{\omega}_5 \wedge \mathbf{v}_{DC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -1.92 \\ -8.41 & -4.625 & 0 \end{vmatrix} = -8.88\hat{\mathbf{i}} + 16.15\hat{\mathbf{j}} \tag{2.295}$$

Substituting these values in Eq. (2.292) and separating the resulting vectors into their components, we obtain Eq. (2.296):

$$\begin{aligned}(1.2\alpha_6\hat{\mathbf{i}} + 392.7\hat{\mathbf{j}}) = (&-172.82\hat{\mathbf{i}} - 40.08\hat{\mathbf{j}}) \\ &+ (4.38\alpha_5\hat{\mathbf{i}} + 2.41\alpha_5\hat{\mathbf{j}}) + (-8.88\hat{\mathbf{i}} + 16.15\hat{\mathbf{j}})\end{aligned}$$

$$\left.\begin{aligned} 1.2\alpha_6 &= -172.82 + 4.38\alpha_5 - 8.88 \\ 392.7 &= -40.08 + 2.41\alpha_5 + 16.15 \end{aligned}\right\} \tag{2.296}$$

Solving the system, we obtain Eq. (2.297):

$$\left.\begin{aligned} \alpha_5 &= 172.9\ \mathrm{rad/s^2} \\ \alpha_6 &= 479.6\ \mathrm{rad/s^2} \end{aligned}\right\} \tag{2.297}$$

The acceleration vectors of points D and E with respect to the frame and the acceleration vector of point D relative to point C (Eq. 2.298) are:

$$\mathbf{a}_D = 575.52\hat{\mathbf{i}} + 392.7\hat{\mathbf{j}} = 696.73\ \mathrm{cm/s^2}\ \angle 34.3^\circ$$

$$\mathbf{a}_{DC} = 748.42\hat{\mathbf{i}} + 432.84\hat{\mathbf{j}} = 864.57\ \mathrm{cm/s^2}\ \angle 30.04^\circ$$

$$\mathbf{a}^n_E = \boldsymbol{\omega}_6 \wedge \mathbf{v}_E = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -18.09 \\ -10.38 & -25.07 & 0 \end{vmatrix} = -453.52\hat{\mathbf{i}} + 187.77\hat{\mathbf{j}}$$

$$\mathbf{a}^t_E = \boldsymbol{\alpha}_6 \wedge \mathbf{r}_{EO_6} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 479.6 \\ 1.5\cos 337.5^\circ & 1.5\sin 337.5^\circ & 0 \end{vmatrix} = 275.29\hat{\mathbf{i}} + 664.73\hat{\mathbf{j}}$$

$$\mathbf{a}_E = \mathbf{a}^n_E + \mathbf{a}^t_E = -178.2\hat{\mathbf{i}} + 852.5\hat{\mathbf{j}} \tag{2.298}$$

$$\mathbf{a}_E = 870.93\,\mathrm{cm/s^2}\,\angle 101.8^\circ \tag{2.299}$$

Finally, we need to find the acceleration in the crank-shaft mechanism formed by links 6, 7 and 8. We define acceleration vectors for point E (Eq. 2.300) and F of link 7:

$$\mathbf{a}_F = \mathbf{a}_F^n + \mathbf{a}_F^t = \mathbf{a}_E^n + \mathbf{a}_E^t + \mathbf{a}_{FE}^n + \mathbf{a}_{FE}^t \tag{2.300}$$

where:

$$\mathbf{a}_F = \mathbf{a}_F^t = a_F\hat{\mathbf{j}} \tag{2.301}$$

$$\mathbf{a}_{FE}^n = \boldsymbol{\omega}_7 \wedge \mathbf{v}_{FE} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \\ & & 0 \end{vmatrix} = 44.24\hat{\mathbf{i}} + 58.6\hat{\mathbf{j}} \tag{2.302}$$

$$\mathbf{a}_{FE}^t = \boldsymbol{\alpha}_7 \wedge \mathbf{r}_{FE} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_7 \\ & & 0 \end{vmatrix} = 1.836\alpha_7\hat{\mathbf{i}} - 1.386\alpha_7\hat{\mathbf{j}} \tag{2.303}$$

Substituting the values of Eqs. (2.301)–(2.303) in the relative acceleration (Eq. 2.300) and breaking the resulting vector into its components, we obtain the equation system of two equations with two unknowns, α_7 and a_F, Eq. (2.304):

$$(a_F\hat{\mathbf{j}}) = (-178.2\hat{\mathbf{i}} + 852.5\hat{\mathbf{j}}) + (44.24\hat{\mathbf{i}} + 58.6\hat{\mathbf{j}}) + (1.836\alpha_7\hat{\mathbf{i}} - 1.386\alpha_7\hat{\mathbf{j}})$$

$$\left.\begin{aligned} 0 &= -178.2 + 44.24 + 1.836\alpha_7 \\ a_F &= 852.5 + 58.6 - 1.386\alpha_7 \end{aligned}\right\} \tag{2.304}$$

Solving the system we obtain:

$$\left.\begin{aligned} \alpha_7 &= 72.96\ \mathrm{rad/s^2} \\ a_F &= 810\,\mathrm{cm/s^2} \end{aligned}\right\}$$

The relative acceleration vector of point F with respect to point E is:

$$\mathbf{a}_{FE} = 178.19\hat{\mathbf{i}} - 42.52\hat{\mathbf{j}} = 183.19\,\mathrm{cm/s^2}\,\angle 346.58^\circ$$

Figure 2.71 represents the acceleration polygon built by drawing the absolute acceleration vectors, all of them starting at the pole of accelerations. Relative acceleration vectors are obtained by joining the extreme points of absolute acceleration vectors. Student are recommended to do this exercise in order to better understand vector directions in this problem.

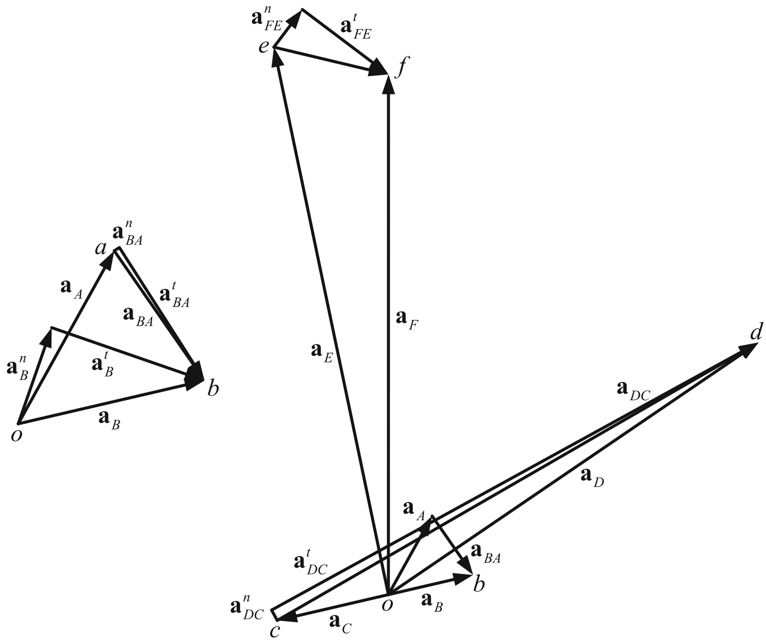


Fig. 2.71 Acceleration polygon of the mechanism shown in Fig. 2.66

Example 16 The mechanism in Fig. 2.72 is part of a calculating machine that carries out the "inverse" ($1/y$) arithmetic operation. Find the solution to the position, velocity and acceleration problems at the instant shown, knowing that the input is equal to $y = 1.87\,\text{cm}$ and that link 4 moves with a constant linear velocity of $0.5\,\text{cm/s}$ in an ascending direction.

We start by solving the position problem using the trigonometric method (Eq. 2.305). The expressions are:

$$\left.\begin{array}{l} \overline{O_2A}\sin\theta_2 = y \\ \overline{O_2A}\cos\theta_2 = \sqrt{3} \end{array}\right\} \tag{2.305}$$

Therefore:

$$\tan\theta_2 = \frac{y}{\sqrt{3}} \Rightarrow \theta_2 = \arctan\frac{1.87}{\sqrt{3}} = 47.19^\circ$$

$$\overline{O_2A} = \frac{y}{\sin\theta_2} = \frac{1.87}{\sin 47.19^\circ} = 2.55\,\text{cm}$$

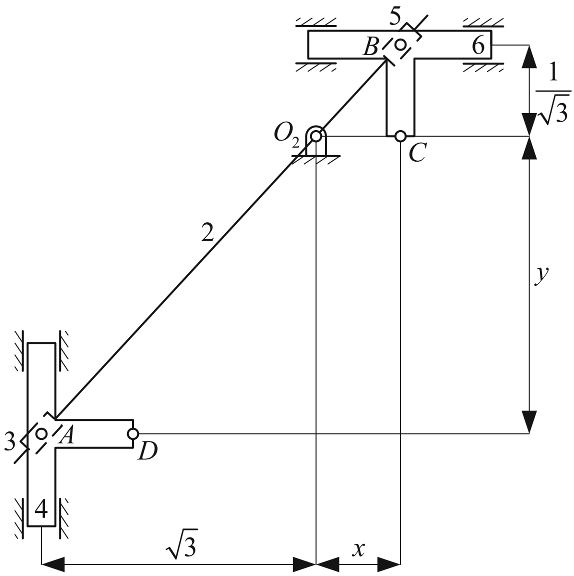


Fig. 2.72 Calculating machine that carries out the "inverse" ($x = 1/y$) arithmetic operation

We analyze the position of point B (Eq. 2.306):

$$\left.\begin{array}{r}\overline{O_2B}\sin\theta_2 = \frac{1}{\sqrt{3}}\\ \overline{O_2B}\cos\theta_2 = x\end{array}\right\} \tag{2.306}$$

From where we obtain:

$$\overline{O_2B} = \frac{1/\sqrt{3}}{\sin\theta_2} = \frac{1/\sqrt{3}}{\sin 47.19^\circ} = 0.79\,\text{cm}$$

So:

$$x = \overline{O_2B}\cos\theta_2 = 0.79\cos 47.19^\circ = 0.535\,\text{cm}$$

It can be verified that the value of y is always the inverse value of x.

Once the position of the links in the mechanism have been defined, we can solve the velocity problem.

Link 4 makes a translational motion and follows a vertical trajectory at a constant velocity of $0.5\,\text{cm/s}$. Therefore, the velocity of point A of link 4 is:

$$\mathbf{v}_A = 0.5\hat{\mathbf{j}}$$

Since point 4 is common to links 3 and 4, it can be denominated A_3. The expression of the relative velocity of the two coincident points of links 2 and 3 (Eq. 2.307) is:

$$\mathbf{v}_{A_3} = \mathbf{v}_{A_2} + \mathbf{v}_{A_3A_2} \tag{2.307}$$

The velocity of point A of link 2 (Eq. 2.308) is:

$$\mathbf{v}_{A_2} = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{AO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \omega_2 \\ 2.55\cos(\theta_2 + 180°) & 2.55\sin(\theta_2 + 180°) & 0 \end{vmatrix} \tag{2.308}$$
$$= 1.87\omega_2\hat{\mathbf{i}} - 1.73\omega_2\hat{\mathbf{j}}$$

The direction of the velocity of point A_3 with respect to point A_2 (Eq. 2.309) is defined by the direction in which link 3 slides over link 2:

$$\mathbf{v}_{A_3A_2} = v_{A_3A_2}\cos 47.19°\hat{\mathbf{i}} + v_{A_3A_2}\sin 47.19°\hat{\mathbf{j}} \tag{2.309}$$

Plugging these values into the relative velocity (Eq. 2.307) and separating the resulting vector into its components yields the system of two algebraic equations with two unknowns, ω_2 and $v_{A_3A_2}$, Eq. (2.310):

$$\left.\begin{array}{l} 0 = 1.87\omega_2 + 0.679v_{A_3A_2} \\ 0.5 = -1.73\omega_2 + 0.734v_{A_3A_2} \end{array}\right\} \tag{2.310}$$

Solving the system we obtain:

$$\left.\begin{array}{l} \omega_2 = -0.13 \text{ rad/s} \\ v_{A_3A_2} = 0.367\,\text{cm/s} \end{array}\right\}$$

With these values we can calculate vectors $\mathbf{v}_{A_2}$ and $\mathbf{v}_{A_3A_2}$:

$$\mathbf{v}_{A_2} = -0.2431\hat{\mathbf{i}} + 0.2249\hat{\mathbf{j}} = 0.331\,\text{cm/s}\,\angle 137.19°$$
$$\mathbf{v}_{A_3A_2} = 0.249\hat{\mathbf{i}} + 0.269\hat{\mathbf{j}} = 0.36\,\text{cm/s}\,\angle 47.19°$$

To find the velocity of links 5 and 6, we have to relate the velocities of the two coincident points at B (Eq. 2.311) (one of link 2 and another of link 5):

$$\mathbf{v}_{B_5} = \mathbf{v}_{B_2} + \mathbf{v}_{B_5B_2} \tag{2.311}$$

Since point B_5 also belongs to link 6 and all the points in this link share the same velocity with horizontal direction, we can assert that:

$$\mathbf{v}_{B_5} = v_{B_5}\hat{\mathbf{i}}$$

Therefore, the velocity of link 2 (Eq. 2.312) is:

$$\mathbf{v}_{B_2} = \boldsymbol{\omega}_2 \wedge \mathbf{r}_{BO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -0.13 \\ \overline{O_2B}\cos\theta_2 & \overline{O_2B}\sin\theta_2 & 0 \end{vmatrix} = 0.075\hat{\mathbf{i}} - 0.069\hat{\mathbf{j}} \tag{2.312}$$

$$\mathbf{v}_{B_2} = 0.102\,\mathrm{cm/s}\,\angle 317.19^\circ$$

The direction of vector $\mathbf{v}_{B_5B_2}$ is defined by the direction in which link 5 slides over link 2 (Eq. 2.313):

$$\mathbf{v}_{B_5B_2} = v_{B_5B_2}\cos\theta_2\hat{\mathbf{i}} + v_{B_5B_2}\sin\theta_2\hat{\mathbf{j}} = 0.679v_{B_5B_2}\hat{\mathbf{i}} + 0.733v_{B_5B_2}\hat{\mathbf{j}} \tag{2.313}$$

Substituting the values obtained in Eq. (2.311) and separating the vectors into their components, we obtain Eq. (2.314):

$$\left.\begin{array}{l} v_{B_5} = 0.075 + 0.679v_{B_5B_2} \\ 0 = -0.069 + 0.733v_{B_5B_2} \end{array}\right\} \tag{2.314}$$

From where we can calculate the magnitudes of $\mathbf{v}_{B_5}$ and $\mathbf{v}_{B_5B_2}$:

$$\left.\begin{array}{l} v_{B_5} = 0.139\,\mathrm{cm/s} \\ v_{B_5B_2} = 0.094\,\mathrm{cm/s} \end{array}\right\}$$

With these values, we can define the velocity vectors:

$$\mathbf{v}_{B_5} = 0.139\hat{\mathbf{i}} = 0.139\,\mathrm{cm/s}\,\angle 0^\circ$$

$$\mathbf{v}_{B_5B_2} = 0.0639\hat{\mathbf{i}} + 0.0688\hat{\mathbf{j}} = 0.094\,\mathrm{cm/s}\,\angle 47.19^\circ$$

Finally, we solve the acceleration problem. We have to take into account that the acceleration of link 4 is null. The relative acceleration (Eqs. 2.315 and 2.316) of the two coincident points, A_2 and A_3, is:

$$\mathbf{a}_{A_3} = \mathbf{a}_{A_2} + \mathbf{a}_{A_3A_2} \tag{2.315}$$

$$\mathbf{a}^n_{A_3} + \mathbf{a}^t_{A_3} = \mathbf{a}^n_{A_2} + \mathbf{a}^t_{A_2} + \mathbf{a}^n_{A_3A_2} + \mathbf{a}^t_{A_3A_2} + \mathbf{a}^c_{A_3A_2} \tag{2.316}$$

Since point A_3 also belongs to link 4:

$$\mathbf{a}_{A_3} = 0$$

The rest of the vectors in Eq. (2.316) are (Eqs. 2.317–2.321):

$$\mathbf{a}^n_{A_2} = \boldsymbol{\omega}_2 \wedge \mathbf{v}_{A_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -0.13 \\ -0.2431 & 0.2249 & 0 \end{vmatrix} = 0.029\hat{\mathbf{i}} + 0.031\hat{\mathbf{j}} \tag{2.317}$$

$$\mathbf{a}_{A_2} = 0.043\,\text{cm/s}^2\,\angle 47.19^\circ$$

$$\begin{aligned} \mathbf{a}^t_{A_2} &= \boldsymbol{\alpha}_2 \wedge \mathbf{r}_{AO_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \alpha_2 \\ 2.55\cos(\theta_2 + 180^\circ) & 2.55\sin(\theta_2 + 180^\circ) & 0 \end{vmatrix} \\ &= 1.87\alpha_2\hat{\mathbf{i}} - 1.73\alpha_2\hat{\mathbf{j}} \end{aligned} \tag{2.318}$$

The relative motion between points A_2 and A_3 follows a straight-line trajectory along link 2. Therefore, the normal component of relative acceleration $\mathbf{a}^n_{A_3A_2}$ is zero and the direction of the tangential component is defined by link 2:

$$\mathbf{a}^n_{A_3A_2} = 0 \tag{2.319}$$

$$\mathbf{a}^t_{A_3A_2} = a_{A_3A_2}\cos\theta_2\hat{\mathbf{i}} + a_{A_3A_2}\sin\theta_2\hat{\mathbf{j}} \tag{2.320}$$

The Coriolis component of the acceleration is given by:

$$\mathbf{a}^c_{A_3A_2} = 2\boldsymbol{\omega}_2 \wedge \mathbf{v}_{A_3A_2} = 2\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -0.13 \\ 0.249 & 0.269 & 0 \end{vmatrix} = 0.07\hat{\mathbf{i}} - 0.0647\hat{\mathbf{j}} \tag{2.321}$$

Substituting the values obtained in Eq. (2.316) and separating the resulting vectors into their components yields (Eq. 2.322):

$$\left.\begin{aligned} 0 &= 0.029 + 1.87\alpha_2 + 0.07 + 0.6796a_{A_3A_2} \\ 0 &= 0.031 - 1.73\alpha_2 - 0.0647 + 0.7336a_{A_3A_2} \end{aligned}\right\} \tag{2.322}$$

$$\left.\begin{aligned} \alpha_2 &= -0.0375\ \text{rad/s}^2 \\ a_{A_3A_2} &= -0.0425\ \text{cm/s}^2 \end{aligned}\right\}$$

In order to calculate the absolute acceleration of point B of link 5 (B_5), we use Eqs. (2.323) and (2.324):

$$\mathbf{a}_{B_5} = \mathbf{a}_{B_2} + \mathbf{a}_{B_5B_2} \tag{2.323}$$

$$\mathbf{a}^n_{B_5} + \mathbf{a}^t_{B_5} = \mathbf{a}^n_{B_2} + \mathbf{a}^t_{B_2} + \mathbf{a}^n_{B_5B_2} + \mathbf{a}^t_{B_5B_2} + \mathbf{a}^c_{B_5B_2} \tag{2.324}$$

We analyze the acceleration components in Eq. (2.324) starting with point B of link 5 (Eq. 2.325). Since point B_5 also belongs to link 6 and all the points in this link follow a horizontal trajectory:

$$\mathbf{a}^n_{B_5} = 0 \tag{2.325}$$

$$\mathbf{a}^t_{B_5} = a_B\hat{\mathbf{i}} \tag{2.326}$$

The acceleration components of point B of link 2 will be:

$$\mathbf{a}^n_{B_2} = \boldsymbol{\omega}_2 \wedge \mathbf{v}_{B_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -0.13 \\ 0.075 & -0.069 & 0 \end{vmatrix} = -0.0089\hat{\mathbf{i}} - 0.0097\hat{\mathbf{j}} \tag{2.327}$$

$$\mathbf{a}^n_{B_2} = 0.013\,\mathrm{cm/s^2}\,\angle 227.19^\circ$$

$$\begin{aligned}\mathbf{a}^t_{B_2} = \boldsymbol{\alpha}_2 \wedge \mathbf{r}_{BO_2} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -0.0375 \\ 0.79\cos 47.19^\circ & 0.79\sin 47.19^\circ & 0 \end{vmatrix} \\ &= 0.0217\hat{\mathbf{i}} - 0.02\hat{\mathbf{j}}\end{aligned} \tag{2.328}$$

$$\mathbf{a}^t_{B_2} = 0.0296\,\mathrm{cm/s^2}\,\angle 317.19^\circ$$

The relative motion of point B_5 with respect to link 2 follows a straight trajectory defined by link 2. Therefore, the normal component of the acceleration of point B_5 relative to point B_2 is zero and the direction of the tangential component is defined by link 2.

$$\mathbf{a}^n_{B_5B_2} = 0 \tag{2.329}$$

$$\mathbf{a}^t_{B_5B_2} = a^t_{B_5B_2}\cos\theta_2\hat{\mathbf{i}} + a^t_{B_5B_2}\sin\theta_2\hat{\mathbf{j}} \tag{2.330}$$

The Coriolis component of the acceleration of point B_5 with respect to point B_2 can be calculated with the following expression:

$$\mathbf{a}^c_{B_5B_2} = 2\boldsymbol{\omega}_2 \wedge \mathbf{v}_{B_5B_2} = 2\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & -0.13 \\ 0.0639 & 0.0688 & 0 \end{vmatrix} = 0.0178\hat{\mathbf{i}} - 0.0166\hat{\mathbf{j}} \tag{2.331}$$

$$\mathbf{a}^c_{B_5B_2} = 0.0243\,\mathrm{cm/s^2}\,\angle 317.19^\circ$$

Plugging the expression of these acceleration components Eqs. (2.325)–(2.331) into Eq. (2.324) and separating each vector into its x and y components, we obtain Eq. (2.332):

$$\left.\begin{array}{l} a_{B_5} = -0.0089 + 0.0217 + 0.0178 + 0.6796 a^t_{B_5 B_2} \\ 0 = -0.0097 - 0.02 - 0.0166 + 0.7336 a^t_{B_5 B_2} \end{array}\right\} \tag{2.332}$$

Solving the system, we find the unknowns:

$$\left.\begin{array}{l} a_{B_5} = 0.0734\,\text{cm/s}^2 \\ a^t_{B_5 B_2} = 0.0631\,\text{cm/s}^2 \end{array}\right\}$$

Hence, the acceleration of link 6 is given by:

$$\mathbf{a}_6 = \mathbf{a}_{B_5} = 0.0734\hat{\mathbf{i}} = 0.0734\,\text{cm/s}^2\ \angle 0^\circ$$

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Fundamentals of Machine Theory and Mechanisms
Simón Mata, A.; Bataller Torras, A.; Cabrera Carrillo, J.A.; Ezquerro Juanco, F.; Guerra Fernández, A.J.; Nadal Martínez, F.; Ortiz Fernández, A.
2016, XI, 409 p. 345 illus., Hardcover
ISBN: 978-3-319-31968-1