Chapter 2
Stability and Stabilization for Continuous-Time Difference Equations with Distributed Delay

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Abstract Motivated by linear hyperbolic conservation laws, we investigate in this chapter new conditions for stability and stabilization for linear continuous-time difference equations with distributed delay. For this, we propose first a state-space realization of networks of linear hyperbolic conservation laws via continuous-time difference equations. Then, based on some recent works, we propose sufficient conditions for exponential stability, which appear also to be necessary and sufficient in some particular cases. Then, the stabilization problem as well as the closed-loop performances are analyzed with constructive methods for state feedback synthesis.

2.1 Introduction

Linear 1-D conservation laws are linear one-dimensional first-order hyperbolic systems of partial differential equations (PDEs) [6]. Conservation laws appear in many engineering applications, as for instance in electrical lines, gas flow in pipelines, open water channel flow, road traffic or heat exchangers. See for instance [2, 3, 24, 30, 42]. When the conservation laws are linearized, they are reduced to transport or propagation equations [5, 8, 15, 21, 43].

From this well known fact, we propose to investigate the stability and performance properties for a set of linear coupled conservation laws with boundary control from

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their transport phenomena, using an exact state-space time-delay realization. The time-delay model is governed by linear continuous-time difference equations, in the form

$$x(t) = Ax(t-r) + Bu(t),\ t \geq 0, \tag{2.1}$$

where $u(t)$ is the input control, $x(t)$ the state, A a constant $n \times n$ matrix, B a constant $n \times m$ matrix and $r > 0$ the transport delay. Based on this model, we investigate exponential stability conditions, as well as boundary control for exponential stabilization and closed-loop performances achievement.

Feedback control for (2.1) may involve static delayed-state feedback, which yields a continuous-time difference equation in closed-loop. More generally, feedback control may involve integral operators in time (and in space), like distributed delay operators. The system (2.1) may be then transformed into the more general class of integral-difference equations in the form

$$x(t) = Ax(t-r) + \int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta,\ t \geq 0, \tag{2.2}$$

where the matrix function $G(\theta)$ has piecewise continuous bounded elements defined for $\theta \in [0, r]$ [5, 9, 31].

Stability and stabilization problems for conservation laws with boundary control have received renewed attention in recent years, related to challenging applications. See for instance [4, 10, 11, 16, 18, 25, 33, 41] and the references therein for Lyapunov-based approaches, [34] for a frequency approach, or [45] for a functional approach. On the other side, stability and stabilization of difference equations in the form (2.1) were studied in various works. In [1, 17, 27–29, 35], sensitivity in the delays for stability and stabilization were analyzed, via delay-independent spectral conditions, Lyapunov-Krasovskii approach was investigated in [7, 12, 20, 22, 40, 44, 46] with numerical tractable conditions for testing stability. Note also that the difference equation (2.1) appears fundamental in neutral time-delay systems. See for instance [23, 26, 32] and in references therein. Integral-difference equations in the form (2.2) have received recently a renewed interest. With constructive methods, stability conditions using Lyapunov-Krasovskii approach were proposed in [13, 37–39].

In this chapter, we propose to analyze linear 1-D conservation laws using continuous-time difference equations (with distributed delay). From a Lyapunov-Krasovskii approach, we give necessary and sufficient conditions for exponential stability in the case of purely difference equations, and sufficient conditions for exponential stability of integral-difference equations in the form (2.2). Then, state-feedback synthesis is studied via constructive numerical algorithms, in order to achieve closed-loop stability or L_2-gain performance.

The chapter is organized as follows: An exact time-delay state-space realization for linear 1-D hyperbolic conservation laws is derived in Sect. 2.2. Essential definitions and stability results are given in Sect. 2.3. In Sect. 2.4, state-feedback synthesis is analyzed. Some concluding remarks and a perspective discussion for large-scale network of conservation laws end the chapter.

Notations: The transpose of a matrix P is denoted by P^T while the smallest and the largest eigenvalues of a symmetric positive (semi)definite matrix P are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively. The standard notation $P \succ 0$ and $P \succeq 0$ are used for symmetric positive definite and positive semidefinite matrices, respectively. Similarly, $P \prec 0$ and $P \preceq 0$ stand for symmetric negative definite and negative semi-definite matrices, respectively.

The space of continuous and bounded functions defined on $[-r, 0)$ is $\mathscr{C}([-r, 0), \mathbb{R}^n)$, while the space of piecewise right-continuous and bounded functions defined on $[-r, 0)$ is denoted by $\mathscr{P}_{\mathscr{C}}([-r, 0), \mathbb{R}^n)$. These spaces are equipped with the uniform norm $\|\varphi\|_r = \sup_{-r\leq\theta\leq 0} \|\varphi(\theta)\|$ and with the L_2 norm $\|\varphi\|_{L_2}^2 = \int_{-r}^{0} \|\varphi(\theta)\|^2 \,\mathrm{d}\theta$, where $\|\varphi(\theta)\|$ stands for the Euclidean norm. For a matrix $G \in \mathbb{R}^{n\times n}$, $\|G\|$ denotes the spectral norm, that is $\|G\|^2 = \lambda_{\max}(G^T G)$. I_n stands for the $n \times n$ identity matrix, and $\rho(A)$ denotes the spectral radius for the matrix A. We denote by $x(t, \varphi)$ the solution of the system under consideration with the initial condition φ and by $x_t(\varphi) = \{x(t + \theta, \varphi) \mid \theta \in [-r, 0)\}$ the partial trajectory of the system. When the initial function is clear from the context, the argument φ is dropped.

2.2 Linear Hyperbolic Conservation Laws

In this section, starting from linear one-dimensional first-order hyperbolic partial differential equations with boundary conditions, we derive an exact state-space realization as a time-delay model.

In order to fix the ideas, let us consider first the simplest example of a scalar linear conservation law, governed by

$$\lambda\partial_t\psi(t, z) + \partial_z\psi(t, z) = 0, \;\; t > 0, z \in (0, \ell], \tag{2.3}$$

with $\lambda > 0$, an initial condition $\psi(0, z) = \psi_0(z)$ for $z \in (0, \ell]$ and a (time-dependent) boundary condition $\psi(t, 0) = u(t)$ for $t > 0$. For any $z \in (0, \ell]$, the unique solution of (2.3) is

$$\psi(t, z) = \begin{cases} \psi_0(z - \frac{t}{\lambda}) & , \text{ for } t \leq \lambda z \\ \psi(t - \lambda z, 0) = u(t - \lambda z) & , \text{ for } t > \lambda z \end{cases}. \tag{2.4}$$

Indeed, the Laplace transform applied on (2.3) leads to $\hat{\psi}(s, z) = \mathrm{e}^{-\lambda z s}\hat{\phi}(s)$, for some function $\phi(\cdot)$. In other words, $\psi(t, z) = \phi(t - \lambda z)$, for any $t \geq 0$. At time $t = 0$, we obtain $\psi_0(z) = \phi(-\lambda z)$, so that

$$\phi(t - \lambda z) = \psi_0(z - \frac{t}{\lambda}), \;\text{ for } t \leq \lambda z.$$

Furthermore, at $z = 0$, $u(t) = \psi(t, 0) = \phi(t)$, which implies that $\psi(t, z) = \psi(t - \lambda z, 0)$ for $t > \lambda z$. It is then a routine to verify that $\psi(t, z)$ is continuous at $t = \lambda z$, and is indeed the solution of (2.3).

The main interest in the formulation (2.4) is the delay-realization of the solution, which unifies the time-space coupled effects in (2.3) into a space-dependent time-delay. For the more general case, the 1-D linear (lossless) conservations laws are of the form [6]

$$\Lambda \partial_t \psi(t, z) + \partial_z \psi(t, z) = 0, \;\; t > 0, z \in (0, \ell), \tag{2.5}$$

where $\psi : \mathbb{R}^+ \times [0, \ell] \to \mathbb{R}^n$, $\psi(t, z) = \left[\begin{smallmatrix} \psi^+(t,z) \\ \psi^-(t,z) \end{smallmatrix}\right]$ with $\psi^+(\cdot, \cdot) \in \mathbb{R}^{n^+}$, $\psi^-(\cdot, \cdot) \in \mathbb{R}^{n^-}$, $n^+ + n^- = n$ and $\Lambda = \left[\begin{smallmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{smallmatrix}\right]$. The matrix Λ is assumed, without loss of generality, to be diagonal, that is $\Lambda^+ = \mathrm{diag}(\lambda_1^+, \ldots, \lambda_{n^+}^+)$, $\Lambda^- = \mathrm{diag}(\lambda_1^-, \ldots, \lambda_{n^-}^-)$, where $\lambda_i^+ > 0$ for $i = 1, \ldots, n^+$ and $\lambda_i^- < 0$ for $i = 1, \ldots, n^-$. The initial condition for (2.5) is $\psi(0, z) = \psi_0(z)$ for $z \in (0, \ell)$, and the boundary conditions are of the form [16]

$$\begin{bmatrix} \psi^+(t, 0) \\ \psi^-(t, \ell) \end{bmatrix} = \begin{bmatrix} \Gamma_{00} & \Gamma_{01} \\ \text{Ł}\Gamma_{10} & \Gamma_{11} \end{bmatrix} \begin{bmatrix} \psi^+(t, \ell) \\ \psi^-(t, 0) \end{bmatrix} + \begin{bmatrix} B_+ \\ B_- \end{bmatrix} u(t), \tag{2.6}$$

for constant matrices Γ_{ij} with appropriate size, $B_+ \in \mathbb{R}^{n^+ \times m}$, $B_- \in \mathbb{R}^{n^- \times m}$ and $u(\cdot) \in \mathbb{R}^m$ some exogenous input signal. From (2.4), it is trivial to see that

$$\psi_i^+(t, z) = \psi_i^+(t - \lambda_i^+ z, 0), \;\; \text{for } t > \lambda_i^+ z, \tag{2.7}$$
$$\psi_i^-(t, z) = \psi_i^-(t + \lambda_i^-(\ell - z), \ell), \;\; \text{for } t > \lambda_i^-(z - \ell), \tag{2.8}$$

where ψ_i^+ and ψ_i^- stand for the entries of ψ^+ and ψ^-, respectively. Introduce the time-delay operators $\sigma_i^+ : f(t) \mapsto f(t - \lambda_i^+ \ell)$ for $i = 1, \ldots, n^+$, $\sigma_i^- : f(t) \mapsto f(t + \lambda_i^- \ell)$ for $i = 1, \ldots, n^-$. With $\sigma^+ = \mathrm{diag}(\sigma_1^+, \ldots, \sigma_{n^+}^+)$ and $\sigma^- = \mathrm{diag}(\sigma_1^-, \ldots, \sigma_{n^-}^-)$, the system (2.5) and (2.6) is transformed into the delay-operator realization

$$\begin{bmatrix} \psi^+(t, 0) \\ \psi^-(t, \ell) \end{bmatrix} = \begin{bmatrix} \Gamma_{00} & \Gamma_{01} \\ \text{Ł}\Gamma_{10} & \Gamma_{11} \end{bmatrix} \begin{bmatrix} \sigma^+ \psi^+(t, 0) \\ \sigma^- \psi^-(t, \ell) \end{bmatrix} + \begin{bmatrix} B_+ \\ B_- \end{bmatrix} u(t). \tag{2.9}$$

For the instantaneous state $x(t) = \left[\begin{smallmatrix} \psi^+(t,0) \\ \psi^-(t,\ell) \end{smallmatrix}\right]$, and if the delays $\lambda_i^+ \ell$ and $-\lambda_i^- \ell$ are rationally dependent, the realization (2.9) is a state-space realization with difference equations in the form (2.1), where the matrix A depends on the length of the delays and on the matrices Γ_{ij}. Remark that (2.9) holds for any $t \geq \max_i\{\lambda_i^+ \ell, -\lambda_i^- \ell\}$. For times less than this lower bound, the solution is determined as in (2.4) from its initial condition. We can then reduce this realization for any $t \geq 0$ with a time translation. From Eqs. (2.7) and (2.8), note also that the stability for $x(t)$ is equivalent to the stability of $\psi(t, z)$ for any $t \geq 0$ and $z \in [0, \ell]$.

2.3 Exponential Stability Analysis

2.3.1 Definitions and Main Properties

Let us consider the linear system governed by difference equations with pointwise commensurate delays and distributed delays

$$x(t) = Ax(t-r) + \int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta \tag{2.10}$$

where $A \in \mathbb{R}^{n\times n}$, $r > 0$, and the matrix function $G(\theta) \in \mathbb{R}^{n\times n}$ has piecewise continuous bounded elements defined for $\theta \in [0, r]$. For any piecewise right-continuous and bounded initial condition $\varphi \in \mathscr{P}_{\mathscr{C}}([-r, 0), \mathbb{R}^n)$, there exists a unique solution $x(t, \varphi)$ of (2.10), for all $t \geq 0$. Such a solution is called the system response of (2.10). This solution is piecewise continuous, in general, and is determined by an iterative scheme in time, with

$$x(0, \varphi) = A\varphi(-r) + \int_0^r G(\theta)\varphi(-\theta)\mathrm{d}\theta.$$

Such equality defines the step discontinuity of the system response at time $t = 0$, which will be propagated in time. In the case of linear 1-D conservation laws, the identity $x(0, \varphi) = \varphi(0^-)$ holds, leading to the so-called initial matching condition. In such a case, if the initial condition $\varphi \in \mathscr{C}([-r, 0), \mathbb{R}^n)$, the system response $x(t, \varphi)$ is also continuous.

Let us remind definitions and some results concerning estimates of this solution. Obviously, these definitions apply also in the case of difference equations (2.1) where $G(\cdot) = 0$ in (2.10).

Definition 1 System (2.10) is said to be exponentially stable if there exists $\gamma \geq 0$ and $\mu > 0$ such that any solution $x(t, \varphi)$ of the system (2.10) satisfies

$$\|x(t, \varphi)\| \leq \gamma \,\|\varphi\|_r \,\mathrm{e}^{-\mu t}, t \geq 0.$$

In such a case, the system response $x(t, \varphi)$ is said to be exponentially stable, with decay rate greater than μ. In the present contribution, we also use the concept of L_2-exponential stability, which is clearly weaker than exponential stability, defined as follows.

Definition 2 System (2.10) is said to be L_2-exponentially stable if there exists $\gamma \geq 0$ and $\mu > 0$ such that any solution $x(t, \varphi)$ of the system (2.10) satisfies

$$\|x_t(\varphi)\|_{L_2} \leq \gamma \,\|\varphi\|_r \,\mathrm{e}^{-\mu t}, t \geq 0.$$

Considering the distributed delay in (2.10) as a perturbation term, a general result for exponential stability was obtained in [37]. Exponential estimate for the system response was retrieved from the analysis of some difference equations with only pointwise delays. In this contribution, while the arguments are indeed similar, another characterization for exponential stability is obtained. For the construction of exponential estimates, we will use the following central result on L_2-exponential stability.

Theorem 1 [12] *Let $x_t(\varphi)$ be a partial trajectory for (2.10), and assume that there exists a continuous functional $v : \mathscr{P}_{\mathscr{C}}([-r, 0), \mathbb{R}^n) \to \mathbb{R}$ such that $v(x_t(\varphi))$ is upper right-hand side differentiable with respect to t along the trajectories of (2.10) and satisfies the following conditions:*

(i) $\alpha_1 \|\varphi\|_{L_2}^2 \leq v(\varphi) \leq \alpha_2 \|\varphi\|_r^2$, *for some constants* $0 < \alpha_1$, $0 \leq \alpha_2$,
(ii) $\frac{\mathrm{d}}{\mathrm{d}t} v(x_t(\varphi)) + 2\mu\, v(x_t(\varphi)) \leq 0$ *for some* $\mu > 0$.

Then system (2.10) is L_2-exponentially stable, and the following exponential estimate

$$\|x_t(\varphi)\|_{L_2} \leq \sqrt{\frac{\alpha_2}{\alpha_1}}\, \|\varphi\|_r\, \mathrm{e}^{-\mu t}$$

holds for all $t \geq 0$.

2.3.2 Stability for Difference Equations

For continuous-time linear difference equations of the form (2.1) with $u(\cdot) = 0$, the exponential stability analysis is trivial. Indeed, its system response is

$$x(t, \varphi) = A^k \varphi(t - kr),\ \forall t \in [(k-1)r, kr),\ k \in \mathbb{N}. \tag{2.11}$$

It follows that the system (2.1) is exponentially stable if and only if $\rho(A) < 1$ (see for instance [19]), or equivalently if and only if there exists a symmetric $n \times n$ positive definite matrix P such that

$$A^T P A - P \prec 0. \tag{2.12}$$

These stability conditions are equivalent to those related to discrete-time linear systems, and show the fruitful realization as a time-delay system for linear conservation laws.

2.3.3 Stability for Difference Equations with Distributed Delay

For integral-difference equations in the form (2.10), our strategy for the exponential stability analysis is based on the following result, leading to a delay-dependent stability condition [14]. Such a condition does not require the assumption that the matrix A is stable.

Theorem 2 *The system (2.10) is exponentially stable if there exist an $n \times n$ symmetric real positive definite matrix $\tilde{P}$, an $n \times n$ symmetric real positive semi-definite matrix $S(\theta)$, for all $\theta \in [0, r]$, and a positive constant $\mu > 0$ such that*

$$P = \tilde{P} - \int_0^r S(\theta)\,\mathrm{d}\theta \succ 0, \tag{2.13}$$

and $M_\mu(\theta) \succeq 0$, for all $\theta \in [0, r]$, where $M_\mu(\theta)$ is given by

$$-M_\mu(\theta) = r\begin{bmatrix} A^T\tilde{P}A - \mathrm{e}^{-2\mu r}P & A^T\tilde{P}G(\theta) \\ G^T(\theta)\tilde{P}A & G^T(\theta)\tilde{P}G(\theta) - \frac{\mathrm{e}^{-2\mu\theta}S(\theta)}{r} \end{bmatrix}. \tag{2.14}$$

Moreover, for any $\varepsilon \in (0, \mu)$, the following exponential estimate of the system response holds

$$\|x(t, \varphi)\| \le \gamma\, \|\varphi\|_r\, \mathrm{e}^{-(\mu-\varepsilon)t}, \tag{2.15}$$

for all $t \ge 0$, where

$$\gamma = \sqrt{\frac{\lambda_{\max}(P) + \alpha + \dfrac{\alpha}{2r\varepsilon\mathrm{e}}}{\lambda_{\min}(P)}}. \tag{2.16}$$

The positive constant α is given by

$$\alpha = \frac{\alpha_2}{\alpha_1} \cdot \sup_{0\le\theta\le r} \lambda_{\max}(S(\theta)),$$

where $\alpha_1 = \mathrm{e}^{-2\mu r}\lambda_{\min}(P)$ and $\alpha_2 = r\lambda_{\max}(P) + \frac{r^2}{2} \cdot \sup_{0\le\theta\le r} \lambda_{\max}(S(\theta))$.

Proof Assume that the conditions of the theorem are fulfilled, and consider the Lyapunov-Krasovskii functional

$$\begin{aligned} v_\mu(x_t(\varphi)) = & \int_{t-r}^{t} \mathrm{e}^{-2\mu(t-\theta)}x^T(\theta)Px(\theta)\,\mathrm{d}\theta \\ & + \int_0^r \int_{t-\theta}^{t} \mathrm{e}^{-2\mu(t-\xi)}x^T(\xi)S(\theta)x(\xi)\,\mathrm{d}\xi\,\mathrm{d}\theta, \end{aligned} \tag{2.17}$$

where P, $S(\theta)$ and μ are described in (2.13) and (2.14). This functional $v_\mu(\cdot)$ is continuous and admits the following lower and upper bounds

$$\alpha_1 \, \|\varphi\|_{L_2}^2 \le v_\mu(\varphi) \le \alpha_2 \, \|\varphi\|_r^2 \,, \tag{2.18}$$

where α_1 and α_2 are given in Theorem 2. The fact that P is symmetric positive definite implies that $\alpha_1 > 0$ and $\alpha_2 > 0$. Furthermore, $v_\mu(x_t(\varphi))$ is upper right-hand differentiable, and its upper right-hand side time derivative along the trajectories of (2.10) satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} v_\mu(x_t(\varphi)) = & - 2\mu \, v_\mu(x_t(\varphi)) + x^T(t)\tilde{P}x(t) - \mathrm{e}^{-2\mu r} x^T(t-r)Px(t-r) \\ & - \int_0^r \mathrm{e}^{-2\mu\theta} x^T(t-\theta)S(\theta)x(t-\theta)\,\mathrm{d}\theta. \end{aligned} \tag{2.19}$$

Substituting (2.10) into (2.19) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} v_\mu(x_t(\varphi)) + 2\mu \, v_\mu(x_t(\varphi)) = & \, 2x^T(t-r)A^T\tilde{P}\int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta \\ & + x^T(t-r)[A^T\tilde{P}A - \mathrm{e}^{-2\mu r}P]x(t-r) \\ & - \int_0^r \mathrm{e}^{-2\mu\theta} x^T(t-\theta)S(\theta)x(t-\theta)\mathrm{d}\theta \\ & + \left(\int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta\right)^T \tilde{P}\int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta. \end{aligned} \tag{2.20}$$

Applying Jensen's integral inequality to the last term in (2.20),

$$\begin{aligned} & \left(\int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta\right)^T \tilde{P}\int_0^r G(\theta)x(t-\theta)\mathrm{d}\theta \\ & \qquad \le r\int_0^r x^T(t-\theta)G^T(\theta)\tilde{P}G(\theta)x(t-\theta)\mathrm{d}\theta, \end{aligned}$$

and substituting it into (2.20), we finally obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t} v_\mu(x_t(\varphi)) + 2\mu \, v_\mu(x_t(\varphi)) \le - \int_0^r \chi^T(\theta)M_\mu(\theta)\chi(\theta)\,\mathrm{d}\theta,$$

where $\chi^T(\theta) = \big[(1/r)x^T(t-r) \;\; x^T(t-\theta)\big]$ and $M_\mu(\theta)$ is given in (2.14). Since $M_\mu(\theta) \succeq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} v_\mu(x_t(\varphi)) + 2\mu \, v_\mu(x_t(\varphi)) \le 0, \;\; \forall t \ge 0. \tag{2.21}$$

From Theorem 1 and (2.18), we conclude that $x_t(\varphi)$ is L_2-exponentially stable, with

$$\|x_t(\varphi)\|_{L_2} \leq \sqrt{\frac{\alpha_2}{\alpha_1}}\, \|\varphi\|_r \, \mathrm{e}^{-\mu t}. \tag{2.22}$$

The inequality (2.22) implies that

$$\int_0^r \mathrm{e}^{-2\mu\theta} x^T(t-\theta) S(\theta) x(t-\theta) \mathrm{d}\theta \leq \alpha \, \|\varphi\|_r^2 \, \mathrm{e}^{-2\mu t}, \tag{2.23}$$

where α is given in Theorem 2. From (2.19) and (2.21), we see that the system response $x(t)$ satisfies

$$\begin{aligned} x^T(t) \tilde{P} x(t) \leq & \, \mathrm{e}^{-2\mu r} x^T(t-r) P x(t-r) \\ & + \int_0^r \mathrm{e}^{-2\mu\theta} x^T(t-\theta) S(\theta) x(t-\theta) \, \mathrm{d}\theta, \end{aligned} \tag{2.24}$$

or in other words, from (2.13) and (2.23),

$$x^T(t) P x(t) \leq \mathrm{e}^{-2\mu r} x^T(t-r) P x(t-r) + \alpha \, \|\varphi\|_r^2 \, \mathrm{e}^{-2\mu t}.$$

This inequality is equivalent to

$$\psi(t) = \mathrm{e}^{-2\mu r} \psi(t-r) + f(t),$$

for $\psi(t) = x^T(t) P x(t)$ and some piecewise continuous function $f(t)$, with $|f(t)| \leq \alpha \, \|\varphi\|_r^2 \, \mathrm{e}^{-2\mu t}$, for all $t \geq 0$. Let $n \in \mathbb{N}$, and take $t = nr + \xi$ where $\xi \in [-r, 0)$. From the standard arguments in [32] (Lemma 6, p. 797), we obtain

$$\begin{aligned} \psi(t) &= \mathrm{e}^{-2\mu n r} \psi(t-nr) + \sum_{k=0}^{n-1} \mathrm{e}^{-2\mu k r} f(t-kr) \\ &\leq \mathrm{e}^{-2\mu n r} \psi(t-nr) + n\alpha \, \|\varphi\|_r^2 \, \mathrm{e}^{-2\mu t}. \end{aligned}$$

Substituting $\psi(t)$ and considering that $t \in [(n-1)r, nr)$, this last inequality leads to

$$\lambda_{\min}(P) \, \|x(t,\varphi)\|^2 \leq \left[\lambda_{\max}(P) + \alpha + \frac{\alpha}{r} t\right] \|\varphi\|_r^2 \, \mathrm{e}^{-2\mu t}.$$

For any $\varepsilon \in (0, \mu)$ (which is possible since $\mu > 0$), we observe that

$$t \, \mathrm{e}^{-2\mu t} = t \, \mathrm{e}^{-2\varepsilon t} \mathrm{e}^{-2(\mu-\varepsilon)t} \leq \frac{1}{2\varepsilon \mathrm{e}} \mathrm{e}^{-2(\mu-\varepsilon)t}.$$

As $\mathrm{e}^{-2\varepsilon t} \leq 1$, we finally get

$$\|x(t,\varphi)\|^2 \leq \frac{\lambda_{\max}(P) + \alpha + \frac{\alpha}{2r\varepsilon\mathrm{e}}}{\lambda_{\min}(P)} \|\varphi\|_r^2 \, \mathrm{e}^{-2(\mu-\varepsilon)t},$$

which gives the desired exponential estimate for the system response of (2.10), for all $t \geq 0$, since such a bound is independent from $n \in \mathbb{N}$.

Remark 1 The conditions described in Theorem 2 ensure that A is a Schur stable matrix. Indeed, it follows from (2.14) that $A^T \tilde{P} A - \mathrm{e}^{-2\mu r} P \preceq 0$. From (2.13), this in turn implies that

$$\begin{aligned} A^T P A - P + (1 - \mathrm{e}^{-2\mu r}) P &= A^T P A - \mathrm{e}^{-2\mu r} P \\ &\preceq A^T \tilde{P} A - \mathrm{e}^{-2\mu r} P \\ &\preceq 0. \end{aligned}$$

Since P is symmetric positive definite and $\mu > 0$, we conclude that $A^T P A - P \prec 0$, which is equivalent to the assertion that A is Schur stable. Clearly, the Schur stability assumption in the Lyapunov-type Theorem 3 of [37] is fulfilled, as well as the functional conditions 1 and 2 therein. Hence Theorem 3 in [37] allows to conclude that system (2.10) is exponentially stable.

Remark 2 Note that the matrix $S(\theta)$, for $\theta \in [0, r]$, is required to be only symmetric positive semi-definite. This fact should be compared to time-delay systems with differential equations [23], where $S(\theta)$ is required, in general, to be positive definite. This remark allows us to apply Theorem 2 in the particular case of difference equations with only pointwise delays, as in [7] or [22]. Indeed, for such systems, $G(\theta) = 0$ for all $\theta \in [0, r]$. Taking $S(\theta) = 0$ on the whole interval, the conditions given in Theorem 2 lead to the existence of $P \succ 0$ and $\mu > 0$ such that $A^T P A - \mathrm{e}^{-2\mu r} P \preceq 0$.

Remark 3 A virtue of Theorem 2 is that, unlike in [37], there is no additional assumption on the stability of the difference operator $\mathscr{A} : \mathscr{P}_{\mathscr{C}}([-r, 0), \mathbb{R}^n) \mapsto \mathbb{R}^n$ defined by

$$\mathscr{A}(x_t) = x(t) - Ax(t - r).$$

It is worthy of mention that a similar feature appears in [20], in the case of neutral type systems. Furthermore, exponential estimates for the system response described in [37] require some knowledge about the decreasing properties of $\|A^k\|$, when $k \to \infty$, $k \in \mathbb{N}$. This fact is not used a priori in the estimate of Theorem 2.

The conditions presented in Theorem 2 are not easy to verify as they depend on the continuous parameter θ in the bounded interval $[0, r]$. Similar conditions to those in Proposition 4 of [37] can be obtained from Theorem 2. But in order to reduce the conservatism of these sufficient conditions, we propose the following tractable conditions.

Lemma 1 *The system (2.10) is exponentially stable if there exist an $n \times n$ symmetric real positive definite matrix $\tilde{P}$, a symmetric positive semi-definite matrix $\tilde{S}$, and some constants $\mu > 0$ and $\beta > 0$ such that*

$$\beta \int_0^r \|G(\theta)\|^2 \, \mathrm{d}\theta \cdot I_n \prec \tilde{P}, \tag{2.25}$$

$$\tilde{S} \preceq \beta \cdot I_n, \tag{2.26}$$

$$0 \preceq \tilde{M}_\mu, \tag{2.27}$$

where $\tilde{M}_\mu$ is given by

$$-\tilde{M}_\mu = r \begin{bmatrix} A^T \tilde{P} A - \mathrm{e}^{-2\mu r} (\tilde{P} - \beta \int_0^r \|G(\theta)\|^2 \, \mathrm{d}\theta \cdot I_n) & A^T \tilde{P} \\ \tilde{P} A & \tilde{P} - \frac{\mathrm{e}^{-2\mu r}}{r} \tilde{S} \end{bmatrix}. \tag{2.28}$$

Moreover, for any $\varepsilon \in (0, \mu)$, the following exponential estimate of the system response holds

$$\|x(t, \varphi)\| \leq \gamma \, \|\varphi\|_r \, \mathrm{e}^{-(\mu - \varepsilon)t}, \tag{2.29}$$

for all $t \geq 0$, where

$$\gamma = \sqrt{\frac{\lambda_{\max}(P) + \alpha + \dfrac{\alpha}{2r\varepsilon \mathrm{e}}}{\lambda_{\min}(P)}} \tag{2.30}$$

and $P = \tilde{P} - \int_0^r G^T(\theta) \tilde{S} G(\theta) \mathrm{d}\theta$. The positive constant α is obtained by

$$\alpha = \frac{\alpha_2 \beta}{\alpha_1} \cdot \sup_{0 \leq \theta \leq r} \lambda_{\max}(G^T(\theta) G(\theta)),$$

where $\alpha_1 = \mathrm{e}^{-2\mu r} \lambda_{\min}(P)$ and $\alpha_2 = r \lambda_{\max}(P) + \frac{r^2}{2} \beta \cdot \sup_{0 \leq \theta \leq r} \lambda_{\max}(G^T(\theta) G(\theta))$.

Proof The proof is divided into two steps. In the first step, we show that (2.25)–(2.27) imply $M_\mu(\theta) \succeq 0$, for all $\theta \in [0, r]$, in Theorem 2. The second step will be devoted to the exponential estimate (2.29).

Assume that conditions (2.25)–(2.27) are satisfied. For any $\theta \in [0, r]$, we have $\mathrm{e}^{-2\mu r} \tilde{S} \preceq \mathrm{e}^{-2\mu\theta} \tilde{S}$. Furthermore, from (2.26),

$$-\beta \int_0^r \|G(\theta)\|^2 \, \mathrm{d}\theta \cdot I_n \preceq -\int_0^r G^T(\theta) \tilde{S} G(\theta) \mathrm{d}\theta.$$

Using these two inequalities as upper bounds in the block-diagonal terms of $\tilde{M}_\mu$ in (2.28), we see that the matrix $N_\mu(\theta)$ defined by

$$-N_\mu(\theta) = r\begin{bmatrix} A^T\tilde{P}A - \mathrm{e}^{-2\mu r}(\tilde{P} - \int_0^r G^T(\theta)\tilde{S}G(\theta)\mathrm{d}\theta) & A^T\tilde{P} \\ \tilde{P}A & \tilde{P} - \frac{\mathrm{e}^{-2\mu\theta}}{r}\tilde{S} \end{bmatrix}, \quad (2.31)$$

is a symmetric real positive semi-definite matrix, for all θ in $[0, r]$. For any θ in $[0, r]$, we define $S(\theta) = G^T(\theta)\tilde{S}G(\theta)$ and

$$P = \tilde{P} - \int_0^r S(\theta)\,\mathrm{d}\theta, \quad (2.32)$$

where $\tilde{P} \succ 0$ is the solution of (2.25)–(2.27). The matrix $S(\theta)$ is symmetric positive semi-definite, while P in (2.32) is positive definite since (2.25) and (2.26) are fulfilled. We see that, with the notations in Theorem 2,

$$-M_\mu(\theta) = \begin{bmatrix} I_n & 0 \\ 0 & G^T(\theta) \end{bmatrix}(-N_\mu(\theta))\begin{bmatrix} I_n & 0 \\ 0 & G(\theta) \end{bmatrix}.$$

Then, we conclude that $M_\mu(\theta) \succeq 0$, for all $\theta \in [0, r]$. The exponential stability of (2.10) and the estimate (2.29) for the system response follow straightforwardly from Theorem 2, since, by construction,

$$\begin{aligned}\lambda_{\max}(S(\theta)) &= \lambda_{\max}(G^T(\theta)\tilde{S}G(\theta)) \\ &\le \beta\,\lambda_{\max}(G^T(\theta)G(\theta)) \\ &= \beta\,\|G(\theta)\|^2 .\end{aligned}$$

2.3.4 Conservatism Analysis

The conditions for exponential stability of (2.10) presented in Lemma 1 are less conservative than those given in [37]. In order to prove this assertion, we consider the two possible cases.

First, let us assume that

$$\sup_{0\le\theta\le r} \|G(\theta)\| > 0, \quad (2.33)$$

and the conditions of Proposition 1 in [37] are fulfilled. Under such assumptions, we construct, in what follows, the solutions of (2.25)–(2.27).

Since

$$\frac{1}{r}\|G_r\|_{L_2}^2 = \frac{1}{r}\int_0^r \|G(\theta)\|^2\,\mathrm{d}\theta \le \sup_{0\le\theta\le r}\|G(\theta)\|^2$$

holds, Proposition 1 in [37] tells us that there exist symmetric positive definite matrices W_0, W_1 and P_1, such that

$$\|G_r\|_{L_2}^2 (P_1 + P_1 A W_0^{-1} A^T P_1) \prec \lambda_{\min}(W_1) \cdot I_n, \tag{2.34}$$

and

$$A^T P_1 A - P_1 = -(W_0 + r W_1). \tag{2.35}$$

By definition, $G(\cdot)$ has piecewise continuous and bounded elements, so that (2.33) implies that $0 < \|G_r\|_{L_2}^2$. By Schur complement, we see from (2.34) that

$$\begin{bmatrix} W_0 & -A^T P_1 \\ -P_1 A & \frac{\lambda_{\min}(W_1)}{\|G_r\|_{L_2}^2} I_n - P_1 \end{bmatrix} \succ 0. \tag{2.36}$$

Moreover, from (2.35),

$$\begin{aligned} W_0 &= -A^T P_1 A + P_1 - r W_1 \preceq -A^T P_1 A + P_1 - r\lambda_{\min}(W_1) \cdot I_n \\ &= -A^T P_1 A + P_1 - r \frac{\lambda_{\min}(W_1)}{\|G_r\|_{L_2}^2} \|G_r\|_{L_2}^2 \cdot I_n. \end{aligned}$$

From this, define

$$\tilde{S} = r \frac{\lambda_{\min}(W_1)}{\|G_r\|_{L_2}^2} \cdot I_n \stackrel{\triangle}{=} \beta \cdot I_n. \tag{2.37}$$

The matrix $\tilde{S}$ is symmetric positive definite, and satisfies (2.26). Defining $\tilde{P} = P_1$ the symmetric positive definite matrix solution of (2.35), we see that

$$r W_1 = -W_0 + \tilde{P} - A^T \tilde{P} A \prec \tilde{P}.$$

This implies $r\lambda_{\min}(W_1) \cdot I_n \prec \tilde{P}$, or equivalently

$$\beta \int_0^r \|G(\theta)\|^2 \,\mathrm{d}\theta \cdot I_n \prec \tilde{P}.$$

Then (2.25) holds. We next show that (2.27) holds for some $\mu > 0$. From (2.36) and the upper bound on W_0, the matrix

$$\tilde{M}_0 = r \begin{bmatrix} -A^T \tilde{P} A + \tilde{P} - \beta \|G_r\|_{L_2}^2 \cdot I_n & -A^T \tilde{P} \\ -\tilde{P} A & \frac{1}{r}\tilde{S} - \tilde{P} \end{bmatrix}$$

is symmetric positive definite, and $\lambda_{\max}(\tilde{P}) \leq \frac{1}{r}\lambda_{\max}(\tilde{S})$. Remarking that $\tilde{M}_0$ satisfies

$$\begin{aligned}\frac{1}{r}\tilde{M}_0 &\preceq \begin{bmatrix}\tilde{P} & 0\\ 0 & \frac{1}{r}\tilde{S}\end{bmatrix} - \begin{bmatrix}A^T & 0\\ I_n & 0\end{bmatrix}\begin{bmatrix}\tilde{P} & 0\\ 0 & \tilde{P}\end{bmatrix}\begin{bmatrix}A & I_n\\ 0 & 0\end{bmatrix}\\ &\preceq \begin{bmatrix}\tilde{P} & 0\\ 0 & \frac{1}{r}\tilde{S}\end{bmatrix} \preceq \frac{1}{r}\lambda_{\max}(\tilde{S})\cdot I_{2n},\end{aligned}$$

we conclude that $\lambda_{\min}(\tilde{M}_0) \leq \lambda_{\max}(\tilde{S})$. In other words, $1 - \frac{\lambda_{\min}(\tilde{M}_0)}{\lambda_{\max}(\tilde{S})} \in [0, 1)$. Define $\mu > 0$ such that

$$0 < \mu \leq -\frac{1}{2r}\ln\left(1 - \frac{\lambda_{\min}(\tilde{M}_0)}{\lambda_{\max}(\tilde{S})}\right).$$

It is then a routine to verify that, for such μ,

$$-\lambda_{\min}(\tilde{M}_0)\cdot I_{2n} \preceq r\begin{bmatrix}(\mathrm{e}^{-2\mu r}-1)\tilde{P} & 0\\ 0 & \frac{(\mathrm{e}^{-2\mu r}-1)}{r}\tilde{S}\end{bmatrix},$$

holds, that is

$$\begin{aligned}-\tilde{M}_\mu &= -\tilde{M}_0 - r\begin{bmatrix}(\mathrm{e}^{-2\mu r}-1)\tilde{P} & 0\\ 0 & \frac{(\mathrm{e}^{-2\mu r}-1)}{r}\tilde{S}\end{bmatrix}\\ &\quad - r\begin{bmatrix}(1-\mathrm{e}^{-2\mu r})\beta\,\|G_r\|_{L_2}^2\cdot I_n & 0\\ 0 & 0\end{bmatrix}\\ &\preceq -\tilde{M}_0 - r\begin{bmatrix}(\mathrm{e}^{-2\mu r}-1)\tilde{P} & 0\\ 0 & \frac{(\mathrm{e}^{-2\mu r}-1)}{r}\tilde{S}\end{bmatrix} \preceq 0.\end{aligned}$$

We have then proved that the conditions of Lemma 1 are implied by Proposition 1 in [37]. The converse is false, as shown in the following counter-example.

Example 1 Let us consider the difference equation

$$x(t) = \mathrm{e}^{-2}x(t-1) - \int_0^1 \mathrm{e}^{-2\theta}x(t-\theta)\mathrm{d}\theta. \tag{2.38}$$

The solutions of the characteristic equation of this system,

$$\lambda = -3,\ \lambda_k = -2 + j2k\pi,\ k \in \mathbb{Z},$$

lie in the open left-half complex plane, so this system is exponentially stable. Proposition 1 requires to find $p > 0$, $w_0 > 0$ and $w_1 > 0$ such that

$$p(1-\mathrm{e}^{-4}) = w_0 + w_1 \ \text{ and } \ pw_0 + p^2\mathrm{e}^{-4} < w_1 w_0.$$

These conditions lead to $p^2 + pw_0 + w_0^2 e^4 < 0$, which is impossible. However, the conditions (2.25)–(2.27) admit a solution

$$\tilde{P} = 10.1540,\ \ \tilde{S} = 32.4619,\ \ \beta = 32.5292,\ \ \mu = 0.46.$$

Hence, from Lemma 1, we conclude that the system response of (2.38) is exponentially stable, that is, for $t \geq 0$,

$$\|x(t,\varphi)\| \leq \sqrt{\frac{609.1465 + \frac{111.6439}{\varepsilon}}{2.1871}}\, \|\varphi\|_r\, \mathrm{e}^{-(0.46-\varepsilon)t},$$

for any $\varepsilon \in (0, 0.46)$.

If we assume that

$$\sup_{0\leq\theta\leq r} \|G(\theta)\| = 0,$$

Proposition 1 leads to the existence of symmetric positive definite matrices W_0, W_1 and P_1 such that

$$A^T P_1 A - P_1 = -(W_0 + rW_1).$$

Take any symmetric positive definite matrix Q, and define the symmetric positive definite matrix

$$\tilde{S} = r(P_1 + P_1 A(W_0 + rW_1)^{-1} A^T P_1) + Q,$$

and $\tilde{P} = P_1$. It is readily seen that (2.25) and (2.26) are fulfilled, with $\beta = \lambda_{\max}(\tilde{S}) > 0$. The rest of the proof follows step by step the previous case.

Example 2 Consider the linear hyperbolic system of balance laws described by

$$\xi_t(z,t) + \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}^{-1} \xi_z(z,t) = 0,\ z \in (0,1],\ t > 0, \tag{2.39}$$

with $r > 0$. The initial condition is $\xi(z, 0) = \xi_0(z)$, for any $z \in [0, 1]$, where $\xi_0(\cdot)$ is some continuous function. The boundary condition is $\xi(0, t) = u(t)$ with $u(t)$ the boundary input control given by

$$u(t) = K_1 \int_0^1 K(\nu)\xi(\nu,t)\mathrm{d}\nu - K_2\xi(1,t),$$

where K_1 and K_2 are two 2×2 constant matrices, and $K(\nu)$ is a 2×2 matrix with piecewise continuous elements in $[0, 1]$. The solution of (2.39) satisfies $\xi(z,t) = \xi(0, t - rz)$, for $t \geq rz$. In closed-loop, the state variable $x(t) = \xi(0, t)$ is then governed by

$$x(t) = -K_2 x(t-r) + K_1 \int_0^r K(r^{-1}\theta)x(t-\theta)\mathrm{d}\theta,$$

for $t \geq r$, with some continuous initial condition on $[0, r)$. With

$$K_2 = -\begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0 \end{bmatrix}, \ K(\theta) = \begin{bmatrix} 0.07 & 0 \\ 0 & \mathrm{e}^{-\theta} \end{bmatrix}, \ K_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{bmatrix},$$

and $r = \sqrt{2}$, a solution of Lemma 1 is

$$\tilde{P} = \begin{bmatrix} 58.3791 & -10.9932 \\ -10.9932 & 68.3308 \end{bmatrix}, \ \tilde{S} = \begin{bmatrix} 333.8131 & 20.7117 \\ 20.7117 & 266.6636 \end{bmatrix}$$

$$\beta = 339.7947, \ \mu = 0.2109.$$

The exponential estimate of the system solution is described by

$$\|x(t, \varphi)\| \leq \sqrt{\frac{200.4865 + \frac{17.5405}{\varepsilon}}{42.4658}} \, \|\varphi\|_r \ \mathrm{e}^{-(0.2109-\varepsilon)t}$$

for any $\varepsilon \in (0, 0.2109)$, $t \geq 0$.

2.4 State-Feedback Synthesis

In the previous section, we were interested with stability analysis. For stabilizing controller synthesis, the previous conditions turn to be useful.
Indeed, let us consider the system (2.1) with the state-feedback

$$u(t) = Fx(t-r). \tag{2.40}$$

The closed-loop system (2.1)–(2.40) is

$$x(t) = (A + BF)x(t-r), \ t \geq 0. \tag{2.41}$$

The stabilizing state-feedback gain F can then be synthesized by the following (necessary and) sufficient condition: If there exist a symmetric positive definite matrix Q and a matrix K such that

$$\begin{bmatrix} Q & QA^T + K^T B^T \\ AQ + BK & Q \end{bmatrix} \succ 0, \tag{2.42}$$

then the state-feedback gain $F = KQ^{-1}$ stabilizes (2.41). Note also that the difference equation (2.1) is useful for performance analysis. To see this, consider the disturbed system

$$x(t) = Ax(t-r) + Bu(t) + Ew(t), \tag{2.43}$$

with the controlled output equation

$$y(t) = Cx(t) + Du(t). \tag{2.44}$$

The input $w(t)$ is a disturbance, and is assumed to be in $L_2([0, \infty), \mathbb{R}^w)$, while the matrices E, C and D are real with appropriate size. The controller synthesis consists in determining a static state-feedback $u(t) = Fx(t-r)$, such that the autonomous closed-loop system

$$x(t) = (A + BF)x(t-r) \tag{2.45}$$

is exponentially stable, and the closed-loop controlled output $y(t)$ satisfies, for null initial conditions in the state $x(t)$, the L_2-gain performance specification

$$\|y(t)\|_{L_2} \leq \gamma \, \|w(t)\|_{L_2}, \tag{2.46}$$

for a given $\gamma \geq 0$. This problem admits an immediate answer.

Theorem 3 *Assume that there exist a symmetric positive definite matrix P and a matrix Y such that the matrix*

$$M = \begin{bmatrix} P & 0 & (AP + BY)^T & (C(AP+BY) + DY)^T \\ * & \gamma^2 \cdot I & E^T & (CE)^T \\ * & * & P & 0 \\ * & * & * & I \end{bmatrix}$$

is positive definite. Then, for the static-state feedback $F = YP^{-1}$, the closed-loop system (2.45) is exponentially stable. Furthermore, for a null state initial condition, the closed-loop controlled output $y(t)$ in (2.44) satisfies (2.46).

For distributed-delay synthesis in (2.10) to get exponential stability, Lemma 1 can be used. Indeed, assume that there exist an $n \times n$ symmetric real positive definite matrix $\tilde{P}$, a symmetric positive semi-definite matrix $\tilde{S}$, and some constants $\mu > 0$ and $\alpha \geq 0$ such that

$$\alpha \cdot I_n \prec \tilde{P}, \tag{2.47}$$

$$0 \preceq \tilde{M}_\mu, \tag{2.48}$$

where $\tilde{M}_\mu$ is given by

$$-\tilde{M}_\mu = r \begin{bmatrix} A^T \tilde{P} A - \mathrm{e}^{-2\mu r}(\tilde{P} - \alpha \cdot I_n) & A^T \tilde{P} \\ \tilde{P} A & \tilde{P} - \frac{\mathrm{e}^{-2\mu r}}{r} \tilde{S} \end{bmatrix}. \tag{2.49}$$

Then, we define $\beta = \lambda_{\max}(\tilde{S})$ and $\frac{\alpha}{\beta} = \int_0^r \|G(\theta)\|^2 \, d\theta$. Note that if $\tilde{S} = 0$, β can be taken as an arbitrary positive constant. With this construction, Lemma 1 is satisfied, so that exponential stability holds. For a construction of $G(\cdot)$, take for instance $G(\theta) = e^{M\theta}$ with M some symmetric matrix. It follows that $\|G(\theta)\|^2 = \rho(e^{D\theta})$, where

$$M + M^T = PDP^{-1}, \; D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

$\lambda_i \in \mathbb{R}$ and for some invertible matrix P. If $\lambda_n = \rho(D)$, it follows by definition that

$$\lambda_n \frac{\alpha}{\beta} = e^{\lambda_n r} - 1. \tag{2.50}$$

It is a routine to see that (2.50) admits a positive solution $\lambda_n > 0$ if $\frac{\alpha}{\beta} > r$. Then, it is possible to determine arbitrary $\lambda_i \leq \lambda_n$ for $i = 1, \ldots, n-1$ and to define $G(\cdot)$ (in a non unique way) as above. Finally, note that the decay rate μ can be obtained by an optimization procedure, or can be imposed in the feedback problem to achieve a specified time-response performance.

2.5 Conclusion

In this chapter, linear 1-D conservation laws with boundary control are analyzed as linear continuous-time difference equations with input control. Exponential stability conditions are proposed, for static state-feedback or state-feedback with distributed delay. A brief discussion on feedback synthesis for performance achievement is also mentioned. A conservatism analysis is also made to show the improvement with respect to conditions which already appear in the literature.

This work has not addressed the difficult question of uncertain constant delays, where the constructive conditions for stability in the literature appear to be quite conservative. A first tentative is proposed in [13]. This is one of the two challenging perspectives of this chapter. The other perspective is to apply the results issued from [36] for large-scale interconnected systems of conservation laws. The main interest in this extension is to obtain conditions for stability which are independent of the size of the interconnection structure. For this, the delay-realization used in this chapter seems to be well suitable for such a generalization.

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