Chapter 2
Algebraic hyperbolicity

2.1 Hyperbolicity and genus of curves

We shall make things here in the absolute case, but everything still works in the more general framework of directed manifolds.

Let X be a compact Kobayashi hyperbolic manifold. Then X is Brody hyperbolic and thus it cannot contain any holomorphic image of $\mathbb{C}$. In particular, from the algebraic point of view, X cannot contain any rational nor elliptic curve (and, more generally, any complex torus). Hence, curves of genus 0 and 1 are prohibited by hyperbolicity. In fact, one can say something stronger.

Proposition 2.1.1 ([17]). *Let X be a compact hermitian manifold, with hermitian metric ω. If X is (infinitesimally) hyperbolic, then there exists a $\varepsilon_0 > 0$ such that for every curve $C \subset X$ one has*

$$-\chi(\widehat{C}) = 2\,g(\widehat{C}) - 2 \geq \varepsilon_0 \deg_\omega C,$$

where $\widehat{C}$ is the normalization of C and $\deg_\omega C = \int_C \omega$.

Proof. Let $C \subset X$ be a curve in X and $\nu \colon \widehat{C} \to C \subset X$ its normalization. Since X is (infinitesimally) hyperbolic and compact, there is an absolute constant $\varepsilon > 0$ such that the infinitesimal Kobayashi pseudometric satisfies a uniform lower bound

$$\mathbf{k}_X(v) \geq \varepsilon ||v||_\omega$$

for every $v \in T_X$. Now, the universal Riemannian cover of $\widehat{C}$ is necessarily the complex unit disc, by the hyperbolicity of X: let it be $\pi \colon \Delta \to \widehat{C}$. We shall endow $\widehat{C}$ by the induced metric of constant negative Gaussian curvature -4 such that

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$$\pi^* \mathbf{k}_{\widehat{C}} = \mathbf{k}_\Delta = \frac{|d\zeta|}{1-|\zeta|^2}.$$

Call $\sigma_\Delta = \frac{i}{2}\, d\zeta \wedge d\overline{\zeta}/(1-|\zeta|^2)^2$ and $\sigma_{\widehat{C}}$ the corresponding area measures. Then the classical Gauss-Bonnet formula yields

$$-4\int_{\widehat{C}} \sigma_{\widehat{C}} = \int_{\widehat{C}} \Theta(T_{\widehat{C}}, \mathbf{k}_{\widehat{C}}) = 2\pi\, \chi(\widehat{C}),$$

where $\Theta(T_{\widehat{C}}, \mathbf{k}_{\widehat{C}})$ is the curvature of $T_{\widehat{C}}$ with respect to the metric $\mathbf{k}_{\widehat{C}}$.

Next, if $\iota\colon C \to X$ is the inclusion, the distance decreasing property of the Kobayashi pseudometric applied to the holomorphic map $\iota \circ \nu\colon \widehat{C} \to X$ gives

$$\mathbf{k}_{\widehat{C}}(\xi) \geq \mathbf{k}_X((\iota\circ\nu)_*\,\xi) \geq \varepsilon\,||(\iota\circ\nu)_*\,\xi||_\omega,$$

for all $\xi \in T_{\widehat{C}}$. From this, we infer that $\sigma_{\widehat{C}} \geq \varepsilon^2(\iota\circ\nu)^*\omega$, hence

$$-\frac{\pi}{2}\,\chi(\widehat{C}) = \int_{\widehat{C}} \sigma_{\widehat{C}} \geq \int \varepsilon^2(\iota\circ\nu)^*\omega = \varepsilon^2\int_C \omega.$$

The assertion follows by putting $\varepsilon_0 = 2\,\varepsilon^2/\pi$. □

In other words, for X a hyperbolic manifold, the ratio between the genus of curves and their degrees with respect to any hermitian metric (or any ample divisor) is bounded away from zero: this, following [17], can be taken as a definition of "algebraic" hyperbolicity.

Definition 2.1.1. Let X be a projective algebraic manifold endowed with any hermitian metric ω (for instance, ω can be taken to be the curvature of any ample line bundle on X). We say that it is *algebraically hyperbolic* if there exists a constant $\varepsilon_0 > 0$ such that for every algebraic curve $C \subset X$ one has

$$2\,g(\widehat{C}) - 2 \geq \varepsilon_0 \deg_\omega C.$$

When $\omega = i\,\Theta(A)$, where A is any hermitian ample line bundle and $i\,\Theta(A)$ its Chern curvature, the right-hand side of the inequality is just the usual degree of a curve in terms of its intersection product $C \cdot A$: in this case the inequality is purely algebraic.

By Riemann-Hurwitz formula, one can take, in the previous inequality of the definition of algebraic hyperbolicity, any finite morphism $f : C \to X$ from a smooth projective curve.

This algebraic counterpart of hyperbolicity satisfies an analogue of the openness property of the Kobayashi hyperbolicity, this time with respect to the Zariski topology.

Proposition 2.1.2. *Let $\mathcal{X} \to S$ be an algebraic family of projective algebraic manifolds, given by a projective morphism. Then the set of $s \in S$ such that the fiber X_s is algebraically hyperbolic is open with respect to the countable Zariski topology of S (by definition, this is the topology for which closed sets are countable unions of algebraic sets).*

Proof. Without loss of generality, we can suppose that the total space $\mathcal{X}$ is quasi-projective. Let ω be the Kähler metric on $\mathcal{X}$ obtained by pulling-back the Fubini-Study metric via an embedding in a projective space. Fix integers $d > 0$ and $g \geq 0$ and call $A_{d,g}$ the set of $s \in S$ such that X_s contains an algebraic 1-cycle $C = \sum m_j C_j$ with $\deg_\omega C = d$ and $g(\widehat{C}) = \sum m_j g(\widehat{C}_j) \leq g$.

This set is closed in S, by the existence of a relative cycle space of curves of given degree and the lower semicontinuity with respect to the Zariski topology of the geometric genus. But then, the set of nonhyperbolic fibers is by definition

$$\bigcap_{k>0} \bigcup_{2g-2<d/k} A_{d,g}.$$

□

An interesting property of algebraically hyperbolic manifolds is

Proposition 2.1.3. *Let X be an algebraically hyperbolic projective manifold and V be an abelian variety. Then any holomorphic map $f\colon V \to X$ is constant.*

Proof. Let m be a positive integer and $m_V\colon V \to V$, $s \mapsto m \cdot s$. Consider $f_m := f \circ m_V$ and A an ample line bundle on X. Let C be a smooth curve in V and $f_m|_C\colon C \to X$. Then

$$2g(C) - 2 \geq \varepsilon\, C \cdot f_m^* A = \varepsilon m^2\, C \cdot f^* A.$$

Letting m go to infinity, we obtain that necessarily $C \cdot f^* A = 0$. Thus f is constant on all curves in V and therefore f is constant on V. □

It is worthwhile here to mention that in the projective algebraic case, Kobayashi hyperbolicity and algebraic hyperbolicity are expected to be equivalent, but not much is known about it. Both of these properties should be equivalent to the following algebraic property.

Conjecture 2.1.1 (Lang). Let X be a projective manifold. Then X is hyperbolic if and only if there are no nontrivial holomorphic maps $V \to X$ where $V = \mathbb{C}^p/\Lambda$ is a compact complex torus.

One may be tempted to extend the conjecture to non-projective manifolds but then it becomes false, as shown by the following

Example 2.1.1 ([11]). Let X be a non-projective $K3$ surface[1] with no algebraic curves (the existence of such a surface is a classical result on K3 surfaces). Then there exists a non-constant entire curve $f\colon \mathbb{C} \to X$. On the other hand, if V is a compact torus, every holomorphic map $F\colon V \to X$ is constant.

Let us justify briefly the claims of the example. The existence of non-constant entire curves is a consequence of the density of Kummer surfaces[2] in the moduli space of $K3$ surfaces. Since Kummer surfaces contain lots of entire curves (inherited from the starting torus), one just has to apply Brody's theorem. The second claim follows from the non-existence of surjective maps $F\colon V \to X$. Indeed, considering Ω a non-vanishing holomorphic 2-form on X, if F is surjective, then $F^*\Omega$ is a non-zero section of the trivial bundle $\Lambda^2 T_V^*$: the rank of this 2-form is therefore constant, equal to 2. Then, one obtains that F factors through a two dimensional compact torus and induces a covering $V \to X$ which contradicts the fact that X is simply connected.

Another characterization of hyperbolicity should be the following.

Conjecture 2.1.2 (Lang). Let X be a smooth projective algebraic manifold. Then X is hyperbolic if and only if all subvarieties of X including X itself are of general type.

In the next section we shall see some partial result in this direction. The latter conjecture should be put in perspective with this other celebrated one.

Conjecture 2.1.3 (Green-Griffiths [31], Lang). Let X be a smooth projective algebraic manifold of general type. Then there should exist a proper algebraic subvariety $Y \subsetneq X$ such that all entire curves $f\colon \mathbb{C} \to X$ have image $f(\mathbb{C})$ contained in Y.

This conjecture is largely open, too. Nevertheless, related to algebraic hyperbolicity we have the following.

Theorem 2.1.4 (Bogomolov [8]). *Let X be a smooth projective surface of general type with $c_1(X)^2 > c_2(X)$. Then there are only finitely many rational or elliptic curves in X.*

Proof. We will see later in some details that the hypothesis on the second Segre number $c_1(X)^2 > c_2(X)$ implies that $h^0(X, S^m T_X^*) \sim m^3$. A nontrivial symmetric differential $\omega \in H^0(X, S^m T_X^*)$ defines a multifoliation on X. Recall that there is an isomorphism (we will come back later on this, too)

$$H^0(X, S^m T_X^*) \cong H^0(P(T_X), \mathcal{O}_{P(T_X)}(m)).$$

[1] A $K3$ surface is a simply connected surface X with irregularity $q(X) = h^1(X, \mathcal{O}_X) = 0$ and trivial canonical bundle $K_X \simeq \mathcal{O}_X$.

[2] Let T be a two dimensional complex torus with a base point chosen. The involution $\iota\colon T \to T$ has exactly 16 fixed points, namely the points of order 2 on T, so that the quotient $T/\langle 1, \iota \rangle$ has sixteen ordinary double points. Resolving the double points we obtain a smooth surface X, the Kummer surface $Km(T)$ of T. Kummer surfaces are special case of $K3$ surfaces.

If $\sigma \in H^0(X, S^m T_X^*)$ and $x \in X$, then $\sigma(x)$ defines naturally a polynomial of degree m on $P(T_{X,x}) \simeq \mathbb{P}^1$. The zeroes of $\sigma(x)$ determine the directions of the multifoliation. Let C be a smooth projective curve and $f: C \to X$. The curve $f(C)$ is a leaf of the multifoliation defined by σ if $f^*\sigma \in H^0(C, T_C^{*\otimes m})$ is trivial. Equivalently if $t_f: C \to P(T_X)$ is the lifting of f, then $f(C)$ is a leaf if $t_f(C)$ lies in the zero locus of $\sigma \in H^0(P(T_X), \mathcal{O}_{P(T_X)}(m))$.

The sections of $\mathcal{O}_{P(T_X)}(m)$ for m large enough provide a rational map $\varphi: P(T_X) \to \mathbb{P}^N$ generically $1-1$ onto its image. Let us denote $Z_m \subset P(T_X)$ the union of the positive dimensional fibers of φ and of the base locus of $\mathcal{O}_{P(T_X)}(m)$.

Let $f: C \to X$ be a curve. Then, $f(C)$ is said to be irregular if $t_f(C) \subset Z_m$, otherwise it is regular. The set of irregular curves can be broken into 2 sets: the curves that are leaves of multifoliations and the curves whose lifts lie on the positive dimensional fibers of φ.

Let C' be a regular curve with normalization $f: C \to C' \subset X$. There is a symmetric differential $\sigma \in H^0(X, S^m T_X^*)$ such that $f^*\sigma \in H^0(C, (T_C^*)^{\otimes m})$ is nontrivial but vanishes somewhere. Hence $\deg_C T_C^{*\otimes m} = m \deg K_C > 0$ and C cannot be rational or elliptic.

Let C' be an irregular curve and write $Z_m = Z_m^1 \cup Z_m^2$ where Z_m^1 is the union of components not dominating X, Z_m^2 is the union of components dominating X. The number of curves that lift in Z_m^1 is clearly finite. The components of Z_m^2 have a naturally defined foliation on them. Curves whose lifts lie in Z_m^2 are leaves of these foliations. By Jouanolou's theorem on compact leaves of foliations, either there are finitely many compact leaves or they are fibers of a fibration. Thus there are finitely many such elliptic or rational curves: X being of general type, the second situation is not possible since a surface of general type cannot be ruled or elliptic. □

In the transcendental case, the only result for a quite general case has been obtained McQuillan in [38], for $\dim X = 2$ and the second Segre number $c_1(X)^2 - c_2(X)$ of X positive. The heart of his proof is

Theorem 2.1.5. *Consider a (possibly singular) holomorphic foliation on a surface of general type. Then any parabolic leaf of this foliation is algebraically degenerate.*

An immediate corollary of the two previous results is a confirmation of the Green-Griffiths conjecture in this situation.

Corollary 2.1.6. *Let X be a smooth projective surface of general type with $c_1(X)^2 > c_2(X)$. Then there are finitely many curves $C \subset X$ such that any non-constant entire curve takes value in one of these curves.*

Unfortunately, these "order one" techniques are insufficient to work with surfaces of degree d in projective 3-space. In this case in fact

$$c_1(X)^2 = d(d-4)^2 < d(d^2 - 4d + 6) = c_2(X), \quad d \geq 3.$$

In higher dimensions, there are few results. For the algebraic version, let us mention the following result of Lu and Miyaoka.

Theorem 2.1.7 ([37]). *Let X be a projective manifold of general type. Then X has only a finite number of nonsingular codimension-one subvarieties having pseudo-effective anticanonical divisor. In particular, X has only a finite number of nonsingular codimension-one Fano, Abelian, and Calabi-Yau subvarieties.*

For some partial result in all dimensions for the transcendental case, we refer to next chapters.

2.2 Algebraic hyperbolicity of generic projective hypersurfaces of high degree

Consider the Grassmannian $\mathbb{G}(1, n+1)$ of projective lines in $\mathbb{P}^{n+1}$ which is canonically identified with the Grassmannian $Gr(2, n+2)$ of 2-planes in $\mathbb{C}^{n+2}$: its complex dimension is $2n$. We are interested in understanding when a generic projective hypersurface $X \subset \mathbb{P}^{n+1}$ contains a line. Fix an integer $d > 0$. Then a projective hypersurface of degree d is an element of the linear system $|\mathcal{O}_{\mathbb{P}^{n+1}}(d)|$ or, equivalently, can be identified with a point in the projectivization $\mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)))$. One has $\dim \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d))) = N_d - 1$, where $N_d = \binom{n+d+1}{n+1} = h^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d))$ is the dimension of homogeneous polynomials of degree d in $n+2$ variables.

Now, consider the incidence variety

$$\mathcal{L} = \{(\ell, X) \in \mathbb{G}(1, n+1) \times \mathbb{P}^{N_d - 1} \mid \text{the line } \ell \text{ is contained in } X\}.$$

By construction, the image of $\mathcal{L}$ in $\mathbb{P}^{N_d-1}$ by the second projection is the set of projective hypersurfaces of degree d which contain at least one line. Of course, if $\dim \mathcal{L}$ is less than $N_d - 1$, then a generic projective hypersurface of degree d does not contain lines, since the second projection cannot be dominant. On the other hand, $\mathcal{L}$ is always mapped onto $\mathbb{G}(1, n+1)$ by the first projection, since every line is always contained in some degree d hypersurface. Next, an easy parameter computation shows that generically a homogeneous polynomial of degree d in $n+2$ variables must satisfy $d+1$ condition in order to contain a line. Therefore, the fiber of the first projection has dimension $N_d - d - 2$ and thus $\dim \mathcal{L} = N_d + 2n - d - 2$.

After all, the second projection maps a variety of dimension $N_d + 2n - d - 2$ to a variety of dimension $N_d - 1$ and so we have proved the following.

Proposition 2.2.1. *If $d \geq 2n$, then a generic projective hypersurface of degree d in $\mathbb{P}^{n+1}$ cannot contain any line.*

This digression shows that if we are interested in hyperbolicity of generic projective hypersurfaces, we surely have to exclude low degree ones. On the other hand, by the Euler short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}} \to \bigoplus_{j=1}^{n+2} \mathcal{O}_{\mathbb{P}^{n+1}}(1) \to T_{\mathbb{P}^{n+1}} \to 0,$$

combined with the classical adjunction formula

$$K_D \simeq (K_Y \otimes \mathcal{O}_Y(D))|_D$$

for smooth divisors $D \subset Y$ in a smooth manifold Y, one finds straightforwardly, by taking determinants, that the canonical bundle of a smooth hypersurface X of degree d in projective $(n+1)$-space is given by

$$K_X = \mathcal{O}_X(d-n-2).$$

So, the higher the degree of the hypersurface X is, the more positive its canonical bundle is. This is somehow consistent with the picture presented at the end of Chapter 1, where hyperbolicity was heuristically linked to the positivity properties of the canonical bundle.

More precisely, Kobayashi made the following.

Conjecture 2.2.1 ([34]). Le $X \subset \mathbb{P}^{n+1}$ be generic projective hypersurfaces of degree d, $n \geq 2$. Then X is Kobayashi hyperbolic if its degree is sufficiently high, say $d \geq 2n+1$.

This conjecture and the bound on the degree are closely related to the conjecture in the case of complements of hypersurfaces.

Conjecture 2.2.2 ([34]). Le $X \subset \mathbb{P}^n$ be generic projective hypersurfaces of degree d. Then $\mathbb{P}^n \setminus X$ is Kobayashi hyperbolic if its degree is sufficiently high, say $d \geq 2n+1$.

One possible explanation for the bounds on the degrees comes, as far as we know, from the following facts. Consider in $\mathbb{P}^n$ with homogeneous coordinates $[Z_1 : \cdots : Z_n]$ the divisor D of degree d defined by the homogeneous equation $P(Z) = 0$. Then, one can construct a cyclic $d:1$ cover of $\mathbb{P}^n$ by taking in $\mathbb{P}^{n+1}$ with homogeneous coordinates $[Z_0 : \cdots : Z_n]$ the divisor X defined by $Z_0^d = P(Z_1, \dots, Z_n)$ together with its projection onto $\mathbb{P}^n$. This covering ramifies exactly along D and thus all holomorphic maps $f\colon \mathbb{C} \to \mathbb{P}^n \setminus D$ lift to X. It is then clear that the hyperbolicity of $\mathbb{P}^n \setminus D$ is intimately correlated with the hyperbolicity of X. On the other hand, if a holomorphic map $f\colon \mathbb{C} \to \mathbb{P}^n$ misses $2n+1$ or more hyperplanes in general position, then it is a constant map; this is the by now classical result of Dufresnoy [28] and Green [32]. Now, just remark that a configuration of d hyperplanes in general position can be seen as a generic completely reducible divisor of degree d.

One has to notice anyway that if one believes to the equivalence of Kobayashi and algebraic hyperbolicity in the projective algebraic setting then, as we shall see in the next section, this bound should probably be $d \geq 2n$, at least for $n \geq 6$ [44]. Anyway the state of the art on the subject is for the moment very far from these optimal bounds, no matter in which one we want to believe.

The rest of this chapter will be devoted to prove several algebraic properties of generic projective hypersurfaces of high degree, such as their algebraic hyperbolicity and the property of their subvarieties of being of general type.

2.2.1 Global generation of the twisted tangent bundle of the universal family

First, given a holomorphic vector bundle $E \to X$ over a compact complex manifold X, we say that E is *globally generated*, if the global sections evaluation maps

$$H^0(X, E) \to E_x$$

are surjective for all $x \in X$, where E_x is the fiber of E over the point x. If a vector bundle is globally generated, so are all its exterior powers, in particular its determinant, as it is easy to verify.

Now, consider the universal family of projective hypersurfaces in $\mathbb{P}^{n+1}$ of a given degree $d > 0$. It is the subvariety $\mathcal{X}$ of the product $\mathbb{P}^{n+1} \times \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)))$ defined by the pairs $([x], X)$ such that $[x] \in X$. The starting point is the following global generation statement.

Proposition 2.2.2 (See [53, 57]). *The twisted tangent bundle*

$$T_{\mathcal{X}} \otimes p^* \mathcal{O}_{\mathbb{P}^{n+1}}(1)$$

is globally generated, where $p \colon \mathcal{X} \to \mathbb{P}^{n+1}$ is the first projection.

Proof. We shall exhibit on an affine open set of $\mathcal{X}$ a set of generating holomorphic vector fields and then show that when extended to the whole space, the pole order of such vector fields in the $\mathbb{P}^{n+1}$-variables is one.

Consider homogeneous coordinates $(Z_j)_{j=0,\dots,n+1}$ and $(A_\alpha)_{|\alpha|=d}$, respectively, on $\mathbb{P}^{n+1}$ and $\mathbb{P}^{N_d-1}$, where $\alpha = (\alpha_0, \dots, \alpha_{n+1}) \in \mathbb{N}^{n+2}$ is a multiindex and $|\alpha| = \sum \alpha_j$. The equation of the universal hypersurface is then given by

$$\sum_{|\alpha|=d} A_\alpha Z^\alpha = 0, \quad Z^\alpha = Z_0^{\alpha_0} \cdots Z_{n+1}^{\alpha_{n+1}}.$$

Next, we fix the affine open set $U = \{Z_0 \neq 0\} \times \{A_{d0\cdots0} \neq 0\} \simeq \mathbb{C}^{n+1} \times \mathbb{C}^{N_d-1}$ in $\mathbb{P}^{n+1} \times \mathbb{P}^{N_d-1}$ with the corresponding inhomogeneous coordinates $(z_j)_{j=1,\dots,n+1}$ and $(a_\alpha)_{|\alpha|=d,\alpha_0<d}$. On this affine open set we have

$$\mathcal{X} \cap U = \left\{ \sum_{|\alpha|=d} a_\alpha z_1^{\alpha_1} \cdots z_{n+1}^{\alpha_{n+1}} = 0 \right\}, \quad a_{d0\cdots0} = 1.$$

Its tangent space in $\mathbb{C}^{n+1} \times \mathbb{C}^{N_d-1} \times \mathbb{C}^{n+1} \times \mathbb{C}^{N_d-1}$ with affine coordinates $(z_j, a_\alpha, z'_j, a'_\alpha)$ is then given by the two equations

$$\begin{cases} \sum_{|\alpha|=d} a_\alpha\, z_1^{\alpha_1} \cdots z_{n+1}^{\alpha_{n+1}} = 0, \quad a_{d0\cdots 0} = 1 \\ \sum_{|\alpha|=d,\alpha_0<d} \sum_{j=1}^{n+1} \alpha_j\, a_\alpha\, z_1^{\alpha_1} \cdots z_j^{\alpha_j - 1} \cdots z_{n+1}^{\alpha_{n+1}}\, z_j' \\ \quad + \sum_{|\alpha|=d,\alpha_0<d} z_1^{\alpha_1} \cdots z_{n+1}^{\alpha_{n+1}}\, a_\alpha' = 0, \end{cases}$$

the second of which is obtained by formal derivation. For any multiindex α with $\alpha_j \geq 1$, set

$$V_\alpha^j = \frac{\partial}{\partial a_\alpha} - z_j \frac{\partial}{\partial a_\alpha^j},$$

where a_α^j is obtained by the multiindex a_α lowering the j-th entry by one. It is immediate to verify that these vector fields are tangent to $\mathcal{X}_0$ and, by an affine change of coordinates, that once extended to the whole $\mathcal{X}$ it becomes rational with pole order equal to one in the z-variables.

Now consider a vector field on $\mathbb{C}^{n+1}$ of the form

$$V_0 = \sum_{j=1}^{n+1} v_j \frac{\partial}{\partial z_j},$$

where $v_j = \sum_{k=1}^{n+1} v_{j,k}\, z_k + v_{j,0}$ is a polynomial of degree at most one in the z-variables. We can then modify it by added some "slanted" direction in order to obtain a vector field tangent to $\mathcal{X}_0$ as follows. Let

$$V = \sum_{|\alpha|=d,\alpha_0<d} v_\alpha \frac{\partial}{\partial a_\alpha} + V_0,$$

where the v_α's have to be determined. The condition to be satisfied in order to be tangent to $\mathcal{X}_0$ clearly is

$$\sum_\alpha v_\alpha\, z^\alpha + \sum_{\alpha,j} a_\alpha\, v_j \frac{\partial z^\alpha}{\partial z_j} \equiv 0$$

and thus it suffices to select the v_α to be constants such that the coefficient in each monomial z^α is zero. Here, an affine change of variables shows that once the extension of V to the whole $\mathcal{X}$ is taken, the pole order is at most one in the z-variables.

It is then straightforward to verify that these packages of vector field generate the tangent bundle, and the poles are compensated by twisting by $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$, since they appear at order at most one and only in the variables living in $\mathbb{P}^{n+1}$. $\square$

2.2.2 Consequences of the twisted global generation

Two remarkable consequences of the twisted global generation of the tangent space of the universal family are the following. First, the very generic projective hypersurface of high degree (is of general type and) admits only subvarieties of general type, that is very generic projective hypersurfaces of high degree satisfy Lang's conjecture stated above, which is conjecturally equivalent to Kobayashi hyperbolicity. Second, very generic projective hypersurfaces are algebraically hyperbolic, which would be implied by their hyperbolicity (and should be in principle equivalent) as we have seen: this can be regarded as another evidence toward Kobayashi's conjecture.

Theorem 2.2.3. *Let $X \subset \mathbb{P}^{n+1}$ be a (very) generic projective hypersurface of degree $d \geq 2n+2$. If $Y \subset X$ is any subvariety, let $\nu\colon \widetilde{Y} \to Y$ be a desingularization. Then*

$$H^0(\widetilde{Y}, K_{\widetilde{Y}} \otimes \nu^*\mathcal{O}_{\mathbb{P}^{n+1}}(-1)) \neq 0.$$

Proof. Let $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d-1}$ be the universal hypersurface of degree d and $\mathcal{Y} \subset \mathcal{X}$ be a subvariety such that the second projection $\mathcal{Y} \to \mathbb{P}^{N_d-1}$ is dominant of relative dimension ℓ. For simplicity, we shall skip here a technical point which consists to allow an étale base change $U \to \mathbb{P}^{N_d-1}$ for the family.

Let $\nu\colon \widetilde{\mathcal{Y}} \to \mathcal{Y}$ be a desingularization and consider an open dense subset $U \subset \mathbb{P}^{N_d-1}$ over which both $\widetilde{\mathcal{Y}}$ and $\mathcal{X}$ are smooth. What we have to show is that

$$H^0(\widetilde{Y}_s, K_{\widetilde{Y}_s} \otimes \nu^*\mathcal{O}_{\mathbb{P}^{n+1}}(-1)) \neq 0,$$

for Y_s the fiber over a generic point $s \in U$. To this aim, observe that, since the normal bundle of a fiber in a family is trivial,

$$K_{\widetilde{Y}_s} \simeq K_{\widetilde{\mathcal{Y}}}|_{\widetilde{Y}_s} = \bigwedge^{k+N_d-1} T^*_{\widetilde{\mathcal{Y}}}\Big|_{\widetilde{Y}_s},$$

by adjunction and that

$$\bigwedge^{k+N_d-1} T^*_{\mathcal{X}}\Big|_{X_s} \simeq K_{X_s} \otimes \bigwedge^{n-k} T_{\mathcal{X}}\Big|_{X_s}$$

by linear algebra and adjunction again.

Therefore, we have to show that $\bigwedge^{k+N_d-1} T^*_{\widetilde{\mathcal{Y}}} \otimes \nu^*\mathcal{O}_{\mathbb{P}^{n+1}}(-1)|_{\widetilde{Y}_s}$ is effective. Now, we have a map

$$\bigwedge^{k+N_d-1} T^*_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(-1)\Big|_{X_s} \to \bigwedge^{k+N_d-1} T^*_{\widetilde{\mathcal{Y}}} \otimes \nu^*\mathcal{O}_{\mathbb{P}^{n+1}}(-1)\Big|_{\widetilde{Y}_s}$$

induced by the generically surjective restriction $T^*_{\mathcal{X}} \to T^*_{\widetilde{\mathcal{Y}}}$, which is non-zero for a generic choice of $s \in U$.

It is then sufficient to prove that $K_{X_s} \otimes \bigwedge^{n-k} T_{\mathcal{X}}\big|_{X_s} \otimes \mathcal{O}_{X_s}(-1)$ is globally generated. Now,

$$K_{X_s} = \mathcal{O}_{X_s}(d-n-2) = \mathcal{O}_{X_s}((n-k) + (d-2n+k-2))$$

and thus

$$K_{X_s} \otimes \bigwedge^{n-k} T_{\mathcal{X}}\bigg|_{X_s} \otimes \mathcal{O}_{X_s}(-1) = \bigwedge^{n-k} T_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(1)\bigg|_{X_s} \otimes \mathcal{O}_{X_s}(d-2n+k-3).$$

By the global generation of $T_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(1)$, the right-hand term is globally generated as soon as $d \geq 2n+3-k$ so that $d \geq 2n+2$ will do the job.

We have thus proved that the theorem holds for the general fiber of the family $\mathcal{Y}$. To conclude, it suffices to let the family $\mathcal{Y}$ vary, that is to let vary the Hilbert polynomial. In this way we obtain the same statement for all subvarieties of X_s outside a countable union of closed algebraic subvarieties of the parameter space U, that is for very generic X. □

Corollary 2.2.4. *Let $X \subset \mathbb{P}^{n+1}$ be a (very) generic projective hypersurface of degree $d \geq 2n+2$. Then any subvariety $Y \subset X$ (and of course X itself) is of general type.*

Proof. This is an immediate consequence of the theorem above: such a subvariety has in fact a desingularization whose canonical bundle can be written as an effective divisor twisted by a big one (the pull-back by a modification of the ample divisor $\mathcal{O}_{\mathbb{P}^{n+1}}(1)$) and hence it is big. □

This corollary can be sharpened as soon as $n \geq 6$, see [44].

Corollary 2.2.5. *A very generic projective hypersurface in $\mathbb{P}^{n+1}$ of degree greater than or equal to $2n+2$ is algebraically hyperbolic.*

Proof. Let $\omega = i\,\Theta(\mathcal{O}_X(1))$ be the reference hermitian metric on X and $C \subset X$ a curve. Consider the finite-to-one normalization morphism $\nu\colon \widetilde{C} \to C$, which is in fact a desingularization, if necessary. Then, the preceding theorem states that $K_{\widetilde{C}} \otimes \nu^*\mathcal{O}_{\mathbb{P}^{n+1}}(-1)$ is effective and so of nonnegative degree on $\widetilde{C}$. By the Hurwitz formula $c_1(K_{\widetilde{C}}) = 2g(\widetilde{C}) - 2$ and thus

$$-\chi(\widetilde{C}) = 2g(\widetilde{C}) - 2 \geq \nu^*\mathcal{O}_{\mathbb{P}^{n+1}}(1) \cdot \widetilde{C} = \int_C \omega.$$

□

Another consequence of the global generation statement is the following result on the non-deformability of entire curves in projective hypersurfaces of high degree.

Theorem 2.2.6 ([25]). *Consider $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}^{N_d-1}$ the universal hypersurface of degree d, $U \subset \mathbb{P}^{N_d-1}$ an open set, and $\Phi\colon \mathbb{C} \times U \to \mathcal{X}$ a holomorphic map such that $\Phi(\mathbb{C} \times \{t\}) \subset X_t$ for all $t \in U$. If $d \geq 2n+2$, the rank of Φ cannot be maximal anywhere.*

In other words, the Kobayashi conjecture may possibly fail only if there is an entire curve on a general hypersurface X which is not preserved by a deformation of X.

Now, let us sketch the proof of the previous result.

Proof. Suppose that $\Phi\colon \mathbb{C} \times U \to \mathcal{X}$ has maximal rank and U is the polydisc $B(\delta_0)^{N_d-1}$. We consider the sequence of maps

$$\Phi_k\colon \mathbb{B}(\delta_0 k)^{N_d} \to \mathcal{X}$$

given by $\Phi_k(z, \xi_1, \dots \xi_{N_d-1}) = \Phi(zk^{N_d-1}, \frac{1}{k}\xi_1, \dots, \frac{1}{k}\xi_{N_d-1})$. The sections

$$J_{\Phi_k}(z,\xi) = \frac{\partial \Phi}{\partial z} \wedge \frac{\partial \Phi}{\partial \xi_1} \wedge \dots \wedge \frac{\partial \Phi}{\partial \xi_{N_d-1}}(z,\xi) \in \Lambda^{N_d} T_{\mathcal{X},\Phi(z,\xi)}$$

are not identically zero and we can assume $J_{\Phi_k}(0)$ non-zero. Thanks to the global generation statement of $T_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(1)$, we can choose $n-1$ vector fields

$$V_1, \dots, V_{n-1} \in T_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(1)$$

such that

$$J_{\Phi_k}(0) \wedge \Phi_k^*(V_1 \wedge \dots \wedge V_{n-1}) \neq 0$$

in $K_{\mathcal{X}}^{-1} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(n-1)_{\Phi_k(0)}$. We consider the sections

$$\sigma_k = J_{\Phi_k} \wedge \Phi_k^*(V_1 \wedge \dots \wedge V_{n-1}),$$

of $\Phi_k^*(K_{\mathcal{X}}^{-1} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(n-1))$ over the polydisc. If $d \geq 2n+2$, the restriction of $K_{\mathcal{X}} \otimes \mathcal{O}_{\mathbb{P}^{n+1}}(1-n)$ over U is ample and we can endow this bundle with a metric h of positive curvature. We consider the sequence of functions $f_k : \mathbb{B}(\delta_0 k)^{N_d} \to \mathbb{R}^+$ defined by

$$f_k(w) = ||\sigma_k(w)||^{2/N_d}_{\Phi_k^* h^{-1}}.$$

The ampleness implies that there exists a positive C such that

$$\Delta \log f_k \geq C f_k.$$

This gives

$$f_k(0) \le Ck^{-2},$$

and therefore $f_k(0) \to 0$ which contradicts the fact that, by construction, there exists a positive constant b such that for all $k, f_k(0) = b$. □

Let us briefly describe the generalization of the above results to the logarithmic case, that is the case of complements of hypersurfaces. If X is an n-dimensional complex manifold and D a normal crossing divisor, *i.e.*, in local coordinates $D = \{z_1 \dots z_l = 0\}$, $l \le n$, we call the pair (X, D) a *log-manifold*.

In the case of complements we have the following notion stronger than hyperbolicity.

Definition 2.2.1. Let (X, D) be a log-manifold and ω a hermitian metric on X. The complement $X \setminus D$ is said to be hyperbolically embedded in X, if there exists $\varepsilon > 0$ such that for every $x \in X \setminus D$ and $\xi \in T_{X,x}$, we have

$$k_X(\xi) \ge \varepsilon ||\xi||_\omega.$$

To generalize to this setting the notion of algebraic hyperbolicity, we need to introduce the following.

Definition 2.2.2. Let (X, D) be a log-manifold, $C \subset X$ a curve not contained in D, and $\nu\colon \widehat{C} \to C$ the normalization. Then we define $i(C, D)$ to be the number of distinct points in $\nu^{-1}(D)$.

Then, we have the next.

Definition 2.2.3. The pair (X, D) is *algebraically hyperbolic* if there exists $\varepsilon > 0$ such that

$$2g(\widehat{C}) - 2 + i(C, D) \ge \varepsilon \deg_\omega(C)$$

for all curves $C \subset X$ not contained in D.

As in the compact case, analytic and algebraic hyperbolicity are closely related.

Proposition 2.2.7 ([47]). *Let (X, D) be a log-manifold such that $X \setminus D$ is hyperbolic and hyperbolically embedded in X. Then (X, D) is algebraically hyperbolic.*

The algebraic version of the Kobayashi conjecture is also verified.

Theorem 2.2.8 ([47]). *Let $X_d \subset \mathbb{P}^n$ be a very generic hypersurface of degree $d \ge 2n + 1$ in $\mathbb{P}^n$. Then $(\mathbb{P}^n, X_d)$ is algebraically hyperbolic.*

2.3 A brief history of the above results

The chronicle of the above results about algebraic hyperbolicity is the following.

First, in [12] it is shown that if X is a generic hypersurface of degree $d \geq 2$ in $\mathbb{P}^{n+1}$, then X does not admit an irreducible family $f\colon \mathcal{C} \to X$ of immersed curves of genus g and fixed immersion degree $\deg f$ which cover a variety of codimension less than $D = ((2-2g)/\deg f) + d - (n+2)$. As an immediate consequence, one gets, for example, that there are no rational curves on generic hypersurfaces X of degree $d \geq 2n+1$ in $\mathbb{P}^{n+1}$.

Two years later, [30] studies the Hilbert scheme of $X \subseteq G$, a generic complete intersection of type $(m_1, \ldots, m_k)$ in the Grassmann variety $G = G(r, n+2)$. As a remarkable corollary one gets that any smooth projective subvariety of X is of general type if $m_1 + m_2 + \cdots + m_k \geq \dim X + n + 2$. It is also proved that the Hilbert scheme of X is smooth at points corresponding to smooth rational curves of " low" degree.

The variational method presented here is due to [57]. By variational method we mean the idea of putting the hypersurfaces in family and to use the positivity property of the tangent bundle of the family itself. The main result of this paper is the following theorem which improves Ein's result in the case of hypersurfaces: let $X \subset \mathbb{P}^{n+1}$ be a hypersurface of degree d. If $d \geq 2n - \ell + 1$, $1 \leq \ell \leq n-2$, then any ℓ-dimensional subvariety Y of X has a desingularization $\widetilde{Y}$ with an effective canonical bundle. Moreover, if the inequality is strict, then the sections of $K_{\widetilde{Y}}$ separate generic points of $\widetilde{Y}$. The bound is now optimal and, in particular, the theorem implies that generic hypersurfaces in $\mathbb{P}^{n+1}$ of degree $d \geq 2n$, $n \geq 3$, contain no rational curves. The method also gives an improvement of a result of [60] as well as a simplified proof of Ein's original result.

Lastly, let us cite [44]: this paper gives the sharp bound $d \geq 2n$ for a general projective hypersurface X of degree d in $\mathbb{P}^{n+1}$ containing only subvarieties of general type, for $n \geq 6$. This result improves the aforesaid results of Voisin and Ein. The author proves the bound by showing that, under some numerical conditions, the locus W spanned by subvarieties not of general type (even more than this) is contained in the locus spanned by lines. This is obtained in two steps. First, with the variational technique inherited by Voisin the author proves that W is contained in the locus spanned by lines with highly nonreduced intersection with X, the so-called bicontact locus. Then the latter is proved to be contained in the locus of lines by using the global generation of certain bundles. Finally, let us mention that similar results have also been obtained independently and at the same time in [13].

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