# Chapter 2
# Banach Spaces and Banach Lattices

We shall now give some background in the theory of normed and Banach spaces, including the key definitions of dual and bidual spaces and of an isomorphism and an isometric isomorphism between two normed spaces. In particular, we shall show how certain bidual spaces can be embedded in other Banach spaces. In §2.3, we shall also recall some basic results and theorems concerning Banach lattices. We shall define complemented subspaces of a Banach space in §2.4, and also we shall discuss, in §2.5, the projective and injective objects in the category of Banach spaces and bounded operators. We shall conclude the chapter by discussing dentability and the Krein–Milman property for Banach spaces in §2.6.

## 2.1 Banach spaces

We now recall the basics of the Banach-space theory that we shall use.

There is a huge literature on the theory of normed and Banach spaces; for example, see [3, 6, 30, 82, 85, 94, 100, 166, 175, 176, 183, 218, 225]. There is a collection of instructive essays on topics in Banach-space theory in [147]. We shall regard the texts of Albiac and Kalton [3], Allan [6], and Rudin [218] as accessible and elementary accounts and shall rarely repeat proofs from those sources.

Let $E$ be a linear space or a real-linear space, with underlying field $\mathbb{K}$, still always $\mathbb{C}$ or $\mathbb{R}$. A *semi-norm* on $E$ is a map $p : E \to \mathbb{R}^+$ such that

$$p(x+y) \leq p(x) + p(y) \quad (x, y \in E), \quad p(\alpha x) = |\alpha|\, p(x) \quad (\alpha \in \mathbb{K}, x \in E);$$

the semi-norm is a *norm* if, further, $p(x) = 0$ if and only if $x = 0$. Then $(E, \|\cdot\|)$ is a *normed space* if $\|\cdot\|$ is a norm on $E$; $(E, \|\cdot\|)$ is a *Banach space* if it is complete with respect to the metric $d_E$ defined by

$$d_E(x,y) = \|x-y\| \quad (x, y \in E).$$



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For example, $(C^b(X), |\cdot|_X)$ is a Banach space for each non-empty topological space $X$.

Let $E$ be a normed space. Then there is a Banach space containing $E$ as a dense subspace (with the same norm); the latter space is the *completion* of $E$. Let $p$ be a semi-norm on a linear space $E$, and set $F = \{x \in E : p(x) = 0\}$. Then we can regard $p$ as a norm on the quotient space $E/F$ and on the completion of $E/F$.

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a linear space $E$ are *equivalent* it there exist constants $m, M > 0$ such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \quad (x \in E),$$

and so the two norms define equivalent metrics and the same topology on $E$. For example, any two norms on the linear space $\mathbb{C}^n$, where $n \in \mathbb{N}$, are equivalent.

Let $F$ be a real linear space, with complexification $E = F \oplus \mathrm{i}F$, so that $E$ is a (complex) linear space. Suppose that $F$ is a normed space. Then $E$ is a normed space for the norm specified by

$$\|x + \mathrm{i}y\| = \sup\{\|x\cos\theta - y\sin\theta\| : \theta \in [0, 2\pi]\},$$

and $F$ is a closed real-linear subspace of $E$; $E$ is a Banach space whenever $F$ is a Banach space. But the above is not always the most appropriate choice of a norm on $F$; indeed, various choices for various different purposes can be made. For a discussion of this point, see [187]. For example, for the norm on the complexification of a Banach lattice, see equation (2.7).

Let $(E, \|\cdot\|)$ be a normed space. We denote by $E_{[1]}$ the *closed unit ball* of $E$; more generally,

$$E_{[r]} = \{x \in E : \|x\| \leq r\} \quad \text{and} \quad B_r(x) = \{y \in E : \|y - x\| < r\}$$

for $r \geq 0$ and $x \in E$; the *unit sphere* of $E$ is

$$S_E = \{x \in E : \|x\| = 1\}.$$

A *barrel* in $E$ is a closed, bounded, absolutely convex, absorbent set; in the case where $E$ is a Banach space, each of these is the closed unit ball of $E$ with respect to a norm on $E$ that is equivalent to the given norm.

Let $F$ be a closed subspace of a normed space $(E, \|\cdot\|)$, with quotient map $\pi : E \to E/F$. Set

$$\|x + F\| = \inf\{\|x + y\| : y \in F\} = \inf\{\|z\| : z \in E, \pi(z) = x + F\} \quad (x \in E).$$

Then $\|\cdot\|$ is the *quotient norm* on $E/F$; always $(E/F, \|\cdot\|)$ is a normed space, called the *quotient space*, and the quotient map $\pi$ is continuous and open; $(E/F, \|\cdot\|)$ is a Banach space whenever $E$ is a Banach space.

A (Hausdorff) *locally convex space* is a linear space $E$ (over $\mathbb{R}$ or $\mathbb{C}$) with a collection $\mathscr{P}$ of semi-norms on $E$ such that $\mathscr{P}$ separates the points of $E$, in the sense that, for each $x \in E$ with $x \neq 0$, there exists $p \in \mathscr{P}$ with $p(x) \neq 0$. We define a topology on $E$ by saying that a subset $U$ of $E$ is open if, for each $x \in U$, there are $p_1, \dots, p_n \in \mathscr{P}$ and $\varepsilon > 0$ such that

$$\{y \in E : p_j(y-x) < \varepsilon \ (j \in \mathbb{N}_n)\} \subset U.$$

A *topological linear space* is a linear space with a Hausdorff topology such that addition and scalar multiplication are continuous. A topological linear space $E$ is a locally convex space if and only if there is a base of neighbourhoods of $0_E$ consisting of convex sets.

Let $E$ be a locally convex space. We denote by $E'$ the *dual* space of $E$, so that $E'$ is the space of all continuous linear functionals on $E$. The action of $\lambda \in E'$ on $x \in E$ gives the complex number $\lambda(x)$ that we shall usually denote by $\langle x, \lambda \rangle$.

In the case where $E$ is a normed space, the dual space $E'$ is itself a Banach space for the norm specified by

$$\|\lambda\| = \sup\{|\langle x, \lambda \rangle| : x \in E_{[1]}\} \quad (\lambda \in E').$$

The dual space $(E')'$ of $(E', \|\cdot\|)$ is denoted by $E''$; it is called the second dual or *bidual space* of $E$. Occasionally, we shall refer to the third dual of $E$; this is $E''' = (E'')'$.

For examples of locally convex spaces, let $E$ be a normed space, and define

$$p_\lambda(x) = |\langle x, \lambda \rangle| \quad (x \in E)$$

for each $\lambda \in E'$. Then each $p_\lambda$ is a semi-norm on $E$, and the family $\{p_\lambda : \lambda \in E'\}$ defines a topology, called $\sigma(E, E')$, with respect to which $E$ is a locally convex space; this topology is the *weak topology* on $E$. Let $(x_\gamma)$ be a net in $E$, and take $x \in E$. Then $\lim_\gamma x_\gamma = x$ weakly (i.e., with respect to the weak topology) if and only if $\lim_\gamma \langle x_\gamma, \lambda \rangle = \langle x, \lambda \rangle$ $(\lambda \in E')$. The closure of a set $S$ in $E$ with respect to the weak topology is called the *weak closure*, etc.

Now define

$$p_x(\lambda) = |\langle x, \lambda \rangle| \quad (\lambda \in E')$$

for each $x \in E$. Then each $p_x$ is a semi-norm on $E'$, and the family $\{p_x : x \in E\}$ defines a topology, called $\sigma(E', E)$, with respect to which $E'$ is a locally convex space. The topology $\sigma(E', E)$ is the *weak$^*$ topology* on $E'$. Clearly $\sigma(E', E) \subset \sigma(E', E'')$; every weakly convergent net in $E'$ is weak$^*$-convergent. We have $(E, \sigma(E, E'))' = E'$ and $(E', \sigma(E', E))' = E$, for example. Later we shall use the weak$^*$ topology $\sigma(E'', E')$ on $E''$.

For a discussion of locally convex spaces and these topologies, see [6, 68, 94, 144, 183, 218], for example.

We shall mention the following class of spaces at a few later points; in particular, see §4.5.

**Definition 2.1.1.** Let $E$ be a Banach space. Then $E$ is a *Grothendieck space*, or $E$ has the *Grothendieck property*, if every weak$^*$-convergent sequence in $E'$ is weakly convergent.

The proto-typical example of a Grothendieck space is the space $C(K)$, where $K$ is a Stonean space [124, Théorème 9, p. 168], as we shall show in Theorem 4.5.6. This class of examples includes the spaces $\ell^\infty(S)$ for each set $S$ as particular instances; a generalization of these examples will be noted in Example 6.7.1. Many characterizations of Grothendieck space are listed, without proofs, in [85, Theorem p. 179]; some of these are proved in [184, Proposition 5.3.10].

The following is a form of the *Hahn–Banach theorem*; see [6, Corollaries 3.4 and 3.27], for example.

**Theorem 2.1.2.** (i) *Let $E$ be a normed space, and suppose that $F$ is a linear subspace of $E$. Take $\lambda \in F'$. Then there exists $\Lambda \in E'$ with $\Lambda \mid F = \lambda$ and $\|\Lambda\| = \|\lambda\|$.*

(ii) *Let $E$ be a real locally convex space, and let $A$ and $B$ be non-empty, convex subsets of $E$ with $A$ compact, $B$ closed, and $A \cap B = \emptyset$. Then there exists $\lambda \in E'$ with*

$$\sup_{x \in A} \langle x, \lambda \rangle < \inf_{x \in B} \langle x, \lambda \rangle .$$

(iii) *Let $E$ be a complex locally convex space, and suppose that $B$ is an absolutely convex, closed subset of $E$ and that $x_0 \in E \setminus B$. Then there exists $\lambda \in E'$ with $|\langle x, \lambda \rangle| \leq 1 \;\; (x \in B)$ and $\langle x_0, \lambda \rangle > 1$.* □

The functional $\Lambda$ in clause (i), above, is a *norm-preserving extension* of $\lambda$.

**Corollary 2.1.3.** *Let $E$ be a normed space, and let $S$ be a circled subset of $S_E$. Then $\overline{\mathrm{co}}\, S = E_{[1]}$ if and only if*

$$\|\lambda\| \leq \sup\{|\langle x, \lambda \rangle| : x \in S\} \quad (\lambda \in E') . \tag{2.1}$$

*Proof.* Suppose that $\overline{\mathrm{co}}\, S = E_{[1]}$, and take $\lambda \in E'$. For each $\varepsilon > 0$, there exist $n \in \mathbb{N}$, $x_1, \dots, x_n \in S$, and $\alpha_1, \dots \alpha_n \in \mathbb{I}$ with $\sum_{i=1}^n \alpha_i = 1$ such that

$$\left| \left\langle \sum_{i=1}^n \alpha_i x_i, \lambda \right\rangle \right| > \|\lambda\| - \varepsilon ,$$

and so $|\langle x_i, \lambda \rangle| > \|\lambda\| - \varepsilon$ for some $i \in \mathbb{N}_n$. Hence (2.1) follows.

Conversely, assume that there exists $x_0 \in E_{[1]}$ with $x_0 \notin \overline{\mathrm{co}}\, S$. Since $S$ is circled, the set $\overline{\mathrm{co}}\, S$ is absolutely convex, and so, by Theorem 2.1.2(iii), there exists $\lambda \in E'$ with $|\langle x, \lambda \rangle| \leq 1 \;\; (x \in S)$, but with $\langle x_0, \lambda \rangle > 1$, a contradiction of equation (2.1). Thus $\overline{\mathrm{co}}\, S = E_{[1]}$. □

Let $E$ be a normed space. It follows from Theorem 2.1.2(i) that, for each $x \in E$, there exists $\lambda \in S_{E'}$ with $\|x\| = \langle x, \lambda \rangle$. The action of $\Phi \in E''$ on $\lambda \in E'$ gives the complex number $\langle \Phi, \lambda \rangle$, and we define the *canonical embedding* $\kappa_E : E \to E''$ by

$$\langle \kappa_E(x), \lambda \rangle = \langle x, \lambda \rangle \quad (x \in E, \lambda \in E') .$$

Clearly $\kappa_E$ is a linear map; by our remark, $\|\kappa_E(x)\| = \|x\| \ (x \in E)$, and so $\kappa_E$ identifies $E$ as a closed subspace of $E''$; the space $E$ is *reflexive* if $E = E''$ under this identification. For example, $\ell^p$ and $L^p(\mathbb{I})$ are both reflexive whenever $1 < p < \infty$.

Here are some further standard theorems; the weak* topology on $E'$ is $\sigma(E', E)$.

**Theorem 2.1.4.** *Let E be a Banach space.*

(i) *The closed unit ball* $E'_{[1]}$ *is weak*-compact and convex.*

(ii) *The space* $\kappa_E(E_{[1]})$ *is weak*-dense in* $E''_{[1]}$.

(iii) *The weak* topology on* $E'_{[1]}$ *is metrizable if and only if* $(E, \|\cdot\|)$ *is separable.*

(iv) *The following conditions on a linear functional* M *on* $E'$ *are equivalent:*

(a) $\mathrm{M} \in \kappa_E(E)$;

(b) M *is weak*-continuous on* $E'$;

(c) M *is weak*-continuous on* $E'_{[1]}$.

(v) *The weak and norm closures of a convex subset of E coincide.*

(vi) *A convex set C in* $E'$ *is weak*-closed if and only if* $C \cap E'_{[r]}$ *is weak*-closed for each* $r > 0$.

(vii) *A subset S of E is relatively weakly compact if and only if each countable, infinite subset of S has a weak limit point in E if and only if each sequence in S has a subsequence converging weakly in E.*

*Proof.* Clause (i) is the *Banach–Alaoglu theorem*; see [6, Theorem 3.21]. Clause (ii) is *Goldstine's theorem*; see [6, Corollary 3.30]. For (iii) and (iv), see [94, Theorems V.5.1, V.5.6], for example. Clause (v) is *Mazur's theorem* [6, Corollary 3.28], clause (vi) is the *Krein–Šmulian theorem* [94, V.5.7], and clause (vii) is the *Eberlein–Šmulian theorem* [3, Theorem 1.6.3]. □

**Proposition 2.1.5.** *Let X be a completely regular topological space. Then there exists a compactification K of X such that each* $f \in C^b(X)$ *has an extension to a function* $f^\beta \in C(K)$.

*Proof.* Set $E = (C^b(X), |\cdot|_X)$, a Banach space. The weak* topology on $E'$ is denoted by $\sigma$, so that $(E'_{[1]}, \sigma)$ is compact by Theorem 2.1.4(i). For $x \in X$, define $\varepsilon(x)$ on $E$ by

$$\varepsilon(x)(f) = f(x) \quad (f \in E).$$

Then $\varepsilon(x) \in E'_{[1]} \ (x \in X)$, and the map $\varepsilon : X \to (E'_{[1]}, \sigma)$ is a continuous injection; since $X$ is completely regular, it is easily seen that $\varepsilon$ is a homeomorphism onto its range, and so we can regard $X$ as a subspace of $(E'_{[1]}, \sigma)$. Take $K$ to be the closure of $X$ in $(E'_{[1]}, \sigma)$, so that $K$ is a compactification of $X$, and, for $f \in C^b(X)$, define $f^\beta$ on $K$ by

$$f^\beta(\lambda) = \langle f, \lambda \rangle \quad (\lambda \in K).$$

Then $f^\beta \in C(K)$ and $f^\beta$ extends $f$, identified with $f \circ \varepsilon$. □

Thus $K = \beta X$ is the Stone–Čech compactification of $X$, as discussed in §1.5.

Recall that $w(X)$ and $d(X)$ are the weight and density character, respectively, of a topological space $X$.

**Proposition 2.1.6.** *Let $E$ be a normed space. Then $d(E) \le d(E')$. In particular, $E$ is separable whenever $E'$ is separable.*

*Proof.* We may suppose that $d(E')$ is infinite.

Let $S$ be the unit sphere of $E'$. Then there is a dense subset, say $\{\lambda_\alpha : \alpha \in A\}$, of $S$ with $|A| = d(E')$. For each $\alpha \in A$, choose $x_\alpha \in E_{[1]}$ with $|\langle x_\alpha, \lambda_\alpha \rangle| > 1/2$, and set $F = \mathrm{lin}\{x_\alpha : \alpha \in A\}$.

Assume towards a contradiction that $F$ is not dense in $E$. By the Hahn–Banach theorem, there exists $\lambda \in S$ with $\lambda \mid F = 0$. There exists $\alpha \in A$ with $\|\lambda_\alpha - \lambda\| < 1/2$, and so

$$\frac{1}{2} < |\langle x_\alpha, \lambda_\alpha \rangle| \le |\langle x_\alpha, \lambda_\alpha - \lambda \rangle| + |\langle x_\alpha, \lambda \rangle| = |\langle x_\alpha, \lambda_\alpha - \lambda \rangle| \le \|\lambda_\alpha - \lambda\| < \frac{1}{2},$$

a contradiction. Thus $\overline{F} = E$.

It follows that linear combinations with coefficients in $\mathbb{Q} + \mathrm{i}\mathbb{Q}$ of the elements $x_\alpha$ constitute a dense subset of $E$ with cardinality $|A|$. Hence $d(E) \le d(E')$. $\square$

**Theorem 2.1.7.** *Let $K$ be a non-empty, compact space. Then:*

(i) *$C(K)$ is separable if and only if $K$ is metrizable;*

(ii) $w(K) = d(C(K))$.

*Proof.* For (i) and (ii), it is clearly sufficient to prove the analogous results for the real Banach space $C_{\mathbb{R}}(K)$; set $E = C_{\mathbb{R}}(K)$.

We regard $K$ as a subset of $(E'_{[1]}, \sigma(E', E))$ by identifying $x \in K$ with $\varepsilon_x \in E'_{[1]}$, where $\varepsilon_x(f) = f(x)$ $(f \in E)$, as above. The restriction of the topology $\sigma(E', E)$ to $K$ is the original topology on $K$.

(i) Suppose that $E$ is separable. By Theorem 2.1.4(iii), $(E'_{[1]}, \sigma(E', E))$ is metrizable, and so $K$ is metrizable.

Conversely, suppose that $d$ is a metric that defines the topology of $K$. Then $(K, d)$ is separable, say $\{x_n : n \in \mathbb{N}\}$ is a dense subset of $K$. For $n \in \mathbb{N}$, define $f_n \in E$ by setting $f_n(x) = d(x, x_n)$ $(x \in K)$. Let $A$ and $B$ be the subsets of $E$ formed by taking all the elements $p(1, f_1, \dots, f_n)$, where $p$ is a polynomial in $n+1$ variables, $n \in \mathbb{N}$, and $p$ has coefficients in $\mathbb{Q}$ and $\mathbb{R}$, respectively. Then $A$ is countable and dense in $B$, and $B$ is a subalgebra of $E$. Further, $B$ contains the constants and separates strongly the points of $K$. By the Stone–Weierstrass theorem, Theorem 1.4.26(i), $B$ is dense in $(E, |\cdot|_K)$, and so $A$ is also dense in this space. Thus $E$ is separable.

(ii) Set $\kappa = d(E)$; necessarily $\kappa$ is infinite.

Let $\{f_\alpha : \alpha < \kappa\}$ be a dense subset of $E$, and define $U_\alpha = \{x \in K : f_\alpha(x) > 0\}$ for $\alpha < \kappa$. It is easy to check that $\{U_\alpha : \alpha < \kappa\}$ is a subbase for the topology of $K$, and so $w(K) \le \kappa$.

Conversely, let $\mathscr{B}$ be a base for the topology of $K$ with $|\mathscr{B}| = w(K)$, and let $\mathscr{A}$ be the family of all pairs $(U,V) \in \mathscr{B} \times \mathscr{B}$ such that $\overline{U} \cap \overline{V} = \emptyset$, so that $|\mathscr{A}| = w(K)$. By Urysohn's lemma, Theorem 1.4.25, for each $(U,V) \in \mathscr{A}$, there exists $f \in C(K,\mathbb{I})$ with $f \mid U = 1$ and $f \mid V = 0$. Form the sets $A$ and $B$ with respect to these functions $f$ as in (i). Then $|A| = w(K)$, and again $A$ is dense in $E$ by the Stone–Weierstrass theorem. Thus $\kappa \leq w(K)$. □

**Corollary 2.1.8.** *Let $E$ be a normed space, and set $B = (E'_{[1]}, \sigma(E',E))$. Then $w(B) = d(E)$.*

*Proof.* The space $B$ is compact. We regard $E$ as a linear subspace of $C(B)$ that separates the points of $B$. By Corollary 1.4.27, $d(E) = d(C(B))$, and, by Theorem 2.1.7(ii), $d(C(B)) = w(B)$. Hence $d(E) = w(B)$. □

The following result was originally proved by Choquet [54, p. 7] by a rather indirect and complicated argument; our simple proof is taken from [104, Proposition 2.9] and [201, Proposition 1.3].

**Proposition 2.1.9.** *Let $K$ be a compact, convex set in a locally convex space $E$. Suppose that the relative topology on $K$ is metrizable. Then* $\operatorname{ex} K$ *is a $G_\delta$-set in $E$. Further,* $\operatorname{ex} K$ *is either countable or has cardinality* $\mathfrak{c}$.

*Proof.* Let $d$ be a metric that gives the relative topology on $K$ from $E$. For $n \in \mathbb{N}$, take $K_n$ to be the set of points $x$ in $K$ such that $2x = y + z$ for some $y, z \in K$ for which $d(y,z) \geq 1/n$. Then each $K_n$ is closed in $K$, and the complement of the union of the sets $K_n$ is a $G_\delta$-set. But this set is exactly $\operatorname{ex} K$.

By Proposition 1.4.14, $\operatorname{ex} K$ is either countable or has cardinality $\mathfrak{c}$. □

Let $E$ be a normed space, and let $* : E \to E$ be an isometric linear involution on $E$. For $\lambda \in E'$, define $\lambda^* \in E'$ by

$$\langle x, \lambda^* \rangle = \overline{\langle x^*, \lambda \rangle} \quad (x \in E).$$

Then the map $* : \lambda \mapsto \lambda^*,\ E' \to E'$, is an isometric linear involution; this map is clearly also continuous with respect to the topology $\sigma(E',E)$. Continuing, we obtain an isometric linear involution $*$ on $E''$; the restriction of this linear involution to the subspace $E$ of $E''$ is the original linear involution.

Let $\{(E_\alpha, \|\cdot\|_\alpha) : \alpha \in A\}$ be a family of normed spaces, defined for each $\alpha$ in a non-empty index set $A$ (perhaps finite). Then we shall consider the following spaces.

First set

$$\bigoplus_\infty E_\alpha = \left\{ (x_\alpha : \alpha \in A) : \|(x_\alpha)\| = \sup_\alpha \|x_\alpha\|_\alpha < \infty \right\}.$$

Similarly, for $p$ with $1 \le p < \infty$, we define

$$\bigoplus_p E_\alpha = \left\{ (x_\alpha : \alpha \in A) : \|(x_\alpha)\| = \left( \sum_\alpha \|x_\alpha\|_\alpha^p \right)^{1/p} < \infty \right\}.$$

Clearly $\bigoplus_\infty E_\alpha$ and $\bigoplus_p E_\alpha$ are normed spaces; they are Banach spaces if each of the spaces $E_\alpha$ is a Banach space. We write

$$E \oplus_\infty F \quad \text{and} \quad E \oplus_p F$$

for the sum of two normed spaces $E$ and $F$ with the appropriate norms, etc., and we write $\ell_n^p(E)$ for $E^n$ with the norm given by

$$\|(x_1, \dots, x_n)\| = \left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p} \qquad (x_1, \dots, x_n \in E).$$

We write $\ell^p(E)$ and $\ell^\infty(E)$ for $\bigoplus_p \{E_n : n \in \mathbb{N}\}$ and $\bigoplus_\infty \{E_n : n \in \mathbb{N}\}$, respectively, when each $E_n$ is equal to $E$.

Take $p$ with $1 \le p < \infty$. Then it is easy to see that we can identify the dual space of the normed space $\bigoplus_p E_\alpha$ with $\bigoplus_q E'_\alpha$, where $q$ is the conjugate index to $p$, with the obvious duality, so that

$$\left( \bigoplus_p E_\alpha \right)' = \bigoplus_q E'_\alpha . \tag{2.2}$$

We shall sometimes refer to $E \times F$, the *product* of two Banach spaces $E$ and $F$; there are many equivalent norms on $E \times F$ making it into a Banach space. For example, we could take the norm to be given by

$$\|(x, y)\| = \|x\| + \|y\| \quad (x \in E, y \in F).$$

Thus we can identify this space with $E \oplus_1 F$ in an obvious way.

**Proposition 2.1.10.** *Let $E$ be a normed space, and suppose that $F$ and $G$ are subspaces such that $E = F \oplus_1 G$. Then $\operatorname{ex} E_{[1]} = \operatorname{ex} F_{[1]} \cup \operatorname{ex} G_{[1]}$.*

*Proof.* First take $x \in E$ with $\|x\| = 1$. We *claim* that there exist $y \in F$ and $z \in G$ with $\|y\| = \|z\| = 1$ and $\alpha, \beta \in \mathbb{I}$ with $\alpha + \beta = 1$ such that $x = \alpha y + \beta z$. Indeed, set $x = y_1 + z_1$ with $y_1 \in F$ and $z_1 \in G$ and $\|y_1\| + \|z_1\| = 1$. We may suppose that $y_1 \neq 0$ and $z_1 \neq 0$, for otherwise the claim is trivial. Set $\alpha = \|y_1\|$ and $\beta = \|z_1\|$, so that $\alpha, \beta > 0$ and $\alpha + \beta = 1$, and set $y = \alpha^{-1} y_1$ and $z = \beta^{-1} z_1$. Then the requirements are satisfied.

To show that $\operatorname{ex} E_{[1]} \subset \operatorname{ex} F_{[1]} \cup \operatorname{ex} G_{[1]}$, take $x \in \operatorname{ex} X_{[1]}$. Then $\|x\| = 1$. If $x \in F$, then trivially $x \in \operatorname{ex} F_{[1]}$, and similarly if $x \in G$. Assume that $x \notin F \cup G$. By the claim, $x$ is a convex combination of two norm 1 elements from $F$ and $G$, with coefficients in $(0, 1)$, a contradiction.

The reverse inclusion is trivial. $\square$

## 2.2 Isomorphisms and isometric isomorphisms

Let $E$ and $F$ be normed spaces over $\mathbb{R}$ or $\mathbb{C}$. Then a linear operator $T$ from $E$ to $F$ is bounded if and only if it is continuous on $E$ if and only if it is continuous at $0_E$, and then

$$\|T\| = \sup\{\|Tx\| : x \in E_{[1]}\}$$

defines the *operator norm* $\|\cdot\|$ of $T$. The linear space $\mathscr{B}(E,F)$ of all bounded linear operators from $E$ to $F$ is itself a normed space with respect to the operator norm; we write $\mathscr{B}(E)$ for $\mathscr{B}(E,E)$, so that $\mathscr{B}(E)$ is a normed algebra with respect to the composition of operators; see §3.1. The space $\mathscr{B}(E,F)$ is a Banach space whenever $F$ is a Banach space. Of course the basic inequality is that

$$\|Tx\| \leq \|T\|\,\|x\| \quad (x \in E,\, T \in \mathscr{B}(E,F))\,.$$

An operator $T \in \mathscr{B}(E,F)$ is a *contraction* if $\|T\| \leq 1$; a projection in $\mathscr{B}(E)$ is a *bounded projections* on $E$.

The following is the famous *uniform boundedness theorem.*

**Theorem 2.2.1.** *Let $E$ be a Banach space, let $\{E_\alpha : \alpha \in A\}$ be a family of normed spaces, and let $T_\alpha : E \to E_\alpha$ be a bounded operator for each $\alpha \in A$. Suppose that* $\sup\{\|T_\alpha x\| : \alpha \in A\} < \infty$ *for each $x \in E$. Then* $\sup\{\|T_\alpha\| : \alpha \in A\} < \infty$. □

**Corollary 2.2.2.** *Let $E$ be a normed space. Then a subset of $E$ is bounded if and only if it is weakly bounded.* □

The following is a form of the *open mapping theorem*, together with *Banach's isomorphism theorem.*

**Theorem 2.2.3.** *Let $E$ and $F$ be Banach spaces, and suppose that $T \in \mathscr{B}(E,F)$ is a surjection. Then $T$ is an open mapping. In particular, in the case where $T$ is a bijection, $T^{-1} \in \mathscr{B}(F,E)$.* □

**Definition 2.2.4.** Let $E$ and $F$ be normed spaces. A bijection $T$ in $\mathscr{L}(E,F)$ is an *isomorphism* or a *linear homeomorphism* if both $T$ and $T^{-1}$ are bounded. Two normed spaces $E$ and $F$ are *isomorphic* if there is an isomorphism from $E$ onto $F$, and in this case we write

$$E \sim F\,.$$

An operator $T \in \mathscr{B}(E,F)$ is an *embedding* if it is an isomorphism onto a subspace of $F$, and $E$ *embeds* in $F$ if there is such an embedding.

Of course, in the case where $E$ and $F$ are Banach spaces, each bijection in $\mathscr{B}(E,F)$ is an isomorphism. An operator $T \in \mathscr{B}(E,F)$ is an embedding if and only if there exists $\delta > 0$ with $\|Tx\| \geq \delta \|x\| \ (x \in E)$. Indeed, when we consider an embedding $T : E \to F$ as an isomorphism onto its range, we see that $T$ has a bounded inverse $T^{-1} : T(E) \to E$; clearly, for $x \in E$, we have $\|x\| \leq \|T^{-1}\| \|Tx\|$, and so, when $E \neq \{0\}$, we can take $\delta = \|T^{-1}\|^{-1}$. (The constant $\delta$ is sometimes called the *embedding constant* of $T$.)

**Definition 2.2.5.** Let $\mathscr{C}$ be a class of Banach spaces. Then a property is an *isomorphic invariant* for the class $\mathscr{C}$ if each Banach space $E$ in $\mathscr{C}$ has the property whenever $E$ is isomorphic to another Banach space in $\mathscr{C}$ that has the property.

For example, it is clear that 'separability' and 'having a separable dual space' are isomorphic invariants of the class of all Banach spaces. Also, the Grothendieck property of Definition 2.1.1 is an isomorphic invariant of this class.

Let $E$ and $F$ be normed spaces, and take $T \in \mathscr{B}(E,F)$. Then, as on page 9, $T$ induces a linear map

$$\overline{T} : x + \ker T \mapsto Tx, \quad E/\ker T \to F,$$

such that $\overline{T} : E/\ker T \to T(E)$ is a linear isomorphism from $E/\ker T$ onto $T(E)$. In our present setting, $\ker T$ is closed in $E$, and $\overline{T}$ is bounded with $\|\overline{T}\| \leq \|T\|$ when $E/\ker T$ has the quotient norm; in the case where $E$ and $F$ are Banach spaces and $T$ has closed range, the map $\overline{T} : E/\ker T \to T(E)$ is an embedding.

Many, but not all, Banach spaces $E$ have the property that $E \sim E \times E$. (This is not true, for example, for the James space, $J$, described in [3, p. 233].) In particular, the following is clear.

**Proposition 2.2.6.** *Take $p$ with $1 \leq p \leq \infty$, and let $E$ be either of the two Banach spaces $\ell^p$ and $L^p(\mathbb{I})$. Then $E \sim E \times E$.* □

We caution that it is possible that two (complex) Banach spaces $E$ and $F$ can fail to be isomorphic (as complex Banach spaces), but to be such that their underlying real spaces are isomorphic (as real Banach spaces): see [43] and, for a more elementary example, [151].

The following definition is given in [3, Definition 7.4.5], for example.

**Definition 2.2.7.** Let $E$ and $F$ be isomorphic normed spaces. Then the *Banach-Mazur distance*, $d(E,F)$, from $E$ to $F$ is given by

$$d(E,F) = \inf\{\|T\| \|T^{-1}\| : T \in \mathscr{B}(E,F) \text{ is an isomorphism}\}.$$

**Definition 2.2.8.** Let $E$ and $F$ be normed spaces. A map $T \in \mathscr{B}(E,F)$ is *isometric* if $\|Tx\| = \|x\| \ (x \in E)$, and then $T$ is a *linear isometry*; $T$ is an *isometric isomorphism* if it is a surjective linear isometry from $E$ onto $F$. When there is such an isometric isomorphism, we say that $E$ and $F$ are *isometrically isomorphic* and write

$$E \cong F\,.$$

A linear isometry from $E$ onto a subspace of $F$ is an *isometric embedding*.

Thus $d(E,F) = 1$ whenever $E \cong F$; the converse is true when $E$ and $F$ are both finite-dimensional spaces, but it is not true in general. An isomorphism $T \in \mathscr{B}(E,F)$ is isometric if and only if $T$ and $T^{-1}$ are both contractions.

Sometimes, with a slight abuse of language, we say that '$E = F$' or '$E$ is $F$' when, strictly, we mean that '$E \cong F$'. For example, in the case where $1 \le p < \infty$, we say that the duals of the Banach spaces $\ell^p$ and $L^p(\mathbb{I})$ are $\ell^q$ and $L^q(\mathbb{I})$, respectively, where $q$ is the conjugate index to $p$. Also, the dual of $c_0$ is $\ell^1$, so that $c_0'' = (\ell^1)' = \ell^\infty = C(\beta\mathbb{N})$.

The difference between the corresponding 'isomorphic' and 'isometric' theories of Banach spaces is of great significance, as we shall see shortly. For example, here is a result that applies in the isometric, but not necessarily in the isomorphic, theory: Let $E$ and $F$ be Banach spaces, and suppose that $T : E \to F$ is an isometric isomorphism. Then $T(\operatorname{ex} E_{[1]}) = \operatorname{ex} F_{[1]}$.

Since we shall be concerned with linear isometries, we give a gem, the *Mazur–Ulam theorem* from 1932; see [30, Chapitre 6, §3].

**Lemma 2.2.9.** *Let $E$ and $F$ be two real Banach spaces. Suppose that a map $\Psi : (E,d_E) \to (F,d_F)$ is isometric and that $\Psi(0_E) = 0_F$. Then $\Psi$ is real-linear.*

*Proof.* Take $x_1, x_2 \in E$ with $x_1 \neq x_2$, and set $x_0 = (x_1 + x_2)/2$. We inductively define subsets $E_n$ of $E$ for $n \in \mathbb{N}$ by setting

$$E_1 = \{x \in E : 2d_E(x,x_1) = 2d_E(x,x_2) = d_E(x_1,x_2)\}$$

and

$$E_{n+1} = \{x \in E_n : 2d_E(x,y) \le \operatorname{diam} E_n \ \ (y \in E_n)\}\,.$$

Then $x_0 \in E_1$ and $\operatorname{diam} E_1 < \infty$. Further $\operatorname{diam} E_{n+1} \le (\operatorname{diam} E_n)/2 \ \ (n \in \mathbb{N})$, and so $\lim_{n\to\infty} \operatorname{diam} E_n = 0$.

We *claim* that, for each $n \in \mathbb{N}$, the point $\overline{y} := x_1 + x_2 - y$ belongs to $E_n$ whenever $y \in E_n$. First suppose that $y \in E_1$. Then

$$d_E(\overline{y},x_1) = d_E(x_2,y) \quad \text{and} \quad d_E(\overline{y},x_2) = d_E(x_1,y)\,,$$

and so $\overline{y} \in E_1$. Now assume that the claim holds for $n \in \mathbb{N}$, and take $y \in E_{n+1}$. For each $z \in E_n$, we have $\overline{z} \in E_n$, and so

$$2d_E(\overline{y},z) = 2d_E(y,\overline{z}) \le \operatorname{diam} E_n\,,$$

and hence $\overline{y} \in E_{n+1}$. Thus the claim follows by induction on $n \in \mathbb{N}$.

We next *claim* that $x_0 \in E_n \ \ (n \in \mathbb{N})$. Clearly $x_0 \in E_1$. Take $n \in \mathbb{N}$ and $y \in E_n$. Then $d_E(y,\overline{y}) \le \operatorname{diam} E_n$. But $2d_E(x_0,y) = d_E(y,\overline{y})$, and so $x_0 \in E_{n+1}$. This gives the claim.

Since $\lim_{n\to\infty} \operatorname{diam} E_n = 0$ and the metric space $(E,d_E)$ is complete, it now follows that $\bigcap_{n=1}^\infty E_n = \{x_0\}$.

Set $y_1 = \Psi(x_1)$, $y_2 = \Psi(x_2)$, and $y_0 = (y_1 + y_2)/2$. We now define subsets $F_n$ of $F$ in an analogous way to the above with respect to $y_1$ and $y_2$, so that $\bigcap_{n=1}^{\infty} F_n = \{y_0\}$. However $\Psi(E_n) = F_n \ (n \in \mathbb{N})$ because $\Psi : (E, d_E) \to (F, d_F)$ is an isometry, and so $\Psi(x_0) = y_0$. It follows that

$$\Psi\left(\frac{1}{2}(x_1 + x_2)\right) = \frac{1}{2}(\Psi(x_1) + \Psi(x_2)) \quad (x_1, x_2 \in E).$$

Further, $\Psi(x/2) = \Psi(x)/2 \ (x \in E)$ because $\Psi(0_E) = 0_F$, and so

$$\Psi(x_1 + x_2) = \Psi(x_1) + \Psi(x_2) \quad (x_1, x_2 \in E).$$

It follows that $\Psi(\alpha x) = \alpha \Psi(x) \ (\alpha \in \mathbb{Q}, x \in E)$, and so $\Psi$ is linear over $\mathbb{Q}$. Since $\Psi$ is continuous, it follows that $\Psi$ is real-linear. □

We cannot say that an isometric map $\Psi$ between two Banach spaces is (complex) linear in the above situation: indeed, the map $\Psi : z \mapsto \overline{z}, \ \mathbb{C} \to \mathbb{C}$, is an isometry with $\Psi(0) = 0$.

**Theorem 2.2.10.** *Let $E$ and $F$ be two real Banach spaces, and suppose that there is an isometry from $E$ onto $F$. Then $E$ and $F$ are isometrically isomorphic.*

*Proof.* Let $\Phi : E \to F$ be an isometry, and set $\Psi(x) = \Phi(x) - \Phi(0_E) \ (x \in E)$. Then $\Psi : E \to F$ is also an isometry with $\Psi(0_E) = 0_F$, and so, by Lemma 2.2.9, $\Psi$ is a real-linear isometry; it is a surjection whenever $\Phi$ is a surjection. □

Let $F$ be a normed space, and let $M$ and $N$ be closed subspaces of $F$ and $F'$, respectively. Define

$$M^\circ = \{\lambda \in F' : \langle x, \lambda \rangle = 0 \ (x \in M)\} \quad {}^\circ N = \{x \in F : \langle x, \lambda \rangle = 0 \ (\lambda \in N)\}.$$

Then $M^\circ$ and ${}^\circ N$ are closed linear subspaces of $(F', \sigma(F', F))$ and $(F, \sigma(F, F'))$, respectively; $M^\circ$ is the *annihilator* of $M$ and ${}^\circ N$ is the *pre-annihilator* of $N$. Clearly, $({}^\circ N)^\circ$ is the $\sigma(F', F)$-closure of $N$ in $F'$, so that $({}^\circ N)^\circ = N$ whenever $N$ is $\sigma(F', F)$-closed.

Now suppose that $F' \cong E$. Then it is standard that $M' \cong E/M^\circ$ and $(F/M)' \cong M^\circ$, and so, in the case where $N$ is $\sigma(E, F)$-closed in $E$, we obtain the following result by setting $M = {}^\circ N$.

**Proposition 2.2.11.** *Let $E$ be a Banach space with $E \cong F'$ for a normed space $F$. Suppose that $N$ is a $\sigma(E, F)$-closed linear subspace of $E$. Then $N \cong (F/{}^\circ N)'$ and $E/N \cong ({}^\circ N)'$.* □

Let $E$ and $F$ be normed spaces. The *dual* (or *adjoint*) of $T \in \mathscr{B}(E, F)$ is the operator $T' \in \mathscr{B}(F', E')$, defined by the formula

$$\langle x, T'\lambda \rangle = \langle Tx, \lambda \rangle \quad (x \in E, \lambda \in F');$$

of course, $\|T'\| = \|T\|$ and $T' : F' \to E'$ is weak*-weak*-continuous. Using dual maps, it is easy to see that $E' \sim F'$ and $E' \cong F'$ whenever $E \sim F$ and $E \cong F$, respectively. Suppose that there is a bounded linear surjection $T$ from a Grothendieck space onto a Banach space $E$. Then, by consideration of $T'$ and $T''$, it is easily seen that $E$ is also a Grothendieck space.

The following standard results are given in [6, §3.16] and [183, pp. 287–293].

**Proposition 2.2.12.** *Let $E$ and $F$ be Banach spaces, and take $T \in \mathscr{B}(E,F)$. Then:*

(i) *$T$ is a surjection if and only if $T' : F' \to E'$ is an embedding if and only if there exists $c > 0$ such that $\|T'\lambda\| \geq c\|\lambda\| \quad (\lambda \in E)$;*

(ii) *$T$ is an injection if and only if $T'(F')$ is weak*-dense in $E'$;*

(iii) *$T$ is an injection with closed range if and only if $T'$ is a surjection;*

(iv) *$T$ is a bijection if and only if $T'$ is a bijection.* □

**Proposition 2.2.13.** *Let $E$ and $F$ be Banach spaces. Then each weak*-weak*-continuous operator from $F'$ to $E'$ has the form $T'$ for some operator $T \in \mathscr{B}(E,F)$.* □

**Proposition 2.2.14.** *Let $E$ be a normed space, and take $B$ to be the weak*-compact space $E'_{[1]}$.*

(i) *The map*

$$J : x \mapsto \kappa_E(x) \mid B, \quad E \to C(B) \subset \ell^\infty(B),$$

*is an isometric embedding.*

(ii) *Suppose that $S$ is a weak*-closed, circled subspace of $B$ with $\overline{\mathrm{co}}(S) = B$. Then the map*

$$J : x \mapsto \kappa_E(x) \mid S, \quad E \to C(S),$$

*is an isometric embedding.*

*Proof.* (i) Clearly $Jx \in C(B) \ (x \in E)$, and the map $J : E \to C(B)$ is linear. Further,

$$|Jx|_B = \sup\{|\langle \kappa_E(x), \lambda\rangle| : \lambda \in B\} = \|x\| \quad (x \in E),$$

and so $T$ is an isometric embedding of $E$ into $C(B)$.

(ii) This follows from Corollary 2.1.3. □

**Definition 2.2.15.** Let $E$ and $F$ be normed spaces, and take $T \in \mathscr{B}(E,F)$. Then $T$ is a *quotient operator* if $T$ maps the open unit ball in $E$ onto the open unit ball in $F$.

**Proposition 2.2.16.** *Let $E$ and $F$ be normed spaces, and take $T \in \mathscr{B}(E,F)$. Then $T$ is a quotient operator if and only if the induced operator $\overline{T} : E/\ker T \to T(E)$ is an isometric embedding into $F$.*

*Proof.* Certainly $\overline{T}$ is always a bounded operator with $\|\overline{T}\| \leq \|T\|$.

It is clear that $\overline{T}$ is an isometric isomorphism onto $T(E)$ whenever $T$ is a quotient operator.

Now suppose that $\overline{T}$ is an isometric isomorphism onto $T(E)$. Then $\|Tx\| < 1$ when $\|x\| < 1$ in $E$. Take $y \in T(E)$ with $\|y\| < 1$. Then there exists $z \in E/F$ with $\|z\| < 1$ and $\overline{T}z = y$, and there exists $x \in E$ with $\|x\| < 1$ and $x + F = z$. We have $Tx = y$, and so $T$ is a quotient operator. □

Each Banach space $E$ is a quotient of a space $\ell^1(\Gamma)$ for some index set $\Gamma$. Indeed, we can take $\Gamma = E_{[1]}$ and define the map

$$\sum \alpha_\gamma \delta_\gamma \mapsto \sum \alpha_\gamma \gamma, \quad \ell^1(\Gamma) \to E.$$

**Proposition 2.2.17.** *Let $E$ be a separable Banach space. Then:*

(i) *there is an isometric embedding of $E$ into $\ell^\infty$;*

(ii) *there is a quotient operator from $\ell^1$ onto $E$;*

(iii) *there is an isometric embedding of $E'$ into $\ell^\infty$.*

*Proof.* (i) Let $S = \{x_n : n \in \mathbb{N}\}$ be a dense subset of $S_E$. For each $n \in \mathbb{N}$, choose $\lambda_n \in E'$ with $\langle x_n, \lambda_n \rangle = \|\lambda_n\| = 1$. Then the map

$$T : x \mapsto (\langle x, \lambda_n \rangle), \quad E \to \ell^\infty.$$

is an isometric embedding.

(ii) Let $S = \{x_n : n \in \mathbb{N}\}$ be a dense subset of $E_{[1]}$. We define

$$T : (\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n x_n, \quad \ell^1 \to E.$$

Then clearly $T$ is a linear contraction with $T\delta_n = x_n \ (n \in \mathbb{N})$.

Now take $x \in E$ with $0 < \|x\| < 1$, say $\|x\| = \eta$, choose $\varepsilon > 0$ with $\varepsilon < 1 - \eta$, and set $y = x/\eta$, so that $y \in S_E$. First choose $n_1 \in \mathbb{N}$ such that $\|y - x_{n_1}\| < \varepsilon$, and then inductively choose a strictly increasing sequence $(n_k)$ in $\mathbb{N}$ such that

$$\left\| y - \left( \sum_{j=1}^{k} \varepsilon^{j-1} x_{n_j} \right) \right\| < \varepsilon^k \quad (k \in \mathbb{N}).$$

Set $\alpha = \sum_{j=1}^{\infty} \varepsilon^{j-1} \delta_{n_j}$, so that $\alpha \in \ell^1$ with $\|\alpha\|_1 = (1 - \varepsilon)^{-1}$ and

$$y = \sum_{j=1}^{\infty} \varepsilon^{j-1} x_{n_j} = T\alpha \in T(\ell^1).$$

Thus $x = T(\eta\alpha)$ with $\|\eta\alpha\|_1 < 1$. This shows that $T : \ell^1 \to E$ is a quotient operator.

(iii) By Proposition 2.2.16, $E$ is isometrically isomorphic to $\ell^1/\ker T$, and then $E'$ is isometrically isomorphic to $(\ker T)^\circ$, a closed subspace of $\ell^\infty$. □

The following notion, which may be new, will be useful in answering questions about when a certain Banach space is not embedded in another.

**Definition 2.2.18.** Let $E$ be a normed space. Then a subset $S$ of $E$ has *bounded finite sums* if there is a constant $M > 0$ such that

$$\left\|\sum\{x : x \in F\}\right\| \leq M$$

for each finite subset $F$ of $S$.

The following is clear.

**Proposition 2.2.19.** *Let $E$ and $F$ be normed spaces. Suppose that $E$ has a subset of cardinality $\kappa$ that has bounded finite sums, and suppose that there is an embedding of $E$ into $F$. Then $F$ has a subset of cardinality $\kappa$ that has bounded finite sums.* □

**Example 2.2.20.** Let $K$ be a non-empty, separable, locally compact space. Then each subset of $C_0(K)$ that has bounded finite sums is countable.

Indeed, take $\{x_n : n \in \mathbb{N}\}$ to be a dense subset of $K$, and take $S$ to be a subset of $C_0(K)$ that has bounded finite sums. For each $n \in \mathbb{N}$, the set $\{f \in S : f(x_n) \neq 0\}$ is countable, and so

$$\bigcup\{\{f \in S : f(x_n) \neq 0\} : n \in \mathbb{N}\}$$

is countable. But the above set is $S \setminus \{0\}$, and so $S$ is countable. □

**Example 2.2.21.** Let $S$ be an infinite set. By Proposition 1.5.5, there is a family $\{S^*_\alpha : \alpha \in A\}$ of non-empty, pairwise-disjoint, clopen subsets of $S^*$, where $|A| = \mathfrak{c}$. The family of characteristic functions in $C(S^*)$ of the sets $S^*_\alpha$ has cardinality $\mathfrak{c}$ and has bounded finite sums. □

**Example 2.2.22.** It is immediate from the above two examples and Proposition 2.2.19 that there is no embedding of $C(\mathbb{N}^*) \cong \ell^\infty/c_0$ into $C(\beta\mathbb{N}) \cong \ell^\infty$. A stronger result will be given in Corollary 2.2.25. □

**Theorem 2.2.23.** *Let $K$ be a non-empty, locally compact space. Then the following conditions on $K$ are equivalent:*

(a) *$K$ does not satisfy* CCC*;*

(b) *there is an uncountable subset of $C_0(K)$ with bounded finite sums;*

(c) *there is an uncountable set $\Gamma$ such that $c_0(\Gamma)$ embeds into $C_0(K)$.*

*Proof.* (a) $\Rightarrow$ (b), (c) Let $\{U_\gamma : \gamma \in \Gamma\}$ be a pairwise-disjoint family of non-empty, open subsets of $K$ such that each $\overline{U_\gamma}$ is compact. For each $\gamma \in \Gamma$, choose $f_\gamma \in C_0(K,\mathbb{I})$ such that $\left|f_\gamma\right|_K = 1$ and supp $f_\gamma \subset U_\gamma$. Then $\{f_\gamma : \gamma \in \Gamma\}$ has bounded finite sums, and, by (a), the set $\Gamma$ is uncountable, giving (b).

Let $\alpha = (\alpha_\gamma : \gamma \in \Gamma)$ be an element of $c_0(\Gamma)$, and set

$$T\alpha = \sum\{\alpha_\gamma f_\gamma : \gamma \in \Gamma\}.$$

Then $T\alpha \in C_0(K)$, and $T : c_0(\Gamma) \to C_0(K)$, is a linear isometry that identifies $c_0(\Gamma)$ with the closed subspace $\overline{\mathrm{lin}}\{f_\gamma : \gamma \in \Gamma\}$ of $C_0(K)$, giving (c).

(c) $\Rightarrow$ (b) The family $\{\chi_{\{\gamma\}} : \gamma \in \Gamma\}$ has bounded finite sums, and so (b) follows from Proposition 2.2.19.

(b) $\Rightarrow$ (a) Let $S := \{f_\gamma : \gamma \in \Gamma\}$ be a family with bounded finite sums, where $\Gamma$ is an uncountable index set and $f_\gamma \neq f_\delta$ when $\gamma, \delta \in \Gamma$ with $\gamma \neq \delta$. By replacing the set $S$ by $\{\Re f_\gamma : \gamma \in \Gamma\}$ or $\{\Im f_\gamma : \gamma \in \Gamma\}$, we may suppose that $S \subset C_{0,\mathbb{R}}(K)$. There exists $\eta > 0$ such that $\{f \in S : |f|_K > \eta\}$ is uncountable, and so we may suppose, by passing to a subset of $\Gamma$ and scaling, that $|f|_K > 1$, and in fact that $\sup\{f(x) : x \in K\} > 1$, for each $f \in S$.

For each $\gamma \in \Gamma$, set

$$U_\gamma = \{x \in K : f_\gamma(x) > 1\},$$

so that $U_\gamma$ is a non-empty, open set in $K$. The assumption that $S$ has bounded finite sums implies that there exists $M \in \mathbb{N}$ such that the intersection of any family of $M$ of the sets $U_\gamma$ is empty. [1]

We shall inductively define a certain family $\{W_\alpha : \alpha < \omega_1\}$ of pairwise-disjoint, non-empty, open subsets of $K$ to satisfy the following properties for each $\alpha < \omega_1$:

(i) for each $\beta < \alpha$, there is a finite subset $\Gamma_\beta$ of $\Gamma$ such that $W_\beta = \bigcap\{U_\gamma : \gamma \in \Gamma_\beta\}$;

(ii) for each $\beta < \alpha$, we have $W_\beta \cap U_\gamma = \emptyset \ (\gamma \in \Gamma \setminus \Gamma_\beta)$;

(iii) for each $\beta_1, \beta_2 < \alpha$ with $\beta_1 \neq \beta_2$, we have $\Gamma_{\beta_1} \cap \Gamma_{\beta_2} = \emptyset$.

First choose a subset $\Gamma_1$ of $\Gamma$ to be maximal with respect to the property that $\bigcap\{U_\gamma : \gamma \in \Gamma_1\} \neq \emptyset$, and set $W_1 = \bigcap\{U_\gamma : \gamma \in \Gamma_1\}$. We observe that $|\Gamma_1| < M$ and that $W_1 \cap U_\gamma = \emptyset \ (\gamma \in \Gamma \setminus \Gamma_1)$.

Now take $\alpha < \omega_1$, and assume that we have defined $W_\beta$ for each $\beta < \alpha$ such that (i), (ii), and (iii) hold. We observe that $\Gamma \setminus \bigcup\{\Gamma_\beta : \beta < \alpha\}$ is uncountable; we then choose $\Gamma_\alpha \subset \Gamma \setminus \bigcup\{\Gamma_\beta : \beta < \alpha\}$ to be maximal with respect to the property that $\bigcap\{U_\gamma : \gamma \in \Gamma_\alpha\} \neq \emptyset$, and set $W_\alpha = \bigcap\{U_\gamma : \gamma \in \Gamma_\alpha\}$. This continues the inductive construction.

In this way, we obtain a family of cardinality $\aleph_1$ of non-empty, pairwise-disjoint, open subsets of $K$. Thus (a) holds. $\square$

**Corollary 2.2.24.** *Let $K$ and $L$ be two non-empty, locally compact spaces. Suppose that $C_0(K)$ embeds in $C_0(L)$ and that $L$ satisfies* CCC. *Then $K$ satisfies* CCC.

[1] An immediate contradiction can be obtained at this point by an appeal to a lemma of Rosenthal (see [131, Proposition 7.21]); we provide a somewhat simpler, self-contained argument here.

*Proof.* This follows from Proposition 2.2.19 and Theorem 2.2.23. □

For more comprehensive versions of Corollary 2.2.24, see [131, Theorem 7.22] and [211, Theorem 4.6].

**Corollary 2.2.25.** *Let $S$ be an infinite set, and let $L$ be a compact space that satisfies* CCC. *Then there is no embedding of $C(S^*)$ into $C(L)$.*

*Proof.* By Proposition 1.5.5, $S^*$ does not satisfy CCC. Thus the claim follows immediately from Corollary 2.2.24. □

**Definition 2.2.26.** Let $X$ be a topological space. Then the *Souslin number* of $X$ is the minimum cardinal number $\kappa$ such that every family of non-empty, pairwise-disjoint, open subsets of $X$ has cardinality at most $\kappa$; it is denoted by $c(X)$.

Thus $X$ satisfies CCC if and only if $c(X) \leq \aleph_0$.

An easy modification of the above argument shows that $c(K) = c(L)$ whenever $K$ and $L$ are two non-empty, locally compact spaces with $C_0(K) \sim C_0(L)$, and so $c(K)$ is an isomorphic invariant of the spaces $C_0(K)$; for further isomorphic invariants of these spaces, see §6.1.

We now introduce a definition that encapsulates a key theme of this work.

**Definition 2.2.27.** Let $E$ be a Banach space. Then a Banach space $F$ is an *isometric predual* of $E$ if $E \cong F'$ and an *isomorphic predual* of $E$ if $E \sim F'$. Similarly, a Banach space $F$ is an *isometric pre-bidual* of $E$ if $E \cong F''$ and an *isomorphic pre-bidual* of $E$ if $E \sim F''$. We say that $E$ is *isomorphically/isometrically a* (*bi*) *dual space* if $E$ has an isomorphic/isometric pre-(bi)dual.

It will be apparent through several later examples that a Banach space $E$ might have many isomorphic preduals, but no isometric preduals. In fact, there is a general result of this nature, due to Davis and Johnson [79]. Let $(E, \|\cdot\|)$ be a Banach space that is not reflexive. Then there is a norm $|||\cdot|||$ on $E$ that is equivalent to $\|\cdot\|$ and such that $(E, |||\cdot|||)$ is not isometrically a dual space. Thus, let $F$ be a non-reflexive Banach space, and set $E = F'$. Then there is a norm $|||\cdot|||$ on $E$ such that $F$ is an isomorphic predual, but not an isometric predual, of $(E, |||\cdot|||)$.

**Theorem 2.2.28.** *Let $E$ be a Banach space.*

(i) *The space $E$ is isometrically a dual space if and only if there is a topology $\tau$ on $E$ such that $(E, \tau)$ is a locally convex space and $(E_{[1]}, \tau)$ is compact.*

(ii) *The space $E$ is isomorphically a dual space if and only if there is a topology $\tau$ on $E$ such that $(E, \tau)$ is a locally convex space and for which $(B, \tau)$ is compact for some barrel $B$ in $E$.*

*Proof.* (i) Suppose that $E \cong F'$ for some Banach space $F$. Then we take $\tau$ to be the topology $\sigma(E,F)$.

Conversely, suppose that there is a topology $\tau$ as specified, and let $F \subset E'$ consist of the $\tau$-continuous functionals on $E$, so that $F$ is a closed subspace of $E'$. There is a natural mapping $j : E \to F'$ defined by

$$\langle \lambda, j(x) \rangle = \langle x, \lambda \rangle \quad (x \in E, \lambda \in F).$$

Clearly $j$ is injective and continuous, with $\|j(x)\| \le \|x\|$ $(x \in E)$. Furthermore, $j$ is continuous from $(E,\tau)$ to $(F',\sigma(F',F))$, and so $j(E_{[1]})$ is a $\sigma(F',F)$-compact subset of $F'_{[1]}$. It follows immediately from Theorem 2.1.2(iii) that $j(E_{[1]}) = F'_{[1]}$. We now see that $j : E \to F'$ is an isometry.

(ii) This follows easily from (i). □

We now consider the uniqueness of isometric preduals.

**Definition 2.2.29.** Let $E$ be a Banach space with an isometric predual $F$. Then $F$ is *unique* if, whenever $G$ is also an isometric predual of $E$, it follows that $F \cong G$. The unique predual of a Banach space $E$ is denoted by $E_*$ when it exists. Further, $F$ is *strongly unique* if, whenever $G$ is also a Banach space and $T : E \to G'$ is an isometric isomorphism, the map $T' : G'' \to F'' = E'$ carries $\kappa_G(G)$ onto $\kappa_F(F)$.

A predual $F$ of a Banach space $E$ is strongly unique if and only if the above map $T : F' \to G'$ is weak*-weak*-continuous. Thus $E$ has a unique predual whenever it has a strongly unique predual. All known examples of Banach spaces with a unique predual actually have a strongly unique predual.

For a fine survey concerning the existence and uniqueness of isometric preduals of Banach spaces, including a discussion of strongly unique preduals, see [115]; see also [50]. The definition of '$E$ has a strongly unique predual' in [115] is that there is a unique bounded projection $\pi : E''' \to E'$ with $\|\pi\| = 1$ and such that $\ker \pi$ is weak*-closed; as noted in [50], this is equivalent to our definition.

It is certainly not the case that every Banach space that is isometrically a dual space has a unique predual; for example, we shall discuss the many isometric preduals of the Banach space $\ell^1$ in §6.3.

We continue this section with a representation theorem for the bidual $E''$ of a Banach space $E$ that we shall use later.

Let $E$ be a Banach space. We shall suppose that there is a subset $S$ of the unit sphere $S_{E'}$ of $E'$ such that, for each $\mu \in S$, there is a closed subspace $F_\mu$ of $E'$ and that the family $\{F_\mu : \mu \in S\}$ of these subspaces has the property that

$$\|\Lambda\| = \sup_{\mu \in S}\{|\langle \Lambda, y\rangle| : y \in (F_\mu)_{[1]}\} \quad (\Lambda \in E''). \tag{2.3}$$

Set

$$F = \bigoplus_1 \{F_\mu : \mu \in S\}, \quad \text{so that} \quad F' = \bigoplus_\infty \{F'_\mu : \mu \in S\}. \tag{2.4}$$

Thus
$$F' = \{(\lambda_\mu) = (\lambda_\mu : \mu \in S) : \lambda_\mu \in F'_\mu,\ \sup \left\|\lambda_\mu\right\| < \infty\},$$
with $\left\|(\lambda_\mu)\right\|_\infty = \sup\{\left\|\lambda_\mu\right\| : \mu \in S\}$. For each $x \in E$ and $\mu \in S$, define $x_\mu \in F'_\mu$ by
$$\langle y, x_\mu \rangle = \langle y, x \rangle \quad (y \in F_\mu).$$
Then it is easy to see that the map $x \mapsto (x_\mu),\ E \to F'$, is a linear isometry. We shall extend this map to a representation of $E''$.

Suppose, further, that $E$ has an isometric linear involution $*$, so that $*$ induces an isometric linear involution on $E'$, and that each $F_\mu$ is a $*$-closed subspace of $E'$. Then we define a linear involution on $F$ coordinatewise, and hence obtain an isometric linear involution on $F$; in turn we obtain an isometric linear involution on $F'$. Clearly, the map $x \mapsto (x_\mu),\ E \to F'$, is $*$-linear and each $F'_\mu$ is a $*$-closed subspace of $F'$.

**Theorem 2.2.30.** *Let $E$ be a Banach space, and let $S$, $\{F_\mu : \mu \in S\}$, and $F$ be as above. For each $\Lambda \in E''$ and $\mu \in S$, define $\Lambda_\mu$ on $F_\mu$ by*
$$\Lambda_\mu(y) = \langle \Lambda, y \rangle \quad (y \in F_\mu).$$
*Then $\Lambda_\mu \in F'_\mu$ with $\left\|\Lambda_\mu\right\| \le \|\Lambda\|$. Further, $(\Lambda_\mu : \mu \in S) \in F'$ with $\left\|(\Lambda_\mu)\right\|_\infty = \|\Lambda\|$. The map*
$$T : \Lambda \mapsto (\Lambda_\mu : \mu \in S), \quad E'' \to F',$$
*is a linear isometry, and $T : (E'', \sigma(E'', E')) \to (F', \sigma(F', F))$ is continuous.*

*Suppose, further, that $E$ has an isometric linear involution $*$ and that $F_\mu$ is a $*$-closed subspace of $E'$. Then $T : E'' \to F'$ is $*$-linear.*

*Proof.* It is clear that, for each $\mu \in S$, we have $\Lambda_\mu \in F_\mu$ with $\left\|\Lambda_\mu\right\| \le \|\Lambda\|$, and so $(\Lambda_\mu : \mu \in S) \in F'$ with $\left\|(\Lambda_\mu)\right\|_\infty \le \|\Lambda\|$. By (2.3), for each $\varepsilon > 0$, there exist $\nu \in S$ and $y \in (F_\nu)_{[1]}$ with $|\langle \Lambda, y \rangle| \ge \|\Lambda\| - \varepsilon$, and so
$$\left\|(\Lambda_\mu)\right\|_\infty \ge \|\Lambda_\nu\| \ge \|\Lambda\| - \varepsilon.$$
This holds for each $\varepsilon > 0$, and hence $\left\|(\Lambda_\mu)\right\|_\infty \ge \|\Lambda\|$. Thus $T : E'' \to F'$ is a linear isometry.

Let $\Lambda_\alpha \to 0$ in $(E'', \sigma(E'', E'))$. Then
$$\langle (\Lambda_\alpha)_\mu, y \rangle = \langle \Lambda_\alpha, y \rangle \to 0 \quad (y \in F_\mu)$$
for each $\mu \in S$, and so $T(\Lambda_\alpha) \to 0$ in $(F', \sigma(F', F))$. This shows that the linear map $T : (E'', \sigma(E'', E')) \to (F', \sigma(F', F))$ is continuous.

We check immediately that $T$ is $*$-linear in the case where the further hypotheses hold. □

Finally in this section we give a technical result that will be used later; see Theorem 4.6.8.

**Proposition 2.2.31.** *Let $G$ be a Banach space, and let $F$ be a separable Banach space. Suppose that $J$ is an uncountable set and that, for each $j \in J$, $L_j$ is a Banach space which does not embed into $G$. Then there is no embedding of the Banach space*

$$\bigoplus_1 \{L_j : j \in J\}$$

*into $F \oplus_1 G$.*

*Proof.* Set $E = \bigoplus_1 \{L_j : j \in J\}$ and regard each $L_j$ as a closed subspace of $E$. Set $H = F \oplus_1 G$, with $F$ and $G$ as stated in the theorem, and take $\pi_F : H \to F$ and $\pi_G : H \to G$ to be the associated bounded projections, so that

$$y = \pi_F(y) + \pi_G(y) \quad (y \in H).$$

Assume towards a contradiction that there is an embedding $T : E \to H$. Then there exists $\delta > 0$ such that $\|Tx\| \geq \delta \|x\|$ $(x \in E)$. For each $j \in J$, there exists $x_j \in L_j$ with $\|x_j\| = 1$ such that $\|\pi_G(Tx_j)\| < \delta/2$. Indeed, otherwise there exists $j \in J$ such that $\|\pi_G(Tx)\| \geq \delta \|x\|/2$ $(x \in L_j)$, so that $(\pi_G \circ T)(L_j)$ is a subspace of $G$ that is isomorphic to $L_j$, contradicting the assumption on $L_j$.

For each $i, j \in J$ with $i \neq j$, we have

$$\|Tx_i - Tx_j\| \geq \delta \|x_i - x_j\| = \delta(\|x_i\| + \|x_j\|) = 2\delta.$$

Now

$$\|y\| = \|\pi_F(y) + \pi_G(y)\| \leq \|\pi_F(y)\| + \|\pi_G(y)\| \quad (y \in H),$$

and so (taking $y = Tx_i - Tx_j$), we have

$$\|\pi_F(Tx_i) - \pi_F(Tx_j)\| \geq \|Tx_i - Tx_j\| - \|\pi_G(Tx_i) - \pi_G(Tx_j)\| > 2\delta - 2(\delta/2) = \delta.$$

Thus there is an uncountable family of mutually disjoint balls in $F$, contradicting the hypothesis that $F$ is separable. $\square$

## 2.3 Banach lattices

We shall require some basic notions in the theory of Banach lattices; for much more on Banach lattices, see [1, 174, 184, 223], for example.

**Definition 2.3.1.** Let $(E, \leq)$ be a Riesz space. A norm $\|\cdot\|$ on $E$ is a *lattice norm* if $\|x\| \leq \|y\|$ whenever $x, y \in E$ with $|x| \leq |y|$. A *normed Riesz space* is a Riesz space equipped with a lattice norm. A *real Banach lattice* is a normed Riesz space which is a Banach space with respect to the norm.

For example, the spaces $L^p_{\mathbb{R}}(\mathbb{I})$ for $1 \leq p \leq \infty$ and the spaces $C_{0,\mathbb{R}}(K)$ for a non-empty, locally compact space $K$ are real Banach lattices with respect to the pointwise lattice operations and the specified norm.

We recall that a linear subspace $F$ of a real Banach lattice $E$ is a sublattice if $x \vee y, x \wedge y \in F$ whenever $x, y \in F$ and a lattice ideal if $x \in F$ whenever $x \in E$ and $|x| \leq |y|$ for some $y \in F$.

Suppose that $E$ is a linear space such that $E = E_{\mathbb{R}} \oplus \mathrm{i}E_{\mathbb{R}}$ for a real Banach lattice $(E_{\mathbb{R}}, \|\cdot\|)$, so that $E$, a linear space over the field $\mathbb{C}$, is a complex Riesz space. Then we make the following definitions. First, set $E^+ = E^+_{\mathbb{R}}$ and

$$E^+_{[r]} = E_{[r]} \cap E^+ \quad (r > 0)\,.$$

Take $z \in E$, say $z = x + \mathrm{i}y$, where $x, y \in E_{\mathbb{R}}$, and define the *modulus* $|z| \in E^+$ of $z$ by

$$|z| = \left(|x|^2 + |y|^2\right)^{1/2} \tag{2.5}$$

(the right-hand side of (2.5) is well defined in $E_{\mathbb{R}}$ by the 'Youdine–Krivine functional calculus'). Alternatively, we can set

$$|z| = |x + \mathrm{i}y| = \sup\{x\cos\theta + y\sin\theta : 0 \leq \theta \leq 2\pi\}\,; \tag{2.6}$$

the supremum always exists in $E^+$ and the two definitions of $|z|$ are consistent. We then define

$$\|z\| = \|\,|z|\,\| \quad (z \in E)\,. \tag{2.7}$$

We see that $\|\cdot\|$ is a norm on $E$ and that $(E, \|\cdot\|)$ is a Banach space. This complexification of a real Banach lattice is defined to be a (complex) *Banach lattice*.

For example, the spaces $L^p(\mathbb{I})$ for $1 \leq p \leq \infty$ and the spaces $C_0(K)$ for a non-empty, locally compact space $K$ are Banach lattices which are the complexifications of the analogous real Banach lattices.

Again, a linear subspace $F$ of a Banach lattice $E$ is a *lattice ideal* if $x \in F$ whenever $x \in E$ and $|x| \leq |y|$ for some $y \in F$.

Let $\{E_\alpha : \alpha \in A\}$ be a family of Banach lattices, and take $p$ with $1 \leq p \leq \infty$. Then the Banach space $\bigoplus_p \{E_\alpha : \alpha \in A\}$ is also a Banach lattice for the obvious operations.

For details of these remarks, including a discussion of the Youdine–Krivine functional calculus, see [1, §3.2], [73], [176, §1.d], [180], [184, §2.2], [223, Chapter II, §11], and [245, §13].

**Definition 2.3.2.** A Banach lattice is *Dedekind complete* (respectively, *Dedekind $\sigma$-complete*) if it is Dedekind complete (respectively, Dedekind $\sigma$-complete) as a complex Riesz space.

Clearly, to show that a Banach lattice $E$ is Dedekind complete, it suffices to show that each increasing net in $E^+$ that is bounded above has a supremum.

The following well-known theorem is proved in [68, Proposition 4.2.29(i)].

**Theorem 2.3.3.** *Let $K$ be a non-empty, compact space. Then $K$ is Stonean if and only if $C(K)$ is Dedekind complete, and $K$ is basically disconnected if and only if $C(K)$ is Dedekind $\sigma$-complete.*

*Proof.* Suppose that $C_{\mathbb{R}}(K)$ is Dedekind complete, and let $U$ be an open set in $K$. Take $\mathscr{F}$ to be the family of functions $f \in C_{\mathbb{R}}(K)$ such that $f(x) = 0 \ (x \in K \setminus U)$ and $0 \leq f \leq 1$. Then $\mathscr{F}$ has a supremum, say $f_0 \in C_{\mathbb{R}}(K)$. Clearly $f_0(x) = 1 \ (x \in U)$ and $f_0(x) = 0 \ (x \in K \setminus \overline{U})$, and so $f_0 = \chi_{\overline{U}}$. Thus $\overline{U}$ is open. This shows that $K$ is Stonean.

Conversely, suppose that $K$ is Stonean, and let $\mathscr{F}$ be a family in $C(K)^+$ which is bounded above, say by 1. For $r \in \mathbb{I}$, define

$$U_r = \bigcup \{\{x \in K : f(x) > r\} : f \in \mathscr{F}\}.$$

Then $U_r$ is open in $K$, and so $V_r := \overline{U_r}$ is also open in $K$. Clearly $V_1 = \emptyset$. Define

$$g(x) = \sup\{r \in \mathbb{I} : x \in U_r\} \in \mathbb{I}.$$

If $g(x) \in (r,s)$, then $x \in V_r \setminus V_s$, and, if $x \in V_r \setminus V_s$, then $g(x) \in [r,s]$.

Take $x_0 \in K$, and take a neighbourhood $V$ of $g(x_0)$. Then there exist $r,s \in \mathbb{R}$ with $g(x_0) \in (r,s) \subset [r,s] \subset V$. Since $V_r \setminus V_s$ is an open set and

$$x_0 \in V_r \setminus V_s \subset g^{-1}([r,s]) \subset g^{-1}(V),$$

we see that $g$ is continuous at $x_0$. Thus $g \in C_{\mathbb{R}}(K)$.

Now take $h \in C_{\mathbb{R}}(K)$ with $h \geq f \ (f \in \mathscr{F})$. Assume that there exists $x_0 \in K$ with $h(x_0) < g(x_0)$. Then $h(x_0) < r$ for some $r$ with $x_0 \in V_r$. Let $W$ be a neighbourhood of $x_0$ with $h(x) < r \ (x \in W)$. Then there exists $x \in W$ with $f(x) > r$ for some $f \in \mathscr{F}$, a contradiction. Thus $h \geq g$, and so $g = \sup \mathscr{F}$. We have shown that $C_{\mathbb{R}}(K)$ is Dedekind complete.

The proof that $K$ is basically disconnected if and only if $C_{\mathbb{R}}(K)$ is Dedekind $\sigma$-complete is a small variation of the above. $\square$

In fact, the term 'Stonean' was used first by Dixmier in the seminal work [91], where a Stonean space was defined to be a compact space $K$ such that $(C_{\mathbb{R}}(K), \leq)$ is Dedekind complete.

The following, related theorem was proved by Seever in [224]; this paper is based on his thesis written under the direction of William Bade. In the proof, we shall use the notation $\prec$ from page 24. See also [225, Theorem 24.7.5].

**Theorem 2.3.4.** *Let $K$ be a non-empty, compact space. Then $K$ is an $F$-space if and only if, whenever $(f_n)$ and $(g_n)$ are sequences in $C_{\mathbb{R}}(K)$ with $f_m \leq g_n \ (m,n \in \mathbb{N})$, there exists $f \in C_{\mathbb{R}}(K)$ with $f_m \leq f \leq g_n \ (m,n \in \mathbb{N})$.*

*Proof.* Suppose that $C_{\mathbb{R}}(K)$ has the stated property, and take disjoint cozero sets $U$ and $V$, say

$$U = \{x \in K : f(x) > 0\} \quad \text{and} \quad V = \{x \in K : g(x) > 0\},$$

where $f,g \in C(K,\mathbb{I})$. For $n \in \mathbb{N}$, set $f_n = 1 \wedge nf$ and $g_n = (1-g)^n$. For $m,n \in \mathbb{N}$, we have $f_m \leq g_n$ in $C(K)^+$, and so there exists $h \in C(K)^+$ with $f_m \leq h \leq g_n \ (m,n \in \mathbb{N})$. Clearly $h(x) = 1 \ (x \in U)$ and $h(x) = 0 \ (x \in V)$, and so $K$ is an $F$-space.

Conversely, suppose that $K$ is an $F$-space, and take $(f_n)$ and $(g_n)$ to be as specified; we may suppose that

$$0 < f_n(x) \leq f_{n+1}(x) \leq g_{n+1}(x) \leq g_n(x) < 1 \quad (x \in K, n \in \mathbb{N}).$$

Let $D$ be the set of dyadic rationals in $[0,1]$, and, for $r \in D$, define

$$U(r) = \bigcup \{\{x \in K : f_n(x) > r\} : n \in \mathbb{N}\}, \quad V(r) = \bigcup \{\{x \in K : g_n(x) < r\} : n \in \mathbb{N}\},$$

so that $U(r)$ and $V(r)$ are disjoint cozero sets, and so $\overline{U(r)} \cap \overline{V(r)} = \emptyset$ because $K$ is an $F$-space. Further, $U(r) \supset U(s)$ and $V(r) \subset V(s)$ when $r,s \in D$ with $r < s$, and $U(1) = V(0) = \emptyset$.

We *claim* that there exist cozero sets $W(r)$ in $K$ for $r \in D$ such that:

(i) $V(r) \prec W(r) \prec K \setminus U(r)$ for $r \in D$;

(ii) $W(r) \prec W(s)$ for $r,s \in D$ with $r < s$.

Indeed, start with $W(0) = \emptyset$ and $W(1) = K$. Now take $n \in \mathbb{N}$, and assume inductively that the sets $W(k/2^n)$ have been defined for $k = 0,\dots,2^n$. Take $k \in \{0,\dots,2^n - 1\}$. We have

$$V((2k+1)/2^{n+1}) \subset V((k+1)/2^n) \prec W((k+1)/2^n)$$

and $W(k/2^n) \prec W((k+1)/2^n)$, and so

$$V((2k+1)/2^{n+1}) \cup W(k/2^n) \prec W((k+1)/2^n).$$

Also, $V((2k+1)/2^{n+1}) \prec K \setminus U((2k+1)/2^{n+1})$ and

$$W(k/2^n) \prec K \setminus U(k/2^n) \subset K \setminus U((2k+1)/2^{n+1}),$$

and so

$$V((2k+1)/2^{n+1}) \cup W(k/2^n) \prec K \setminus U((2k+1)/2^{n+1}).$$

Thus

$$V((2k+1)/2^{n+1}) \cup W(k/2^n) \prec W((k+1)/2^n) \cap (K \setminus U((2k+1)/2^{n+1})).$$

By a remark on page 24, there is a cozero set $U$ with

$$V((2k+1)/2^{n+1}) \cup W(k/2^n) \prec U \prec W((k+1)/2^n) \cap (K \setminus U((2k+1)/2^{n+1}));$$

we take $W((2k+1)/2^{n+1})$ to be this set $U$. This completes the definition of the sets $W(k/2^{n+1})$ for $k = 0,\dots,2^{n+1}$. We see that the recursion continues.

Now define $f(x) = \inf\{r \in D : x \in W_r\} \ (x \in K)$. As in Theorem 2.3.3, $f \in C(K)^+$.

Fix $n \in \mathbb{N}$. For each $x \in K$ and $\varepsilon > 0$, choose $r, s \in D$ with

$$f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon\,.$$

Then $x \notin W(r)$. By (i), $x \notin V(r)$, and so $g_n(x) \geq r$. Hence $g_n(x) \geq f(x) - \varepsilon$. This holds true for each $\varepsilon > 0$, and so $g_n(x) \geq f(x)$, whence $f \leq g_n$. Similarly, $f \geq f_n$. Thus $f$ has the required properties. □

Let $E$ and $F$ be Banach lattices that are the complexifications of the real Banach lattices $E_{\mathbb{R}}$ and $F_{\mathbb{R}}$, respectively. An operator $T \in \mathscr{B}(E,F)$ is a *Banach-lattice homomorphism* or *Banach-lattice isomorphism* if $T \mid E_{\mathbb{R}} : E_{\mathbb{R}} \to F_{\mathbb{R}}$ is a Riesz homomorphism or a Riesz isomorphism, respectively; it is a *Banach-lattice isometry* if, further, $T$ is a linear isometry. The Banach lattices $E$ and $F$ are *Banach-lattice isomorphic* or *Banach-lattice isometric* if there is a Banach-lattice isomorphism or isometry, respectively, between them; an *isometric lattice embedding* is an isometric embedding that is a lattice homomorphism.

Let $E$ and $F$ be Banach lattices, and take $T \in \mathscr{B}(E,F)$. Then $T$ is *positive* if $T(E^+) \subset F^+$. It is clear that an isomorphism $T \in \mathscr{B}(E,F)$ such that $T$ and $T^{-1}$ are positive operators is a lattice isomorphism from $E_{\mathbb{R}}$ onto $F_{\mathbb{R}}$, and so is a Banach-lattice isomorphism.

The following remark establishes some consistency in our terminology.

**Proposition 2.3.5.** *Let $E$ and $F$ be Banach lattices, and suppose that $T \in \mathscr{B}(E,F)$ is a Banach-lattice isomorphism such that $\|Tx\| = \|x\| \ \ (x \in E^+)$. Then*

$$|Tz| = T(|z|) \quad (z \in E)\,,$$

*and $T : E \to F$ is a Banach-lattice isometry.*

*Proof.* Take $z = x + \mathrm{i}y \in E$, where $x, y \in E_{\mathbb{R}}$, and set

$$S = \{x\cos\theta + y\sin\theta : 0 \leq \theta \leq 2\pi\}\,.$$

Then $T(S) = \{(Tx)\cos\theta + (Ty)\sin\theta : 0 \leq \theta \leq 2\pi\}$, and, by (2.6), $\sup S = |z|$ and $\sup T(S) = |Tz|$. Since $T$ is a lattice isomorphism, $T(\sup S) = \sup T(S)$, i.e., $T(|z|) = |Tz|$. Hence $\|Tz\| = \|\,|Tz|\,\| = \|T(|z|)\| = \|\,|z|\,\| = \|z\|$, and so $T : E \to F$ is a linear isometry. □

Let $E$ be a real Banach lattice, with dual space $E'$. Then $E'$ is ordered by the requirement that $\lambda \in E'$ belongs to $(E')^+$ if and only if $\langle x, \lambda\rangle \geq 0 \ (x \in E^+)$ (*cf.* page 9). One checks easily that this ordering gives a lattice ordering, and so $E'$ becomes a real Banach lattice. The equations that define the lattice operations are the following; they are called the *Riesz–Kantorovich formulae*. Take $\lambda, \mu \in E'$. Then $\langle x, \lambda \vee \mu\rangle$ and $\langle x, \lambda \wedge \mu\rangle$ are defined for $x \in E^+$ by

$$\begin{cases} \langle x, \lambda \vee \mu\rangle = \sup\{\langle y, \lambda\rangle + \langle z, \mu\rangle : y, z \in E^+, y + z = x\}\,, \\ \langle x, \lambda \wedge \mu\rangle = \inf\{\langle y, \lambda\rangle + \langle z, \mu\rangle : y, z \in E^+, y + z = x\}\,, \end{cases} \tag{2.8}$$

and then $\lambda \vee \mu$ and $\lambda \wedge \mu$ are extended linearly to all of $E'$. The dual of a real Banach lattice $E$ is also a real Banach lattice for these operations; this is the *dual Banach lattice* of $E$.

It is standard that a dual Banach lattice is Dedekind complete. Indeed, let $E$ be a Banach lattice, and take $\mathscr{F}$ to be a non-empty, bounded subset of $(E')^+$. Consider the net $\mathscr{G}$ of finite subsets of $\mathscr{F}$, and, for each $x \in E^+$, set

$$\lambda(x) = \lim\left\{\left\langle x, \bigvee S\right\rangle : S \in \mathscr{G}\right\}.$$

Then $\lambda$ is additive, homogeneous, and positive on $E^+$, and thus $\lambda$ extends uniquely to an element, also $\lambda$, of $E'$. Clearly $\lambda = \bigvee\{F : F \in \mathscr{F}\}$, and so $E'$ is Dedekind complete.

Let $F$ be a real Banach lattice, and set $E = F \oplus \mathrm{i}F$, its complexification. Let $\lambda$ be a continuous, real-linear functional on $F$. Then $\lambda$ extends to a continuous, complex-linear functional on $E$: indeed, we define

$$\lambda(x+\mathrm{i}y) = \lambda(x) + \mathrm{i}\lambda(y) \quad (x, y \in F),$$

and so we may regard $F'$ as a real-linear subspace of $E'$. For each $\lambda$ in $E'$, there exist $\lambda_1$ and $\lambda_2$ in $F'$ such that

$$\lambda(x) = \lambda_1(x) + \mathrm{i}\lambda_2(x) \quad (x \in F),$$

and so $E'$ is isomorphic as a complex Banach space to the complexification $F' \oplus \mathrm{i}F'$. In fact, this identification is isometric; the details of this are given in [1, Corollary 3.26] and [184, Proposition 2.2.6], for example. Thus we obtain the *dual Banach lattice* of a Banach lattice.

Let $E$ be a Banach lattice, and take $\lambda \in E'$. Then clearly $(E')^+$ is weak$^*$-closed in $E'$. We have

$$\|\lambda\| = \sup\{\langle x, \lambda\rangle : x \in E^+_{[1]}\} \quad (\lambda \in (E')^+).$$

Further, take $x \in E$. Then $x \in E^+$ if and only if $\langle x, \lambda\rangle \geq 0$ $(\lambda \in (E')^+)$; this follows from the Hahn–Banach theorem.

The bidual $E''$ of a Banach lattice $E$ is also a Banach lattice, and the embedding $\kappa_E : E \to E''$ is an isometric lattice embedding. It also follows from the Hahn–Banach theorem that $E^+_{[1]}$ is weak$^*$-dense in $(E'')^+_{[1]}$.

**Proposition 2.3.6.** *Let $F$ be a real Banach lattice which is isometrically the dual of a real Banach space. Then the complexification of $F$ is also isometrically a dual space.*

*Proof.* Set $E = F \oplus \mathrm{i}F$, the complexification of $F$; we recall that $E' = F' \oplus \mathrm{i}F'$.

Suppose that $F \cong G'$ for a real Banach space $G$, and regard $G$ as a closed subspace of $F'$; set $H = G \oplus \mathrm{i}G$, so that $H$ is a closed subspace of $E' \cong G'' \oplus \mathrm{i}G'' = H''$, and hence $H$ is a Banach space. We shall show that $H' \cong E$, which will give the result.

Take $z \in E$, and set $\lambda(h) = \langle z, h\rangle$ $(h \in H)$. Then $\lambda \in H'$ with $\|\lambda\| \leq \|z\|$, and the map $S : z \mapsto \lambda,\ E \to H'$, is a linear contraction.

Take $\lambda \in H'$, and set $\lambda_1 = \Re\lambda \mid G$ and $\lambda_2 = \Im\lambda \mid G$, so that $\lambda_1$ and $\lambda_2$ are bounded, real-linear functionals on $G$ with $\lambda = \lambda_1 + \mathrm{i}\lambda_2$. Thus there exist unique elements $x$ and $y$ in $F$ such that $\lambda_1(g) = \langle x, g\rangle$ and $\lambda_2(g) = \langle y, g\rangle$ for $g \in G$. Set $z = x + \mathrm{i}y \in E$. Then, for each $g_1, g_2 \in G$, we have

$$\begin{aligned}\lambda(g_1 + \mathrm{i}g_2) &= (\lambda_1 + \mathrm{i}\lambda_2)(g_1 + \mathrm{i}g_2) = \langle x, g_1\rangle - \langle y, g_2\rangle + \mathrm{i}(\langle y, g_1\rangle + \langle x, g_2\rangle)\\ &= \langle x + \mathrm{i}y, g_1 + \mathrm{i}g_2\rangle = \langle z, g_1 + \mathrm{i}g_2\rangle\,,\end{aligned}$$

and so $\lambda = Sz$. Thus $S : E \to H'$ is a surjection.

Now fix $\varepsilon > 0$. Since $H$ is weak*-dense in $H''$, there exists $h \in H$ with $\|h\| = 1$ and $|\langle z, h\rangle| > \|z\| - \varepsilon$, and hence $\|\lambda\| > \|z\| - \varepsilon$. This holds for each $\varepsilon > 0$, and so $\|\lambda\| \geq \|z\|$. We have shown that $S : E \to H'$ is an isometric isomorphism. □

A somewhat more general version of the above result is given in [187, Proposition 7]. We shall prove the converse of the above theorem in the special case where $F = C_{\mathbb{R}}(K)$ in Proposition 6.2.5; we do not know whether the converse holds in general.

Finally, we define some special types of Banach lattices.

**Definition 2.3.7.** A (real or complex) Banach lattice $(E, \|\cdot\|)$ is an *AL-space* (or *abstract L-space*) if

$$\|x + y\| = \|x\| + \|y\| \quad \text{whenever} \quad x, y \in E^+ \quad \text{with} \quad x \wedge y = 0\,,$$

and an *AM-space* (or *abstract M-space*) if

$$\|x \vee y\| = \max\{\|x\|, \|y\|\} \quad \text{whenever} \quad x, y \in E^+ \quad \text{with} \quad x \wedge y = 0\,.$$

For example, each space of the form $L^1(\Omega, \mu)$, where $(\Omega, \mu)$ is a measure space, is an $AL$-space, and each space $C_0(K)$, where $K$ is a non-empty, locally compact space, is an $AM$-space.

Let $E$ be a Banach lattice. Then it is standard that $E$ is an $AL$-space if and only if

$$\|x + y\| = \|x\| + \|y\| \quad (x, y \in E^+)\,,$$

and an $AM$-space if and only if

$$\|x \vee y\| = \max\{\|x\|, \|y\|\} \quad (x, y \in E^+)\,.$$

The following duality result is [5, Theorem 4.23] or [184, Proposition 1.4.7], for example.

**Theorem 2.3.8.** *Let $E$ be a Banach lattice, with dual Banach lattice $E'$. Then $E$ is an AL-space if and only if $E'$ is an AM-space, and $E$ is an AM-space if and only if $E'$ is an AL-space.* □

The following central representation theorem is proved in [1, Theorems 3.5 and 3.6], [5, Theorems 4.27 and 4.29], [174, II. §1.b], and [184, Theorems 2.1.3 and 2.7.1], for example. The proofs are usually given for real Banach lattices, but the complex versions are valid; the technique for the complex version is illustrated in [1, Theorem 3.20]. We shall call this result '*Kakutani's theorem*'; detailed attributions for the various statements are given in [1].

An *AM-unit* in a Banach lattice $E$ is an element $e \in E$ with $e > 0$ such that $E_{[1]} = \{x \in E : |x| \leq e\}$. Thus $\|x\| = \inf\{r \in \mathbb{R} : |x| \leq re\}$ for $x \in E$, and so $E$ is an *AM*-space. An *AM*-unit is an order unit in the ordered linear space $(E_{\mathbb{R}}, \leq)$.

**Theorem 2.3.9.** (i) *A Banach lattice is an AL-space if and only if it is Banach-lattice isometric to a Banach lattice of the form $L^1(\Omega,\mu)$ for some measure space $(\Omega,\mu)$.*

(ii) *A Banach lattice is an AM-space if and only if it is Banach-lattice isometric to a closed sublattice of a space $C(K)$ for some non-empty, compact space $K$.*

(iii) *A Banach lattice with an AM-unit is Banach-lattice isometric to a space $C(K)$ for some non-empty, compact space $K$.* □

## 2.4 Complemented subspaces of Banach spaces

We first define complemented subspaces of a normed space; earlier we defined complemented subspaces of a linear space.

**Definition 2.4.1.** Let $E$ be a normed space. A closed subspace $F$ of $E$ is *complemented* in $E$ if there is a closed subspace $G$ of $E$ such that $E = F \oplus G$.

In the case that a Banach space $E$ is such that $E = F \oplus G$ for closed subspaces $F$ and $G$, we have $E \sim F \times G$ and $E/F \sim G$.

It is elementary that finite-dimensional subspaces and subspaces of finite codimension in a normed space $E$ are complemented in $E$, but we shall see soon that there are closed subspaces of a Banach space that are not complemented. It is remarkable that there is an infinite-dimensional Banach space $E$ such that the *only* closed subspaces that are complemented in $E$ are those that are either of finite dimension or of finite codimension; see page 195.

The following result is a standard consequence of the closed graph theorem.

**Proposition 2.4.2.** *Let $E$ be a Banach space, and suppose that $F$ and $G$ are closed subspaces of $E$ such that $E = F \oplus G$. Then there is a unique projection $P \in \mathscr{B}(E)$ with $P(E) = F$ and $(I_E - P)(E) = G$.* □

It follows immediately from the preservation of the Grothendieck property by bounded linear surjections (see page 59) that a closed, complemented subspace of a Grothendieck space is also a Grothendieck space.

**Definition 2.4.3.** A closed, complemented subspace $F$ of a Banach space $E$ is $\lambda$*-complemented* (for $\lambda \geq 1$) if there is a projection $P \in \mathscr{B}(E)$ with $P(E) = F$ and $\|P\| \leq \lambda$.

Thus a closed, complemented subspace of a Banach space is $\lambda$-complemented for some $\lambda \geq 1$.

**Proposition 2.4.4.** *Let $E$ be a normed space. Then $E'$ is 1-complemented in $E'''$.*

*Proof.* The required bounded projection from $E'''$ to $E'$ is the dual of the canonical injection $\kappa_E : E \to E''$; it is called the *Dixmier projection*. □

**Corollary 2.4.5.** *Let $E$ be a Banach space such that $E$ is isomorphically a dual space. Then $E$ is complemented in $E''$.* □

**Proposition 2.4.6.** *Let $K$ be an infinite, locally compact space.*

(i) *The space $C_0(K)$ contains a subspace that is isometrically isomorphic to $c_0$.*

(ii) *Suppose that $K$ contains a convergent sequence of distinct points. Then $C_0(K)$ contains a 2-complemented subspace that is isometrically isomorphic to $c_0$.*

*Proof.* (i) The space $K$ contains sequences $(x_n)$ of distinct points and $(U_n)$ of pairwise-disjoint, open subsets such that $x_n \in U_n$ and $\overline{U_n}$ is compact for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists $f_n \in C_0(K)^+$ with $f_n(x_n) = |f_n|_K = 1$ and supp $f_n \subset U_n$. Essentially as in the proof of Theorem 2.2.23, (a) $\Rightarrow$ (c), set

$$T\alpha = \sum_{n=1}^{\infty} \alpha_n f_n \quad (\alpha = (\alpha_n) \in c_0).$$

Then $T : c_0 \to C_0(K)$ is a linear isometry that identifies $c_0$ with the closed subspace $\overline{\mathrm{lin}}\{f_n : n \in \mathbb{N}\}$ of $C_0(K)$.

(ii) Let $(x_n)$ be a convergent sequence of distinct points in $K$, say $x_n \to x_0$ as $n \to \infty$; we may suppose that $x_n \neq x_0$ $(n \in \mathbb{N})$. Choose neighbourhoods of each $x_n$ as in (i) such that $x_0 \notin U_n$ $(n \in \mathbb{N})$, and let $(f_n)$ and $T : c_0 \to C_0(K)$ be as in (i), so that $T$ is an isometric embedding.

For $g \in C_0(K)$, set

$$Pg = \sum_{n=1}^{\infty} (g(x_n) - g(x_0)) f_n .$$

Then $Pg \in C_0(K)$ $(g \in C_0(K))$ and $P \in \mathscr{B}(C_0(K))$ is a projection onto $T(c_0)$ with $\|P\| = 2$. Hence $T(c_0)$ is 2-complemented in $C_0(K)$. □

In fact, the following result concerning complemented copies of $c_0$ in $C(K)$–spaces is given in [184, Corollary 5.3.12 and Proposition 5.3.6], for example; Grothendieck spaces were defined in Definition 2.1.1.

**Proposition 2.4.7.** *Let $K$ be a non-empty, compact space. Then the Banach space $C(K)$ is a Grothendieck space if and only if $C(K)$ contains no complemented subspace that is isomorphic to $c_0$.* □

**Proposition 2.4.8.** *The Banach space $\ell^1$ is isometrically isomorphic to a 1-complemented subspace of $L^1(\mathbb{I})$.*

*Proof.* Let $\{I_n : n \in \mathbb{N}\}$ be a family of pairwise-disjoint, closed intervals in $\mathbb{I}$, and, for each $n \in \mathbb{N}$, let $\chi_n$ be the characteristic function of $I_n$, $\ell_n$ the length of $I_n$, and $f_n = \chi_n/\ell_n$, so that $\|f_n\|_1 = 1$. Then take $E = \overline{\mathrm{lin}}\{f_n : n \in \mathbb{N}\}$, so that $E$ is a closed subspace of $L^1(\mathbb{I})$.

Take $\alpha = (\alpha_n) \in \ell^1$. Then it is clear that the map

$$\alpha \mapsto \sum_{n=1}^{\infty} \alpha_n f_n\,, \quad \ell^1 \to E\,,$$

is an isometric embedding, and so $E \cong \ell^1$.

Define the map

$$P : f \mapsto \sum_{n=1}^{\infty} \left( \int_{I_n} f \right) f_n\,, \quad L^1(\mathbb{I}) \to L^1(\mathbb{I})\,.$$

Clearly $P$ is a linear map with

$$Pf_n = f_n \quad (n \in \mathbb{N}) \quad \text{and} \quad \|Pf\|_1 \le \|f\|_1 \quad (f \in L^1(\mathbb{I}))\,.$$

Thus $P$ is a bounded projection onto $E$ with $\|P\| = 1$, and so $E$ is a 1-complemented subspace in $L^1(\mathbb{I})$. □

A similar argument [3, Proposition 6.4.1] shows that, for each $p$ with $1 < p < \infty$, the Banach space $\ell^p$ is isometrically isomorphic to a 1-complemented subspace of $L^p(\mathbb{I})$.

We also remark that, for $r, p > 1$, the Banach space $\ell^r$ is isomorphic to a complemented subspace of $L^p(\mathbb{I})$ if and only if $r = p$ or $r = 2$ [3, Proposition 6.4.21]. Now take $r \ge 1$. For $p$ with $1 \le p \le 2$, the Banach space $\ell^r$ is isomorphic to a closed subspace of $L^p(\mathbb{I})$ if and only if $p \le r \le 2$, and, for $2 < p < \infty$, the space $\ell^r$ is isomorphic to a closed subspace of $L^p(\mathbb{I})$ if and only if $r = 2$ or $r = p$ [3, Proposition 6.4.19].

We now present a beautiful result of Pełczyński from [196]; it will be used later. It is called the *Pełczyński decomposition method*. Our proof is taken from [3, Theorem 2.2.3].

**Theorem 2.4.9.** *Let $E$ and $F$ be normed spaces such that both $E$ and $F$ are isomorphic to complemented subspaces of the other. Further, suppose that either $E \sim E \times E$ and $F \sim F \times F$ or that $E \sim \ell^\infty(E)$. Then $E \sim F$.*

*Proof.* There exist normed spaces $G$ and $H$ such that $F \sim E \oplus G$ and $E \sim F \oplus H$, so that $F \sim E \times G$ and $E \sim F \times H$.

In the first case, we have

$$E \sim F \times H \sim (F \times F) \times H \sim F \times (F \times H) \sim F \times E$$

and, similarly, $F \sim E \times F$. But $E \times F \cong F \times E$, and so $E \sim F$.

In the second case, we have $E \sim E \times E$, and so $F \sim E \times F$, as before. But now

$$E \sim \ell^\infty(E) \sim \ell^\infty(F \times H) \sim \ell^\infty(F) \times \ell^\infty(H) \sim F \times \ell^\infty(F) \times \ell^\infty(H) \sim F \times E\,,$$

and so we again see that $E \sim F$. □

Since we shall discuss complemented subspaces of Banach spaces of the form $C(K)$, it is important to note that not all such closed subspaces are complemented; indeed, the most famous counter-example to this possibility is given by *Phillips' theorem* that $c_0$ is not complemented in $\ell^\infty$. A slightly stronger version of this theorem already follows easily from a previous result. Indeed, assume towards a contradiction that $c_0$ is complemented in $\ell^\infty$. Then there is an embedding of $\ell^\infty/c_0$ into $\ell^\infty \cong C(\beta\mathbb{N})$. However it follows from Example 2.2.22 that there is no such embedding. See also [148, p. 19].

Nevertheless, we wish to give the classical, elementary proof of Phillips' theorem; it is taken from [240]. See also [3, Theorem 2.5.5] and [183, Theorem 3.2.20].

**Definition 2.4.10.** Let $E$ be a Banach space. A subset $T$ of $E'$ is *total* if $x = 0$ whenever $x \in E$ and $\langle x, \lambda\rangle = 0$ $(\lambda \in T)$; a Banach space $E$ has *property* (T) if $E'$ contains a countable, total subset.

Note that property (T) is preserved under isomorphisms and under the passage to closed subspaces.

**Theorem 2.4.11.** *The subspaces $c_0$ and $c$ are not complemented in $\ell^\infty$.*

*Proof.* First, assume towards a contradiction that there is a closed subspace $F$ of $\ell^\infty$ such that $\ell^\infty = c_0 \oplus F$. We regard $F$ as a Banach space by setting

$$\|x\| = d(x, c_0) \quad (x \in F)\,,$$

the distance from $x$ to $c_0$ in $\ell^\infty$, thus identifying $F$ with the quotient space $\ell^\infty/c_0$.

Clearly $\{\delta_n : n \in \mathbb{N}\}$ is a countable, total subset of $(\ell^\infty)'$, and so $(F, \|\cdot\|)$ has property (T).

Let $\{S_\alpha : \alpha \in A\}$ be a family of subsets of $\mathbb{N}$ as specified in Proposition 1.5.5, and, for $\alpha \in A$, let $f_\alpha$ be the coset in $F$ that corresponds to $\chi_{S_\alpha}$, so that $\|f_\alpha\| = 1$ $(\alpha \in A)$.

Take $\lambda \in F'$. We *claim* that the set $\{\alpha \in A : \langle f_\alpha, \lambda\rangle \neq 0\}$ is countable. For this, it suffices to show that, for each $n \in \mathbb{N}$, the set

$$C_n := \{\alpha \in A : |\langle f_\alpha, \lambda\rangle| \geq 1/n\}$$

is finite. Indeed, fix $n \in \mathbb{N}$, and then, for $m \in \mathbb{N}$ with $m \le |C_n|$, choose distinct elements $\alpha_1, \dots, \alpha_m \in C_n$; set $g_i = f_{\alpha_i}$ $(i \in \mathbb{N}_m)$ and

$$\beta_i = \operatorname{sgn}\langle g_i, \lambda \rangle \quad (i \in \mathbb{N}_m) \quad \text{and} \quad g = \sum_{i=1}^{m} \beta_i g_i \,.$$

Then there exists a number $N \in \mathbb{N}$ such that $S_{\alpha_i} \cap S_{\alpha_j} \subset \mathbb{N}_N$ for $i, j \in \mathbb{N}_m$ with $i \neq j$, and so $\|g\| = 1$, regarding $g$ as an element of $F$. Thus

$$\|\lambda\| \ge |\langle g, \lambda \rangle| = \sum_{i=1}^{m} |\langle g_i, \lambda \rangle| \ge \frac{m}{n} \,,$$

and so $|C_n| \le n \|\lambda\| + 1$. Hence $C_n$ is finite, and the claim follows.

Now suppose that $\Lambda$ is a countable set in $(F, \|\cdot\|)'$. Then there are only countably many values of $\alpha \in A$ such that $\langle f_\alpha, \lambda \rangle \neq 0$ for some $\lambda \in \Lambda$, and so there exists an index $\alpha \in A$ with $\langle f_\alpha, \lambda \rangle = 0$ for all $\lambda \in \Lambda$. Thus the set $\Lambda$ is not total in $(F, \|\cdot\|)'$, a contradiction of the fact that $F$ has property (T).

It follows that $c_0$ is not complemented in $\ell^\infty$. Clearly $c$ is not complemented in $\ell^\infty$: if $c$ were so complemented, then $c_0$ would be complemented in $\ell^\infty$ because it is complemented in $c$. □

The following generalization by Conway of Phillips' theorem is taken from [64].

**Theorem 2.4.12.** *Let $K$ be a non-empty, locally compact space that is not pseudo-compact. Then $C_0(K)$ is not complemented in $C^b(K)$.*

*Proof.* There is a function $f \in C(K, \mathbb{R}^+) \setminus C^b(K)$. Choose $x_1 \in K$ with $f(x_1) > 1$, and then inductively choose $(x_n)$ in $K$ such that $f(x_{n+1}) > f(x_n) + 4$ for each $n \in \mathbb{N}$; set

$$U_n = \{x \in K : |f(x) - f(x_n)| < 1\} \quad (n \in \mathbb{N}) \,.$$

For each $n \in \mathbb{N}$, choose $f_n \in C_0(K, \mathbb{I})$ with $f_n(x_n) = 1$ and $\operatorname{supp} f_n \subset U_n$, and define

$$T\alpha = \sum_{n=1}^{\infty} \alpha_n f_n \quad (\alpha = (\alpha_n) \in \ell^\infty) \,.$$

For each $x \in K$, the neighbourhood $\{y \in K : |f(y) - f(x)| < 1\}$ of $x$ has non-empty intersection with at most one set $U_n$, and it follows easily from this that $T\alpha \in C^b(K)$ for each $\alpha \in \ell^\infty$. We see that $T(c_0) \subset C_0(K)$ and that $T : \ell^\infty \to C^b(K)$ is a linear isometry.

Define $Sg = (g(x_n))$ $(g \in C_0(K))$. Since the sequence $(x_n)$ has no accumulation point in $K$, each compact subset of $K$ contains at most finitely many points of this set, and so $Sg \in c_0$ $(g \in C_0(K))$. Clearly $S : C_0(K) \to c_0$ is a linear isometry and $(S \circ T)(\alpha) = \alpha$ $(\alpha \in c_0)$.

Assume to the contrary that there is a bounded projection $P : C^b(K) \to C_0(K)$. Then the map $S \circ P \circ T : \ell^\infty \to c_0$ is a bounded projection. But this is a contradiction of Theorem 2.4.11. Thus $C_0(K)$ is not complemented in $C^b(K)$. □

An elementary special case of the above is the following.

**Corollary 2.4.13.** *Let $\Gamma$ be an infinite set. Then $c_0(\Gamma)$ is not complemented in $\ell^\infty(\Gamma)$.* □

The space $[0,\omega_1)$ is pseudo-compact. Here $C_0([0,\omega_1))$ has codimension 1, and so is complemented, in $C^b([0,\omega_1)) \cong C([0,\omega_1])$.

The following result, called *Sobczyk's theorem*, is taken from [3, Theorem 2.5.8], [20, Theorem 2.3], and [175, Theorem 2.f.5]; the elegant proof is due to Veech [238].

**Theorem 2.4.14.** *Let $E$ be a separable Banach space containing $c_0$ as a closed subspace. Then $c_0$ is 2-complemented in $E$.*

*Proof.* Since $E$ is separable, it follows from Theorem 2.1.4(iii) that there is a metric, say $d$, giving the weak* topology on $E'_{[1]}$.

Let $n \in \mathbb{N}$. Then the map $\delta_n : (\alpha_m) \mapsto \alpha_n$ is a continuous linear functional on $c_0$ with $\|\delta_n\| = 1$. Let $\lambda_n \in E'$ be a norm-preserving extension of $\delta_n$, and set

$$S = \{\lambda \in E'_{[1]} : \lambda \mid c_0 = 0\}\,.$$

Since each weak*-limit point of $\{\lambda_n : n \in \mathbb{N}\}$ belongs to $S$, $\lim_{n\to\infty} d(\lambda_n, S) = 0$, and so there is sequence $(\mu_n)$ in $S$ with $\lim_{n\to\infty} d(\lambda_n, \mu_n) = 0$. Since $\lim_{n\to\infty}(\lambda_n - \mu_n) = 0$ in $(E'_{[1]}, \sigma(E',E))$, the map $P : x \mapsto (\langle x, \lambda_n - \mu_n\rangle)$, $E \to c_0$, is a bounded projection onto $c_0$, and clearly $\|P\| \le 2$. □

For interesting extensions of Sobczyk's theorem, see [14]. In fact, it is a theorem of Zippin that a Banach space that is complemented in every separable Banach space that contains the space as a closed subspace is isomorphic to $c_0$ [246, 247]. For an entertaining essay on Sobczyk's theorem and Phillips' theorem, see [48].

**Theorem 2.4.15.** *Let $E$ be a Banach space containing $c_0$ as a closed, complemented subspace. Then $E$ is not complemented in $E''$ and $E$ is not isomorphically a dual space. In particular, $c_0$ is not isomorphically a dual space.*

*Proof.* There is a bounded projection $P$ of $E$ onto $c_0$. Assume that there is a bounded projection $Q$ of $E''$ onto $E$. We may regard the spaces $c_0$ and $\ell^\infty = c_0''$ as closed subspaces of $E''$, and then $(P \circ Q) \mid \ell^\infty$ is a bounded projection of $\ell^\infty$ onto $c_0$, a contradiction of Theorem 2.4.11. Thus $E$ is not complemented in $E''$. By Corollary 2.4.5, $E$ is not isomorphically a dual space. □

**Corollary 2.4.16.** *Let $E$ be a separable Banach space containing $c_0$ as a closed subspace. Then $E$ is not complemented in $E''$ and $E$ is not isomorphically a dual space.*

*Proof.* By Theorem 2.4.14, $c_0$ is complemented in $E$, and so this follows from the theorem. □

**Corollary 2.4.17.** *Let $K$ be a locally compact space that contains a convergent sequence of distinct points. Then $C_0(K)$ is not complemented in $C_0(K)''$ and $C_0(K)$ is not isomorphically a dual space.*

*Proof.* By Proposition 2.4.6(ii), $C_0(K)$ contains $c_0$ as a closed, complemented subspace, and so the result follows from Theorem 2.4.15. □

In particular, the above corollary covers the cases where $K$ is an infinite, compact, metrizable space, where $K = [0, \alpha]$ for an ordinal $\alpha \geq \omega$, and where $K = \mathbb{Z}_2^\kappa$, the Cantor cube of weight $\kappa$: in each of these cases, it is easy to see that the space contains a convergent sequence of distinct points.

**Definition 2.4.18.** Let $E$ be a Banach space. Then $E$ is *prime* if every complemented, infinite-dimensional, closed subspace of $E$ is isomorphic to $E$.

Clause (i) of the following theorem is a famous result of Pełczyński [3, Theorem 2.2.4]; clause (ii) is a theorem of Lindenstrauss [3, Theorem 5.6.5].

**Theorem 2.4.19.** (i) *The spaces $c_0$ and $\ell^p$, for $1 \leq p < \infty$, are prime Banach spaces.*

(ii) *The space $\ell^\infty$ is a prime Banach space.* □

**Definition 2.4.20.** Let $E$ be a Banach space. Then $E$ is *primary* if, whenever $E$ is isomorphic to the direct sum of two Banach spaces, $E$ is isomorphic to one of the two summands.

As stated in [3, p. 122], $L^1(\mathbb{I})$ and $C(\mathbb{I})$ are not prime, but both are primary. In fact, each space $L^p(\mathbb{I})$ for $1 \leq p \leq \infty$ is primary [176, Theorem 2.d.11].

It is easily seen that $C(\mathbb{N}^*)$ is isomorphic to $C(\mathbb{N}^*) \oplus \ell^\infty$, and so we can regard (a copy of) $\ell^\infty$ as a complemented, infinite-dimensional, closed subspace of $C(\mathbb{N}^*)$. However, by Example 2.2.22, $\ell^\infty$ is not isomorphic to $C(\mathbb{N}^*)$, and so $C(\mathbb{N}^*)$ is not prime. It is known that, with CH, $C(\mathbb{N}^*)$ is primary [92], but it is not known whether this is a theorem of ZFC. Incidentally, we note that it is proved in [92] that, with CH, $C(\mathbb{N}^*) \sim \ell^\infty(C(\mathbb{N}^*))$ and in [46] that it is consistent with ZFC that $C(\mathbb{N}^*)$ is not isomorphic to $\ell^\infty(E)$ for any Banach space $E$.

A major result in this area is the following solution of the *complemented subspace problem*, due to Lindenstrauss and Tzafriri [173]. For a proof of this theorem, see [3, §12.4].

**Theorem 2.4.21.** *Let $E$ be an infinite-dimensional Banach space such that every closed subspace of $E$ is complemented in $E$. Then $E$ is isomorphic to a Hilbert space.* □

## 2.5 Projection properties and injective Banach spaces

We now consider the appropriate versions of projectivity and injectivity in the category of Banach spaces and bounded operators that we are considering.

**Definition 2.5.1.** A Banach space $E$ has the *projection property* if, whenever $F$ is a closed subspace of a Banach space $G$ that is isometrically isomorphic to $E$, the space $F$ is complemented in $G$. More generally, a Banach space $E$ is a $P_\lambda$*-space* (for $\lambda \geq 1$) if such a space $F$ is $\lambda$-complemented in $G$.

Suppose that $E$ is a $P_\lambda$-space for some $\lambda \geq 1$. Then the *projection constant* of $E$ is the infimum of the numbers $\lambda$ such that $E$ is a $P_\lambda$-space.

We represent the above situation with the following commutative diagram:

$$\begin{tikzcd} & G \arrow[d, dashed, bend right, "P"'] \\ E \arrow[r, Leftrightarrow] & F \arrow[u, hook] \end{tikzcd} .$$

The following is an immediate property of $P_\lambda$-spaces. Let $E$ be a $P_\lambda$-space, and suppose that $E$ is a closed subspace of a Banach space $G$, that $F$ is a Banach space, and that $T \in \mathscr{B}(E,F)$. Then there is an extension $\widetilde{T}$ of $T$ in $\mathscr{B}(G,F)$ such that $\left\|\widetilde{T}\right\| \leq \lambda \|T\|$. Indeed, let $P : G \to E$ be a bounded projection with $\|P\| \leq \lambda$, and set $\widetilde{T} = T \circ P$.

We represent the above situation with the following commutative diagram:

$$\begin{tikzcd} G \arrow[dr, dashed, "\widetilde{T}"] & \\ E \arrow[u, hook] \arrow[r, "T"'] & F \end{tikzcd} .$$

It is proved in [171, Theorem 6.10] that a real Banach space which is a $P_{1+\varepsilon}$-space for each $\varepsilon > 0$ is already a $P_1$-space. It seems to be unknown whether the same result holds for complex Banach spaces. However an example in [143] shows that a (real) Banach space which is a $P_{2+\varepsilon}$-space for each $\varepsilon > 0$ is not necessarily a $P_2$-space.

The next definition gives a similar concept with the spaces $E$ and $F$ 'the other way round'.

**Definition 2.5.2.** A Banach space $E$ is *injective* if, for every Banach space $G$, every closed subspace $F$ of $G$, and every $T \in \mathscr{B}(F,E)$, there is an extension $\widetilde{T} \in \mathscr{B}(G,E)$ of $T$; the space $E$ is $\lambda$*-injective* if, further, we can always find such a $\widetilde{T}$ such that $\left\|\widetilde{T}\right\| \leq \lambda \|T\|$.

We represent this situation with the following commutative diagram:

$$\begin{tikzcd} G \arrow[dr, dashed, "\widetilde{T}"] & \\ F \arrow[u, hook] \arrow[r, "T"'] & E\,. \end{tikzcd}$$

For a discussion of injective spaces, see [20, Chapter 1].

Clearly an injective space is complemented in any Banach space that contains it as a closed subspace, and injectivity is an isomorphic invariant for the class of all Banach spaces. For example, by Theorem 2.4.12 and Corollary 2.4.17, respectively, $C_0(K)$ is not injective whenever $K$ is a non-empty, locally compact space that is not pseudo-compact and whenever $K$ is a compact space that contains a convergent sequence of distinct points.

We see that a real Banach space is injective if and only if its complexification is injective.

We shall use the following obvious remark.

**Proposition 2.5.3.** *A complemented subspace of an injective space is injective; a* 1*-complemented subspace of a* 1*-injective space is* 1*-injective.* □

The next proposition is immediate from Theorem 2.4.9.

**Proposition 2.5.4.** *Let $E$ and $F$ be injective Banach spaces such that $E \sim E \times E$ and $F \sim F \times F$ and such that both $E$ and $F$ are isomorphic to closed subspaces of the other. Then $E \sim F$.* □

The following result was first noted by Phillips in [202, Corollary 7.2].

**Proposition 2.5.5.** *The space $\ell^\infty(S) = C(\beta S)$ is* 1*-injective for each non-empty set $S$.*

*Proof.* Take a Banach space $G$, a closed subspace $F$, and $T \in \mathscr{B}(F, \ell^\infty(S))$. For each $s \in S$, the functional $\lambda_s : x \mapsto (Tx)(s)$ on $F$ is continuous with $\|\lambda_s\| \leq \|T\|$. By the Hahn–Banach theorem, Theorem 2.1.2(i), each $\lambda_s$ has a norm-preserving extension $\tilde{\lambda}_s$ to $G$. Set

$$(\widetilde{T}x)(s) = \langle x, \tilde{\lambda}_s \rangle \quad (s \in S, x \in G)\,.$$

Then $\widetilde{T} \in \mathscr{B}(G, \ell^\infty(S))$ is an extension of $T$ with $\left\|\widetilde{T}\right\| = \|T\|$. □

**Corollary 2.5.6.** *Let $E$ be a Banach space. Then $E$ is isometrically isomorphic to a subspace of a* 1*-injective space.*

*Proof.* By Proposition 2.2.14(i), $E$ is isometrically isomorphic to a closed subspace of a space of the form $\ell^\infty(S)$. □

Take $p \in \mathbb{N}^*$, and set $M_p = \{f \in C(\beta\mathbb{N}) : f(p) = 0\}$. Then $M_p$ is a complemented subspace of $C(\beta\mathbb{N})$, and so $M_p$ is injective. This gives an example of a non-compact space $K = \beta\mathbb{N} \setminus \{p\}$ such that $C_0(K)$ is injective; see also Example 6.9.1 for a slightly stronger fact. Of course, as in Example 1.5.3(ii), $K$ is a pseudo-compact space.

On the other hand, the following result is immediate from Theorem 2.4.15.

**Proposition 2.5.7.** *Let $E$ be a Banach space containing $c_0$ as a closed, complemented subspace. Then $E$ is not injective.* □

**Proposition 2.5.8.** *Let $E$ be a separable, infinite-dimensional Banach space. Then $E$ is not injective.*

*Proof.* By Proposition 2.2.17(i), there is an isometric embedding of $E$ into $\ell^\infty$.

Assume to the contrary that $E$ is injective. Then $E$ is complemented in $\ell^\infty$. But, by Theorem 2.4.19(ii), $\ell^\infty$ is prime, and so $E$ is isomorphic to $\ell^\infty$. But $\ell^\infty$ is not separable, a contradiction. □

It follows from Theorem 2.1.7(i) that $C(K)$ is not injective whenever $K$ is an infinite, compact, metrizable space; a stronger result was given in Corollary 2.4.17.

**Proposition 2.5.9.** *Take $\lambda \geq 1$. Then a Banach space is $\lambda$-injective if and only if it is a $P_\lambda$-space, and it is injective if and only if it has the projection property.*

*Proof.* Suppose that the Banach space $E$ is $\lambda$-injective. Take $F$ to be a closed subspace of a Banach space $G$ such that $E \cong F$, and let $T : F \to E$ be a linear isometry. Then there is an extension $S \in \mathscr{B}(G,E)$ of $T$ with $\|S\| \leq \lambda$. Set $P = T^{-1} \circ S : G \to F$. Then $P$ is a bounded projection with $\|P\| \leq \lambda$, and so $E$ is a $P_\lambda$-space.

Now suppose that $E$ is a $P_\lambda$-space. Take $F$ to be a closed subspace of a Banach space $G$, and take $T \in \mathscr{B}(F,E)$. By Corollary 2.5.6 , we can identify $E$ as a closed subspace of a 1-injective space, say $H$. There is a bounded projection $P$ from $H$ onto $E$ with $\|P\| \leq \lambda$, and, since $T \in \mathscr{B}(F,H)$, there is a norm-preserving extension, say $L \in \mathscr{B}(G,H)$, of $T$. Set $\widetilde{T} = P \circ L$ to obtain the required extension of $T$.

Similarly, $E$ is injective if and only if it has the projection property. □

**Proposition 2.5.10.** *A Banach space with the projection property is a $P_\lambda$-space for some $\lambda \geq 1$.*

*Proof.* It is easy to see that $F_0$ is a $P_{\lambda\mu}$-space whenever $F$ is a $P_\lambda$-space, and there is a bounded projection of norm $\mu$ from $F$ onto the subspace $F_0$. By Corollary 2.5.6, each Banach space $E$ is a closed subspace of a 1-injective space $F$. In the case where $E$ has the projection property, there is a bounded projection $P : F \to E$, and so $E$ is $P_\lambda$-space with $\lambda = \|P\|$. □

It follows that an injective Banach space is $\lambda$-injective for some $\lambda \geq 1$.

Let $K$ and $L$ be two non-empty, compact spaces. First, let $\eta : K \to L$ be a continuous map, and define

$$\eta^\circ : f \mapsto f \circ \eta\,, \quad C(L) \to C(K)\,. \tag{2.9}$$

Then $\eta^\circ$ is a bounded operator with $\|\eta^\circ\| = 1$. Further, $\eta^\circ$ is a surjection if and only if $\eta$ is an injection, and $\eta^\circ$ is an injection if and only if $\eta$ is a surjection if and only if $\eta^\circ$ is isometric. In particular, let $(G_L, \pi_L)$ be the Gleason cover of $L$, as in Theorem 1.6.5. Then the map $\pi_L^\circ : C(L) \to C(G_L)$ is an isometric embedding.

We first generalize Proposition 2.5.5.

**Theorem 2.5.11.** *Let $K$ be a non-empty, Stonean space. Then $C(K)$ is 1-injective. Further, $C(K)$ is isometrically isomorphic to a complemented subspace of $C(\beta K_d)$, which is isometrically a bidual space.*

*Proof.* By Theorem 1.6.3, (a) ⇒ (b), there is a retraction $\theta : \beta K_d \to K$.

Let $\eta : K \to \beta K_d$ be the natural embedding, so that $\theta \circ \eta$ is the identity on $K$. Then the map $\theta^\circ : C(K) \to C(\beta K_d)$ is an isometry and $\eta^\circ : C(\beta K_d) \to \theta^\circ(C(K))$ is a linear surjection with $\|\eta^\circ\| = 1$. Since $\eta^\circ \circ \theta^\circ$ is the identity on $C(K)$, the map $\eta^\circ$ is a bounded projection. By Proposition 2.5.5, $C(\beta K_d)$ is 1-injective, and so $C(K)$ is 1-injective.

Of course, $C(\beta K_d)$ is isometrically the bidual of $C_0(K_d)$. □

We shall see in Theorem 6.8.3 that, conversely, $K$ is Stonean whenever $C(K)$ is 1-injective. Indeed, Question 3 on page 212 will raise the possibility that the only injective Banach spaces are those isomorphic to $C(K)$ for $K$ a Stonean space.

**Corollary 2.5.12.** *Let $K$ be a non-empty, compact space. Then $C(G_K)$ is 1-injective.*

*Proof.* By Theorems 1.6.5, $G_K$ is a Stonean space. □

There is a closely related theory of extensions of Banach spaces. Some of these results will be used in the characterization of 1-injective Banach spaces to be given in Theorem 6.8.6. The next few results are based on Bade's notes [23, 24]; see also [166, §11].

**Definition 2.5.13.** Let $E$ be a closed subspace of a Banach space $F$. Then:

(i) $F$ is an *essential extension* of $E$ if, for each Banach space $G$ and each contraction $T \in \mathscr{B}(F, G)$ such that $T \mid E$ is an isometry, $T$ is also an isometry;

(ii) $F$ is a *rigid extension* of $E$ if, for each contraction $T \in \mathscr{B}(F)$ such that $T \mid E = I_E$, necessarily $T = I_F$.

**Proposition 2.5.14.** *Let $E$ be a Banach space. Then each essential extension of $E$ is rigid.*

*Proof.* Let $F$ be an essential extension of $E$, and assume towards a contradiction that $F$ is not rigid. Then there are a contraction $T \in \mathscr{B}(F)$ and $y \in F$ such that $T \mid E = I_E$ and $Ty \neq y$, say $z = y - Ty$, so that $z \neq 0$; we may suppose that $\|y\| = 1$. Set $M = \mathbb{C}z$. Then the quotient map $q : F \to F/M$ is a contraction that is not an isometry because $q(z) = 0$.

We *claim* that $q \mid E$ is an isometry; this will give the required contradiction. Indeed, assume that $q \mid E$ is not an isometry. Then there exist $x \in E$ with $\|x\| = 1$ and $\delta > 0$ such that $\|x + \delta z\| < 1$. There exists $\eta > 0$ such that $\|w + \delta z\| < \|w\|$ whenever $\|w - x\| \leq \eta$. For such an element $w$, we have

$$\|w\| < \|w - \eta z\| , \tag{2.10}$$

for otherwise

$$\|w\| \leq \frac{\eta}{\delta + \eta} \|w + \delta z\| + \frac{\delta}{\delta + \eta} \|w - \eta z\| < \|w\| .$$

We apply (2.10) with $w = x + \eta y$, so that $w - \eta z = x + \eta Ty = T(x + \eta y)$, to see that $\|x + \eta y\| < \|T(x + \eta y)\|$; this is a contradiction of the fact that $T$ is a contraction. Thus $q \mid E$ is an isometry. □

**Proposition 2.5.15.** *Let $E$ be a closed subspace of a Banach space $(F, \|\cdot\|)$. Then the following are equivalent:*

(a) *for each semi-norm $p$ on $F$ with $p(x) = \|x\| \ (x \in E)$ and $p(y) \leq \|y\| \ (y \in F)$, necessarily $p(y) = \|y\| \ (y \in F)$;*

(b) *$F$ is an essential extension of $E$.*

*Proof.* (a) $\Rightarrow$ (b) Let $G$ be a Banach space, and suppose that $T : F \to G$ is a contraction such that $T \mid E$ is an isometry. Set $p(y) = \|Ty\| \ (y \in F)$. Then $p$ is a semi-norm on $F$ satisfying the conditions in (a), and so $p(y) = \|y\| \ (y \in F)$, whence $T$ is an isometry.

(b) $\Rightarrow$ (a) Let $p$ be a semi-norm on $F$ satisfying the conditions in (a), and set $K = \{y \in F : p(y) = 0\}$. Take $q$ to be the quotient map from $F$ onto the space $F/K$, let $F/K$ have the norm induced by $p$, and take $G$ to be the completion of this space. Then $q : F \to G$ is a contraction and $q \mid E$ is an isometry, and so, by (b), $q$ is an isometry. It follows that $p(y) = \|y\| \ (y \in F)$. □

**Theorem 2.5.16.** *Let $E$ be a closed subspace of a Banach space $F$, and suppose that $F$ is a 1-injective space. Then there is a closed subspace $G$ of $F$ containing $E$ such that $G$ is a 1-injective space and $G$ is a rigid extension of $E$.*

*Proof.* Let $\mathscr{F}$ be the family of semi-norms $p$ on $F$ such that $p(x) = \|x\| \ (x \in E)$ and $p(y) \leq \|y\| \ (y \in F)$. For $p, q \in \mathscr{F}$, set $p \leq q$ if $p(y) \leq q(y) \ (y \in F)$. Then $(\mathscr{F}, \leq)$

is a partially ordered space. Clearly each chain in $(\mathscr{F}, \leq)$ has a lower bound, and so $(\mathscr{F}, \leq)$ has a minimal element, say $p_0$. Let $H$ be the completion of $F/\ker p_0$, and let $\pi : F \to F/\ker p_0$ be the quotient map.

We can regard $E$ as a closed subspace of $H$. Since $F$ is 1-injective, there is a contraction $T : H \to F$ with $T \mid E = I_E$. Set $P = T \circ \pi$, so that $Px = x \ (x \in E)$ and $\|P\| = 1$.

Set $p_1(y) = \|Py\| \ \ (y \in F)$. Then $p_1 \leq p_0$ in $(\mathscr{F}, \leq)$, and so $p_1 = p_0$ by the minimality of $p_0$. Next define

$$p_2(y) = \limsup_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} P^i y \right\| \quad (y \in F).$$

Then $p_2 \leq p_1$ in $(\mathscr{F}, \leq)$, and so $p_2 = p_0$. Further,

$$p_2(y - Py) = \limsup_{n\to\infty} \left\| \frac{1}{n} \left(Py - P^{n+1}y\right) \right\| = 0 \quad (y \in F),$$

and so $\left\|Py - P^2y\right\| = p_1(y - Py) = p_2(y - Py) = 0 \ (y \in F)$. This shows that $P^2 = P$ in $\mathscr{B}(F)$.

Set $G = P(F)$. Then $G$ is a closed subspace of $F$ containing $E$ and $G$ is a 1-injective space.

Finally, we show that $G$ is an essential extension of $E$; for this, we verify clause (a) of Proposition 2.5.15. Indeed, let $p$ be a semi-norm on $G$ such that $p(x) = \|x\| \ \ (x \in E)$ and $p(y) \leq \|y\| \ \ (y \in G)$. Then $p \circ P \in \mathscr{F}$, and

$$(p \circ P)(y) \leq \|Py\| = p_1(y) = p_0(y) \quad (y \in F),$$

and so $p \circ P = p_0$ and $p(y) = \|y\| \ \ (y \in G)$, as required.

By Proposition 2.5.14, $G$ is a rigid extension of $E$. □

The rigid extension $G$ of $E$ clearly has the property that, for each 1-injective subspace $H$ of $G$ with $E \subset H$, necessarily $H = G$. Further, suppose that $H$ has the same properties as $G$. Then $H$ is isometrically isomorphic to $G$ by a map that is the identity on $E$. The space $G$ is the *injective envelope* of $E$; we shall see in Theorem 6.8.6 that an injective envelope of a Banach space has the form $C(K)$ for a certain Stonean space $K$.

Recall from page 15 that $\Delta$ denotes the Cantor set.

**Proposition 2.5.17.** *Let $E$ be a separable Banach space. Then there is an isometric embedding of $E$ into $C(\Delta)$.*

*Proof.* By Proposition 2.2.14(i), there is a non-empty, compact, metrizable space $B$ and an isometric isomorphism $T : E \to C(B)$. By Proposition 1.4.6(i), there is a continuous surjection $\eta : \Delta \to B$. Thus $\eta^\circ : C(B) \to C(\Delta)$ is an isometric embedding. The map $\eta^\circ \circ T : E \to C(\Delta)$ is also an isometric embedding. □

The above results say that $C(\Delta)$ is *universal* in the class of separable Banach spaces. It follows easily that $C(\mathbb{I})$ is also universal in the class of separable Banach spaces. This is the *Banach–Mazur theorem*, already given in [30, Chapitre XI, §8]; see also [3, Theorem 1.4.3] and [225, Theorem 8.7.2]. These results are contained in [23, Chapter 4]; early texts in which they appeared are [82, p. 123] and [225], and the standard account is [175, §2f]. For example, it is proved in [175, Theorem 2.f.3] that every infinite-dimensional injective Banach space contains a closed subspace that is isomorphic to $\ell^\infty$. For a more recent discussion of these properties, see [247]. It is stated in [172, p. 337] that a Banach space is injective if and only if it is a so-called $\mathscr{L}_\infty$ space and is isomorphic to a complemented subspace of a dual space.

We have noted in equation (1.6) that $w(\mathbb{N}^*) = d(C(\mathbb{N}^*)) = |C(\mathbb{N}^*)| = \mathfrak{c}$. By a famous theorem of Parovichenko (see [99, p. 236] and [239, p. 81]), every compact (Hausdorff) space of weight at most $\aleph_1$ is a continuous image of $\mathbb{N}^*$. Recall from Proposition 2.2.14(i) that each Banach space $E$ is isometrically embedded in the space $C(B)$, where $B = E'_{[1]}$ and that $d(E) = w(B)$ by Corollary 2.1.8. Hence every Banach space of density at most $\aleph_1$ can be isometrically embedded in $C(\mathbb{N}^*)$, and so, with CH, $C(\mathbb{N}^*)$ is universal in the class of Banach spaces of density $\mathfrak{c}$. However this is not a result of the theory ZFC: it is consistent with ZFC that there is no isometrically universal Banach space of density $\mathfrak{c}$ [226]. For further related and stronger results, see [45, 46]. For example, it is consistent with ZFC that the Banach space $C([0,\mathfrak{c}])$ does not embed into $C(\mathbb{N}^*)$.

There is an extension of the notion of an injective space. A Banach space $E$ is *separably injective* if, for every separable Banach space $G$, every closed subspace $F$ of $G$, and every $T \in \mathscr{B}(F,E)$, there is an extension $\widetilde{T} \in \mathscr{B}(G,E)$ of $T$. Obviously, every injective space is separably injective. By Zippin's theorem, mentioned above, the only separable and separably injective Banach space is $c_0$. The idea of extending the notion of separably injective spaces to non-separable spaces was introduced by Rosenthal in [214]. Examples of non-separable spaces which are separably injective but not injective are certain Banach spaces $\ell^\infty_c(\Gamma)$, to be discussed below at Example 6.7.1, and $C(\mathbb{N}^*)$ (due to Lindenstrauss). For accounts of separably injective Banach spaces, including these examples, see [19, 20] and [247, p. 1722].

We shall discuss the injectivity of $C(K)$-spaces further in §6.8.

Although it is not strictly relevant to our work, we briefly introduce the dual concept to that of an injective space.

**Definition 2.5.18.** A Banach space $E$ is *projective* if, for every Banach space $G$, every quotient Banach space $F$ of $G$ with quotient map $q : G \to F$, and every operator $T \in \mathscr{B}(E,F)$, there is a *lifting* $\widetilde{T} \in \mathscr{B}(E,G)$ of $T$, in the sense that $T = q \circ \widetilde{T}$; the space $E$ is $\lambda$*-projective* (for $\lambda \geq 1$) if, further, we can always find such a $\widetilde{T}$ with $\left\|\widetilde{T}\right\| \leq \lambda \|T\|$.

We represent the above situation with the following commutative diagram:

$$\begin{tikzcd} & G \arrow[d, two heads, "q"] \\ E \arrow[ur, dashed, "\widetilde{T}"] \arrow[r, "T"'] & F. \end{tikzcd}$$

Each projective Banach space is $\lambda$-projective for some $\lambda \geq 1$. The following results give characterizations of projective Banach spaces.

**Theorem 2.5.19.** *A Banach space is $(1+\varepsilon)$-projective for each $\varepsilon > 0$ if and only if it is isometrically isomorphic to a Banach space of the form $\ell^1(\Gamma)$ for a non-empty set $\Gamma$.*

*Proof.* This is proved in [166, Theorem 9, p. 178] and in [225, Theorem 27.4.2]; that a 1-projective space has the form $\ell^1(\Gamma)$ is due to Grothendieck [125]. □

**Theorem 2.5.20.** *A Banach space is 1-projective if and only if it is isometrically isomorphic to a Banach space of the form $L^1(\Omega,\mu)$ for a measure space $(\Omega,\mu)$.*

*Proof.* This is proved in [166, Corollary to Theorem 8, p. 178]. □

**Theorem 2.5.21.** *A Banach space is projective if and only if it is isomorphic to a Banach space of the form $\ell^1(\Gamma)$ for a non-empty set $\Gamma$.*

*Proof.* For this, see [175, p. 108]; the result is due to Köthe [162]. □

## 2.6 The Krein–Milman and Radon–Nikodým properties

We shall be concerned with the extreme points of the closed unit ball and other bounded subsets of a Banach space; we shall discuss, rather briefly, the seminal notions of Banach spaces having the Krein–Milman property and the Radon–Nikodým property.

The first result is the famous *Krein–Milman theorem*; see [6, Theorem 3.31] or [218, Theorem 3.23], for example.

**Theorem 2.6.1.** *Let $L$ be a non-empty, compact, convex subset of a locally convex space over $\mathbb{R}$ or $\mathbb{C}$. Then $L = \overline{\mathrm{co}}(\mathrm{ex}\,L)$.* □

**Corollary 2.6.2.** *Let $E$ be a normed space. Then the set $\mathrm{co}(\mathrm{ex}\,E'_{[1]})$ is weak$^*$-dense in $E'_{[1]}$.* □

**Corollary 2.6.3.** *Let $E$ be a Banach space such that* $\operatorname{ex} E_{[1]} = \emptyset$*. Then $E$ is not isometrically a dual space.*

*Proof.* Assume that $E \cong F'$ for a Banach space $F$. Set $L = E_{[1]}$, so that $L$ is a non-empty, compact, convex subset of the locally convex space $(E, \sigma(E,F))$, and hence, by the theorem, $\operatorname{ex} L \neq \emptyset$, a contradiction. □

We shall see in Example 6.9.1 that there are Banach spaces $E$ such that $\operatorname{ex} E_{[1]} = \emptyset$ and $E$ is isomorphically a dual space.

**Corollary 2.6.4.** *Let $E$ be Banach space, and set $B = E'_{[1]}$. Suppose that $L$ is a closed subset of $B$ such that* $\operatorname{ci} L = \overline{\operatorname{ex}} B$*. Then the map*

$$J : x \mapsto \kappa_E(x) \mid L, \quad E \to C(L),$$

*is an isometric embedding.*

*Proof.* The set $\operatorname{ci} L$ is a circled subspace of $B$ with $\overline{\operatorname{co}}(\operatorname{ci} L) = B$, and so it follows from Proposition 2.2.14(ii) that, for each $x \in E$, there exist $\lambda \in L$ and $\zeta \in \mathbb{T}$ such that $\|x\| = |\langle x, \zeta\lambda \rangle|$. But then $\|x\| = |\langle x, \lambda \rangle|$, and so $J$ is an isometry. □

We now give a geometric property, that of 'dentability', of subsets of a Banach space. This is a notion that was introduced by Rieffel in [208].

**Definition 2.6.5.** Let $E$ be a Banach space. Then a bounded subset $S$ of $E$ is *dentable* if, for each $\varepsilon > 0$, there exists $x \in S$ such that $x \notin \overline{\operatorname{co}}(S \setminus B_\varepsilon(x))$.

The next theorem, Theorem 2.6.7, is due to Rieffel [208, Theorem 3]; it will be used in the proof of Corollary 2.6.12.

**Lemma 2.6.6.** *Let $E$ be a Banach space, and let $S$ be a bounded subset of $E$. Suppose that $\overline{\operatorname{co}} S$ is dentable. Then $S$ is dentable.*

*Proof.* Take $\varepsilon > 0$. Then there exists $x_0 \in (\overline{\operatorname{co}} S) \setminus Q$, where

$$Q = \overline{\operatorname{co}}((\overline{\operatorname{co}} S) \setminus B_{\varepsilon/2}(x_0)).$$

Assume that $S \subset Q$. Then $\overline{\operatorname{co}} S \subset Q$ and $x_0 \in Q$, a contradiction. So $S \not\subset Q$, and there exists an element $x_1 \in S \setminus Q$; necessarily $x_1 \in B_{\varepsilon/2}(x_0)$. Thus $B_{\varepsilon/2}(x_0) \subset B_\varepsilon(x_1)$, and so $S \setminus B_\varepsilon(x_1) \subset Q$, whence $\overline{\operatorname{co}}(S \setminus B_\varepsilon(x_1)) \subset Q$. This shows that

$$x_1 \in S \setminus \overline{\operatorname{co}}(S \setminus B_\varepsilon(x_1)),$$

and so $S$ is dentable. □

**Theorem 2.6.7.** *Let $\Gamma$ be any non-empty set. Then every non-empty, bounded subset of $\ell^1(\Gamma)$ is dentable.*

*Proof.* We shall work in the underlying real-linear space of $\ell^1(\Gamma)$.

By Lemma 2.6.6, it suffices to show that every non-empty, closed, convex, bounded set in $\ell^1(\Gamma)$ is dentable. Let $S$ be such a set, and suppose without loss of generality that $\sup\{\|f\|_1 : f \in S\} = 1$. Take $\varepsilon > 0$.

Choose $f \in S$ with $\|f\|_1 > 1 - \varepsilon/6$. Then there is a finite subset $F$ of $\Gamma$ such that $\sum_{\gamma \in F} |f(\gamma)| > 1 - \varepsilon/6$. Let $P : \ell^1(\Gamma) \to \ell^1(F)$ be the natural projection, so that $\|Pf\|_1 > 1 - \varepsilon/6$.

The set $P(S)$ is convex and bounded in the finite-dimensional space $\ell^1(F)$, and so $\overline{P(S)}$ is convex and compact. By the Krein–Milman theorem, Theorem 2.6.1, there is an extreme point $g_0$ of $\overline{P(S)}$ with $\|g_0\|_1 > 1 - \varepsilon/6$, and so $g_0 \notin \overline{\mathrm{co}}\,(P(S) \setminus B_{\varepsilon/6}(g_0))$. By the Hahn–Banach theorem, Theorem 2.1.2(ii), there is a real-linear functional $\lambda$ in the underlying real-linear space of $\ell^\infty(\Gamma)$ such that

$$\langle g_0, \lambda \rangle > 1 \quad \text{and} \quad \langle g, \lambda \rangle < 1 \quad (g \in P(S) \setminus B_{\varepsilon/6}(g_0))\,.$$

Choose $g \in S$ with $\|Pg - g_0\|_1 < \varepsilon/6$ and $\langle Pg, \lambda \rangle > 1$. We *claim* that

$$g \notin \overline{\mathrm{co}}\,(S \setminus B_\varepsilon(g))\,. \tag{2.11}$$

Indeed, take $h \in S$ with $\langle Ph, \lambda \rangle \geq 1$. Then $\|Ph - g_0\|_1 \leq \varepsilon/6$, and so we have $\|Ph - Pg\|_1 \leq \varepsilon/3$ and $\|Ph\|_1 > 1 - \varepsilon/3$; also, $\|Pg\|_1 > 1 - \varepsilon/3$. Since

$$\|Pg\|_1 + \|g - Pg\|_1 = \|g\|_1 \leq 1\,,$$

we have $\|g - Pg\|_1 < \varepsilon/3$; similarly, $\|h - Ph\|_1 < \varepsilon/3$. Thus $\|g - h\|_1 < \varepsilon$. It follows that $\langle Ph, \lambda \rangle < 1$ for each $h \in S \setminus B_\varepsilon(g)$, and hence $\langle Ph, \lambda \rangle < 1$ for each element $h \in \overline{\mathrm{co}}\,(S \setminus B_\varepsilon(g))$. Since $\langle Pg, \lambda \rangle > 1$, our claim that (2.11) holds is valid.

It follows that $S$ is dentable. □

We remark that the following related theorem of Rieffel is proved in [190]; see also [123, Appendix 2].

**Theorem 2.6.8.** *Let $E$ be a separable Banach space. Then every weakly compact, convex subset of $E$ is dentable.* □

**Definition 2.6.9.** Let $K$ be a closed, bounded, convex set in a Banach space $E$. Then $K$ has the *Krein–Milman property* if $L = \overline{\mathrm{co}}(\mathrm{ex}\,L)$ for every closed, convex subset $L$ of $K$. A Banach space $E$ has the *Krein–Milman property* if $E_{[1]}$ has the Krein–Milman property.

Suppose that $E$ has the Krein–Milman property. Then every closed, bounded, convex set in $E$ has the Krein–Milman property. Suppose, further, that $F$ is a Banach space with $F \sim E$. Then $F$ has the Krein–Milman property; the Krein–Milman property is an isomorphic invariant.

The study of the Krein–Milman property is assisted by the *Bishop–Phelps theorem* from [37]; we state an extension of the theorem given by Bollobás [40]. For a proof, see [85, VII, Theorem 4] and [100, Theorem 7.41], for example.

**Theorem 2.6.10.** *Let $E$ be a real Banach space. Suppose that $x \in S_E$, that $\lambda \in S_{E'}$, and that $\varepsilon > 0$. Then there exist $y \in S_E$ and $\mu \in S_{E'}$ such that $\langle y, \mu \rangle = 1$, such that $\|\mu - \lambda\| < \varepsilon$, and such that $\|y - x\| < \varepsilon + \varepsilon^2$.* □

A short, direct proof of the following theorem, using the Bishop–Phelps theorem, is given in the Handbook article of Johnson and Lindenstrauss [148, p. 35] and in [85, Theorem 5, p. 190]; see also [100, Theorem 11.3].

**Theorem 2.6.11.** *Let $E$ be a Banach space for which every non-empty, bounded subset is dentable. Then $E$ has the Krein–Milman property.* □

**Corollary 2.6.12.** *Let $\Gamma$ be a non-empty set. Then $\ell^1(\Gamma)$ has the Krein–Milman property.*

*Proof.* This follows from Theorem 2.6.7 and the above theorem. □

The above results give examples of Banach spaces that do have the Krein–Milman property. We shall now show, in Theorem 2.6.15, that the spaces $C_0(K)$ never have the Krein–Milman property whenever $K$ is infinite.

**Proposition 2.6.13.** *Let $K$ be a non-empty, locally compact space, and suppose that $f \in C_0(K)$. Then $f \in \operatorname{ex} C_0(K)_{[1]}$ if and only if $|f(x)| = 1 \ (x \in K)$.*

*Proof.* Set $B = C_0(K)_{[1]}$. Suppose that $f \in B$ and that there exists $x_0 \in K$ such that $|f(x_0)| < 1$. Set $\varepsilon = (1 - |f(x_0)|)/2$. Then there exists $U \in \mathcal{N}_{x_0}$ with $|f(x)| < 1 - \varepsilon$ for $x \in U$. Take $g \in C_{\mathbb{R}}(K)$ such that $0 \le g \le \chi_U$ and $g(x_0) = 1$. Then $f \pm \varepsilon g \in B$ and

$$f = \frac{1}{2}(f + \varepsilon g) + \frac{1}{2}(f - \varepsilon g),$$

and so $f \notin \operatorname{ex} B$.

It is easy to see that each $f \in C_0(K)$ with $|f(x)| = 1 \ (x \in K)$ belongs to $\operatorname{ex} B$. □

**Corollary 2.6.14.** *Let $K$ be a locally compact space that is not compact. Then $C_0(K)_{[1]}$ has no extreme points, and $C_0(K)$ is not isometrically a dual space.* □

In particular, we see again that $c_0$ is not isometrically a dual space.

**Theorem 2.6.15.** *Let $K$ be an infinite, locally compact space. Then $C_0(K)$ does not have the Krein–Milman property.*

*Proof.* By Corollary 2.6.14, we may suppose that $K$ is compact. Since $K$ is infinite, there is a non-isolated point, say $x_0$, of $K$. Consider the set

$$\{f \in C(K)_{[1]} : f(x_0) = 0\} :$$

this set is closed, bounded, and convex in $C(K)$, but it follows from Proposition 2.6.13 that it has no extreme points. □

The results in the remainder of this section require more background than our guidelines indicate, and so we shall omit most proofs.

The main theorem relating the above properties is the following.

**Theorem 2.6.16.** *Let $E$ be a Banach space. Then the following conditions on $E$ are equivalent:*

(a) *$E'$ has the Krein–Milman property;*

(b) *each bounded subset of $E'$ is dentable;*

(c) *each separable subspace of $E$ has a separable dual space.*

*Proof.* The implication (b) $\Rightarrow$ (a) follows from Theorem 2.6.11. For proofs of the other implications, see [85, pp. 190, 198], where histories of the theorems are also given. A key original source is a paper of Stegall [230]; see also [100, Theorem 11.14]. □

**Corollary 2.6.17.** *Let $E$ be a separable Banach space. Then $E'$ has the Krein–Milman property if and only if $E'$ is separable.*

*Proof.* This follows from the equivalence (a) $\Leftrightarrow$ (c) of the above theorem. □

**Corollary 2.6.18.** *Let $\Gamma$ be a non-empty set. Then $\ell^1(\Gamma)$ is isomorphically the dual of a separable Banach space if and only if $\Gamma$ is countable.*

*Proof.* We have $\ell^1(\Gamma) \cong (c_0(\Gamma))'$ and $c_0(\Gamma)$ is separable whenever $\Gamma$ is countable.

Now suppose that $\ell^1(\Gamma) \sim E'$ for a separable Banach space $E$. By Corollary 2.6.12, $\ell^1(\Gamma)$ has the Krein–Milman property, and so $E'$ has this property. By Corollary 2.6.17, $E'$ is separable, and so $\ell^1(\Gamma)$ is separable. Hence $\Gamma$ is countable. □

**Corollary 2.6.19.** *Let $E$ be a separable Banach space such that $\operatorname{ex} E_{[1]} = \emptyset$. Then $E$ is not isomorphically a dual space.*

*Proof.* Assume that $E \sim F'$ for a Banach space $F$. By Proposition 2.1.6, $F$ is separable, and so, by Corollary 2.6.17, $F'$ has the Krein–Milman property, and hence $E$ has this property. In particular, $\operatorname{ex} E_{[1]} \neq \emptyset$, a contradiction. □

We outline, without defining terms, a proof of one implication in Corollary 2.6.17, namely, of the fact that $E'$ has the Krein–Milman property whenever $E'$ is separable; this implication will be used in the proof of Theorem 4.4.17(i). The proof uses an idea of Bessaga and Pełczyński [35] concerning a re-norming theorem of Kadec and Klee for spaces with a separable dual. The full proofs are available in readily accessible texts, but this argument may not be as well known as some others.

The first step is as follows.

**Proposition 2.6.20.** *Let $E$ be a Banach space such that $E'$ is separable. Then $E$ admits an equivalent norm which is Fréchet differentiable at every $x \in E$ with $x \neq 0$.*

*Proof.* An explicit formula for such an equivalent norm on $E'$ is given in [33, Theorem 4.13, p. 89]. This norm is shown to be the dual of the desired equivalent Fréchet-differentiable norm on $E$. □

**Proposition 2.6.21.** *Let $E$ be a Banach space whose norm is is Fréchet differentiable at every $x \in E$ with $x \neq 0$. Then $E'$ is dentable.*

*Proof.* This is also a standard result; it is again a straightforward application of the Bishop–Phelps theorem, Theorem 2.6.10. See [100, Proposition 8.11, p. 391], for example. □

The stated implication in Corollary 2.6.17 now follows from Theorem 2.6.11.

There is another elegant proof of the above proposition due to Namioka [189]. This article introduced and crystallized the important concept of points of weak$^*$-to-norm continuity of the identity map on a dual Banach space (although the concept was already implicit in Bessaga–Pełczyński [35]). Namioka's proof is reproduced in [83, p. 159]; the original article is not cited in [83].

Let $E$ be a Banach space. The *Radon–Nikodým property* for $E$ delineates when there is an $E$-valued version of the standard Radon–Nikodým theorem: see [85, III.1] and [100], for example. The fine text [85] contains many different characterizations of the Radon–Nikodým property. See Chapters III, IV, and VII of [85] for a discussion of this property and some of its variants; in particular, pages 217/218 summarize many equivalent formulations of this property, and pages 218/219 specify many spaces that do and do not have the property. Each Banach space with the Radon–Nikodým property has the Krein–Milman property; it is not known whether the converse of this statement holds.

It is shown in [85, pp. 190, 198] and [100, Theorem 11.14] that the three clauses in Theorem 2.6.16 are also equivalent to the condition that $E'$ have the Radon–Nikodým property.

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