

## Chapter 2

# Prototype Examples of Complex Turbulent Dynamical Systems

Here we introduce three different prototype models of complex turbulent dynamical systems with the structure in (1.5)–(1.7). The first is the basic one layer geophysical model for the atmosphere or ocean with the effects of rotation, stratification, topography, and both deterministic and random forcing plus various dissipative mechanisms [124, 130, 171]; without geophysical effects this model reduces to the 2-dimensional Navier–Stokes equation but all these geophysical effects are a very rich source of new and important phenomena in the statistical dynamics far beyond ordinary 2-D flow [130, 171]. The second model is a 40-dimensional turbulent dynamical system due to Lorenz [98] which mimics weather waves of the mid-latitude atmosphere called the L-96 model. This qualitative model is an important test model for new strategies and algorithms for prediction, UQ, and state estimation, and is widely used for these purposes in the geoscience community [73, 99, 117, 118, 144, 159]. The third models discussed in some detail here are stochastic triad models [121–123] which are the elementary building blocks of complex turbulent systems with energy conserving nonlinear interactions like those in (1.5)–(1.7). All three examples will be used throughout the article. The chapter concludes with a list and brief discussion of some other important examples of complex turbulent dynamical systems.

### 2.1 Turbulent Dynamical Systems for Complex Geophysical Flows: One-Layer Model

Turbulence in idealized geophysical flows is a very rich and important topic with numerous phenomenological predictions and idealized numerical experiments. The anisotropic effects of explicit deterministic forcing, the  $\beta$ -effect due to the earth's curvature, and topography together with random forcing all combine to produce a

remarkable number of realistic phenomena; see the basic textbooks [130, 153, 171]. These include the formation of coherent jets and vortices, and direct and inverse turbulent cascades as parameters are varied [130, 153, 171]. It is well known that careful numerical experiments indicate interesting statistical bifurcations between jets and vortices as parameters vary [133, 135, 172, 161, 163, 167], and it is a contemporary challenge to explain these with approximate statistical theories [13, 45, 46, 163]. However, careful numerical experiments and statistical approximations are only possible or valid for large finite times so the ultimate statistical steady state of these turbulent geophysical flows remain elusive. Recently Majda and Tong [126] contribute to these issues by proving with full mathematical rigor that for any values of the deterministic forcing, the  $\beta$ -plane effect, and topography and with precise minimal stochastic forcing for any finite Galerkin truncation of the geophysical equations, there is a unique smooth invariant measure which attracts all statistical initial data at an exponential rate, that is geometric ergodicity. The rate constant depends on the geophysical parameters and could involve a large pre-constant.

Next we introduce the equations for geophysical flows which we consider in this article. Here we investigate geophysical flow on a periodic domain  $\mathbb{T}^2 = [-\pi, \pi]^2$ , with general dissipation,  $\beta$ -plane effect, stratification effect, topography, deterministic forcing and a minimal stochastic forcing. The model [130] is given by

$$\begin{aligned} \frac{dq}{dt} + \nabla^\perp \psi \cdot \nabla q &= D(\Delta) \psi + f(\mathbf{x}) + \dot{W}_t, \\ q &= \Delta \psi - F^2 \psi + h(\mathbf{x}) + \beta y. \end{aligned} \quad (2.1)$$

In the equation above:

- $q$  is the potential vorticity.  $\psi$  is the stream functions. It determines the vorticity by  $\omega = \Delta \psi$ , and the flow by  $u = \nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi)$ . Here  $\mathbf{x} = (x, y)$  denotes the spatial coordinate.
- The operator  $D(\Delta) \psi = \sum_{j=0}^l (-1)^j \gamma_j \Delta^j \psi$  stands for a general dissipation operator. We assume here  $\gamma_j \geq 0$  and at least one  $\gamma_j > 0$ . This term can include: (1) Newtonian (eddy) viscosity,  $\nu \Delta^2 \psi$ ; (2) Ekman drag dissipation,  $-d \Delta \psi$ ; (3) radiative damping,  $d \psi$ ; (4) hyperviscosity dissipation, which could be a higher-order power of  $\Delta$  and any positive combination of these. All versions are often utilized in these models in the above references.
- Here  $f(\mathbf{x})$  is the external deterministic forcing. The random forcing  $W_t$  is a Gaussian random field. Its spectral formulation will be given explicitly later.
- $\beta y$  is the  $\beta$ -plane approximation of the Coriolis effect and  $h(\mathbf{x})$  is the periodic topography.
- The constant  $F = L_R^{-1}$ , where  $L_R = \sqrt{g H_0 / f_0}$  is the Rossby radius which measures the relative strength of rotation to stratification [124].

Note if one considers for example the atmospheric wind stress on the ocean, the equation in (2.1) naturally has both deterministic and stochastic components to the forcing. The remarkable effects of topography and the  $\beta$ -effect on dynamics are discussed in detail in [130, 171]. The general mathematical framework of turbulent

dynamical systems will be shown later to apply to this model. If we ignore geophysical effects with  $F, \beta, h \equiv 0$  and use viscosity, (2.1) becomes the 2-D Navier–Stokes equations.

## 2.2 The L-96 Model as a Turbulent Dynamical System

The large dimensional turbulent dynamical systems studied here have fundamentally different statistical character than in more familiar low dimensional chaotic dynamical systems. The most well known low dimensional chaotic dynamical system is Lorenz’s famous three-equation model [97] which is weakly mixing with one unstable direction on an attractor with high symmetry. In contrast, as discussed earlier, realistic turbulent dynamical systems have a large phase space dimension, a large dimensional unstable manifold on the attractor, and are strongly mixing with exponential decay of correlations. The simplest prototype example of a turbulent dynamical system is also due to Lorenz and is called the L-96 model [98, 99]. It is widely used as a test model for algorithms for prediction, filtering, and low frequency climate response [102, 130], as well as algorithms for UQ [117, 159]. The L-96 model is a discrete periodic model given by the following system

$$\frac{du_j}{dt} = (u_{j+1} - u_{j-2})u_{j-1} - u_j + F, \quad j = 0, \dots, J-1, \quad (2.2)$$

with  $J = 40$  and with  $F$  the forcing parameter. The model is designed to mimic baroclinic turbulence in the midlatitude atmosphere with the effects of energy conserving nonlinear advection and dissipation represented by the first two terms in (2.2). For sufficiently strong forcing values such as  $F = 6, 8, 16$ , the L-96 model is a prototype turbulent dynamical system which exhibits features of weakly chaotic turbulence ( $F = 6$ ), strong chaotic turbulence ( $F = 8$ ), and strong turbulence ( $F = 16$ ) [102] as the strength of forcing,  $F$ , is increased. In order to quantify and compare the different types of turbulent chaotic dynamics in the L-96 model as  $F$  is varied, it is convenient to rescale the system to have unit energy for statistical fluctuations around the constant mean statistical state,  $\bar{u}$  [102]; thus, the transformation  $u_j = \bar{u} + E_p^{1/2} \tilde{u}_j$ ,  $t = \tilde{t} E_p^{-1/2}$  is utilized where  $E_p$  is the energy fluctuation [102]. After this normalization, the mean state becomes zero and the energy fluctuations are unity for all values of  $F$ . The dynamical equation in terms of the new variables,  $\tilde{u}_j$ , becomes

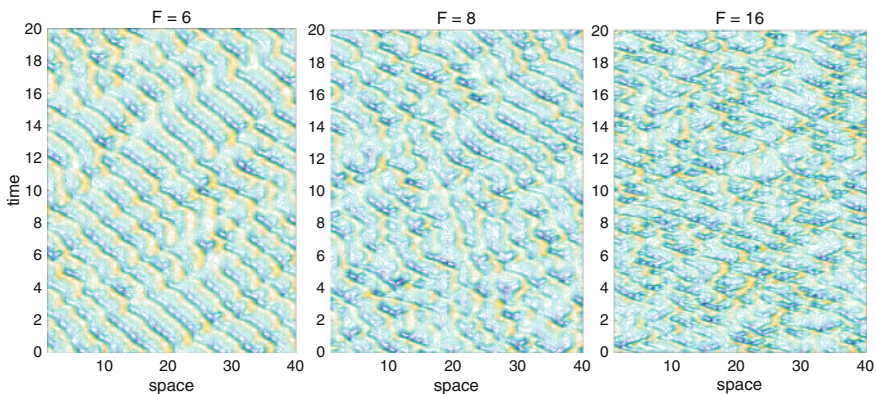
$$\frac{d\tilde{u}_j}{d\tilde{t}} = (\tilde{u}_{j+1} - \tilde{u}_{j-2})\tilde{u}_{j-1} + E_p^{-1/2} ((\tilde{u}_{j+1} - \tilde{u}_{j-2})\bar{u} - \tilde{u}_j) + E_p^{-1} (F - \bar{u}). \quad (2.3)$$

Table 2.1 lists in the non-dimensional coordinates, the leading Lyapunov exponent,  $\lambda_1$ , the dimension of the unstable manifold,  $N^+$ , the sum of the positive Lyapunov exponents (the KS entropy), and the correlation time,  $T_{\text{corr}}$ , of any  $\tilde{u}_j$  variable with itself as  $F$  is varied through  $F = 6, 8, 16$ . Note that  $\lambda_1$ ,  $N^+$  and KS increase

**Table 2.1** Dynamical properties of L-96 model for weakly chaotic regime ( $F = 6$ ), strongly chaotic regime ( $F = 8$ ) and fully turbulent regime ( $F = 16$ )

|                  | $F$ | $\lambda_1$ | $N^+$ | KS    | $T_{\text{corr}}$ |
|------------------|-----|-------------|-------|-------|-------------------|
| Weakly chaotic   | 6   | 1.02        | 12    | 5.547 | 8.23              |
| Strongly chaotic | 8   | 1.74        | 13    | 10.94 | 6.704             |
| Fully turbulent  | 16  | 3.945       | 16    | 27.94 | 5.594             |

Here,  $\lambda_1$  denotes the largest Lyapunov exponent,  $N^+$  denotes the dimension of the expanding subspace of the attractor, KS denotes the Kolmogorov–Sinai entropy, and  $T_{\text{corr}}$  denotes the decorrelation time of energy-rescaled time correlation function

**Fig. 2.1** Space-time of numerical solutions of L-96 model for weakly chaotic ( $F = 6$ ), strongly chaotic ( $F = 8$ ), and fully turbulent ( $F = 16$ ) regime

significantly as  $F$  increases while  $T_{\text{corr}}$  decreases in these non-dimensional units; furthermore, the weakly turbulent case with  $F = 6$  already has a twelve dimensional unstable manifold in the forty dimensional phase space. Snapshots of the time series for (2.1) with  $F = 6, 8, 16$ , as depicted in Figure 2.1, qualitatively confirm the above quantitative intuition with weakly turbulent patterns for  $F = 6$ , strongly chaotic wave turbulence for  $F = 8$ , and fully developed wave turbulence for  $F = 16$ . It is worth remarking here that smaller values of  $F$  around  $F = 4$  exhibit the more familiar low-dimensional weakly chaotic behavior associated with the transition to turbulence.

## 2.3 Statistical Triad Models, the Building Blocks of Complex Turbulent Dynamical Systems

Statistical triad models are special three dimensional turbulent dynamical systems with quadratic nonlinear interactions that conserve energy. For  $\mathbf{u} = (u_1, u_2, u_3)^T \in \mathbb{R}^3$ , these equations can be written in the form of (1.5)–(1.7) with a slight abuse of

notation as

$$\frac{d\mathbf{u}}{dt} = L \times \mathbf{u} + D\mathbf{u} + B(\mathbf{u}, \mathbf{u}) + \mathbf{F} + \sigma \dot{\mathbf{W}}_t, \quad (2.4)$$

where ‘ $\times$ ’ is the cross-product,  $L \in \mathbb{R}^3$ , and the nonlinear term

$$B(\mathbf{u}, \mathbf{u}) = \begin{pmatrix} B_1 u_2 u_3 \\ B_2 u_3 u_1 \\ B_3 u_1 u_2 \end{pmatrix},$$

with  $B_1 + B_2 + B_3 = 0$ , so that  $\mathbf{u} \cdot B(\mathbf{u}, \mathbf{u}) = 0$ . They are the building blocks of complex turbulent dynamical systems since a three-dimensional Galerkin truncation of many complex turbulent dynamics in (1.5)–(1.7) have the form in (2.4), in particular the models from Sections 2.1 and 2.2. A nice paper illustrating the fact for many examples in the geosciences is [53]; the famous three-equation chaotic model of Lorenz is a special case of this procedure. The random forcing together with some damping represents the effect of the interaction with other modes in a turbulent dynamical system that are not resolved in the three dimensional subspace [121–123]. Stochastic triad models are qualitative models for a wide variety of turbulent phenomena regarding energy exchange and cascades and supply important intuition for such effects. They also provide elementary test models with subtle features for prediction, UQ, and state estimation [49, 51, 105, 156, 157].

Elementary intuition about energy transfer in such models can be gained by looking at the special situation with  $L = D = F = \sigma \equiv 0$  so that there are only the nonlinear interactions in (2.4). We examine the linear stability of the fixed point,  $\bar{\mathbf{u}} = (\bar{u}_1, 0, 0)^T$ . Elementary calculations show that the perturbation  $\delta u_1$  satisfies  $\frac{d\delta u_1}{dt} = 0$  while the perturbations  $\delta u_2, \delta u_3$  satisfy the second-order equation

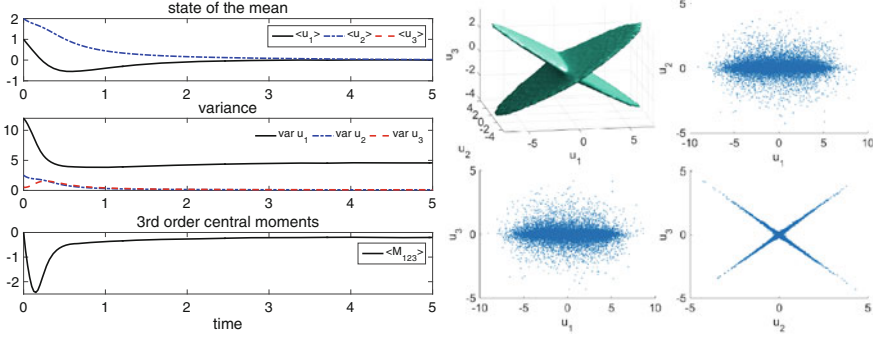
$$\frac{d^2}{dt^2} \delta u_2 = B_2 B_3 \bar{u}_1^2 \delta u_2, \quad \frac{d^2}{dt^2} \delta u_3 = B_2 B_3 \bar{u}_1^2 \delta u_3,$$

so that

$$\begin{aligned} &\text{there is instability with } B_2 B_3 > 0 \text{ and} \\ &\text{the energy of } \delta u_2, \delta u_3 \text{ grows provided } B_1 \text{ has} \\ &\text{the opposite sign of } B_2 \text{ and } B_3 \text{ with } B_1 + B_2 + B_3 = 0. \end{aligned} \quad (2.5)$$

The elementary analysis in (2.5) suggests that we can expect a flow or cascade of energy from  $u_1$  to  $u_2$  and  $u_3$  where it is dissipated provided the interaction coefficient  $B_1$  has the opposite sign from  $B_2$  and  $B_3$ .

We illustrate this intuition in a simple numerical experiment in a nonlinear regime with a statistical cascade. For the nonlinear coupling we set  $B_1 = 2, B_2 = B_3 = -1$  so that (2.5) is satisfied and  $L \equiv 0, F \equiv 0$  for simplicity. We randomly force  $u_1$  with a large variance  $\sigma_1^2 = 10$  and only weakly force  $u_2, u_3$  with variances  $\sigma_2^2 = \sigma_3^2 = 0.01$  while we use diagonal dissipation  $D$  with  $d_1 = -1$  but the stronger



**Fig. 2.2** Triad model simulation in strongly nonlinear regime with energy cascade: full-system statistics predicted with direct Monte Carlo using triad system (2.4). The time evolutions of the mean, variance, and third-order interaction are shown in the left; in the right plots the steady state conditional probability density functions of  $p_{u_1 u_2 u_3}$  are shown as well as 2D scatter plots

damping  $d_2 = d_3 = -2$  for the other two modes. A large Monte Carlo simulation with  $N = 1 \times 10^5$  is used to generate the variance of the statistical solution and the probability distribution function (PDF) along the coordinates in Figure 2.2. These results show a statistical steady state with much more variance in  $u_1$  than  $u_2$  and  $u_3$  reflecting the above intuition below (2.5) on energy cascades. Intuitively the transfer of energy in this triad system in each component separately is reflected by the third moment,  $\langle u_1 u_2 u_3 \rangle := \langle M_{123} \rangle$ , and this is negative and non-zero reflecting the non-Gaussian energy transfer in this system from  $u_1$  to  $u_2$  and  $u_3$  (see Proposition 3.2 and Theorem 3.1). This illustrates the use of the triad model for gaining intuition about complex turbulent dynamics.

It is worth remarking that the degenerate stochastic triad model in (2.4) with  $B_1 \equiv 0$  and  $B_2 = -B_3$  and  $L \neq 0$  is statistically exactly solvable, has non-Gaussian features and mimics a number of central issues for geophysical flows and is an important unambiguous test model for prediction and state estimation [49–51, 112].

## 2.4 More Rich Examples of Complex Turbulent Dynamical Systems

We briefly list and mention other important examples where the subsequent theory, techniques, and ideas in this article can be applied currently or in the near future. We begin with quantitative models and end with a list of judicious qualitative models. We also mention recent applications for prediction, UQ, and state estimation.

### 2.4.1 *Quantitative Models*

- (A) The truncated turbulent Navier–Stokes equations in two or three space dimensions with shear and periodic or channel geometry [143].
- (B) Two-layer or even multi-layer stratified flows with topography and shears in periodic, channel geometry or on the sphere [94, 130, 171]. These models include more physics like baroclinic instability for transfer of heat and generalize the one-layer model discussed in Section 2.1. There has been promising novel multiscale methods in two-layer models for the ocean which overcome the curse of ensemble size for statistical dynamics and state estimation called stochastic superparameterization. See [110] for a survey and for the applications [63–68] for state estimation and filtering. The numerical dynamics of these stochastic algorithms is a fruitful and important research topic. The end of Chapter 1 of [130] contains the formal relationship of these more complex models to the one-layer model in Section 2.1.
- (C) The rotating and stratified Boussinesq equations with both gravity waves and vortices [94, 124, 171].

There are even more models with clouds and moisture which could be listed. Next is the list of qualitative models with insight on the central issues for complex turbulent dynamical systems.

### 2.4.2 *Qualitative Models*

- (A) The truncated Burgers–Hopf (TBH) model: Galerkin truncation of the inviscid Burgers equation with remarkable turbulent dynamics with features predicted by simple statistical theory [119, 120, 125]. The models mimic stochastic backscatter in a deterministic chaotic system [2].
- (B) The MMT models of dispersive wave turbulence: One-dimensional models of wave turbulence with coherent structure, wave radiation, and direct and inverse turbulent cascades [23, 116]. Recent applications to multi-scale stochastic superparameterization [66], a novel multi-scale algorithm for state estimation [62], and extreme event prediction [35] are developed.
- (C) Conceptual dynamical models for turbulence: There are low-dimensional models capturing key features of complex turbulent systems such as non-Gaussian intermittency through energy conserving dyad interactions between the mean and fluctuations in a short self-contained paper [115]. Applications as a test model for non-Gaussian multi-scale filtering algorithms for state estimation and prediction [91] will be discussed in Section 5.4.

It is very interesting and accessible to develop a rigorous analysis of these models and also the above algorithms.

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