Preface to the first edition

These are lecture notes of a course held at IMPA, Rio de Janeiro, in September 2010: the purpose was to present recent results on Kobayashi hyperbolicity in complex geometry.

This area of research is very active, in particular because of the fascinating relations between analytic and arithmetic geometry. After Lang and Vojta, we have precise conjectures between analytic and arithmetic hyperbolicity; e.g. existence of Zariski dense entire curves should correspond to the density of rational points. Our ultimate goal is to describe the results obtained in [23] on questions related to the geometry of entire curves traced in generic complex projective hypersurfaces of high degree.

For the convenience of the reader, this survey tries to be as self-contained as possible. Thus, we start by recalling the basic definitions and concepts of complex hyperbolic geometry. Our presentation will focus later on the concepts of jet bundles and jet differentials which turn out to be the crucial tools that have been applied successfully in the last decades. These ideas date back to the work of Bloch [7] and have been developed later by many other people (let us cite, e.g., Green and Griffiths [31], Siu and Yeung [55], Demailly [17], etc.).

The presentation of the main techniques is certainly inspired by the notes [17], but many progress have been achieved since these notes were written and it seemed to us quite useful to update them.

Let us now describe the contents of this survey. In Chapter 1, we introduce the classical Poincaré distance on the complex unit disc, and, following Kobayashi, we use it to construct an invariant pseudodistance on any complex space X by means of chains of holomorphic discs. The complex space X will be said (Kobayashi) hyperbolic if this pseudodistance is actually a true distance. We then present an infinitesimal form of this pseudodistance which reveals to be very useful in order to characterize the hyperbolicity of compact complex manifold in terms of the existence or not of entire curves (nonconstant holomorphic maps from the entire complex plane) in it: this is the content of the famous Brody's criterion for hyperbolicity. We then end the chapter with a couple of applications of Brody's criterion to deformations of compact complex manifolds and to hyperbolicity of complex tori and with a general discussion about uniformization and hyperbolicity in complex dimension one in order to put in perspective the new difficulties which come up in higher dimension.

Chapter 2 deals with the notion of algebraic hyperbolicity. In the case of projective varieties, people have looked for a characterization of hyperbolicity depending only on algebraic subvarieties. Here, we focus on the so-called algebraic hyperbolicity as in Demailly [17], which is by definition a uniform lower bound on the genus of algebraic curves in terms of their degree once a polarization is fixed. We first discuss a nowadays classical result of Bogomolov about the finiteness of rational and elliptic curves on algebraic surfaces of general type with positive Segre class. Then, motivated by the Kobayashi conjecture which predicts the hyperbolicity of generic projective hypersurfaces of high degree, we explain an algebraic analogue of this conjecture which has been proved in the works of Clemens [12], Ein [30], Voisin [57] and Pacienza [44]. We focus here on the approach coming from ideas of Voisin which makes an essential use of the universal family $\mathcal{X} \subset \mathbb{P}^{n+1} \times \mathbb{P}(H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d)))$ of projective hypersurfaces in $\mathbb{P}^{n+1}$ of a given degree $d > 0$. This object turns out to be very useful because of the positivity properties of its tangent bundle. The existence of sufficiently many vector fields with controlled pole order is used to prove that generic projective hypersurfaces satisfy the conjecture of Lang claiming that a projective manifold is hyperbolic if and only if all its subvarieties are of general type.

Starting from Chapter 3, we enter in the core of this survey, turning to the study of transcendental objects. We describe in detail the constructions of jet bundles (introduced in this formalism by [31]) following closely the presentation of projectivized jet bundles of Demailly [17] as an inductive procedure in the category of directed manifolds (X, V) where V is a holomorphic subbundle of the tangent bundle T_X. This tower of projectivized bundles is naturally endowed with tautological line bundles at each stage. Considering the sheaf of sections of the direct images of these line bundles leads to the concepts of (invariant) jet differentials which are more concretely interpreted as algebraic differential operators $Q(f', f'', \dots, f^{(k)})$ acting on jets of germs of holomorphic curves. The algebraic structure of these vector bundles $E_{k,m}T_X^*$ of invariant jet differentials leads to interesting (and difficult) questions in invariant theory which were intensively investigated recently [5, 40, 49].

In Chapter 4, we begin by recalling classical notions of Hermitian geometry, such as curvature and positivity of Hermitian line bundles on complex manifolds. A basic idea is that Kobayashi hyperbolicity is somehow related with suitable properties of negativity of the curvature of the manifold even in dimension greater than one. We formalize this heuristic concept by means of the Ahlfors-Schwarz lemma in connection with invariant jet differentials: we illustrate the general philosophy whose key point is that global jet differentials vanishing along an ample divisor provide algebraic differential equations which every entire curve must satisfy.

It is then possible to state a general strategy which leads to sufficient conditions in order to have algebraic degeneracy of entire curves in a given compact complex manifold. The first step consists in finding a global section of the bundle of jet differentials vanishing on an ample divisor. The second step should produce much more differential equations, enough to impose sufficiently many conditions on the entire curves to force their algebraic degeneracy. One way to do this is to generalize the ideas described in Chapter 2 about vector fields. Following the strategy of Siu [53], one should now consider vector fields tangent to the jet space. As a jet differential is after all a function on the jet space, one can differentiate it with vector fields and obtain new jet differentials. Of course, one has to guarantee that these new differential operators still vanish on an ample divisor. So, one is forced to have a precise control of the pole order of the vector fields constructed (which, e.g., in the case of projective hypersurface, should not depend on the degree of the hypersurface itself).

The general strategy presented in Chapter 4 is not directly applicable to deal with projective hypersurfaces. To illustrate the modification needed in order to be able to run it, we present in Chapter 5 the solution of the Kobayashi conjecture for generic surfaces in projective 3-space, after [16, 39] and [45]. In particular, we show how to find global invariant jet differentials vanishing along an ample divisor on a projective surface of general type by means of Riemann-Roch-type computations together with a vanishing theorem for the higher cohomology groups by Bogomolov. Then we explain in great detail how to produce meromorphic vector fields of controlled pole order on the universal family of degree d surfaces in $\mathbb{P}^3$. Finally, with these two ingredients available, we adapt the aforesaid general strategy to obtain the conclusion that very generic projective surfaces of degree greater than or equal to 90 in projective 3-space are Kobayashi hyperbolic. This is far from being an optimal bound and it is even far from the bound obtained independently by Mc Quillan, Demailly-El Goul and Păun, but the strategy presented here is the only one which we were able to generalize in higher dimension.

The last chapter is devoted to the recent result on algebraic degeneracy of entire curves in generic projective hypersurfaces of high degree obtained in [23]. In the higher-dimensional case, the non-vanishing of the higher cohomology groups creates new conceptual difficulties. On the other hand, the extension to all dimensions of the existence of lots of meromorphic vector fields with controlled pole order presents "only" new technical difficulties, while the conceptual nature of the construction remains the same of the one described in Chapter 5. Therefore, we have decided to concentrate ourself more on the general proof of the existence of global invariant jet differentials—first in dimension three and then in the general case.

One way to control the cohomology is to use the holomorphic Morse inequalities of Demailly. If one can compute the Euler characteristic of the bundle of jet differentials $E_{k,m}T_X^*$ and then find upper bounds for the higher even cohomology groups $H^{2i}(X, E_{k,m}T_X^*)$ using the weak Morse inequalities, the first step is achieved as in dimension three [49]. Unfortunately, in general, the control of the cohomology is quite involved; thus one tries to apply directly the strong Morse inequalities to twisted tautological bundles on the projectivized jet bundles. This permits to obtain global jet differentials on hypersurfaces of sufficiently high degree [22] in every dimension.

Then Siu's strategy of exhibiting vector fields is realized on the jet spaces of the universal hypersurfaces [41, 50]. Finally, the full strategy is used to obtain the algebraic degeneracy of entire curves in generic projective hypersurfaces of degree larger than 2^{n^5} [23].

Last but not least, we would like to warmly thank Alcides Lins Neto, Jorge Vitório Pereira, Paulo Sad and all the people of the IMPA for having organized this course and our stay in Rio. These have been very stimulating, interesting and, why not, funny days.

Preface to the second edition

Since the first publication of the present monograph in 2011, several important advances have been made in the theory of Kobayashi hyperbolicity of projective manifolds. These will not be included in this edition, but we would like nevertheless to take the opportunity here to give an overview of the main results together with some references. We shall restrict our attention only on issues concerning existence of jet differentials and hyperbolicity of complete intersection varieties.

As we hope it is clear from the present monograph, one of the crucial steps in order to prove hyperbolicity-type statements for projective manifolds is to construct (many) global jet differentials vanishing on an ample divisor on the given manifold X. The general philosophy spread out by Lang conjectures suggests that one should be able to prove such an existence statement only under some positivity condition of the canonical bundle K_X of X, namely, its bigness. In the special case of hypersurfaces in $\mathbb{P}^{n+1}$, people have concentrated their effort to construct global invariant jet differentials of order n (this is the smallest possible order). In order to have such existence theorems, one is forced to look at (very) high-degree hypersurfaces, while the canonical bundle of the hypersurface is big—in fact ample—as soon as the degree is greater than $n + 2$. In order to have an optimal existence theorem, as far as the degree of the hypersurface is concerned, one needs thus to look for jet differentials of higher and higher order. This is achieved in [43], where the author is able—thanks to a real *tour de force*—to control the composition series of the graded ring associated to the Green-Griffiths jet differential vector bundle, as well as the growth of its higher cohomology.

A real breakthrough was subsequently made by J.-P. Demailly in his paper [19]. Demailly provides a full answer in complete generality to the question of existence of global jet differentials. A simplified version of his main result may be stated as follows: for every projective manifold of general type, there exists an order $k \gg 1$ such that the growth of the space of global jet differentials of order k and weighted degree m is maximal with respect to m. This shows in particular that as soon as a projective manifold X is of general type, every entire curve traced in X must satisfy some constraints given in terms of algebraic differential equations.

This circle of ideas and also the (highly nontrivial) techniques employed in the proofs were very clearly explained in the Bourbaki talk given by M. Păun in 2013. We warmly recommend the corresponding article [46] for all the details (the interested reader will find therein also a somewhat simplified proof of the crucial effective estimate of [23]).

Unfortunately, a series of examples given in [26] show that it is not possible in general to obtain algebraic degeneracy-type statements for manifolds of general type using purely jet differentials techniques. A new idea is thus needed to attack the Green-Griffiths-Lang conjecture!

Concerning the Kobayashi conjecture for (very) general projective hypersurfaces of high degree, we explain in great detail in this monograph how in [23] the authors are able to pursue Siu's strategy to obtain hyperbolicity with an effective control of the degree (although just obtaining the weaker statement of algebraic degeneracy). The size of the bound in *ibid.* is a double exponential, namely, 2^{n^5}, and it has been subsequently improved to n^{9n} in [4] (see also [3]) and to $\left\lfloor \frac{n^4}{3}\big(n\log\big(n\log(24n)\big)\big)^n \right\rfloor$ in [20].

Siu's strategy was only very briefly outlined in his survey [53], and the author referred there to an unpublished preprint for the details. This preprint was finally available on the ArXiv in 2012 and published only very recently [54]. Let us also mention that the bound on the degree following [54] are only in principle effective, but hard to make explicit.

Let us finish this brief update talking about general complete intersection in projective space. In principle, a (very) generic complete intersection should have more and more hyperbolicity properties (here in a vague sense) when its codimension increases (being obtained by cutting more and more with projective hypersurfaces). For instance, inspired by an analogue for complete intersections in abelian varieties, O. Debarre proposes in his paper [14] (see also [15]) the following conjecture: *the cotangent bundle of the intersection in $\mathbb{P}^n$ of at least $n/2$ general hypersurfaces of sufficiently high degree is ample* (see also [27] for a related more general conjecture involving the bundle of invariant jet differentials). This implies of course (in a very strong sense) hyperbolicity of the complete intersections in question.

The first important results in this direction were obtained by D. Brotbek in [10], where he was able to prove Debarre's conjecture for complete intersection surfaces in $\mathbb{P}^n$, $n \geq 4$. In *ibid.*, he obtained also bigness statements for these cotangent bundles in all dimensions, as well as the hyperbolicity of the intersection of at least $n/3$ general hypersurfaces of sufficiently high degree. Subsequently, Brotbek was also able to prove (among other things) in [9] a special case of the above-mentioned conjecture, namely, the ampleness of the cotangent bundle of the intersection of at least $\frac{3n-2}{4}$ general hypersurfaces *of the same* high degree.

The very explicit methods introduced in [9] have been pushed forward so far by S.-Y. Xie that he was recently able to give a proof of the Debarre conjecture, in its complete generality; the corresponding (94 pages long!) preprint [58] (see also [59]) can be found on the ArXiv.

Let us finally cite the preprint [2], where the authors prove the following very interesting statement: in any smooth projective variety X, for each $n \leq \dim X/2$, there exists a smooth subvariety of dimension n with ample cotangent bundle.

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Added in proof. In the very recent preprint [61], D.Brotbek proposes a proof of the Kobayashi conjecture (unfortunately without giving an effective bound on the degree). His statement is thus the following: *a general sufficiently ample hypersurface in a smooth projective variety is hyperbolic*. Even without an explicit lower bound for the degree of the hypersurface, if his proof reveals to be correct, this will be of course a central and major achievement in the subject.

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