Chapter 2
Change of Time Methods: Definitions and Theory

"To know, is to know that you know nothing. That is the meaning of true knowledge".—Socrates.

Abstract In this chapter, we consider the general theory of a change of time method (CTM). One of probabilistic methods which is useful in solving stochastic differential equations (SDEs) arising in finance is the "*change of time method*". We give the definition of CTM and describe CTM in martingale, semimartingale, and the SDEs settings. We also point out the association of CTM with subordinators and stochastic volatilities.

2.1 Change of Time Methods: Definitions, Properties, and Theory

2.1.1 A Change of Time Process: Definition and Properties

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ be a filtered probability space with a sample space Ω, σ-algebra $\mathscr{F}$ of subsets of Ω and probability measure P. The filtration $\mathscr{F}_t, t \geq 0$, is a nondecreasing right-continuous family of sub-σ-algebras of $\mathscr{F}$.

Definition of a Change of Time Process. A *change of time process* is a right-continuous increasing, $[0, +\infty]$-valued and $\mathscr{F}_t$-adapted process $(T_t)_{t \in R_+}$ such that $\lim_{t \to +\infty} T_t = +\infty$. T_t is also a stopping time for any $t \in R_+$.

By $\hat{\mathscr{F}}_t := \mathscr{F}_{T_t}$, we define the *time-changed filtration* $(\hat{\mathscr{F}}_t)_{t \in R_+}$. The *inverse time change* $(\hat{T}_t)_{t \in R_+}$ is defined as $\hat{T}_t := \inf\{s \in R_+ : T_s > t\}$. We note that $\hat{T}_t$ is an increasing process and that $\lim_{t \to +\infty} \hat{T}_t = +\infty$. Furthermore, $\hat{T}_t$ is an $\mathscr{F}_t$-stopping time. Let X_t be an $\mathscr{F}_t$-adapted process. By this, we may define $X_{\hat{T}_t}$. Then $X_{\hat{T}_t}$ is an $\hat{\mathscr{F}}_t$-adapted process, and this process is called the *time change of X_t by T_t*.

A. Swishchuk, *Change of Time Methods in Quantitative Finance*,
SpringerBriefs in Mathematics, DOI 10.1007/978-3-319-32408-1_2

One of the examples of change of time is the following:

Let A_t be an $\mathscr{F}_t$-adapted, increasing, right-continuous random process with $A_0 = 0$. Define the following process:

$$\hat{T}_t = \inf\{s : A_s > t\}, \quad t \geq 0.$$

Then the process $\hat{T}_t$ is a change of time process. We call A_t the *process generating the change of time* $\hat{T}_t$. We note that the process T_t (see definition above) coincides with A_t in this case. It means that change of time processes T_t and $\hat{T}_t$ is a mutually inverse process—someone may construct $\hat{T}_t$ using T_t,

$$\hat{T}_t = \inf\{s : T_s > t\},$$

or may construct T_t using $\hat{T}_t$

$$T_t = \inf\{s : \hat{T}_s > t\}.$$

We also note that

$$T_{\hat{T}_t} = t \quad \textit{and} \quad \hat{T}_{T_t} = t.$$

We would also like to mention the change of time in Lebesgue-Stieltjes integrals, which is well known from calculus. If we take $A_t, A_0 = 0$ as a deterministic increasing continuous function and $f(t)$ as a nonnegative Borel function on $[0, +\infty)$, we may put

$$\hat{A}_t = \inf\{s : A_s > t\},$$

and then we have

$$\int_0^{\hat{A}_a} f(t) dA_t = \int_0^a f(\hat{A}_t) dt, \quad a > 0.$$

We note that $A_t = \int \{s : \hat{A}_s > t\}$ and $A_{\hat{A}_t} = t$. The latter expression can be written in the symmetric form as well:

$$\int_0^{A_a} f(t) d\hat{A}_t = \int_0^a f(A_t) dt, \quad a > 0.$$

There are many stochastic generalizations of the last two relationships for the case of A and f being some stochastic processes. One of them is the following: let $f(t, \omega)$ be a progressively measurable nonnegative stochastic process, and let $B_t(\omega)$ be an $\mathscr{F}_t$-measurable right-continuous process with bounded variation. Then

$$\int_0^{\hat{T}_a} f(t, \omega) dB_t(\omega) = \int_0^a f(\hat{T}_t, \omega) dB_{\hat{T}_t}(\omega),$$

where $\hat{T}_t$ is the inverse change of time process. For example, if $f(t, \omega) = F(T(t))$, where $T_t = A_t$, and A_t is a continuous and strictly increasing process generating the change of time $\hat{T}_t$ (see above), then

$$\int_0^{\hat{T}_a} F(T_t)dB_t(\omega) = \int_0^a F(t)dB_{\hat{T}_t}(\omega).$$

Of course, if $B_t = t$, then

$$\int_0^{\hat{T}_a} F(T_t)dt = \int_0^a F(t)d\hat{T}_t,$$

and if $B_t = T_t$, then

$$\int_0^{\hat{T}_a} F(T_t)dT_t = \int_0^a F(t)dt.$$

See Ikeda and Watanabe (1981) and Barndorff-Nielsen and Shiryaev (2010) for more details.

2.1.2 CTM: Martingale and Semimartingale Settings

The general theory of time changes for martingale and semimartingale theories is well known (see Ikeda and Watanabe 1981). We will give a brief overview of those results.

The following result on martingales and a change of time process belongs to Dambis (1965) and Dubins and Schwartz (1965): Suppose M_t is a square integrable local continuous martingale such that $\lim_{t\to+\infty} < M > (t) = +\infty$ a.s., and define $\hat{T}_t := \inf\{u :< M >_u > t\}$ and $\hat{\mathscr{F}}_t = \mathscr{F}_{\hat{T}_t}$. Then the time-changed process $B(t) := M_{\hat{T}_t}$ is an $\hat{\mathscr{F}}_t$-Brownian motion. Also, $M_t = B(< M >_t)$. Thus, M_t can be presented by an $\hat{\mathscr{F}}_t$-Brownian motion $B(t)$ and an $\hat{\mathscr{F}}_t$-stopping time $< M >_t$. Here, $< \cdot >$ defines a predictable quadratic variation. One of the examples of this result was considered in section 1.2. for a continuous local martingale $M_t = \int_0^t \sigma_s(\omega)dB(s)$, where B_t is a Brownian motion and $\sigma_t(\omega)$ is a positive process such that $\int_0^t \sigma_s^2(\omega)ds < +\infty$. In this case, $\hat{T}_t = \inf\{s : \int_0^s \sigma_u^2(\omega)du \geq t\}$ and $T_t = \int_0^t \sigma_s^2(\omega)ds$.

This result was generalized by Knight (1971) for a d-dimensional case: Let M_t^i be square integrable local continuous martingales, $i = 1,2,...,d$, such that $< M^i, M^j >_t = 0$ if $i \neq j$ and $\lim_{t\to+\infty} < M^i >_t = +\infty$ a.s. If $\hat{T}_t^i = \inf\{u :< M^i >_u > t\}$, then $\mathbf{B}(t) = (B^1(t), B^2(t), ..., B^d(t))$ is a d-dimensional Brownian motion, where $B^i(t) = M^i_{\hat{T}_t^i}$, $i = 1,2,...,d$.

One of the main properties of the semimartingale X_t with respect to the CTM is the following (see Liptser and Shiryaev 1989): If X_t is a semimartingale with respect to a filtration $\mathscr{F}_t$, then the changed time process $X_{\hat{T}_t}$ is also a semimartingale with respect to the filtration $\hat{\mathscr{F}}_t$ (see sec. 1.1).

If we have the triplet of predictable characteristics (B_t, C_t, ν) for a semimartingale X_t,, then the triplet of the time-changed semimartingale $X_{\hat{T}_t}$ is determined as $(B_{\hat{T}_t}, C_{\hat{T}_t}, I_G \nu_{\hat{T}_t})$ (see Kallsen and Shiryaev 2002).

The connection of semimartingales, Brownian motions, and CTM is described by the Monroe result (see Monroe 1978): if X_t is a semimartingale, then there exists a filtered probability space with Brownian motion $\hat{B}_t$ and a change of time T_t on it such that the distribution of X_t coincides with the distribution of $\hat{B}_{T_t}$, i.e.

$$X_t =^{law} \hat{B}_{T_t}. \tag{2.1}$$

Let us now consider a counting process N_t with respect to the filtration $\mathscr{F}_t$ and with the continuous compensator A_t such that $N_t = A_t + M_t$, where M_t is a local martingale. Here, $< M >= A$. Let us then define time change as $\hat{T}_t = \inf\{s :< M >_s > t\}$. If we suppose that $< M >_{+\infty} = +\infty$, then the following process:

$$\hat{N}_t := N_{\hat{T}_t}$$

is a standard Poisson process with the intensity parameter $\lambda = 1$. We note that the initial counting process N_t can be expressed in the following way:

$$N_t = \hat{N}_{T_t},$$

where $T_t =< M >_t$. Here, we note that $M_t = \hat{M}_{T_t}$, where $\hat{M}_t = \hat{N}_t - t$ is a Poisson martingale (see Liptser and Shiryaev 2001 for more details).

Suppose that we have a nondecreasing Lévy process X_t and a Brownian motion $\hat{B}_t$ independent of X_t. Then we can find a change of time T_t such that

$$X_t = \hat{B}_{T_t} \tag{2.2}$$

holds with a probability one. This change of time T_t can be found as

$$T_t = \inf\{s : \hat{B}_s = X_t\}.$$

We mention that a semimartingale X_t can be presented in the form of (2.1) with continuous change of time T_t if and only if the process X_t is a continuous local martingale (see Huff 1969 and Cherny and Shiryaev 2002 for more details).

2.1.3 CTM: Subordinators and Stochastic Volatility

We note that if the process $\hat{T}_t$ (see sec. 1.1) is a Lévy process, then $\hat{T}_t$ is called a *subordinator*. Feller (1966) introduced a subordinated process X_{τ_t} for a Markov process X_t and τ_t a process with independent increments. τ_t was called a "randomized operational time". Increasing Lévy processes can also be used as a time change for other Lévy processes (see Applebaum 2004; Barndorff-Nielsen et al. 2001; Barndorff-Nielsen et al. 2003; Bertoin (1996); Cont and Tankov 2004; Schoutens 2003). Lévy processes of this kind are called subordinators. They are very important ingredients for building Lévy-based models in finance (see Cont and Tankov 2004; Schoutens 2003). If S_t is a subordinator, then its trajectories are almost surely increasing, and S_t can be interpreted as a "time deformation" and used to "time change" other Lévy processes. Roughly, if $(X_t)_{t\geq 0}$ is a Lévy process and $(S_t)_{t\geq 0}$ is a subordinator independent of X_t, then the process $(Y_t)_{t\geq 0}$ defined by $Y_t := X_{S_t}$ is a Lévy process (see Cont and Tankov 2004). This time scale has the financial interpretation of business time (see Geman et al. 2001), that is, the integrated rate of information arrival. Using the subordinator S_t and a Brownian motion $\hat{B}_t$ that is independent of S_t, we can construct many stochastic processes such as $X_t = \hat{B}_{S_t}$. For example, for the Cauchy process $S_t = \inf\{s : B_s > t\}$, where B_s is a standard Brownian motion independent of $\hat{B}_t$; for generalized hyperbolic Lévy processes, S_t is generated by the nonnegative infinitely divisible random variable having generalized inverse Gaussian distribution (the normal inverse Gaussian and hyperbolic Lévy processes are particular cases of the generalized hyperbolic Lévy processes).

The time change method was used to introduce stochastic volatility into a Lévy model to achieve the leverage effect and a long-term skew (see Carr et al. 2003). In the Bates (1996) model, the leverage effect and long-term skew were achieved using correlated sources of randomness in the price process and the instantaneous volatility. The sources of randomness are thus required to be Brownian motions. In the Barndorff-Nielsen et al. (2001, 2002) model, the leverage effect and long-term skew are generated using the same jumps in the price and volatility without a requirement for the sources of randomness to be Brownian motions. Another way to achieve the leverage effect and long-term skew is to make the volatility govern the time scale of the Lévy process driving jumps in the price. Carr et al. (2003) suggested the introduction of stochastic volatility into an exponential-Lévy model via a time change. The generic model here is $S_t = \exp(X_t) = \exp(Y_{v_t})$, where $v_t := \int_0^t \sigma_s^2 ds$. The volatility process should be positive and mean-reverting (i.e. an Ornstein-Uhlenbeck or Cox-Ingersoll-Ross processes). Barndorff-Nielsen et al. (2003) reviewed and placed in the context some of their recent work on stochastic volatility models including the relationship between subordination and stochastic volatility.

In general setting, the connection between stochastic volatility and change of time can be described in the following way :

Let $X_t = \int_0^t H_s dB_s$, where H_s is the adapted process such that $\int_0^t H_s^2 ds < +\infty$, $\int_0^{+\infty} H_s^2 ds = +\infty$ and B_t is a Brownian motion. Then the process $\hat{B}_t := X_{\hat{T}_t}$, where $\hat{T}_t = \inf\{s :< X >_s > t\}$, is a Brownian motion. Moreover, the process X_t has the following representation $X_t = \hat{B}_{T_t}$, where $T_t =< X >_t = \int_0^t H_s^2 ds$.

In the case of α-stable processes Y_t^α instead of Brownian motion B_t in the integral $X_t = \int_0^t H_s dY_s^\alpha$, we have a similar result for $T_t = \int_0^t |H_s|^\alpha ds < +\infty$ and $\int_0^{+\infty} |H_s|^\alpha ds = +\infty$. If we set $\hat{T}_t = \inf\{s : T_s >> t\}$, then $\hat{Y}_t^\alpha = X_{\hat{T}_t}$ is an α-stable process and $X_t = \hat{Y}_{T_t}^\alpha$.

The main difference between the change of time method and the subordinator method is that in the former case, the change of time process T_t depends on the process X_t, but in the latter case, the subordinator S_t and Lévy process X_t are independent.

2.1.4 CTM: Stochastic Differential Equations (SDEs) Setting

2.1.4.1 General Result

We consider the following generalization of the previous results to the SDE of the following form (without a drift):

$$dX(t) = \alpha(t, X(t))dW(t), \tag{2.1}$$

where $W(t)$ is a Brownian motion and $\alpha(t, X)$ is a continuous and measurable (by t and X) function on $[0, +\infty) \times R$.

The reason to consider this equation is the following: if we solve the equation, then we can solve a more general equation with the drift $\beta(t, X)$ by *drift transformation method* or *Girsanov transformation* (see Ikeda and Watanabe 1981, Chapter 4, Section 4).

Theorem 2.1 (Ikeda and Watanabe 1981, Chapter IV, Theorem 4.3). *Let $\tilde{W}(t)$ be an one-dimensional $\mathscr{F}_t$-Wiener process with $\tilde{W}(0) = 0$, given on a probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ and let $X(0)$ be an $\mathscr{F}_0$-adopted random variable. Define a continuous process $V = V(t)$ by the equality*

$$V(t) = X(0) + \tilde{W}(t). \tag{2.2}$$

Let T_t be the change of time process (see Section 2.1.1):

$$T_t = \int_0^t \alpha^{-2}(T_s, X(0) + \tilde{W}(s))ds. \tag{2.3}$$

If

$$X(t) := V(\hat{T}_t) = X(0) + \tilde{W}(\hat{T}_t), \tag{2.4}$$

where

$$\hat{T}_t = \int_0^t \alpha^2(s, X(0) + \tilde{W}(\hat{T}_s))ds,$$

and $\tilde{\mathscr{F}}_t := \mathscr{F}_{\hat{T}_t}$, then there exists a $\tilde{\mathscr{F}}_t$-adopted Wiener process $W = W(t)$ such that $(X(t), W(t))$ is a solution of (2.1) on the probability space $(\Omega, \mathscr{F}, \tilde{\mathscr{F}}_t, P)$. Here, $\hat{T}_t$ is the inverse process of T_t in (2.3).

Proof. of this theorem may be found in Ikeda and Watanabe (1981), Chapter IV, Theorem 4.3.

We note that in this case,

$$M(t) := \tilde{W}(\hat{T}_t) \tag{2.5}$$

is a martingale with quadratic variation

$$< M > (t) = \hat{T}_t = \int_0^{\hat{T}_t} \alpha^2(T_s, X) dT_s = \int_0^t \alpha(s, X)^2 ds, \tag{2.6}$$

and $\hat{T}_t$ satisfies the equation

$$\hat{T}_t = \int_0^t \alpha^2(s, X(0) + \tilde{W}(\hat{T}_s)) ds. \tag{2.7}$$

We also remark that

$$W(t) = \int_0^t \alpha^{-1}(s, X(s)) d\tilde{W}(\hat{T}_s) = \int_0^t \alpha^{-1}(s, X(s)) dM(s) \tag{2.8}$$

and

$$X(t) = X(0) + \int_0^t \alpha(s, X) dW(s).$$

2.1.4.2 Corollary

The solution of the following SDE

$$dX(t) = a(X(t)) dW(t) \tag{2.9}$$

may be presented in the following form

$$X(t) = X(0) + \tilde{W}(\hat{T}_t),$$

where $a(X)$ is a continuous measurable function, $\tilde{W}(t)$ is a one-dimensional $\mathscr{F}_t$-Wiener process with $\tilde{W}(0) = 0$, given on a probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \geq 0}, P)$ and $X(0)$ is an $\mathscr{F}_0$-adapted random variable. In this case,

$$T_t = \int_0^t a^{-2}(X(0) + \tilde{W}(s)) ds, \tag{2.10}$$

and

$$\hat{T}_t = \int_0^t a^2(X(0) + \tilde{W}(\hat{T}_s)) ds. \tag{2.11}$$

(See Ikeda and Watanabe 1981, Chapter IV, Example 4.2).

We note that

$$M(t) := \tilde{W}(\hat{T}_t)$$

is a martingale with a quadratic variation

$$< M > (t) = \hat{T}_t = \int_0^{\hat{T}_t} a^2(X) dT_s = \int_0^t a(X)^2 ds.$$

We also remark that

$$W(t) = \int_0^t a^{-1}(X(s)) d\tilde{W}(\hat{T}_s) = \int_0^t a^{-1}(X(0) + \tilde{W}(\hat{T}_s))) d\tilde{W}(\hat{T}_s)$$

and

$$X(t) = X(0) + \int_0^t a(X(s)) dW(s).$$

2.1.4.3 One-Factor Diffusion Models and Their Solutions Using CTM

In this section, we introduce well-known one-factor diffusion models (used in finance) described by SDEs and driven by a Brownian motion (so-called Gaussian models).

For one-factor Gaussian models, we define the following well-known processes:

1. *The geometric Brownian motion*: $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$;
2. *The continuous-time GARCH process*: $dS(t) = \mu(b - S(t))dt + \sigma S(t)dW(t)$;
3. *The Ornstein-Uhlenbeck (1930) process*: $dS(t) = -\mu S(t)dt + \sigma dW(t)$;
4. *The Vasićek (1977) process*: $dS(t) = \mu(b - S(t))dt + \sigma dW(t)$;
5. *The Cox et al. (1985) process*: $dS(t) = k(\theta - S(t))dt + \gamma\sqrt{S(t)}dW(t)$;
6. *The Ho and Lee (1986) process*: $dS(t) = \theta(t)dt + \sigma dW(t)$;
7. *The Hull and White (1987) process*: $dS(t) = (a(t) - b(t)S(t))dt + \sigma(t)dW(t)$;
8. *The Heath et al. (1992) process*: Define the forward interest rate $f(t,s)$, for $t \le s$, characterized by the following equality $P(t,u) = \exp[-\int_t^u f(t,s)ds]$ for any maturity u. $f(t,s)$ represents the instantaneous interest rate at time s as "anticipated" by the market at time t. It is natural to set $f(t,t) = r(t)$. The process $f(t,u)_{0\le t\le u}$ satisfies an equation

$$f(t,u) = f(0,u) + \int_0^t a(v,u)dv + \int_0^t b(f(v,u))dW(v),$$

where the processes a and b are continuous. We note that the last SDE may be written in the following form: $df(t,u) = b(f(t,u))(\int_t^u b(f(t,s)))ds + b(f(t,u))$ $d\hat{W}(t)$, where $\hat{W}(t) = W(t) - \int_0^t q(s)ds$ and
$q(t) = \int_t^u b(f(t,s))ds - \frac{a(t,u)}{b(f(t,u))}$.

We use the change of time method to get the solutions of the SDEs mentioned above.

$W(t)$ below is a standard Brownian motion, and $\hat{W}(t)$ is a $(\hat{T}_t)_{t\in R_+}$-adapted standard Brownian motion on $(\Omega, \mathscr{F}, (\hat{\mathscr{F}}_t)_{t\in R_+}, P)$.

1. The geometric Brownian motion: $dS(t)=\mu S(t)dt+\sigma S(t)dW(t)$. Solution $S(t)=e^{\mu t}[S(0)+\hat{W}(\hat{T}_t)]$, where $\hat{T}_t=\sigma^2\int_0^t[S(0)+\hat{W}(\hat{T}_s)]^2ds$.
2. The continuous-time GARCH process: $dS(t)=\mu(b-S(t))dt+\sigma S(t)dW(t)$. Solution $S(t)=e^{-\mu t}(S(0)-b+\hat{W}(\hat{T}_t))+b$, where $\hat{T}_t=\sigma^2\int_0^t[S(0)-b+\hat{W}(\hat{T}_s)+e^{\mu s}b)^2ds$.
3. The Ornstein-Uhlenbeck process: $dS(t)=-\mu S(t)dt+\sigma dW(t)$, solution $S(t)=e^{-\mu t}[S(0)+\hat{W}(\hat{T}_t)]$, where $\hat{T}_t=\sigma^2\int_0^t(e^{\mu s}[S(0)+\hat{W}(\hat{T}_s)])^2ds$.
4. The Vasićek process: $dS(t)=\mu(b-S(t))dt+\sigma dW(t)$, solution $S(t)=e^{-\mu t}[S(0)-b+\hat{W}(\hat{T}_t)]$, where $\hat{T}_t=\sigma^2\int_0^t(e^{\mu s}[S(0)-b+\hat{W}(\hat{T}_s)]+b)^2ds$.
5. The Cox-Ingersoll-Ross process: $dS^2(t)=k(\theta-S^2(t))dt+\gamma S(t)dW(t)$, solution $S^2(t)=e^{-kt}[S_0^2-\theta^2+\hat{W}(\hat{T}_t)]+\theta^2$, where $T_t=\gamma^{-2}\int_0^t[e^{kT_s}(S_0^2-\theta^2+\hat{W}(s))+\theta^2e^{2kT_s}]^{-1}ds$.
6. The Ho and Lee process: $dS(t)=\theta(t)dt+\sigma dW(t)$. Solution $S(t)=S(0)+\hat{W}(\sigma^2t)+\int_0^t\theta(s)ds$.
7. The Hull and White process: $dS(t)=(a(t)-b(t)S(t))dt+\sigma(t)dW(t)$.
   Solution $S(t)=\exp[-\int_0^t b(s)ds][S(0)-\frac{a(s)}{b(s)}+\hat{W}(\hat{T}_t)]$, where $\hat{T}_t=\int_0^t\sigma^2(s)[S(0)-\frac{a(s)}{b(s)}+\hat{W}(\hat{T}_s)+\exp[\int_0^s b(u)du]\frac{a(s)}{b(s)}]^2ds$.
8. The Heath, Jarrow, and Morton process: $f(t,u)=f(0,u)+\int_0^t a(v,u)dv+\int_0^t b(f(v,u))dW(v)$. Solution $f(t,u)=f(0,u)+\hat{W}(\hat{T}_t)+\int_0^t a(v,u)dv$, where $\hat{T}_t=\int_0^t b^2(f(0,u)+\hat{W}(\hat{T}_s)+\int_0^s a(v,u)dv)ds$.

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http://www.springer.com/978-3-319-32406-7

Change of Time Methods in Quantitative Finance
Swishchuk, A.
2016, XV, 128 p. 11 illus., 10 illus. in color., Softcover
ISBN: 978-3-319-32406-7