# Preface

Let $K$ be a locally compact space, and denote by $C_0(K)$ the Banach space of all continuous functions on $K$ that vanish at infinity, taken with the uniform norm. This fundamentally important and very familiar Banach space has been studied for many decades, and it arises in a vast number of applications in mathematical analysis. This book is devoted to the study of certain aspects of this space.

Indeed we shall address the rather specific questions:

(I) *When is* $(C_0(K), |\cdot|_K)$ *isometrically isomorphic or isomorphic* (*i.e. linearly homeomorphic*) *to the dual of a Banach space? If so, how unique is the predual?*

(II) *When is* $(C_0(K), |\cdot|_K)$ *isometrically isomorphic or isomorphic to the bidual of a Banach space?*

A more general theme that informs our work is the following question:

(III) *What are the relations between topological properties of the locally compact space K and Banach-space properties of* $C_0(K)$*?*

These questions have a long history, developed over around 80 years, some of which we shall record. Nevertheless, it seems that answers to even some basic questions are not complete, and at best are rather scattered in the literature. Sometimes existing proofs seem to be more complicated than is necessary.

We aim to give a coherent survey account of these matters; we shall recall, and sometimes clarify, necessary background from topology, measure theory, functional analysis, and other relevant areas of mathematics. Our approach will be close in spirit to the theory of Boolean algebras and ultrafilters, and there will be little mention of approaches through the theory of representations of $C^*$-algebras as algebras of operators on Hilbert spaces.

As we shall recount, some of the seminal work on these topics was brought together by Professor William Bade of Berkeley in mimeographed, unpublished lecture notes as early as around 1957.

We shall include some new results and examples. (However, some unattributed remarks are 'well known' and not original to us.) We shall offer more straightforward proofs of some known theorems and shall raise a number of open questions, some of which have withstood the test of time. We shall recall quickly some quite elementary results in topology, measure theory, and functional analysis, and a reader may wish to skim these pages, referring back to them when necessary. In general, we do not repeat proofs of theorems that are available in easily accessible and standard texts, but we do give proofs of some results that are basic to our work or which seem to be somewhat obscure or in less accessible sources.

We now give some more information on particular results that appear in the text; each chapter has a more detailed introduction to its contents.

Chapter 1 gives background in topology and Boolean algebras. Themes that will emerge include those of Stonean spaces, ultrafilters and the Stone space of a lattice, covers of locally compact spaces, the Stone–Čech compactification of a completely regular topological space, Gleason's characterization of projective compact spaces, and the Boolean algebras of regular-open and Borel subsets of a topological space.

Chapter 2 recalls background in Banach spaces and Banach lattices; we are particularly concerned to determine when two Banach spaces are mutually isomorphic and when they are mutually isometrically isomorphic, noting that there is a large difference between these two notions. We shall define what it means for a Banach space to be isomorphically/isometrically a (bi)dual space, and we shall discuss representations of the bidual of a given Banach space.

Let $K$ be a compact space. A key fact for us is the classic result that the Banach lattice $C_{\mathbb{R}}(K)$ is Dedekind complete if and only if $K$ is Stonean. In §2.5, we shall introduce $\lambda$-injective Banach spaces; later, in §6.8, we shall prove the famous result that $C(K)$ is 1-injective if and only if $K$ is Stonean; indeed, every 1-injective Banach space is isometrically isomorphic to $C(K)$ for some Stonean space $K$. It is one of the long-standing open questions to determine whether every injective Banach space is isomorphic to a 1-injective space.

The Krein–Milman property that is introduced in §2.6 will mainly be used to show that $C(K)$ is not isometrically a dual space for certain compact spaces $K$.

The topic of Chapter 3 is that of Banach algebras and $C^*$-algebras; of course the spaces $C_0(K)$, for locally compact spaces $K$, are the generic examples of commutative $C^*$-algebras. Let $A$ be a Banach algebra. Then the bidual space $A''$ is a Banach algebra containing $A$ as a closed subalgebra for two, sometimes distinct, products, $\Box$ and $\Diamond$, called the Arens products; the algebra $A$ is said to be Arens regular if these two products coincide on $A''$. It is another famous classical result that each $C^*$-algebra $A$ is Arens regular and that $(A'', \Box)$ is itself a $C^*$-algebra; it is the enveloping von Neumann algebra of $A$. By using the Gel'fand–Naimark theorem, it follows that the bidual space $C_0(K)''$ has the form $C(\widetilde{K})$ for a uniquely determined compact space $\widetilde{K}$, which we shall call the hyper-Stonean envelope of $K$ in §5.4. For example, the hyper-Stonean envelope of $\mathbb{N}$ is the Stone–Čech compactification $\beta\mathbb{N}$. However, we do not follow this abstract approach: in §5.4, we shall give a more explicit 'construction' of $\widetilde{K}$. Indeed we give three somewhat different constructions of $\widetilde{K}$. This will enable us, in §6.5, to give a topological characterization of $\widetilde{K}$ for each uncountable, compact, metrizable space $K$, and, eventually, in §6.6, to determine the cardinalities of various subsets of this space $\widetilde{K}$. An earlier, rather simple, proof of the fact that each space $C_0(K)$ is Arens regular will be given in §4.5.

In §3.3, we shall discuss some commutative $C^*$-algebras that are the Baire classes on certain topological spaces; this topic will be developed further in §6.7. In the final section of Chapter 3, we shall make some remarks on the extensions of our theory from the commutative $C^*$-algebras $C_0(K)$ to more general, non-commutative $C^*$-algebras; however, we shall say very little on the vast topic of the representation theory of $C^*$-algebras on Hilbert spaces.

In Chapter 4, we shall first recall some theory of measures on a locally compact space, with a brief mention of more general measure spaces. Let $K$ be a locally compact space. Then the Banach space $M(K)$ of complex-valued, regular Borel measures on $K$ is identified with the dual space of $C_0(K)$. In §4.4, we shall recall properties of the Banach spaces $L^p(K,\mu)$, where $K$ is a locally compact space, $\mu \in M(K)^+$, and $1 \leq p \leq \infty$, and in §4.5 we shall consider rather briefly when spaces of the form $C(K)$, for $K$ a compact space, are Grothendieck spaces; we shall also note that all injective Banach spaces are Grothendieck spaces. Maximal singular families of measures on $K$, defined in §4.6, are a key ingredient in one construction of $\widetilde{K}$. In §4.7, we shall define the space $N(K)$ of normal measures on a locally compact space $K$, and we shall give various examples of compact spaces $K$ for which $N(K)$ is and is not equal to $\{0\}$; in particular, we shall prove a new result of Plebanek that shows that there is connected, compact space $K$ such that $N(K) \neq \{0\}$.

The hyper-Stonean spaces of Chapter 5 are Stonean spaces with 'many' normal measures. Let $K$ be a locally compact space, and take $\mu \in M(K)^+$. Then the compact character space of the unital $C^*$-algebra $L^\infty(K,\mu)$ is denoted by $\Phi_\mu$; the Gel'fand transform $\mathscr{G}_\mu : L^\infty(K,\mu) \to C(\Phi_\mu)$ is a $C^*$-isomorphism and a Banach-lattice isometry, and $\Phi_\mu$ is a hyper-Stonean space identified with the Stone space of the Boolean algebra $\mathfrak{B}_\mu$ that is a natural quotient of the Boolean algebra of Borel subsets of $K$. We shall use the spaces $\Phi_\mu$ to 'build' the space $\widetilde{K}$. We shall then give a new 'construction' of $\widetilde{K}$ as $\beta S_K$, where $S_K$ is the Stone space of a Boolean ring $M(K)^+/\sim$; we shall also give two further representations of the space $\widetilde{K}$, one involving $L$-decompositions, whose theory is introduced in §5.5. The analogous theory for general $C^*$-algebras is sketched in §5.6.

Our main study of the Banach spaces $C_0(K)$ is given in Chapter 6. As a preliminary, we shall give in §6.1 some isomorphic invariants of $C(K)$-spaces; these will include the cardinality of $K$. Then we shall give in §6.2 some easy examples of locally compact spaces $K$ such that $C_0(K)$ is not (either isometrically or isomorphically) a dual space. In §6.3, many Banach spaces that are isomorphic preduals of the Banach space $\ell^1$, but are not isomorphic to each other, will be described. The question when a space $C(K)$, for compact $K$, is isometrically a dual space is then fully determined in §6.4. For example, this is the case if and only if the space $K$ is hyper-Stonean, and then the unique isometric predual $C(K)_*$ of $C(K)$ is identified with the space $N(K)$ of normal measures on $K$. This result combines classic theorems of Dixmier and of Grothendieck; aspects of the proof were first expounded by William Bade.

It is apparently much harder to characterize the locally compact spaces $K$ such that $C_0(K)$ is isomorphically a dual space, and we do not have a full answer to this question. Of course this holds whenever $K$ is hyper-Stonean. We shall give several examples of compact spaces, even of Stonean spaces, $K$ such that $C(K)$ is not isomorphically a dual space, and other examples, even of a totally disconnected space which is not Stonean, such that $C(K)$ is isomorphically a dual space. Each injective space of the form $C_0(K)$, and hence each space $C_0(K)$ that is isomorphically a dual space, is such that $K$ contains a dense, open, extremely disconnected subset and so has infinitely many components, but we do not know whether $K$ must be totally disconnected.

In §6.10, we shall discuss when a space $C(X)$, for an infinite compact space $X$, is isometrically a bidual space. Our aim in this section was to establish the conjecture that there is then a compact space $K$ such that $X$ is homeomorphic to $\widetilde{K}$; we can at least show that there is a compact space $K$ such that $X$ is homeomorphic to a clopen subspace of $\widetilde{K}$. In the case where $C(X)$ is isometrically the bidual of a separable Banach space, we can resolve the conjecture. Indeed, there are only two possibilities for $X$: either $X$ is homeomorphic to $\beta\mathbb{N}$, and $C(X)$ is isometrically isomorphic to $C(\beta\mathbb{N}) = c_0''$ or $X$ is homeomorphic to $\widetilde{\mathbb{I}}$, and $C(X)$ is isometrically isomorphic to $C(\mathbb{I})''$.

In §6.11, we shall summarize some results that we have obtained concerning the question when a space $C_0(K)$ is injective, when it is (isomorphically or isometrically) the dual of a Banach space, and when it is the bidual of a Banach space. There is a list of open questions in §6.12.

We have striven to eliminate errors in our text, but some are likely to remain. Readers are invited to send comments or errors to CK-Banachduals-book@cecat.chapman.edu. Corrections and some new results will be posted on the CECAT home page, http://mathcs.chapman.edu/CECAT.

We trust that this volume will stimulate new research in this attractive area, by graduate students and many researchers.

H.G. Dales, Lancaster, UK
F.K. Dashiell, Jr., Los Angeles, CA, USA
A.T.-M. Lau, Edmonton, AB, Canada
D. Strauss, Leeds, UK

http://www.springer.com/978-3-319-32347-3

Banach Spaces of Continuous Functions as Dual Spaces

Dales, H.G.; Dashiell, Jr., F.K.; Lau, A.T.-M.; Strauss, D.

2016, XIV, 277 p. 6 illus., Hardcover
ISBN: 978-3-319-32347-3