Preface

Nothing is lost, nothing is created out of nothing, everything is transformed.
Lavoisier

Mathematics has been created by human beings for their needs. The teaching mathematician must remain a teacher of action.
Henri Lebesgue

I hear, I forget; I see, I remember; I do, I understand!
Chinese proverb

The purpose of this book is to present some basic results in analysis that can be used to solve various problems, so it may serve as a kind of toolbox. We do not intend to give a complete panorama of the field. In particular, we concentrate on real analysis and leave complex analysis aside; however we consider complex functions now and then. Although we cannot claim that this volume is devoted to applied analysis, our choice of topics is driven by our wish to present results that can be applied to concrete problems. It is in this sense that this book could be called "Motivated Analysis". This expression is due to J.-P. Aubin and means that we shall gather some results of analysis that may be useful in applications, even if the nature of these applications is not the focus of the text. Thus, our route is not the choice of some important problems that occur in applications, as in [22, 62, 87, 88, 101, 129, 130, 140, 178, 191, 236, 241, 245], but neither is it the panorama of deep advances in mathematical concepts presented in [100] and [101].

A leading thread throughout this book is a strategy often advised by mathematicians, given in many instances: if a problem seems to be difficult, try to change it into a more tractable problem. This can be done in various ways. A first approach consists in reformulating the question, keeping its crucial features and dropping inessential details as much as possible (in a sense, mathematics can be considered as an art of strip tease). In doing so, one is often led to a more general problem that is no more difficult to solve. On the contrary, because its framework is rather bare and simple, its solution is often easier to reach. Of course, some knowledge of general mathematics may help.

Another means of solving a problem consists in using a transformation. This approach can been combined with the simplification just mentioned. Quite often, a hidden or rather intricate operation is interchanged with a simple operation via the transformation. For instance, a convolution may be transformed into an addition or a product. Then the problem may appear as much more tractable. Many transformations have been proposed by mathematicians. Among them are those due to Cole-Hopf, Fenchel, Fourier, Hilbert, Laplace, Legendre, Mellin, Radon and Vaslov. For this reason, we focus our attention on some useful transforms, even if we just give an introduction to them. Their applications to economics, engineering, mathematics, medicine and physics are numerous and important. The passage from monotone operators to convex functions (and the inverse passage) is another example of such a transformation that is not yet classical, but it is fruitful and it has attracted a number of mathematicians recently. We give some attention to it in Chap. 9.

The different chapters of this book may serve as separate courses. However, we consider it is important not to neglect the interdependence of the subjects. This is revealed in several instances throughout the book. Integration can be taught without a knowledge of abstract measure theory, but the latter can serve as an important foundation and it is the basis of probability theory. Nemytskii operators play a crucial role in nonlinear partial differential equations. Functional analysis permeates almost all of the topics considered in the book. Elementary differential calculus can be set in Euclidean spaces rather than in normed vector spaces. However the main lines may be hidden by a heavy use of partial derivatives and components and some applications would be lost if we confined differential calculus to such a restricted framework.

The balance between generality and simplicity is not easy to reach. Take the notion of convergence for example. It can be considered as the keystone of analysis. For that reason we present the main lines of the concept, but it could be expounded more thoroughly. In fact, we encourage the reader to prune rather than to develop what is presented here, since we have given more than the essentials of what is needed in practice. Also, the theory of distributions could be given a more prominent role than the one we offer it in our presentation of Sobolev spaces; incidentally the convergence approach is certainly simpler than the topological approach to distributions considered as elements of a dual space. However, we prefer to give a rather direct treatment, even if the concept of transposition is central to the understanding of generalized derivatives and generalized solutions.

The difficulty in choosing the presentation of a subject is well reflected in the following quotation from Julian Barnes, *Staring at the Sun*, (Jonathan Cape, London, 1986) mentioned by I. Smith, in Bulletin of the American Mathematical Society 52 (3) p. 415 (2015):

> ...everything you wanted to say required a context. If you gave the full context, people thought you a rambling old fool. If you didn't give the context, people thought you a laconic old fool.

In his remarkable book [117] L.C. Evans writes "notation is a nightmare". It certainly presents difficulties, in particular when one tries to conciliate various uses.

However, this difficulty is no greater than the challenge posed by the choice of topics and the writing of clear and concise proofs. Our experience with texts that were written one or more lifetimes ago showed us that the need of rigour and precision has increased and is likely to increase more in the future. Thus, we avoid some common abuses such as confusing a function f or $f(\cdot)$ with its value $f(x)$, a sequence (x_n) with its general term x_n or its set of values $\{x_n\}$, and a space with its dual. We distinguish the adjoint A^* of a continuous linear map A and its transpose $A^\intercal$ and we distinguish the derivative Df of a function and its gradient ∇f. If you are not convinced by such distinctions, you are challenged to give at once the higher derivatives of a composite function. Also, we refrain from using the notation f_{x_1} to denote the partial derivative of a function f with respect to its first variable x_1 because f_{x_1} may denote the function $f(x_1,\cdot)$ of the variables other than x_1, the variable x_1 being "frozen". On the other hand, our position is not rigid, as the next two examples show. We do not follow the choice of C. Zalinescu in [265] who denotes by $\alpha^\#$ the especial conjugate of a function α defined on $\mathbb{R}_+$ rather than $\mathbb{R}$; albeit his choice is wise, we prefer the reader to see at first glance the link with convex conjugacy, even if there is a risk of confusion (but this risk is limited since we only use nonnegative values of the arguments). Also, we retain the classical terminology concerning "locally uniformly convex" norms despite the fact that the notion is a pointwise notion rather than a local one.

Our general choice of discarding ambiguity may lead to unusual expressions or notations. We hope the reader will not be disturbed by such novelties and that authors will support them. Most mathematicians are unaware that the notation they use was once considered as shocking. Rarely is credit given to their inventors or promoters, such as Oresme (for coordinates and exponents), Recorde (for the sign $=$), Leibniz (for derivatives, products, quotients and $\int$, as the first letter of the Latin *summa* or sum was denoted at that time), and Peano (for inclusion...). In this respect we must say that the notation $\mathbb{P}$ used here for the set of positive numbers (more present in analysis than the set $\mathbb{Q}$ of rational numbers) is, to our knowledge, due to J.M. Borwein. The scalar product of two vectors u, v is denoted here by the compromise $\langle u \mid v\rangle$ between the mathematicians' and the physicists' uses; but, from time to time, we use the dot notation $u\cdot v$ in view of its simplicity. We avoid the notation (u, v) which may mean either a pair or an open interval. In general, we try to avoid ambiguity since mathematics aims at being a clear language. For that reason, we sometimes depart from the most common terminology in order to avoid confusion. But, consciously or unconsciously, we also use some abuses of notation that tradition has made acceptable (until they are rejected?).

In preparing this book, we have benefited from the teaching of our masters, some of whom were actors in the clarification or the setting up of the subject. We also benefited from several excellent books. The readers will easily detect these sources. Our choices were dictated by our wish to present the most elegant proofs. The word "elegant" may seem a strange choice for a mathematical book. However, mathematicians are often sensitive to aesthetics when considering proofs. Under this term they often appreciate bold, clear, concise proofs. A frequent drawback is the effort required from the reader. But the reward is a better understanding of the nature of the result and of its possible applications. For that reason, we encourage the reader to approach proofs with a paper and a pen in order to devise variants or developments or to mark the decisive steps. Experiencing proofs is the best apprenticeship for a field. We are so convinced of the value of proofs that sometimes we give two proofs of the same result; on the other hand we skip some proofs, either because they are too involved or because they are outside the scope of the book. Exercises are often augmented with hints, but not complete solutions. The most difficult exercises are marked with an asterisk; they are included as complements rather than for training. An asterisk also marks results or sections that can be omitted on a first reading.

It is a great pleasure to thank my colleagues and friends who kindly read and criticized parts of the successive versions of this book; in particular, the contributions of Luc Barbet, Marc Dambrine, Marc Durand, Emmanuel Giner, Dena Kazerani, Alexander Kruger, Khadra Nachi, Steve Robinson, Lionel Thibault, Constantin Zălinescu, Nadia Zlateva have been precious for reducing the number of mistakes or misprints and for bringing me some encouragement.

I hope the reader will find this toolbox useful and will enjoy the rich legacy of our predecessors. Comments and criticisms will be welcome at penotj@ljll.math.upmc.fr.

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