On the (1 + 3) Threading of Spacetime

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Dedicated to Jaime Muñoz Masqué on the occasion of his 65th birthday

Abstract We develop a $(1+3)$ threading formalism of the spacetime with respect to a non-normalized timelike vector field. It is worth mentioning that in our approach the spatial distribution is not necessarily integrable. Thus, this formalism is suitable for general Lorentz metrics from both the theory of black holes and perturbation theory. Also, the simple form of the $(1+3)$ decomposition of Einstein Field Equations stated in the paper, might have an important impact on the work of discovering new inhomogeneous cosmological models.

Keywords (1+3) Threading formalism · (1 +3) Decomposition of Einstein field equations · Riemannian spatial connection · Spatial tensor fields

1 Introduction

In cosmology, in order to relate the physics and geometry to the observations, it is frequently used the $(1+3)$ threading of spacetime. Namely, it is taken a unit 4-velocity field **u** which is tangent to a preferred congruence of world lines. Then, the study of both physics and geometry of the spacetime is developed by considering (provided they exist), orthogonal hypersurfaces to **u**. This was successfully applied to the study of the Friedmann–Lemaître–Robertson–Walker universe (cf. [4]). Also, the gravito-electromagnetism and the splitting of Einstein Field Equations (EFE)

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have been intensively studied (cf. [6, 8–10]). Two conditions have been imposed on the geometric objects of the spacetime:

(i) **u** must be a unit vector field.

(ii) The distribution that is orthogonal to the congruence determined by **u**, must be integrable.

Recently, we developed a new point of view on the $(1+3)$ threading of spacetime, where we removed both conditions above (cf. [1, 2]). In this general setting we obtained in a covariant form, the fully general $3D$ equations of motion and a $3D$ identity satisfied by the geodesics of a spacetime. Also, we applied this general method to the study of Kerr-Newman black holes.

The main purpose of this paper is to state the $(1+3)$ decomposition of EFE with respect to arbitrary timelike vector field and spatial distribution. The study is based on both the Riemannian spatial connection and the spatial tensor fields defined in [2]. It is worth mentioning that each group of the EFE given by (7.3) is invariant with respect to the transformations of coordinates on the spacetime.

Now, we outline the content of the paper. In Sect. 2 we introduce the kinematic quantities determined by a non-normalized timelike vector field ξ. Note that in [2] we put on Φ given by (2.2a) the condition that it is independent of time. This condition is satisfied by all stationary black holes (cf. [3, 5]). However, in perturbation theory (cf. [7, 11]), Φ is not, in general, independent of time. For the sake of general applications of our study, we remove the above condition on Φ. In Sect. 3 we present the Riemannian spatial connection and express the Levi-Civita connection of the spacetime (M, g) in terms of spatial tensor fields (cf. (3.4)). The local coefficients of the Riemannian spatial connection and the kinematic quantities are used in Sect. 4 to express the fully general equations of motion in (M, g). In particular, we obtain a geometric characterization of the spatial geodesics. In Sect. 5 we obtain the structure equations for the spatial distribution (cf. (5.1)), which lead us to the decomposition (6.6) of the Ricci tensor of (M, g). Also, in Sect. 6 we deduce the Raychaudhuri equation (6.8) for the $(1+3)$ threading formalism determined by ξ, and find the local components of the stress-energy-momentum tensor field with respect to the threading frame field (cf. (6.12)). Finally, we state the $(1+3)$ decomposition of the EFE (cf. (7.3)).

2 Kinematic Quantities in a Spacetime with Respect to a Non-Normalized Timelike Vector Field

Let (M, g) be a $4D$ spacetime, and ξ be a timelike vector field on M which is not necessarily normalized. Then, we have

$$TM = VM \oplus SM, \tag{2.1}$$

where VM is the *time distribution* spanned by ξ and SM is the *spatial distribution* that is complementary orthogonal to VM in TM.

Throughout the paper we use the ranges of indices: $i, j, k, \dots \in \{1, 2, 3\}$ and $a, b, c, \dots \in \{0, 1, 2, 3\}$. Also, for any vector bundle E over M, denote by $\Gamma(E)$ the $\mathscr{F}(M)$-module of smooth sections of E, where $\mathscr{F}(M)$ is the algebra of smooth functions on M.

Now, we consider a coordinate system (x^a) on M such that $\xi = \partial/\partial x^0$. Then, we put

$$\begin{aligned} &(a)\quad \xi_0 = g\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0}\right) = -\Phi^2, \quad (b)\quad \xi_i = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^0}\right), \\ &(c)\quad g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \end{aligned} \tag{2.2}$$

where Φ is a non-zero function that is globally defined on M. The decomposition (2.1) enables us to use the *threading frame* $\{\partial/\partial x^0, \delta/\delta x^i\}$ and the *threading coframe field* $\{\delta x^0, dx^i\}$, where we put (cf. [1, 2])

$$(a)\quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + \Phi^{-2}\xi_i \frac{\partial}{\partial x^0}, \quad (b)\quad \delta x^0 = dx^0 - \Phi^{-2}\xi_i dx^i. \tag{2.3}$$

The Lie brackets of the vector fields from the threading frame are expressed as follows:

$$(a)\quad \left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right] = 2\omega_{ij}\frac{\partial}{\partial x^0}, \quad (b)\quad \left[\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^i}\right] = a_i \frac{\partial}{\partial x^0}, \tag{2.4}$$

where we put

$$\begin{aligned} &(a)\quad \omega_{ij} = \Phi^{-2}\left\{c_i\xi_j - c_j\xi_i + \frac{1}{2}\left(\frac{\delta \xi_i}{\delta x^j} - \frac{\delta \xi_j}{\delta x^i}\right)\right\}, \\ &(b)\quad c_i = \Phi^{-1}\frac{\delta \Phi}{\delta x^i}, \qquad (c)\quad a_i = \Phi^{-2}\left\{\frac{\partial \xi_i}{\partial x^0} - 2\Psi\xi_0\right\}, \\ &(d)\quad \Psi = \Phi^{-1}\frac{\partial \Phi}{\partial x^0}. \end{aligned} \tag{2.5}$$

As SM is integrable if and only if $\omega_{ij} = 0$ for all $i, j \in \{1, 2, 3\}$, we say that $\{\omega_{ij}\}$ are the local components of the *vorticity tensor field* with respect to the threading frame.

Next, we denote by h_{ij} the local components of the Riemannian metric induced by g on SM with respect to the threading frame, and deduce that

$$h_{ij} = g\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = g_{ij} + \Phi^{-2}\xi_i\xi_j. \tag{2.6}$$

Thus the line element of the Lorentz metric g on M with respect to the threading coframe is expressed as follows:

$$ds^2 = -\Phi^2(\delta x^0)^2 + h_{ij}dx^i dx^j. \tag{2.7}$$

By using h_{ij} and the entries h^{ij} of the inverse of the 3×3 matrix $[h_{ij}]$, we define the expansion tensor field $\{\Theta_{ij}\}$, the expansion function Θ, and the shear tensor field $\{\sigma_{ij}\}$, as follows

$$(a)\ \ \Theta_{ij} = \frac{1}{2}\frac{\partial h_{ij}}{\partial x^0}, \quad (b) \quad \Theta = h^{ij}\Theta_{ij}, \quad (c) \quad \sigma_{ij} = \Theta_{ij} - \frac{1}{3}\Theta h_{ij}. \tag{2.8}$$

Raising and lowering indices $i, j, k, \ldots$ are done by using h^{ij} and h_{ij}, as for example:

$$(a) \quad \omega_j^{\ k} = h^{ki}\omega_{ij}, \quad (b) \quad \omega_{ij} = h_{ik}\omega_j^{\ k}, \quad (c) \quad \omega^{kl} = h^{ki}h^{lj}\omega_{ij}. \tag{2.9}$$

In earlier literature, spatial tensor fields have been introduced as projections on SM of the tensor fields on M (cf. [4, 6, 8]). In our approach, *a spatial tensor field* of type (p, q) is locally given by 3^{p+q} locally defined functions $T_{i\cdots}^{k\cdots}$, satisfying

$$T_{i\cdots}^{k\cdots}\frac{\partial \tilde{x}^h}{\partial x^k}\cdots = \tilde{T}_{j\cdots}^{h\cdots}\frac{\partial \tilde{x}^j}{\partial x^i}\cdots,$$

with respect to the coordinate transformations $\tilde{x}^a = \tilde{x}^a(x^0, x^i)$ on M. It is worth mentioning that $\{h_{ij}, \omega_{ij}, \Theta_{ij}, \sigma_{ij}\}$ and $\{a_i, c_i\}$ are spatial tensor fields of type $(0, 2)$ and $(0, 1)$, respectively.

3 The Riemannian Spatial Connection on a Spacetime

Let ∇ be the Levi-Civita connection on the spacetime (M, g). Then the *Riemannian spatial connection* on M is a metric linear connection $\nabla^\star$ on SM, given by

$$(a) \quad \nabla^\star_X sY = s\nabla_X sY, \quad \forall\ X, Y \in \Gamma(SM), \tag{3.1}$$

where s is the projection morphism of TM on SM. Locally, $\nabla^\star$ is given by

$$(a)\ \ \nabla^\star_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i} = \Gamma^{\star\ k}_{i\ j}\frac{\delta}{\delta x^k}, \quad (b) \quad \nabla^\star_{\frac{\partial}{\partial x^0}}\frac{\delta}{\delta x^i} = \Gamma^{\star\ k}_{i\ 0}\frac{\delta}{\delta x^k}, \tag{3.2}$$

where we put

$$\begin{aligned} &(a) \quad \Gamma^{\star\ k}_{i\ j} = \frac{1}{2} h^{kl} \left\{ \frac{\delta h_{lj}}{\delta x^i} + \frac{\delta h_{li}}{\delta x^j} - \frac{\delta h_{ij}}{\delta x^l} \right\}, \\ &(b) \quad \Gamma^{\star\ k}_{i\ 0} = \Theta^k_i + \Phi^2 \omega^k_i. \end{aligned} \tag{3.3}$$

Remark 3.1 The Riemannian spatial connection $\nabla^\star$ is different from the *three-dimensional operator* $\bar{\nabla}$ that has been used in earlier literature (cf. (4.19) of [4]). Note that $\nabla^\star$ is a metric linear connection on SM, and therefore defines covariant derivatives of spatial tensor fields with respect to vector fields on M. On the contrary, $\bar{\nabla}$ is an operator which acts on tensor fields on M, but it does not define a linear connection on M. □

Next, by using (3.1)–(3.3), we express the Levi-Civita connection on (M, g), as follows:

$$\begin{aligned} &(a) \quad \nabla_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} = \Gamma^{\star\ k}_{i\ j} \frac{\delta}{\delta x^k} + \left(\omega_{ij} + \Phi^{-2}\Theta_{ij}\right) \frac{\partial}{\partial x^0}, \\ &(b) \quad \nabla_{\frac{\partial}{\partial x^0}} \frac{\delta}{\delta x^i} = \left(\Theta^k_i + \Phi^2 \omega^k_i\right) \frac{\delta}{\delta x^k} + b_i \frac{\partial}{\partial x^0}, \\ &(c) \quad \nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial x^0} = \left(\Theta^k_i + \Phi^2 \omega^k_i\right) \frac{\delta}{\delta x^k} + c_i \frac{\partial}{\partial x^0}, \\ &(d) \quad \nabla_{\frac{\partial}{\partial x^0}} \frac{\partial}{\partial x^0} = \Phi^2 b^k \frac{\delta}{\delta x^k} + \Psi \frac{\partial}{\partial x^0}, \end{aligned} \tag{3.4}$$

where we put

$$b_i = a_i + c_i. \tag{3.5}$$

Denote by $R^\star$ the curvature tensor field of $\nabla^\star$ and put

$$\begin{aligned} &(a) \quad R^\star\left(\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^i}\right) = R^{\star j}_{i\ kh} \frac{\delta}{\delta x^j}, \\ &(b) \quad R\left(\frac{\delta}{\delta x^0}, \frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^i}\right) = R^{\star j}_{i\ k0} \frac{\delta}{\delta x^j}. \end{aligned}$$

Then by using (3.1)–(3.3), (2.4) and the well-known formula for $R^\star$, we obtain

$$\begin{aligned} &(a) \quad R^{\star j}_{i\ kh} = \frac{\delta \Gamma^{\star j}_{i\ k}}{\delta x^h} - \frac{\delta \Gamma^{\star j}_{i\ h}}{\delta x^k} + \Gamma^{\star l}_{i\,k} \Gamma^{\star j}_{l\,h} - \Gamma^{\star l}_{i\ h} \Gamma^{\star j}_{l\ k} - 2\omega_{kh} \Gamma^{\star j}_{i\ 0}, \\ &(b) \quad R^{\star j}_{i\ k0} = \frac{\partial \Gamma^{\star j}_{i\ k}}{\partial x^0} - \Gamma^{\star j}_{i\ 0|k} - a_k \Gamma^{\star j}_{i\ 0}. \end{aligned} \tag{3.6}$$

Here, and in the sequel, the vertical bar "|" represents covariant derivative with respect to the Riemannian spatial connection.

4 3*D* Equations of Motion in a 4*D* Spacetime

Let C be a smooth curve in M given by parametric equations

$$x^a = x^a(t), \quad a \in \{0, 1, 2, 3\}, \quad t \in [\alpha, \beta],$$

where (x^a) is the special coordinate system introduced by the $(1+3)$ threading of (M, g). The tangent vector field d/dt to C is expressed as follows:

$$\frac{d}{dt} = \frac{dx^i}{dt}\frac{\delta}{\delta x^i} + \frac{\delta x^0}{\delta t}\frac{\partial}{\partial x^0}, \tag{4.1}$$

where we put

$$\frac{\delta x^0}{\delta t} = \frac{dx^0}{dt} - \Phi^{-2}\xi_i\frac{dx^i}{dt}.$$

By direct calculations, using (4.1) and (3.4) we deduce that C is a geodesic of (M, g), if and only if,

$$\begin{aligned} &(a) \quad \frac{d^2x^k}{dt^2} + \Gamma^{\star}{}_{i}{}^{k}{}_{j}\frac{dx^i}{dt}\frac{dx^j}{dt} + 2\frac{\delta x^0}{\delta t}\frac{dx^i}{dt}(\Theta_i^k + \Phi^2\omega_i^k) + \Phi^2\left(\frac{\delta x^0}{\delta t}\right)^2 b^k = 0, \\ &(b) \quad \frac{d}{dt}\left(\frac{\delta x^0}{\delta t}\right) + \Phi^{-2}\Theta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{\delta x^0}{dt}(b_i + c_i)\frac{dx^i}{dt} + \left(\frac{\delta x^0}{dt}\right)^2\Psi = 0. \end{aligned} \tag{4.2}$$

We note that Eq. (4.2) represent the splitting of the fully general equations of motion of the spacetime. We call (4.2a) the 3*D* *equations of motion* in the 4*D* spacetime (M, g). It is worth mentioning that these equations are related to the equations of autoparallel curves of the Riemannian spatial connection. To show this we introduce a special class of geodesics in (M, g). A geodesic C of (M, g) is called a *spatial geodesic*, if it satisfies one of the following conditions:

$$(a) \quad \frac{\delta x^0}{\delta t} = 0 \quad \text{or} \quad (b) \quad \frac{d}{dt} = \frac{dx^i}{dt}\frac{\delta}{\delta x^i}. \tag{4.3}$$

Taking into account (4.2) and (4.3), we deduce that a curve C is a spatial geodesic, if and only if, (4.3) and the following equations are satisfied:

$$\begin{aligned} &(a) \quad \frac{d^2x^k}{dt^2} + \Gamma^{\star k}_{i\ j}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0, \\ &(b) \quad \Theta_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0. \end{aligned} \tag{4.4}$$

Now, we say that a curve C in M is autoparallel for the Riemannian spatial connection $\nabla^{\star}$, if it satisfies (4.3) and

$$\nabla^{\star}_{\frac{d}{dt}} \frac{d}{dt} = 0. \tag{4.5}$$

By using (4.3b) and (3.2a) into (4.5) we infer that C is an autoparallel for $\nabla^{\star}$, if and only if, (4.3a) and (4.4a) are satisfied. Now, from (3.4a) we see that

$$K_{ij} = \omega_{ij} + \Phi^{-2}\Theta_{ij}, \tag{4.6}$$

can be thought as local components of the second fundamental form of SM. Then we say that a curve C in M is an asymptotic line for SM if it satisfies (4.3) and the following equation

$$K_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0. \tag{4.7}$$

Taking into account that ω_{ij} define a skew-symmetric spatial tensor field, and using (4.6) into (4.7), we deduce that C is an asymptotic line for SM, if and only if, it satisfies (4.3) and (4.4b). Summing up these results, we can state the following:

A curve C in a spacetime (M, g) is a spatial geodesic, if and only if, the following conditions are satisfied:

(i) *C is autoparallel for the Riemannian spatial connection.*
(ii) *C is an asymptotic line for the spatial distribution.*

5 Structure Equations for the Spatial Distribution

In this section, R stands for both curvature tensor fields of types (0,4) and (1,3) of ∇, related by

$$R(X, Y, Z, U) = g(R(X, Y, U), Z).$$

Locally, R is completely determined by the following local components with respect to the threading frame field $\{\partial/\partial x^0, \delta/\delta x^i\}$:

$$R_{ijkh} = R\Big(\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\Big),$$

$$R_{i0kh} = R\Big(\frac{\delta}{\delta x^h}, \frac{\delta}{\delta x^k}, \frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^i}\Big),$$

$$R_{i0k0} = R\Big(\frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^k}, \frac{\partial}{\partial x^0}, \frac{\delta}{\delta x^i}\Big).$$

Then, by direct calculations, using (3.4) and (2.4), we deduce that

$$
\begin{aligned}
(a)\quad R_{ijkh} &= R^{\star}_{ijkh} + (\Theta_{ik} + \Phi^2\omega_{ik})(\omega_{jh} + \Phi^{-2}\Theta_{jh}) \\
&\quad - (\Theta_{ih} + \Phi^2\omega_{ih})(\omega_{jk} + \Phi^{-2}\Theta_{jk}), \\
(b)\quad R_{i0kh} &= \Theta_{ih|_k} - \Theta_{ik|_h} + \Theta_{ik}c_h - \Theta_{ih}c_k \\
&\quad + \Phi^2\left\{\omega_{ih|_k} - \omega_{ik|_h} + \omega_{ih}c_k - \omega_{ik}c_h + 2\omega_{kh}b_i\right\}, \\
(c)\quad R_{i0k0} &= \Phi^2\Big\{b_ib_k + \frac{1}{2}(b_{i|k} + b_{k|i})\Big\} + \Psi\Theta_{ik} - \Theta_{ik|0} \\
&\quad - \Theta_{il}\Theta^l_k - \Phi^4\omega_{il}\omega^l_k.
\end{aligned}
\tag{5.1}
$$

We call (5.1) the *structure equations* for the spatial distribution SM on the spacetime (M, g). Note that these equations are obtained in the most general spacetime, that is, SM is not necessarily an integrable distribution and ξ is not necessarily a unit vector field. Such general spacetimes are intensively studied in perturbation theory (cf. [7, 11]), and the theory of black holes [3, 5].

In particular, suppose that ξ is a unit vector field, that is, we have $\Phi^2 = 1$. Then by using (2.5) and (3.5), we deduce that

$$
\Psi = 0, \quad c_i = 0, \quad b_i = a_i = \frac{\partial \xi_i}{\partial x^0}, \quad \forall\, i \in \{1, 2, 3\}. \tag{5.2}
$$

Hence, in this particular case, the above structure equations become

$$
\begin{aligned}
(a)\quad R_{ijkh} &= R^{\star}_{ijkh} + (\Theta_{ik} + \omega_{ik})(\Theta_{jh} + \omega_{jh}) \\
&\quad - (\Theta_{ih} + \omega_{ih})(\Theta_{jk} + \omega_{jk}), \\
(b)\quad R_{i0kh} &= \omega_{ih|_k} - \omega_{ik|_h} + \Theta_{ih|_k} - \Theta_{ik|_h} + 2\omega_{kh}a_i, \\
(c)\quad R_{i0k0} &= a_ia_k + \frac{1}{2}(a_{i|_k} + a_{k|_i}) - \Theta_{ik|_0} - \Theta_{il}\Theta^l_k - \omega_{il}\omega^l_k.
\end{aligned}
\tag{5.3}
$$

If moreover, SM is an integrable distribution, that is, the vorticity tensor field vanishes identically on M, then (5.3) becomes

$$
\begin{aligned}
(a)\quad R_{ijkh} &= R^{\star}_{ijkh} + \Theta_{ik}\Theta_{jh} - \Theta_{ih}\Theta_{jk}, \\
(b)\quad R_{i0kh} &= \Theta_{ih|_k} - \Theta_{ik|_h}, \\
(c)\quad R_{i0k0} &= a_ia_k + \frac{1}{2}\left(a_{i|_k} + a_{k|_i}\right) - \Theta_{ik|_0} - \Theta_{il}\Theta^l_k.
\end{aligned}
\tag{5.4}
$$

Finally, note that (5.4) refers to a spacetime more general than the Friedmann–Lemaître–Robertson–Walker (FLRW) universe, where we have $a_i = 0$ and $\theta_{ij} = a(x^0)h_{ij}$.

6 Ricci Tensor and Stress-Energy-Momentum Tensor

The purpose of this section is to express both the Ricci tensor of (M, g) and the stress-energy-momentum tensor in terms of spatial tensor fields. First, we consider an orthonormal frame field $\{E_k\}$ in $\Gamma(SM)$:

$$E_k = E^i_k \frac{\delta}{\delta x^i}, \tag{6.1}$$

and obtain

$$h^{ij} = \sum_{k=1}^{3} E^i_k E^j_k. \tag{6.2}$$

The Ricci tensor Ric of (M, g) is given by (cf. [12], p. 87)

$$Ric(X, Y) = \sum_{k=1}^{3} R(E_k, X, E_k, Y) - \Phi^{-2} R\Big(\frac{\partial}{\partial x^0}, X, \frac{\partial}{\partial x^0}, Y\Big), \tag{6.3}$$

for all $X,\ Y \in \Gamma(TM)$. Then, by using (6.1)–(6.3), we obtain

$$\begin{aligned} &(a)\ \ R_{ik} = h^{jl} R_{ijkl} - \Phi^{-2} R_{i0k0}, \quad (b)\ \ R_{i0} = h^{jl} R_{j0li}, \\ &(c)\quad R_{00} = h^{jl} R_{j0l0}, \end{aligned} \tag{6.4}$$

where we put

$$\begin{aligned} &(a)\quad R_{ik} = Ric\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^k}\right), \quad (b)\quad R_{i0} = Ric\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^0}\right), \\ &(c)\quad R_{00} = Ric\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0}\right). \end{aligned} \tag{6.5}$$

Next, by direct calculations using (5.1) and (6.4), we deduce that the Ricci tensor of (M, g) is given by

$$\begin{aligned} &(a)\quad R_{ik} = R^{\star}_{ik} - b_i b_k - \frac{1}{2}\left(b_{i|_k} + b_{k|_i}\right) \\ &\qquad\qquad + \Phi^{-2}\left\{\Theta_{ik|_0} + (\Theta - \Psi)\Theta_{ik}\right\}, \\ &(b)\quad R_{i0} = \Theta^{j}_{i|_j} - \Theta_{|_i} + \Theta c_i - \Theta^j_i c_j \\ &\qquad\qquad + \Phi^2\left\{\omega^{j}_{i\ |_j} + \omega^j_i c_j + 2\omega^j_i b_j\right\}, \\ &(c)\quad R_{00} = -\Theta_{|_0} - \Theta_{ij}\Theta^{ij} + \Psi\Theta + \Phi^2\left\{b_j b^j + b^j_{\ |_j} + \Phi^2\omega^2\right\}, \end{aligned} \tag{6.6}$$

where we put

$$R^{\star}_{ik} = \frac{1}{2}\left(R^{\star l}_{i\ kl} + R^{\star l}_{k\ il}\right), \quad \Theta_{|i} = \frac{\delta\Theta}{\delta x^i}, \quad \Theta_{|0} = \frac{\partial\Theta}{\partial x^0}, \quad \omega^2 = \omega_{ij}\omega^{ij}.$$

Taking into account (2.8b) and (2.8c), we deduce that

$$\Theta_{ij}\Theta^{ij} = \sigma^2 + \frac{1}{3}\Theta^2, \tag{6.7}$$

where we put

$$\sigma^2 = \sigma_{ij}\sigma^{ij}.$$

Due to (6.7), we see that (6.6c) becomes

$$\Theta_{|_0} = -\sigma^2 - \frac{1}{2}\Theta^2 + \Psi\Theta + \Phi^2\left\{b_j b^j + b^j{}_{|_j} + \Phi^2\omega^2\right\} - R_{00}. \tag{6.8}$$

According to the usual terminology, we call (6.8) the *Raychaudhuri equation* for the $(1+3)$ threading formalism determined by the non-normalized timelike vector field $\xi = \partial/\partial x^0$.

Next, we express the local components of the stress-energy-momentum tensor T with respect to a threading frame field, in terms of the quantities measured by an observer moving with unit 4-velocity

$$\mathbf{u} = \Phi^{-1}\frac{\partial}{\partial x^0}.$$

First, we note that

$$\rho = T(\mathbf{u}, \mathbf{u}), \tag{6.9}$$

is the relativistic energy density measured by the observer. Then, we put:

$$\begin{aligned} &(a)\quad T_{ij} = T\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right), \quad (b)\quad T_{i0} = T\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^0}\right), \\ &(c)\quad T_{00} = T\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0}\right), \end{aligned} \tag{6.10}$$

and define

$$(a)\quad p = \frac{1}{3}T_{ij}h^{ij}, \quad (b)\quad q_i = -\Phi^{-1}T_{i0}, \quad (c)\quad \pi_{ij} = T_{ij} - ph_{ij}. \tag{6.11}$$

Note that q_i and π_{ij} define spatial tensor fields of types $(0, 1)$ and $(0, 2)$, respectively. Moreover, comparing with the quantities defined on [4, p. 92], it is easy to see that p is the relativistic pressure, while q_i and π_{ij} determine completely the relativistic momentum density and the relativistic anisotropic (trace-free) stress tensor field, respectively. Finally, we conclude that the local components of the stress-energy-momentum tensor field T with respect to the threading frame field $\{\partial/\partial x^0, \delta/\delta x^i\}$, are given by

$$(a) \quad T_{ij} = \pi_{ij} + ph_{ij}, \quad (b) \quad T_{i0} = -\Phi q_i, \quad (c) \quad T_{00} = \Phi^2 \rho. \tag{6.12}$$

7 The (1 + 3) Decomposition of Einstein Field Equations

Based on the (1 + 3) decomposition of both the Ricci tensor field and the stress-energy-momentum tensor field from the previous section, we express in a simple and elegant form the Einstein Field Equations (EFE).

Let the EFE given by (cf. [4, p. 65]):

$$Ric = 8\pi G(T - \frac{1}{2}\mathbf{T}g) + \Lambda g, \tag{7.1}$$

where G is the Newton constant, Λ is the cosmological constant, and $\mathbf{T}$ is the trace of T. Then, by applying the tensor fields to pairs of vector fields from the threading frame field $\{\partial/\partial x^0, \delta/\delta x^i\}$, and using (6.12), we deduce that (7.1) is equivalent to

$$\begin{aligned}
&(a) \quad R_{ik} = \{4\pi G(\rho - p) + \Lambda\}h_{ik} + 8\pi G\pi_{ik},\\
&(b) \quad R_{i0} = -8\pi G\Phi q_i,\\
&(c) \quad R_{00} = \Phi^2\{4\pi G(\rho + 3p) - \Lambda\}.
\end{aligned} \tag{7.2}$$

Finally, using (6.6a), (6.6b) and (6.8) into (7.2), we obtain

$$\begin{aligned}
&(a) \quad R^{\star}_{ik} - b_i b_k - \frac{1}{2}(b_{i|k} + b_{k|i}) + \Phi^{-2}\{\Theta_{ik|0} + (\Theta - \Psi)\Theta_{ik}\}\\
&\qquad\qquad - \{4\pi G(\rho - p) + \Lambda\}h_{ik} - 8\pi G\pi_{ik} = 0,\\
&(b) \quad \Theta^j_{i|j} - \Theta_{|i} + \Theta c_i - \Theta^j_i c_j + \Phi^2\{\omega^j_{i|j} + \omega^j_i c_j + 2\omega^j_i b_j\}\\
&\qquad\qquad + 8\pi G\Phi q_i = 0,\\
&(c) \quad \Theta_{|0} + \sigma^2 + \frac{1}{3}\Theta^2 - \Psi\Theta\\
&\qquad\qquad - \Phi^2\{b^j{}_{|j} + b^2 + \Phi^2\omega^2 + 4\pi G(\rho + 3p) - \Lambda\} = 0,
\end{aligned} \tag{7.3}$$

where we put

$$b^2 = b_j b^j.$$

In spite of the huge literature on the $(1+3)$ threading of spacetime (cf. [4]), the Eq. (7.3), as far as we know, are stated here for the first time. This is because these equations apply for any Lorentz metric regardless the threading vector field and the integrability of the spatial distribution. Most of the literature on this matter presented these equations in the case of a unit vector field ξ which is also hypersurface orthogonal. This particular case applies to the Friedmann–Lemaître–Robertson–Walker universe, but it fails in any attempt to study metrics given by (2.7), with $\Phi \neq 0$ and $\omega \neq 0$. Such metrics are specific to both the black holes theory (cf. [3, 5]) and perturbation theory (cf. [7, 11]) where the splitting of the EFE given by (7.3) can be easy handled into the study.

References

1. Bejancu, A.: A new point of view on $(1+3)$ threading of spacetime. Adv. Math.: Sci. J. **4**(1), 21–42 (2015)
2. Bejancu, A., Călin, C.: On the (1+3) threading of spacetime with respect to an arbitrary timelike vector field. Eur. Phys. J. C **75**, 159 (2015)
3. Chandrasekhar, S.: The Mathematical Theory of Black Holes. Clarendon Press, Oxford (1983)
4. Ellis, G.F.R., Maartens, R., MacCallum, M.A.H.: Relativistic Cosmology. Cambridge University Press, Cambridge (2012)
5. Frolov, V.P., Novikov, I.D.: Black Hole Physics. Basic Concepts and New Developments. Kluwer, Dordrecht (1998)
6. Jantzen, R.T., Carini, P., Bini, D.: The many faces of gravitoelectromagnetism. Ann. Phys.-New York **215**, 1–50 (1992)
7. Kodama, H., Sasaki, M.: Cosmological perturbation theory. Prog. Theor. Phys. Supp. **78**, 1–166 (1984)
8. Maartens, R.: Linearization instability of gravity waves? Phys. Rev. D **55**, 463–467 (1997)
9. Mashhoon, B., McClune, J.C., Quevedo, H.: On the gravitoelectromagnetic stress-energy tensor. Classical Quant. Grav. **16**, 1137–1149 (1999)
10. Massa, E.: Space Tensors in General Relativity III. General Relativ. Gravit. **11**, 715–736 (1974)
11. Mukhanov, V.F., Feldman, H.A., Brandenberger, R.H.: Theory of cosmological perturbations. Phys. Rep. **215**, 203–333 (1992)
12. O'Neill, B.: Semi-Riemannian Geometry and Applications to Relativity. Academic Press, New York (1983)

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