

Chapter 2

Linear Strain Theory

Linear Hookean elasticity formulated in terms of linear strain measures is the simplest way of modeling deformation in self-gravitating terrestrial bodies. In this case the solutions for the stresses, the strains, and the displacements can be presented in *closed* form. This chapter is dedicated, first, to a thorough presentation of the underlying theory. Second, solutions to the resulting equations will be obtained. Third, the equations will be evaluated by using physical data of various objects, such as terrestrial planets, moons, and asteroids. The latter will show that under certain circumstances the displacements may be enormous. Consequently, the limits of linear strain theory will become evident.

As a special feature we will also leave the canonical pathway of linear elasticity, where it is conventionally assumed that the body forces are applied to the undeformed configuration. Therefore, in contrast to conventional (engineering) literature, we will present an “extended model” and study the influence of linear terms of displacement gradients in the body force density. In fact, this approach may serve as a bridge between linear elasticity at small strains and elasticity at large deformations. Moreover, it has the advantage of still leading to closed-form solutions.

The chapter ends with an extension of the theory to the case of an originally spherical, self-gravitating body undergoing rotation and finally coming to a stationary state. It will be shown that self-gravity is dominant and fully decouples from the effect of centrifugal accelerations. The so-called flattening parameter will be calculated and it will be shown that it is fully independent of the shrinkage due to self-gravity.

2.1 Pure Self-Gravity: Fundamental Relations

2.1.1 *Equilibrium of Forces and Hooke’s Law*

In this chapter we will investigate static equilibrium of a self-gravitating, linear elastic sphere. In this case the balance of momentum degenerates into the following form:

$$\nabla \cdot \boldsymbol{\sigma} = -\rho \mathbf{f}, \quad (2.1.1)$$

where ρ denotes the local current mass density, and $\boldsymbol{\sigma}$ is the Cauchy stress tensor. The specific body force $\mathbf{f}$, i.e., the gravitational acceleration, is conservative and originates from self-gravity. Hence a gravitational potential $U^g(\mathbf{x})$ exists, where $\mathbf{x}$ denotes an arbitrary (current) position within the body, and we may write:

$$\mathbf{f}(\mathbf{x}) = -\nabla U^g(\mathbf{x}). \quad (2.1.2)$$

The gravitational potential obeys Poisson's equation¹:

$$\Delta U^g(\mathbf{x}) = 4\pi G \rho(\mathbf{x}). \quad (2.1.3)$$

For the stress tensor we assume that the isotropic Hooke's law holds:

$$\boldsymbol{\sigma} = \lambda \operatorname{Tr} \boldsymbol{\epsilon} \mathbf{1} + 2\mu \boldsymbol{\epsilon}, \quad (2.1.4)$$

where the *linear* strain tensor has been used:

$$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top). \quad (2.1.5)$$

$\mathbf{u}$ refers to the displacement vector, i.e., to $\mathbf{u} = \mathbf{x} - \mathbf{X}$, $\mathbf{X}$ being the reference position of a material point of the sphere. λ and μ are Lamé's elastic constants. Recall that all nabla operators indicate differentiation w.r.t. the *current* spatial position, $\mathbf{x}$. This is how the theory works for “linear elasticians”: There is only one gradient in linear theory of elasticity at small deformations, namely that one. Hence for them putting an emphasis on it sounds trivial. However, there is a world outside of linear elasticity as understood by [22] or [24], in order to quote just two references of that denomination. Indeed, it is possible to understand linear elasticity as a limit case of nonlinear materials theory. Then Hooke's law results in a natural way written in terms of gradients with respect to the reference configuration, $\mathbf{X}$, see [27], p. 170, or [17], p. 72. However, in the next breath it is said, see [26], Sects. 57 and 301, that it does not really matter, and these gradients can be replaced by derivatives w.r.t. the current position, since the deformations are so small. What they do not say, though, is that it *does matter* from a principal, didactic point-of-view.

Moreover, from the standpoint of linear elasticity at small deformations, the set of Eqs. (2.1.1)–(2.1.5) serves only one purpose: It allows us to calculate the equilibrium displacement $\mathbf{u}(\mathbf{x})$ provided the mass density $\rho(\mathbf{x})$ is known. In fact, the mass density is determined from the local balance of mass, which can be integrated in time to yield (as a reminder see [7], p. 79 or [18], p. 64):

$$\rho(\mathbf{x}) = \frac{\rho_0}{\det \mathbf{F}(\mathbf{x})} \quad (2.1.6)$$

¹A rational derivation of the Poisson equation for the case of Newtonian gravity can be found in the Appendix.

and $\mathbf{F}(\mathbf{x}) \equiv \nabla_X \mathbf{x}$ is the deformation gradient pertinent to a material point, where ∇_X denotes the nabla operator of the reference configuration, i.e., differentiation w.r.t. $\mathbf{X}$. Of course, Eq. (2.1.6) is *not* the result of a linear theory. It is the result of the physical principle of (local) mass conservation and geometry, i.e., kinematics, and it holds for arbitrary deformations. If we wish to study small deformations we must replace Eq. (2.1.6) by (see [11], p. 233, [4], p. 194, or [18], p. 138):

$$\rho(\mathbf{x}) \approx \rho_0 [1 - \text{Tr } \epsilon(\mathbf{x})]. \quad (2.1.7)$$

In linear elasticity of small deformations this result is interpreted as follows: Once the linear strain $\epsilon(\mathbf{x})$ is known the current mass density distribution of the strained body is calculated from Eq. (2.1.7). However, the strains or rather the displacements are calculated from Eqs. (2.1.1)–(2.1.5) *after* the current mass density has been replaced by the reference mass density, which in the most simple case is assumed to be constant. This is a very subtle point. We may rephrase it in the jargon of technical mechanics by saying that the forces are applied to the undeformed structure and a first order theory is used to calculate the resulting deformation. From the standpoint of the purist such an approach seems questionable. However, we shall see shortly that it still leads to useful results, at least within certain limits.

In the same context it should be mentioned that Kienzler and Schröder explicitly list *three* prerequisites for linear elasticity theory in the introduction to their textbook on strength of materials [10], namely, first, a linear relationship between stress and strain, second, strains and displacements are small and, third, equilibrium of an *undeformed* element. The latter implies the use of a constant mass density for the body force. However, it is fair to say that these requirements are rarely expressed so compactly and stringently. Timoshenko and Goodier hide this important fact between many lines of text [24], p. 7: “... and base our calculations on initial dimensions and initial shape of body.” If we look at the more mathematically oriented textbooks on linear elasticity, the situation is even worse. It takes Sokolnikoff [22] 72 pages to state finally: “The equations of equilibrium (24.1) contain the components F_i of the body forces $\mathbf{F}$, and they are assumed to be prescribed functions of the coordinates x_i of the undeformed body.” If, in addition, we take a look at the physics oriented literature, more specifically at Landau and Lifshitz’ textbook on elasticity [12], we find no explicit reference according to which the forces must be applied to the undeformed configuration. However, there is an indirect hint regarding this issue in a footnote on p. 5 of their book: “Strictly speaking, to determine the total force on a deformed portion of the body we should integrate, not over the old coordinates x_i , but over the coordinates x'_i of the points of the deformed body. ... However, in view of the smallness of the deformation, the derivatives with respect to x_i and x'_i differ only by higher order quantities, and so the derivatives can be taken with respect to the coordinates x_i .” We conclude it is necessary to point out these facts in the community even if they might be obvious to some of us.

Finally in this section some discussion is in order: Our idea is to start from a spherical, initially homogeneous, unloaded sphere (the planet in *statu nascendi*), which is eventually subjected to self-gravity. We must ask as to whether static equilibrium

is possible, how it is reached, and what exactly is modeled by the afore-mentioned equations. Two scenarios come to mind.

First, imagine that gravity is “suddenly switched on.” Then we will essentially face a situation similar to that of a moving masspoint connected in parallel to a linear-elastic Hookean spring and a dashpot: Due to the inertial terms in the equations of motion and due to the fact that the gravitational potential in a radially symmetric, self-gravitating sphere is formally identical to that of a linear oscillator, this sphere will begin to shrink below the radius determined by static equilibrium of forces. While doing so, stress-related forces will build up so that the sphere will finally start to rebound. Provided there is no dissipation it will reach its initial radius again. This will happen over and over if we assume the material of the sphere to be perfectly elastic without internal friction and without heat conduction, so that isothermal conditions prevail. In other words, without dissipation there will be a constant exchange between the elastic energy, the gravitational potential, and the kinetic energies: The motion of the self-gravitating matter would never come to a standstill.

Of course, in reality there will be dissipative processes present. The shrinking will be accompanied by dissipation in terms of visco-elastic or visco-plastic deformation, and there will be heat conduction. All of this will, in the end, bring motion to a standstill, and the sphere will arrange itself in thermomechanical equilibrium, i.e., there will be equilibrium of gravitational and (linear) elastic forces in a state of homogeneous temperature. Clearly, the temporal transition to final equilibrium is *not* modeled by the aforementioned equations: There is no inertial term in Eq. (2.1.1), Hooke’s law (2.1.4) does not include viscous effects, and the balance of energy has not been mentioned at all, so that there is no heat conduction. Only the final mechanical state is modeled by these equations.

Alternatively, and in a certain way even more artificial than the previous way of modeling, we may imagine that gravitation is switched on “slowly.” This way inertial forces in the equations of motion can be neglected and the sphere would quasistatically and isothermally move into its final state of deformation. Such a situation is frequently conjured up in so-called $p\,dV$ -thermodynamics, for example, if we allow the pressure on a piston to change very slowly so that the gas which is trapped in the corresponding container has time to accommodate pressure- and temperature-wise.

This in mind we shall now proceed as follows: In this chapter, Chaps. 3 and 4 we will concentrate on describing only the final state of equilibrium, first, on the basis of linear elasticity with small deformations, second by using linear elastic stress-strain relationships with nonlinear deformation measures and, third, for self-gravitating, (in-)compressible, fluidic spheres. In all these chapters the temporal development of deformation characteristics will be no issue. We will only study the final state of equilibrium and not model how to get there. However, in Chap. 5 an attempt will be made in this direction. The idea is to model a quasistatic approach toward equilibrium. We will look at a linear viscoelastic model of the Kelvin-Voigt type for small deformations. In other words, the balance of momentum will still be given by

(2.1.1) but the stress-strain relationship (2.1.4) will be replaced by a time-dependent relation.

2.1.2 Solution in Spherical Coordinates

Due to the geometrical nature of our problem we rewrite the equations of the last section in *physical* spherical coordinates. Poisson's equation (2.1.3) becomes (for the Laplace operator in spherical coordinates see [18], p. 116 or [30]):

$$\begin{aligned} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U^g}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial U^g}{\partial \vartheta} \right) \\ + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 U^g}{\partial \varphi^2} = 4\pi G \rho(r, \vartheta, \varphi), \quad U^g = U^g(r, \vartheta, \varphi). \end{aligned} \quad (2.1.8)$$

For reasons of symmetry the mass density should be a function of the current radial position r only, i.e., $\rho = \rho(r)$, because the matter within the sphere will eventually come to a state of rest long after gravity has been “switched on.” Thus Eq. (2.1.8) simplifies and can be integrated as follows:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dU^g(r)}{dr} \right) = 4\pi G \rho(r) \Rightarrow \frac{dU^g}{dr} = G \frac{m(r)}{r^2}, \quad (2.1.9)$$

where $m(r)$ denotes the total mass within a spherical region of radial extension r :

$$m(r) = 4\pi \int_{\tilde{r}=0}^{\tilde{r}=r} \rho(\tilde{r}) \tilde{r}^2 d\tilde{r}, \quad 0 \leq r \leq r_o, \quad (2.1.10)$$

and r_o stands for the current outer radius of the spherical body. Consequently, according to Eq. (2.1.2), the body force density is given by:

$$\rho(r) \mathbf{f}(r) = -G \frac{\rho(r)m(r)}{r^2} \mathbf{e}_r. \quad (2.1.11)$$

This represents the well-known high school result according to which the gravitational force at a distance r within a homogeneous sphere is given by Newton's law of gravity for point masses: The attracting mass is given by all the matter below the position r , i.e., $m(r)$, and can be thought of as being concentrated in the origin of the sphere, i.e., $r = 0$. The to-be-attracted mass is given by $dm = \rho(r) dV$, dV being the corresponding volume element to be used for multiplication in Eq. (2.1.11). Moreover, the gravitational force is attractive, as indicated by the negative direction of the current radial unit vector, $\mathbf{e}_r$. It should be pointed out that the mass density

within the sphere does not necessarily have to be homogeneous. Rather it can be a function of the current radius, $\rho(r)$.

Conservation of mass allows us to write²:

$$m(r) = M(R) \Leftrightarrow 4\pi \int_{\tilde{r}=0}^{\tilde{r}=r} \rho(\tilde{r}) \tilde{r}^2 d\tilde{r} = \frac{4\pi}{3} \rho_0 R^3, \quad (2.1.12)$$

where R is the counterpart to r in the stress-free reference configuration, i.e., a radial position in the *homogeneous* sphere before gravity was “switched on.” As outlined before, it is customary in linear elasticity to approximate the body force as follows:

$$\rho(r) \mathbf{f}(r) \approx -G \frac{\rho_0 m(r)}{r^2} \mathbf{e}_r \approx -\frac{4\pi G \rho_0^2}{3} r \mathbf{e}_r. \quad (2.1.13)$$

Note that a two-step approximation is involved. First, the current mass density, $\rho(r)$, in Eq. (2.1.11) is replaced by the reference mass density, ρ_0 . Second, no distinction is made between the current and the reference radius on the right hand side of Eq. (2.1.12), i.e., $R \equiv r$. We will see later what happens if we do not make these extreme approximations but keep linear terms of the displacement and of its derivatives instead. However, for the time being we will use the approximation (2.1.13) in Eq. (2.1.1), which reads in spherical coordinates (for the convenience of the reader, see, e.g., [18], p. 116):

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\vartheta}}{\partial \vartheta} + \frac{1}{r \sin \vartheta} \frac{\partial \sigma_{r\varphi}}{\partial \varphi} + \frac{2\sigma_{rr} - \sigma_{\vartheta\vartheta} - \sigma_{\varphi\varphi} + \sigma_{r\vartheta} \cot \vartheta}{r} &= \frac{4\pi G \rho_0^2}{3} r, \\ \frac{\partial \sigma_{r\vartheta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\vartheta\vartheta}}{\partial \vartheta} + \frac{1}{r \sin \vartheta} \frac{\partial \sigma_{\vartheta\varphi}}{\partial \varphi} + \frac{3\sigma_{r\vartheta} + (\sigma_{\vartheta\vartheta} - \sigma_{\varphi\varphi}) \cot \vartheta}{r} &= 0, \\ \frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\vartheta\varphi}}{\partial \vartheta} + \frac{1}{r \sin \vartheta} \frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{3\sigma_{r\varphi} + 2\sigma_{\vartheta\varphi} \cot \vartheta}{r} &= 0. \end{aligned} \quad (2.1.14)$$

Note that the left hand sides of Eq. (2.1.14) contain no approximations and hold for the current configuration. We will be able to reuse them for large deformations. Moreover, Hooke’s law reads in spherical coordinates (as a reminder see, e.g., [18], p. 137):

$$\begin{aligned} \sigma_{rr} &= \lambda(\epsilon_{\vartheta\vartheta} + \epsilon_{\varphi\varphi}) + (\lambda + 2\mu)\epsilon_{rr}, \quad \sigma_{\vartheta\vartheta} = \lambda(\epsilon_{rr} + \epsilon_{\varphi\varphi}) + (\lambda + 2\mu)\epsilon_{\vartheta\vartheta}, \\ \sigma_{\varphi\varphi} &= \lambda(\epsilon_{rr} + \epsilon_{\vartheta\vartheta}) + (\lambda + 2\mu)\epsilon_{\varphi\varphi}, \\ \sigma_{r\vartheta} &= 2\mu \epsilon_{r\vartheta}, \quad \sigma_{r\varphi} = 2\mu \epsilon_{r\varphi}, \quad \sigma_{\vartheta\varphi} = 2\mu \epsilon_{\vartheta\varphi}. \end{aligned} \quad (2.1.15)$$

²Recall that capitol symbols refer to the reference configuration. Hence $M(R)$ is the mass within a sphere of radius R , where R is the image of the current r in the reference configuration.

And finally the linear strain tensor is linked with spatial derivatives of the displacements by (again, for convenience, see among many others [18], p. 137):

$$\begin{aligned}\epsilon_{rr} &= \frac{\partial u_r}{\partial r}, \quad \epsilon_{\vartheta\vartheta} = \frac{1}{r} \frac{\partial u_{\vartheta}}{\partial \vartheta} + \frac{u_r}{r}, \quad \epsilon_{\varphi\varphi} = \frac{1}{r \sin \vartheta} \frac{\partial u_{\varphi}}{\partial \varphi} + \frac{u_r}{r} + \frac{\cot \vartheta}{r} u_{\vartheta}, \\ \epsilon_{r\varphi} &= \frac{1}{2} \left(\frac{1}{r \sin \vartheta} \frac{\partial u_r}{\partial \varphi} + \frac{\partial u_{\varphi}}{\partial r} - \frac{u_{\varphi}}{r} \right), \\ \epsilon_{r\vartheta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \vartheta} + \frac{\partial u_{\vartheta}}{\partial r} - \frac{u_{\vartheta}}{r} \right), \quad \epsilon_{\vartheta\varphi} = \frac{1}{2} \left(\frac{1}{r \sin \vartheta} \frac{\partial u_{\vartheta}}{\partial \varphi} + \frac{1}{r} \frac{\partial u_{\varphi}}{\partial \vartheta} - \frac{\cot \vartheta}{r} u_{\varphi} \right).\end{aligned}\tag{2.1.16}$$

We now proceed to solve these equations. To this end we make use of the semi-inverse method. Because of symmetry it seems reasonable to look for solutions with the following *ansatz*:

$$u_r = u_r(r), \quad u_{\vartheta} = 0, \quad u_{\varphi} = 0.\tag{2.1.17}$$

Consequently, we find for the linear strains:

$$\epsilon_{rr} = u'_r(r), \quad \epsilon_{\vartheta\vartheta} \equiv \epsilon_{\varphi\varphi} = \frac{u_r}{r}, \quad \epsilon_{r\vartheta} = \epsilon_{r\varphi} = \epsilon_{\vartheta\varphi} \equiv 0,\tag{2.1.18}$$

where the dash refers to a differentiation w.r.t. r . Because of that Hooke's law (2.1.15) reduces to:

$$\sigma_{rr} = (\lambda + 2\mu)u'_r + 2\lambda \frac{u_r}{r}, \quad \sigma_{\vartheta\vartheta} \equiv \sigma_{\varphi\varphi} = \lambda u'_r + 2(\lambda + \mu) \frac{u_r}{r},\tag{2.1.19}$$

$$\sigma_{r\vartheta} = \sigma_{r\varphi} = \sigma_{\vartheta\varphi} \equiv 0.$$

Thus the angular components of the balance of momentum shown in Eq. (2.1.14) are identically satisfied and the first one in radial direction results in an ordinary differential equation of second order (a dash refers to differentiation with respect to the radius, r):

$$u''_r + 2 \frac{u'_r}{r} - 2 \frac{u_r}{r^2} = \frac{4\pi\rho_0^2 G}{3(\lambda + 2\mu)} r.\tag{2.1.20}$$

The general solution consists of the full solution to the homogeneous part and one particular solution of the inhomogeneous case. It reads with two constants of integration, A and B , respectively:

$$u_r = Ar + \frac{B}{r^2} + \frac{4\pi\rho_0^2 G}{30(\lambda + 2\mu)} r^3.\tag{2.1.21}$$

Two conditions are required to determine the two constants of integration. For the case of a solid sphere we require, first, the solution not to become singular at

$r = 0$ and, second, the traction to be continuous at the outer radius, r_o , of the sphere. Hence, $\sigma_{rr}|_{r=r_o} = 0$, because the influence of the outer atmospheric pressure of roughly 1 bar is negligibly small when it comes to the deformation of a solid. With Eq. (2.1.19)₁ we find that:

$$B = 0, \quad A = -\frac{4\pi G \rho_0^2}{30(\lambda + 2\mu)} \frac{3 - \nu}{1 + \nu} r_o^2 \equiv -\frac{4\pi G \rho_0^2}{90k} \frac{3 - \nu}{1 - \nu} r_o^2, \quad (2.1.22)$$

because $\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$, E being Young's modulus and ν Poisson's ratio, respectively. Moreover, the bulk modulus is given by $k = \frac{E}{3(1-2\nu)}$. Hence we arrive at:

$$u_r = -\frac{2\pi G \rho_0^2 r_o^2}{15(\lambda + 2\mu)} \left(\frac{3 - \nu}{1 + \nu} - \frac{r^2}{r_o^2} \right) r \quad (2.1.23)$$

or, alternatively, rewritten:

$$u_r = -\frac{\alpha}{20} \left(\frac{3 - \nu}{1 + \nu} - \frac{r^2}{r_o^2} \right) r \equiv -\frac{\alpha_k}{30} \frac{1 + \nu}{1 - \nu} \left(\frac{3 - \nu}{1 + \nu} - \frac{r^2}{r_o^2} \right) r \quad (2.1.24)$$

with

$$\alpha = \frac{8\pi G \rho_0^2 r_o^2}{3(\lambda + 2\mu)}, \quad \alpha_k = \frac{4\pi G \rho_0^2 r_o^2}{3k}. \quad (2.1.25)$$

These two dimensionless factors will play an important role during the convergence study of the nonlinear deformation case, which is why we have decided to introduce them here already.

The non-vanishing stresses follow from Eq. (2.1.19):

$$\sigma_{rr} = -\frac{2\pi G \rho_0^2 r_o^2}{15} \frac{3 - \nu}{1 - \nu} \left(1 - \frac{r^2}{r_o^2} \right), \quad (2.1.26)$$

$$\sigma_{\vartheta\vartheta} \equiv \sigma_{\varphi\varphi} = -\frac{2\pi G \rho_0^2 r_o^2}{15} \frac{3 - \nu}{1 - \nu} \left(1 - \frac{1 + 3\nu}{3 - \nu} \frac{r^2}{r_o^2} \right).$$

2.1.3 Numerical Evaluation

For a numerical analysis it is best to work with dimensionless quantities. In the case of a solid sphere the outer radius, r_o , is the only length parameter in the problem. Hence, there is no other choice but to define a dimensionless distance and a dimensionless displacement by:

$$x = \frac{r}{r_o}, \quad u = \frac{u_r}{r_o}. \quad (2.1.27)$$

The differential equation (2.1.20) then assumes the form

$$u'' + 2\frac{u'}{x} - 2\frac{u}{x^2} = \frac{\alpha}{2}x, \quad (2.1.28)$$

where the dash now means differentiation w.r.t. the dimensionless radius, x . The solution for the displacement from Eq. (2.1.24) reads:

$$u = -\frac{\alpha}{20} \left(\frac{3-\nu}{1+\nu} - x^2 \right) x \equiv -\frac{\alpha_k}{30} \frac{1+\nu}{1-\nu} \left(\frac{3-\nu}{1+\nu} - x^2 \right) x. \quad (2.1.29)$$

Whilst the appearance of α is a straightforward consequence of Eq. (2.1.23) the need for α_k must be explained. To this end note that the dimensionless expressions in the parentheses of Eq. (2.1.29) still contain Poisson's ratio. However, Poisson's ratio of a terrestrial planet is *not* an immediately accessible parameter. A homogenization technique has to be applied in order to find out which effective elastic parameters such an object has. On the other hand, if we evaluate the parentheses in this equation at the surface of the planet, i.e., at $x = 1$, we obtain twice the fraction $\frac{1-\nu}{1+\nu}$. Hence Poisson's ratio disappears completely in the expression for the normalized displacement if we use the dimensionless factor α_k . In this case one only needs to know the effective compressibility of the planet, a parameter that can only vary within certain physically reasonable bounds. And what is more, since $u(x = 1)$ can be interpreted as an average strain³ characterizing the state of deformation of the planet, which we wish to access numerically, it is very useful to have one elastic parameter less to worry about. A final comment is in order in context with Eq. (2.1.29). Since u must be small and small strain theory applies, it obviously provides a restriction to the size of α . We will get back to this issue later.

Moreover, later we shall be interested in a strain-based failure criterion. Hence it is useful to know the strains, which are dimensionless to begin with, explicitly:

$$\epsilon_{rr} = -\frac{\alpha}{20} \left(\frac{3-\nu}{1+\nu} - 3x^2 \right), \quad \epsilon_{\vartheta\vartheta} \equiv \epsilon_{\varphi\varphi} = -\frac{\alpha}{20} \left(\frac{3-\nu}{1+\nu} - x^2 \right), \quad (2.1.30)$$

or

$$\epsilon_{rr} = -\frac{\alpha_k}{30} \frac{1+\nu}{1-\nu} \left(\frac{3-\nu}{1+\nu} - 3x^2 \right), \quad \epsilon_{\vartheta\vartheta} \equiv \epsilon_{\varphi\varphi} = -\frac{\alpha_k}{30} \frac{1+\nu}{1-\nu} \left(\frac{3-\nu}{1+\nu} - x^2 \right). \quad (2.1.31)$$

We now turn to the stresses given by Eq. (2.1.26). We shall see that in contrast to the case of length related quantities there are *various* possibilities for their normalization. An immediate, "natural" choice comes to mind when looking at Eq. (2.1.26): The stresses should be normalized by their common factor, $\frac{2\pi G \rho_0^2 r_0^2}{15}$. The question should be asked as to whether this factor can be interpreted intuitively. For an answer, note

³Note that according to Eq. (2.1.18)₁ we may write $\epsilon_{rr} = \frac{du_r}{dr}$. On the other hand we have by definition from Eq. (2.1.28)₂ that $u(x = 1) = \frac{u_r(r=r_0)}{r_0}$. And if we compare both we see that the latter is nothing else but a strain average over the full domain.

that (within the approximations made) the total mass of the gravitating sphere, m_0 , the gravitational acceleration on its surface, g , and its outer surface area, A_o , are given by:

$$m_0 = \frac{4\pi}{3} \rho_0 r_o^3, \quad g = \frac{Gm_0}{r_o^2}, \quad A_o = 4\pi r_o^2. \quad (2.1.32)$$

Hence we may write for the normalization factor for the stresses:

$$\frac{3m_0 g}{10A_o} \equiv \frac{2\pi G \rho_0^2 r_o^2}{15} \quad (2.1.33)$$

and, with the exception of the factor $3/10$, interpret it as an average “pressure,” namely the ratio between the “total” gravitational force, $m_0 g$, distributed over the total outer surface area, A_o .

However, alternatively, we may use combinations of (effective) elastic constants for stress normalization. One possibility is to use the factor $\lambda + 2\mu$, which is related to the velocity of seismic P-waves ($v_p = \sqrt{\frac{\lambda+2\mu}{\rho_0}}$), and, hence, a physically relevant quantity. Moreover, we may turn to the compressibility k , which is a direct measure of the resistance of a planet’s response to its own self-gravity. Hence we may write either:

$$\frac{\sigma_{rr}}{\lambda + 2\mu} = -\frac{\alpha}{20} \frac{3 - \nu}{1 - \nu} (1 - x^2), \quad (2.1.34)$$

$$\frac{\sigma_{\vartheta\vartheta}}{\lambda + 2\mu} \equiv \frac{\sigma_{\varphi\varphi}}{\lambda + 2\mu} = -\frac{\alpha}{20} \frac{3 - \nu}{1 - \nu} \left(1 - \frac{1 + 3\nu}{3 - \nu} x^2\right),$$

or:

$$\frac{\sigma_{rr}}{k} = -\frac{\alpha_k}{10} \frac{3 - \nu}{1 - \nu} (1 - x^2), \quad (2.1.35)$$

$$\frac{\sigma_{\vartheta\vartheta}}{k} \equiv \frac{\sigma_{\varphi\varphi}}{k} = -\frac{\alpha_k}{10} \frac{3 - \nu}{1 - \nu} \left(1 - \frac{1 + 3\nu}{3 - \nu} x^2\right).$$

Figure 2.1 illustrates the dependence of the normalized displacement, $u \equiv \frac{u_r}{r_o}$, and the radial strain, ϵ_{rr} , on $x \equiv \frac{r}{r_o}$ for three different choices of Poisson’s ratio, $\nu = 0$ (red), $\nu = 0.3$ (green), and $\nu = 0.5$ (blue). Note that, as it should be, the radial displacement is negative and that the curves show a minimum. Because of Eq. (2.1.16)₁ we may interpret the slope of the curves as radial strain. Hence the only positive radial strains can be found to the right of the minimum. The transition point between positive and negative strains, identifiable by locating the minimum, is a.k.a. the *Love radius* and given by:

$$r_{\text{Love}} = r_o \sqrt{\frac{3 - \nu}{3(1 + \nu)}}. \quad (2.1.36)$$

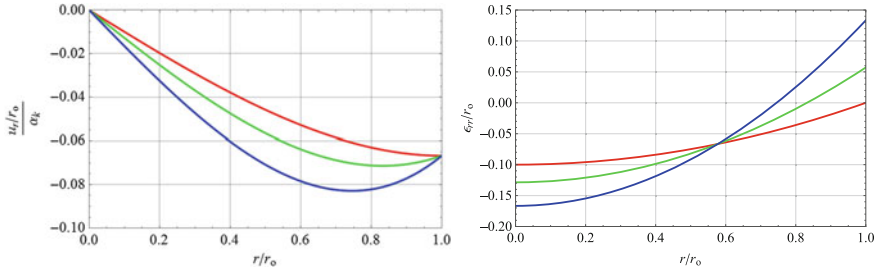


Fig. 2.1 Normalized displacement and radial strain versus dimensionless radius ($\nu = 0$ (red), $\nu = 0.3$ (green) and $\nu = 0.5$ (blue))

This result was first mentioned by Love in his books on linear elasticity, namely in Article 127 of [14] and later in Article 98 of [15] or [16]. An intuitive explanation for the necessity of its occurrence is as follows: Unlike a homogeneous, isotropic sphere subjected to a constant external pressure, the state of strain in our case is not homogeneous and not isotropic. We face a non-constant “external hydrostatic pressure,” so-to-speak, given by an effective gravitational force depending linearly on the distance from the center. However, there is also Poisson’s effect, i.e., the ability of a radial strain “making up” for the lateral contractions, $\epsilon_{\vartheta\vartheta}$ and $\epsilon_{\varphi\varphi}$, which are purely compressive in nature everywhere. The contractive force is proportional to r , the stretching stress (compensation of the lateral contraction) is proportional to r^2 . Hence from some r onward the force is not strong enough for compression, resulting in a transition from the negative to the positive, in other words in the existence of a Love radius. In his later editions Love does no longer comment on the physical significance of his result. However, in the first edition [14] we find a clue: “The greatest extension⁴ has place at the surface, and is equal to”:

$$\epsilon_{rr}|_{x=1} = \frac{\alpha}{5} \frac{\nu}{1+\nu}. \quad (2.1.37)$$

And he continues: “According to the theory of Poncelet and Saint-Venant (art. 57) the sphere will be certainly unable to resist the strain arising from its own gravitation if the breaking stress T_0 of the material be less than” $E \frac{\alpha}{5} \frac{\nu}{1+\nu}$. This makes it perfectly clear that Love is aiming at a failure criterion, namely specifically at what is known today as *maximum principal strain theory*, which goes back to Saint-Venant (cf., [9]). This is quite a daring concept, because Earth (say) is heterogeneous, and surely not perfectly linear elastic, and the materials it is made of might not be susceptible to strain-based failure, etc. But even if we accept his idea in principal, what is the proper Young’s modulus to be used for a heterogeneous object like Earth? On second glance, however, note that the factor α also contains Young’s modulus in its denominator (cf., Eq. (2.1.25)₁) since $\lambda + 2\mu = \frac{(1-\nu)E}{(1-2\nu)(1+\nu)}$. Thus we do not need to know E but

⁴Love refers to the radial strain as “extension.” Also note that we have rewritten the equations of the citation in modern form.

only Poisson's ratio, ν , which runs within well known bounds, namely $0 \leq \nu \leq 0.5$. Hence, for later evaluation, we rewrite Love's result as follows:

$$T_0 = \frac{\nu(1-2\nu)}{1-\nu} \frac{8\pi G \rho_0^2 r_o^2}{15}. \quad (2.1.38)$$

In conclusion we may say that the Love radius is a demarcation line beyond which potential failure in the celestial body may arise, which explains the interest in it in later research work (see Sect. 3.1.10).

A numerical examination of the influence of the α parameter for various terrestrial objects becomes possible with the physical data shown in Table 2.1. The data was first used to create the plots for the dimensionless displacements shown in Fig. 2.2. Note that the normalized displacement was evaluated at the surface, i.e., at r_o . As it was explained before it can be interpreted as an average radial strain. For this position the dependence on Poisson's ratio shown in the α_k containing expression of Eq. (2.1.29) vanishes *completely* and we do not need to worry about it. However, the dependence on k remains (see Eq. (2.1.25)₂) and opens up the question which effective bulk modulus (in other words which compressibility) for that particular object should actually be used. Since this parameter is not known we have decided to plot the dimensionless displacement for a reasonable range of bulk moduli. It should also be mentioned that most of the bodies mentioned in the table rotate. Because of that they turn from a sphere into a spheroid. The effect is very small. Thus we

Table 2.1 Necessary planetary data [29]

	a [m]	m [kg]	α [–]
Planets			
Mercury	2.4395E+06	3.300E+23	0.347
Venus	6.0520E+06	4.870E+24	1.993
Earth	6.3780E+06	5.970E+24	2.428
Mars	3.3960E+06	6.420E+23	0.349
Jupiter	7.1492E+07	1.898E+27	n/a
Saturn	6.0268E+07	5.680E+26	n/a
Uranus	2.5559E+07	8.680E+25	n/a
Neptune	2.4764E+07	1.020E+26	n/a
Pluto	1.1950E+06	1.310E+22	0.009
Moons			
Moon	1.738E+06	7.34E+22	0.066
Io	1.82E+06	8.932E+22	0.082
Europa	1.56E+06	4.8E+22	0.044
Ganymede	2.63E+06	1.482E+23	0.052
Callisto	2.41E+06	1.08E+23	0.039
Titan	2.576E+06	1.3452E+23	0.046

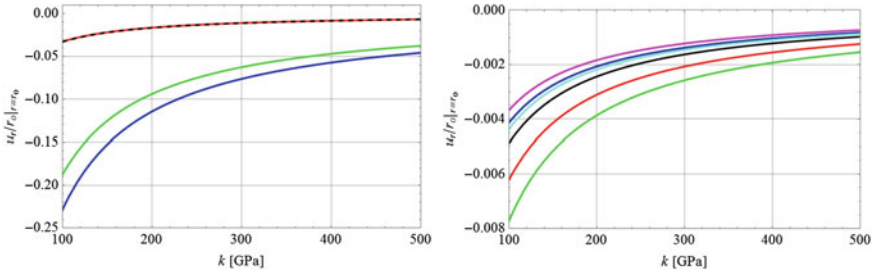


Fig. 2.2 Normalized displacement (average radial strain) on the periphery of various terrestrial bodies versus effective compressibility (for color code see text)

may identify the quantity a from the table, which is nothing else but the major axis of the corresponding spheroid, with the outer radius, r_0 . In order to evaluate Eqs. (2.1.26) and (2.1.29) we also approximate the reference density by the average density, $\rho_0 \approx \bar{\rho} = m_0/\frac{4\pi}{3}r_0^3$. As we shall see this is a reasonable approximation as long as the body is not too massive.

On the left of Fig. 2.2 we see curves for the inner planets (Mercury and Mars in red and dashed-black, respectively, nearly coinciding, Venus in green, and Earth in blue). On the right we see results for Earth's Moon (red), Io (green), Europa (blue), Ganymede (black), Callisto (magenta), and Titan (cyan). For the moons, Mercury, and Mars the normalized displacements are small enough in order to accept linear small strain elasticity as a viable tool for computing the deformation. In the case of Venus and of Earth the strains are *not* small. Rather they turn out to be of the order of 10 % and more, which is alarmingly high, to say the least.

Figure 2.3 illustrates the dependence of the breaking stress on Poisson's ratio according to Love, cf., Eq. (2.1.38). The color code in both insets corresponds exactly to that of Fig. 2.2. The conclusions are similar to the ones for the displacements before: The predicted stress is totally unrealistic for the planets Venus and Earth. For Mercury, Mars and for the various moon we obtain breaking stresses that are still rather high. Note that the tensile and flexural strength of most metals and ceramics is

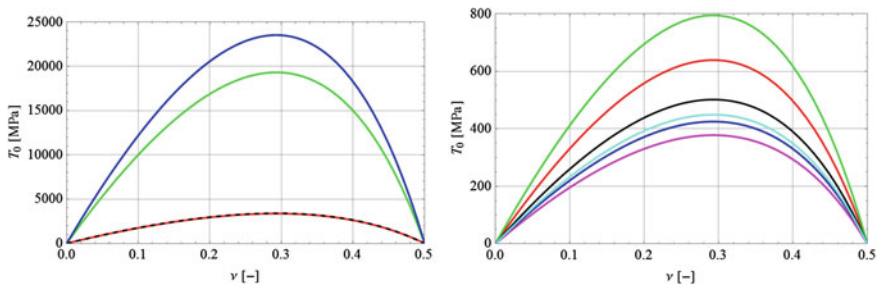


Fig. 2.3 Breaking stress versus Poisson's ratio (color code as in Fig. 2.2)

in the order of several hundred MPa, at most [1], so that failure due to self-gravitation according to Love's theory should be immanent. This is obviously not the case. We may say, that Love's efforts should be considered as the beginning of a future, more realistic failure prognosis.

Figure 2.4 illustrates the dependence of the normalized stresses on $x \equiv \frac{r}{r_o}$ for three different choices of Poisson's ratio, $\nu = 0$ (red), $\nu = 0.3$ (green), and $\nu = 0.5$ (blue). The maximum compression is at the body's center. Interestingly there is a cross-over point of the angular stresses. It is independent of Poisson's ratio and located at $r/r_o = \sqrt{1/2}$.

As to be expected, all principal stresses are of compressive nature. Their absolute value is dictated by the factor $\frac{3gm_0}{10A_0} \equiv \frac{2\pi G\rho_0^2 r_o^2}{15}$. In other words, they are of the same magnitude as the breaking stress, T_0 , from Eq. (2.1.38) and extremely large, in particular near the center of the body. However, unlike T_0 there is some physical significance to the size of the principal stresses: Matter deep inside a planet is indeed under an enormous pressure. In fact, according to the PREM model [3], pressures of more than 1400 GPa must be expected. In order to explore this a little further from the standpoint of linear elasticity we, first, define the pressure by:

$$p = -\frac{1}{3} (\sigma_{rr} + \sigma_{\vartheta\vartheta} + \sigma_{\varphi\varphi}), \quad (2.1.39)$$

and, second, evaluate this expression by using the relations from Eq. (2.1.26) in combination with Table 2.1. The results (for planet Earth) are shown in Fig. 2.5. Obviously the pressure at the center remains well below the one predicted by the PREM model. This is because PREM uses a non-homogeneous mass density distribution and our simple model does *not*.

We finally take a look at the resulting mass density. Eqs. (2.1.7), (2.1.30), and (2.1.31) allow us to compute the spatial distribution of the current mass density resulting from an originally homogeneous distribution, ρ_0 , after "gravity has been switched on:"

$$\frac{\rho}{\rho_0} = 1 + \frac{\alpha}{20} \left(3 \frac{3 - \nu}{1 + \nu} - 5x^2 \right). \quad (2.1.40)$$

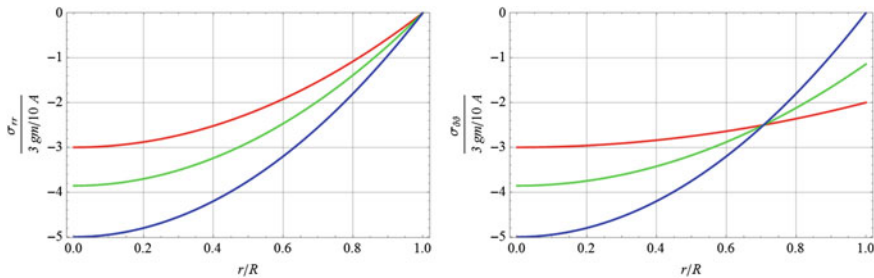


Fig. 2.4 Normalized stress components for three values of Poisson's ratio versus normalized radius (color code as in Fig. 2.1)

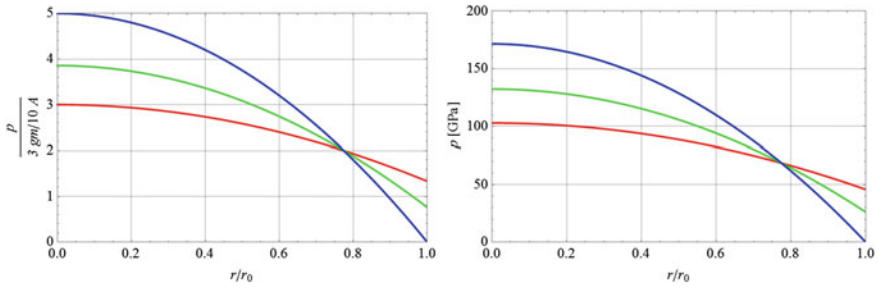


Fig. 2.5 Normalized and absolute pressure for three values of Poisson's ratio versus normalized radius (color code as in Fig. 2.1)

Note that the assumption underlying this result was a body force density completely based on the reference mass density, cf., Eq. (2.1.13). Due to the factor α , the current density depends on several parameters, namely mass and radius of the considered object, its effective Young's modulus, and Poisson's ratio. We choose the elastic constants of iron, i.e., $E = 210$ GPa and $\nu = 0.3$ in combination with the physical data of Table 2.1. The result is shown in Fig. 2.6. For comparison the corresponding density plot of the PREM model is shown in Fig. 2.7. On first glance one would think that the amplification of the density at the center of the Earth predicted by linear elasticity is a little smaller than the one from the PREM model (factors 1.75 and 2.4, respectively) but of similar magnitude. However, there is a fundamental difference between both plots. In Fig. 2.6 the *current* density was normalized by a homogeneous *reference* density, ρ_0 , whereas in Fig. 2.7 the *current average* density, $\rho_{\text{ave}} \equiv \bar{\rho}^E$, of Earth is used for normalization. Note that the reference density is *not* known. Indeed, we do know the mass of the Earth, which is a conserved quantity. On the other hand, we do not know the outer reference radius of Earth, R_0 , which, because of the self-gravitational contraction, will certainly be greater than r_0 . In short, a value for the ratio, $\rho_0 \equiv m_E / \frac{4\pi}{3} R_0^3$ cannot be calculated. On the other hand, if linear elasticity were to be trusted, we could argue that both outer radii are not too

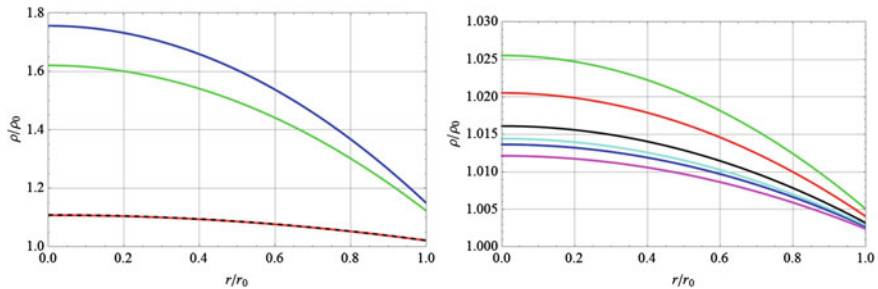
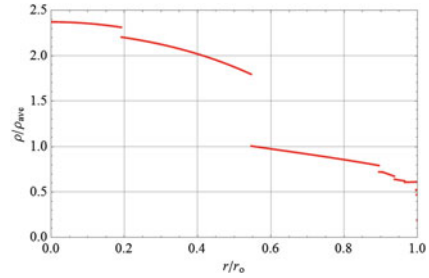


Fig. 2.6 Normalized current density versus normalized radius (color code as in Fig. 2.2)

Fig. 2.7 Normalized density distribution of the PREM model versus normalized radius



different and we might write $\rho_0 \approx m_E / \frac{4\pi}{3} r_o^3$. In fact, this is the normalizing factor that was used in the PREM model, $\rho_0 \equiv \rho_{\text{ave}} = 5514 \frac{\text{kg}}{\text{m}^3}$ with $m_E = 5.973 \times 10^{24} \text{ kg}$ and $r_o = 6371 \text{ km}$. However, this is an underestimate of the true circumstances: As we shall see the Earth is too massive to make this approximation. A homogeneous reference density would be considerably smaller than the current average density. This means that the amplification factor at the center shown in Fig. 2.7 should be multiplied by a number smaller than one, which would bring it closer to the relative density value of 1.75 predicted by a linear elastic model.

Moreover, there is another fundamental difference between the plots shown in Figs. 2.6 and 2.7. Creating a plot of the type shown in Fig. 2.7 requires a piece-wise constant reference density $\rho_0(X)$, i.e., more than a constant value is required. Such a multi-sphere approach is more complex, in particular in non-linear continuum mechanics, and we will get back to this issue in Chap. 5. However, for the time being we will now investigate the case of a bimaterial linear-elastic sphere.

2.1.4 The Bimaterial Sphere

Recall the general solution for the radial displacement from Eq. (2.1.21). The case of a bimaterial sphere obeys similar relations. They are the result of *two* differential equations, one for the inner core, $0 \leq r \leq r_i$, and one for the outer spherical shell, $r_i \leq r \leq r_o$. We start by integrating Poisson's equation (2.1.9) for the inner and outer region with constant mass densities $\rho_{0,i}$ and $\rho_{0,o}$, respectively:

$$\rho(r) \mathbf{f}(r) \approx -\frac{4\pi G \rho_{0,i}^2}{3} r \mathbf{e}_r, \quad 0 \leq r \leq r_i, \quad (2.1.41)$$

and

$$\rho(r) \mathbf{f}(r) \approx -\frac{4\pi G \rho_{0,o}^2}{3} \left[1 - \left(1 - \frac{\rho_{0,i}}{\rho_{0,o}} \right) \frac{r_i^3}{r^3} \right] r \mathbf{e}_r, \quad r_i \leq r \leq r_o. \quad (2.1.42)$$

The differential equations for the radial displacement follow analogously to Eqs. (2.1.14)–(2.1.20). They read in dimensionless form:

$$u'' + 2\frac{u'}{x} - 2\frac{u}{x^2} = \frac{\alpha_i}{2\xi^2}x, \quad 0 \leq x \leq \xi, \quad (2.1.43)$$

and

$$u'' + 2\frac{u'}{x} - 2\frac{u}{x^2} = \frac{\alpha_o}{2} \left[1 - (1 - \rho) \frac{\xi^3}{x^3} \right] x, \quad \xi \leq x \leq 1, \quad (2.1.44)$$

where the following abbreviations have been used:

$$\alpha_{i/o} = \frac{8\pi G \rho_{0,i/o}^2 r_{i/o}^2}{3(\lambda_{i/o} + 2\mu_{i/o})}, \quad x = \frac{r}{r_o}, \quad u = \frac{u_r}{r_o}, \quad \rho = \frac{\rho_{0,i}}{\rho_{0,o}}, \quad \xi = \frac{r_i}{r_o}. \quad (2.1.45)$$

At this point a remark is in order. Judging from the differential equations we may once more conclude that the gravitational force at a radial distance r is dictated by all the mass below that point. Obviously, in the case of the outer shell, this includes the total mass of the core. Hence, the question arises why a *local* field equation like Eq. (2.1.44) “knows” that this mass must be included. After all, this is a quantity undefined in the range of validity of the equation. The reason is that, strictly speaking, we have to solve the coupled problem of the balances of mass and momentum including Hooke’s law, plus Poisson’s equation, all of them specified for both regions. Both sets of equations then need to be complemented by suitable boundary and transition conditions, for the displacement, the radial stress, and for the gravitational potential. Due to the approximation shown in Eqs. (2.1.41) and (2.1.42) we were able to avoid the problem of finding the potentials and concentrate on the equations of equilibrium of forces alone, where continuity of the body force, i.e., the gradient of the potential at the transition point of both regions, was required. The latter also put a constraint on the right hand sides of Eqs. (2.1.43) and (2.1.44), such that the gravitational force within the shell “knows” about the presence of the core. Summarizing we may say that Newtonian gravity introduces a “long distance” effect in the mechanical continuum equations, which, in analogous form, is known in higher gradient theories of the stress tensor (for example). Hence, the body force has turned into a true constitutive equation.

In complete analogy to Eq. (2.1.21) the general solutions of the two differential equations read:

$$u(x) = A_i x + \frac{B_i}{x^2} + \frac{\alpha_i}{20} \frac{x^3}{\xi^2}, \quad 0 \leq x \leq \xi, \quad (2.1.46)$$

and

$$u(x) = A_o x + \frac{B_o}{x^2} + \frac{\alpha_o}{20} [x^3 + 5\xi^3 (1 - \rho)], \quad \xi \leq x \leq 1. \quad (2.1.47)$$

The four constants of integration, A_i , B_i , A_o , B_o , must now be adjusted suitably. They follow from the requirements for (i) regularity of the inner solution at $x = 0$, (ii) and (iii) continuity of the displacements and of the radial stresses at the transition between the core and the hollow sphere, i.e., at $x = \xi$, and (iv) a vanishing radial

stress at $x = 1$. In principal the solution can be obtained in closed form. However, it is quite unwieldy and does not immediately reveal the effects of the various parameters involved, such as the size of the inner core, and the elastic constants of both regions. We therefore refrain from writing it down explicitly. In order to demonstrate that the impact on displacements and stresses in a bimaterial sphere can be severe when compared to the homogeneous case, we assume that the inner core, with $r_i \approx 3480$ km ([3], p. 308 or [21], p. 76, $r_o = 6371$ km), is made of steel, $E_i = 210$ GPa, $\nu_i = 0.3$, $\rho_i = 10 \times 10^3 \frac{\text{kg}}{\text{m}^3}$, and the outer sphere consists of silicate rock, $E_o = 100$ GPa, $\nu_o = 0.17$, $\rho_o = 4.6 \times 10^3 \frac{\text{kg}}{\text{m}^3}$. The numerical result is shown in Fig. 2.8.

However, there is a special case for which a concise analytical solution can be obtained: The *rigid* core. In this case only one set of elastic parameters survives, namely those for the outer shell. Only two boundary conditions are required here, which read in the nomenclature of dimensionless parameters of Eq. (2.1.45):

$$u(\xi) = 0, \quad \sigma_{rr}(r_o) = 0 \quad \Rightarrow \quad u'(1) + \frac{2\nu}{1-\nu}u(1) = 0. \quad (2.1.48)$$

They allow us to obtain within the elastic shell region $\xi \leq x \leq 1$ assuming that $\rho_i = \rho_o$:

$$u(x) = \frac{\alpha}{20} \left(x^3 - \frac{3-\nu+2(1-2\nu)\xi^5}{1+\nu+2(1-2\nu)\xi^3} x + \frac{3-\nu-(1+\nu)\xi^2}{1+\nu+2(1-2\nu)\xi^3} \frac{\xi^3}{x^2} \right). \quad (2.1.49)$$

It is easy to verify that for $\xi = 0$ the old solution from Eq. (2.1.29)₁ results. In this context it should be mentioned that the case of a rigid core surrounded by a hollow sphere of linear elastic material has been subject of various publications. Pan [20] investigates the influence of linearly varying elastic material parameters for various Earth density models. Samanta [21] adds the complication of rotation and Chakravorty [2] allows for more complicated spatial variations of the shear modulus.

Figure 2.9 shows a comparison between the location of the Love radius as predicted for the homogeneous case by Eq. (2.1.36) and from a numerical solution of

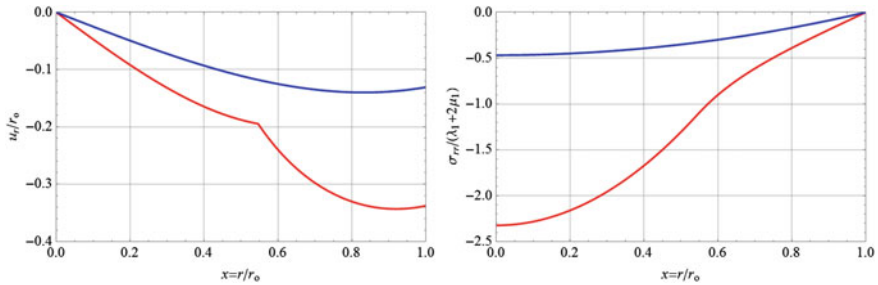
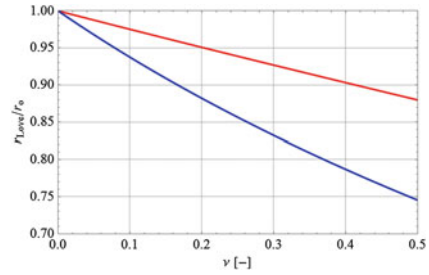


Fig. 2.8 Radial displacement and stress versus normalized radius: Homogeneous case (*blue*) and bimaterial sphere (*red*); for data see text

Fig. 2.9 Love radius versus Poisson's ratio:
Homogeneous case (*blue*)
and rigid core solution (*red*)



Eq. (2.1.49) for the aforementioned choice $r_i \approx 3480$ km. The difference is obviously considerable.

2.1.5 Body Force with Current Mass Density

In this subsection we shall walk on very thin ice, so-to-speak, because we shall now depart from one of the crucial prerequisites of the linear theory of elasticity at small deformations: We shall no longer insist that the mass density in the expression for the body force is a constant. In agreement with the principles of a consistently linearized theory, as outlined in context with Eq. (1.2.9), but departing from the idea that the forces act on the undeformed configuration, we will now take *linear* terms of the displacement and its derivative on the right hand side of the momentum balance into account according to Eqs. (2.1.7) and (2.1.11). Thus Eq. (2.1.13) will be replaced by:

$$\begin{aligned} \rho(r) \mathbf{f}(r) &\approx -\frac{4\pi G \rho_0^2}{3} r \left(1 - u'_r - 5 \frac{u_r}{r}\right) \mathbf{e}_r \\ &\equiv -\frac{\lambda+2\mu}{r_0} \frac{\alpha}{2} x \left(1 - u' - 5 \frac{u}{x}\right) \mathbf{e}_r \end{aligned} \quad (2.1.50)$$

with

$$\alpha = \frac{8\pi G \rho_0^2 r_0^2}{3(\lambda + 2\mu)}, \quad x = \frac{r}{r_0}, \quad u = \frac{u_r}{r_0}. \quad (2.1.51)$$

The equivalent to Eqs. (2.1.20) or (2.1.28) then reads:

$$u'' + \left(\frac{2}{x} + \frac{\alpha}{2}x\right) u' - \left(\frac{2}{x^2} - \frac{5\alpha}{2}\right) u = \frac{\alpha}{2}x. \quad (2.1.52)$$

This differential equation deserves a detailed comment. Its original left hand side (cf., Eq. (2.1.28)) consisted of three small quantities of first order, namely of u and its spatial derivatives. Consequently, its right hand side should also be small, i.e., consist of first order terms comparable in magnitude to $u(x)$, $u'(x)$, and $u''(x)$. This

concerns first of all the term $\frac{\alpha}{2}x$, and we must ask the question as to whether it is “small” or not. Unfortunately the size of α is dictated by physics and not by mathematics. It cannot really be chosen freely and it is as large as required by the celestial object we wish to model. Inspection of the last column of Table 2.1 (which was generated for $E = 210$ GPa and $\nu = 0.3$) shows that only Pluto and a few of the moons show α -values in the order of a few percent, which is “small” in the sense of linear elasticity theory. Under these circumstances the α -terms in the parentheses of Eq. (2.1.52) are of second order and should rightfully be omitted. This would take us back to Eq. (2.1.28), which shows that it is indeed the proper equation for linear elasticity with small strains and for small amounts of self-gravity. However, if self-gravity is as strong as in the case of Mercury (say) then the two additional terms in the parentheses should be included as first order terms. But then the right hand of Eq. (2.1.52) is not of zeroth order and no longer “linear,” which renders the validity and applicability Eq. (2.1.52) questionable.

Of course, the question must be asked why under such circumstances we should not directly turn to the corresponding large strain equation: After all, Eq. (2.1.52) does not make a distinction between current and reference positions, r and R , respectively, and this is most definitely as important as the α -issue. However, there is an answer to that: Eq. (2.1.52) can still be solved in closed form, which makes a numerical evaluation for arbitrary values of α easily possible. As we shall learn in Chap. 3 this is not so in the case of nonlinear deformations. We shall see later that, first, only a numerical solution is possible and, second, good convergence can be achieved easily only if α is not too large. In short, it is for this reason why we will now study the closed-form solutions of Eq. (2.1.52) as a model that goes one step beyond pure linear elasticity at small deformations. Following the tradition of mathematicians we may want to speak of an “extended” model.

The solution is similar to Eq. (2.1.21): Two constants of integration appear, one of which, B , must be equal to zero in order to avoid singularities. Hence, only one constant, A , remains and we find that:

$$u = \frac{1}{96\alpha} \frac{1}{x^2} \left[2x (6 + 5\alpha x^2 + 6A (2 - \alpha x^2)) - 3(1 + 2A) (4 + 4\alpha x^2 - \alpha^2 x^4) \exp\left(-\frac{\alpha x^2}{4}\right) \int_{\tau=0}^{\tau=x} \exp\left(\frac{\alpha \tau^2}{4}\right) d\tau \right]. \quad (2.1.53)$$

In order to compare this result with the previous solution shown in Eq. (2.1.21) we expand in α and find:

$$u = Ax + \frac{\alpha}{20} (1 - 6Ax^3) + \mathcal{O}[\alpha^{3/2}]. \quad (2.1.54)$$

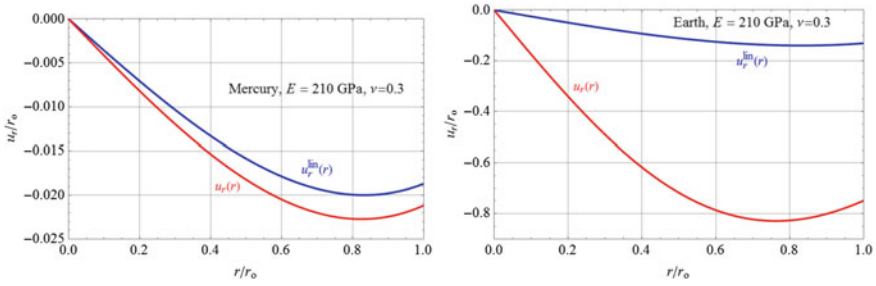


Fig. 2.10 Radial displacement versus normalized radius: fully linear solution (*blue*) and “extended” solution (*red*); for data see text

In other words: As it should be, we obtain the old result provided α is small since A is supposed to be small in linear elasticity. The remaining constant A follows from the requirement of vanishing traction, i.e., vanishing radial stress at the outside, r_o , of the sphere, $u'|_{x=1} + \frac{2\nu}{1-\nu} u|_{x=1} = 0$. The result is a very lengthy expression, which we decided not to present here. However, it should be mentioned that in the limit we obtain:

$$A = -\frac{\alpha}{20} \frac{3-\nu}{1+\nu} + \mathcal{O}[\alpha^{3/2}], \quad (2.1.55)$$

in other words, we end up with the old result from Eqs. (2.1.22)₂ and (2.1.29), as it should be.

Figure 2.10 gives us a foretaste of what to expect if the gravitating mass becomes really large. Two curves are presented, the fully linear solution according to Eq. (2.1.29) in blue and the “extended” solution based on Eq. (2.1.53) in red. The equations were evaluated for Mercury and for Earth according to the data presented in Table 2.1. Although α is already equal to 0.347 in the case of Mercury (i.e., it is *not* small but in the range of more than 30 percent) the difference the two predictions for the normalized displacement are close together, and in the range of 2%. This is completely different for the case of Earth. Both predictions are far apart and we should expect normalized displacements of almost 90%. We will get back to this issue in Chap. 3.

2.2 Rotating Objects: Fundamental Relations

So far we have studied a problem of perfect spherical symmetry: Newtonian self-gravitation being the only force acting on initially homogeneously arranged matter within a sphere. Note that the spherical coordinates used in the equations presented in Sect. 2.1.2 could refer either to an inertial frame, e_i , or to a frame, e'_i , co-moving with the rotating Earth. The equations would retain their form since classical Newtonian

gravity is unaffected by the state of motion. For simplicity, in both cases the center of the frames would be located in the sphere's center of mass.

However, since the Earth is spinning at a roughly 24h time period, centrifugal accelerations will redistribute matter in combination with self-gravitation. We shall see that the deformations associated with rotation are much smaller than those due to self-gravity. For this reason a linear-elastic analysis of the rotational effect is adequate and sufficient. In fact, we may consider the deformation due to self-gravity, no matter if it is calculated by a linear or a non-linear approach, as a new state of reference onto which additional deformation due to centrifugal accelerations is imposed.

2.2.1 Kinematics and Dynamics of Self-Gravity and “Inertial Forces”

Consider a Euclidian transformation $\mathbf{x}' = \mathbf{Q} \cdot \mathbf{x} + \mathbf{b}'$, $\mathbf{Q}$ being an orthogonal, time-dependent rotation, $\mathbf{Q} \cdot \mathbf{Q}^\top = \mathbf{Q}^\top \cdot \mathbf{Q} = \mathbf{1}$, which maps the orthogonal unit base of an inertial system, $\mathbf{e}_k$, onto that of a non-inertial frame, $\mathbf{e}'_i(t)$, $i, k \in \{1, 2, 3\}$ (cf., Fig. 2.11; see the classic textbooks [26], Sect. 143 and [25], Sect. 19, or the modern references [6], p. 31, [8], Sect. 4.3, [13], Sect. 2.6, or [18], p. 183 for this and for the following remarks). The origins of the two systems are separated by the vector $\mathbf{b}'$ as shown in the figure. We may write for the relations between the velocities, $\mathbf{v} = \dot{\mathbf{x}}$ and $\mathbf{v}' = \dot{\mathbf{x}'}$, and the accelerations, $\mathbf{a} = \ddot{\mathbf{x}}$ and $\mathbf{a}' = \ddot{\mathbf{x}'}$, in both systems:

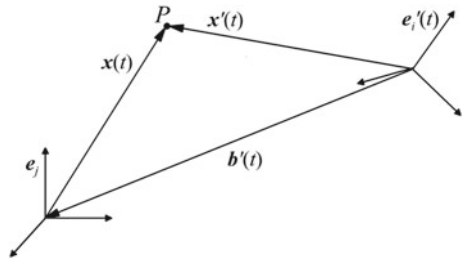
$$\mathbf{x}' = \mathbf{Q} \cdot \mathbf{x} + \mathbf{b}' \Rightarrow \mathbf{v}' = \mathbf{Q} \cdot \mathbf{v} + \dot{\mathbf{Q}} \cdot (\mathbf{x}' - \mathbf{b}') + \dot{\mathbf{b}}' \Rightarrow \quad (2.2.1)$$

$$\mathbf{a}' = \mathbf{Q} \cdot \mathbf{a} + 2\dot{\mathbf{Q}} \cdot (\mathbf{v}' - \dot{\mathbf{b}}') + (\ddot{\mathbf{Q}} - \dot{\mathbf{Q}} \cdot \dot{\mathbf{Q}}) \cdot (\mathbf{x}' - \mathbf{b}') + \ddot{\mathbf{b}}',$$

where the spin tensor, $\dot{\mathbf{Q}} := \dot{\mathbf{Q}} \cdot \mathbf{Q}^\top$, has been introduced.

It is customary to introduce the dual to the spin tensor, the angular velocity vector, $\boldsymbol{\omega}$, by $\dot{\mathbf{Q}} \cdot \mathbf{y}' = \boldsymbol{\omega} \times \mathbf{y}'$, where $\mathbf{y}'$ is an arbitrary vector, e.g., those related to the moving system from above. Then we obtain:

Fig. 2.11 Euclidian transformation



$$\begin{aligned}\mathbf{x}' &= \mathbf{Q} \cdot \mathbf{x} + \mathbf{b}' \Rightarrow \mathbf{v}' = \mathbf{Q} \cdot \mathbf{v} + \boldsymbol{\omega} \times (\mathbf{x}' - \mathbf{b}') + \dot{\mathbf{b}}' \Rightarrow \\ \mathbf{a}' &= \mathbf{Q} \cdot \mathbf{a} + 2\boldsymbol{\omega} \times (\mathbf{v}' - \dot{\mathbf{b}}') + \dot{\boldsymbol{\omega}} \times (\mathbf{x}' - \mathbf{b}') - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{x}' - \mathbf{b}')) + \ddot{\mathbf{b}}'.\end{aligned}\quad (2.2.2)$$

Thus the balance of momentum for the non-inertial system reads (see, e.g., [26], p. 437 or [13], p. 59):

$$\rho' \frac{d\mathbf{v}'}{dt} = \nabla' \cdot \boldsymbol{\sigma}' + \rho' (\mathbf{f}' + \mathbf{i}'), \quad (2.2.3)$$

where the inertial accelerations were collected in the following term:

$$\mathbf{i}' = 2\boldsymbol{\Omega} \cdot (\mathbf{v}' - \dot{\mathbf{b}}') + (\dot{\boldsymbol{\Omega}} - \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}) \cdot (\mathbf{x}' - \mathbf{b}') + \ddot{\mathbf{b}}', \quad (2.2.4)$$

or if a representation with the angular velocity vector is preferred:

$$\mathbf{i}' = 2\boldsymbol{\omega} \times (\mathbf{v}' - \dot{\mathbf{b}}') + \dot{\boldsymbol{\omega}} \times (\mathbf{x}' - \mathbf{b}') - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{x}' - \mathbf{b}')) + \ddot{\mathbf{b}}'. \quad (2.2.5)$$

We assume that the Earth rotates at a constant angular speed, i.e., $\dot{\boldsymbol{\omega}} = \mathbf{0}'$, as well as stationary conditions, i.e., $\mathbf{v}' = \mathbf{0}'$. Moreover, the origins of the two systems shall coincide, i.e., $\mathbf{b}' = \mathbf{0}'$. Consequently:

$$\nabla' \cdot \boldsymbol{\sigma}' + \rho' (\mathbf{f}' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}')) = \mathbf{0}'. \quad (2.2.6)$$

Potentials can be used to obtain the gravitational as well as the centrifugal acceleration by differentiation w.r.t. position, i.e., $\mathbf{f}' = -\nabla' U^g$ and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}') = \nabla' U'^\omega$ and therefore:

$$\nabla' \cdot \boldsymbol{\sigma}' = \rho' \nabla (U^g + U'^\omega). \quad (2.2.7)$$

We use Cartesian coordinates in the non-inertial frame (i.e., co-moving ones, which explains the dash), such that $\boldsymbol{\omega} = \omega_0 \mathbf{e}'_3$, $\omega_0 = \text{const.}$ Note that in our case we have in particular $\mathbf{e}'_3 \equiv \mathbf{e}_3$, independently of time. In the end we obtain (cf., [5], p. 42):

$$\nabla' U'^\omega = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}') = -\omega_0^2 (x'_1 \mathbf{e}'_1 + x'_2 \mathbf{e}'_2) \Rightarrow \quad (2.2.8)$$

$$U'^\omega = -\frac{1}{2}\omega_0^2 (x_1'^2 + x_2'^2) \equiv -\frac{1}{2}\omega_0^2 r'^2 \sin^2 \vartheta',$$

where in the last step we switched from co-moving Cartesian to co-moving spherical coordinates.

The Poisson equation in the moving system reads:

$$\Delta' U^g(\mathbf{x}') = 4\pi G \rho'(\mathbf{x}'), \quad (2.2.9)$$

and since the mass density is a Euclidean scalar, $\rho'(\mathbf{x}') = \rho(\mathbf{x})$, where $\mathbf{x}'$ and $\mathbf{x}$ represent the same material point, so is the gravitational potential, $U^g(\mathbf{x}') = U^g(\mathbf{x})$. In other words, we expect the functional form for the potential stemming from the

solution of Eq. (2.2.9) in the moving system to be the same as in the inertial one stemming from a solution of Eq. (2.1.3). Note that due to the spinning the masses of the originally spherical Earth will redistribute and form an ellipsoid. This means that the solution shown in Eq. (2.1.9) is not applicable any more and a dependence of the polar angle will appear. However, recall that within the framework of linear elasticity we must ignore this effect in order to comply with the principle of using body forces applied to the *undeformed* system. Hence, we will still make use of Eq. (2.1.13):

$$\rho'(r') \mathbf{f}'(r') \approx -G \frac{\rho_0 m(r')}{r'^2} \mathbf{e}'_r \approx -\frac{4\pi G \rho_0^2}{3} r' \mathbf{e}'_r. \quad (2.2.10)$$

For convenience we will from now on no longer add a dash to the symbols. The evaluation in a co-moving frame is automatically implied.

Due to the symmetry of the problem, derivatives w.r.t. φ cannot prevail and Eq. (2.1.14) must now be replaced by:

$$\begin{aligned} & \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\vartheta}}{\partial \vartheta} + \frac{1}{r} (2\sigma_{rr} - \sigma_{\vartheta\vartheta} - \sigma_{\varphi\varphi} + \sigma_{r\vartheta} \cot \vartheta) \\ &= \rho_0 \left(\frac{4\pi G \rho_0}{3} - \omega_0^2 \right) r + \rho_0 \omega_0^2 r \cos^2 \vartheta, \\ & \frac{\partial \sigma_{r\vartheta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\vartheta\vartheta}}{\partial \vartheta} + \frac{1}{r} [3\sigma_{r\vartheta} + (\sigma_{\vartheta\vartheta} - \sigma_{\varphi\varphi}) \cot \vartheta] \\ &= -\rho_0 \omega_0^2 r \sin \vartheta \cos \vartheta, \\ & \frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\vartheta\varphi}}{\partial \vartheta} + \frac{1}{r} (3\sigma_{r\varphi} + 2\sigma_{\vartheta\varphi} \cot \vartheta) = 0. \end{aligned} \quad (2.2.11)$$

Similarly, Hooke's law into which kinematic conditions have been inserted reduces to the form:

$$\begin{aligned} \sigma_{rr} &= \lambda \Delta + 2\mu \frac{\partial u_r}{\partial r}, \quad \sigma_{\vartheta\vartheta} = \lambda \Delta + \frac{2\mu}{r} \left(\frac{\partial u_{\vartheta}}{\partial \vartheta} + u_r \right), \\ \sigma_{\varphi\varphi} &= \lambda \Delta + \frac{2\mu}{r} (u_r + u_{\vartheta} \cot \vartheta), \quad \sigma_{r\vartheta} = \mu \left[\frac{\partial u_{\vartheta}}{\partial r} - \frac{1}{r} \left(u_{\vartheta} - \frac{\partial u_r}{\partial \vartheta} \right) \right], \\ \sigma_{\vartheta\varphi} &= \frac{\mu}{r} \left(\frac{\partial u_{\varphi}}{\partial \vartheta} - u_{\varphi} \cot \vartheta \right), \quad \sigma_{r\varphi} = \mu \left(\frac{\partial u_{\varphi}}{\partial r} - \frac{1}{r} u_{\varphi} \right), \end{aligned} \quad (2.2.12)$$

with the following abbreviation:

$$\Delta = \frac{1}{r^2 \sin \vartheta} \left[\frac{\partial}{\partial r} (r^2 u_r \sin \vartheta) + \frac{\partial}{\partial \vartheta} (r u_{\vartheta} \sin \vartheta) \right]. \quad (2.2.13)$$

Mutual insertion results in three, partially coupled, partial differential equations which can be solved by means of Legendre series. The details can be found in Müller and Lofink [19].

2.2.2 Explicit Forms for the Stresses and Displacements

The final results for the stresses read:

$$\begin{aligned}\sigma_{rr} &= - \left[\frac{1}{10} \frac{3-\nu}{1-\nu} \left(\frac{g}{a_c} - \frac{2}{3} \right) + \frac{2}{3} \frac{3+2\nu}{7+5\nu} P_2 \right] \left(1 - \frac{r^2}{r_o^2} \right) \rho_0 \omega_0^2 r_o^2, \\ \sigma_{\vartheta\vartheta} &= - \left(\frac{1}{10} \frac{3-\nu}{1-\nu} \left(\frac{g}{a_c} - \frac{2}{3} \right) \left(1 - \frac{1+3\nu}{3-\nu} \frac{r^2}{r_o^2} \right) \right. \\ &\quad \left. + \frac{3+2\nu}{3(7+5\nu)} \left[2P_2 \left(1 - \frac{1}{3+2\nu} \frac{r^2}{r_o^2} \right) + \frac{d^2 P_2}{d\vartheta^2} \left(1 - \frac{2+\nu}{3+2\nu} \frac{r^2}{r_o^2} \right) \right] \right) \rho_0 \omega_0^2 r_o^2, \\ \sigma_{\varphi\varphi} &= - \left(\frac{1}{10} \frac{3-\nu}{1-\nu} \left(\frac{g}{a_c} - \frac{2}{3} \right) \left(1 - \frac{1+3\nu}{3-\nu} \frac{r^2}{r_o^2} \right) \right. \\ &\quad \left. + \frac{1}{3} \left[\left(\frac{2(1+\nu)}{4+5\nu} P_2 + \frac{2+\nu}{7+5\nu} \right) \frac{r^2}{r_o^2} - \frac{3+2\nu}{7+5\nu} \right] \right) \rho_0 \omega_0^2 r_o^2 \\ \sigma_{r\vartheta} &= - \frac{3+2\nu}{3(7+5\nu)} \frac{dP_2}{d\vartheta} \left(1 - \frac{r^2}{r_o^2} \right) \rho_0 \omega_0^2 r_o^2, \sigma_{r\varphi} = 0, \sigma_{\vartheta\varphi} = 0,\end{aligned}\tag{2.2.14}$$

and for the displacements:

$$\begin{aligned}u_r &= - \frac{\rho_0 \omega_0^2 r_o^2}{E} \frac{(1+\nu)(1-2\nu)}{1-\nu} \left[\frac{1}{10} \left(\frac{g}{a_c} - \frac{2}{3} \right) \left(\frac{3-\nu}{1+\nu} - \frac{r^2}{r_o^2} \right) \right. \\ &\quad \left. + \frac{2(1-\nu)}{3(1-2\nu)} P_2 \left(\frac{3+2\nu}{7+5\nu} - \frac{1+\nu}{7+5\nu} \frac{r^2}{r_o^2} \right) \right] r, \\ u_{\vartheta} &= - \frac{\rho_0 \omega_0^2 r_o^2}{3E} (1+\nu) \frac{dP_2}{d\vartheta} \left(\frac{3+2\nu}{7+5\nu} - \frac{2+\nu}{7+5\nu} \frac{r^2}{r_o^2} \right) r, \quad u_{\varphi} = 0,\end{aligned}\tag{2.2.15}$$

where $P_2(\cos\vartheta) = \frac{1}{2}(3\cos^2\vartheta - 1)$ is the Legendre polynomial of second degree. For further investigations it is advantageous to use Young's modulus, E , and Poisson's ratio, ν , instead of the Lamé constants, $\lambda = \frac{\nu}{(1-2\nu)(1+\nu)} E$, $2\mu = \frac{E}{1+\nu}$. Moreover, we have defined the gravitational and centrifugal accelerations at the outer (equatorial) surface, $r = r_o$, by:

$$g = \frac{Gm}{r_o^2}, \quad a_c = r_o \omega_o^2. \quad (2.2.16)$$

As we shall see it is instructive to divide the stresses and displacements into purely gravitational and centrifugal bits identifiable by superscripts ‘g’ and ‘c’, respectively:

$$\sigma_{ij} = \sigma_{ij}^g + \sigma_{ij}^c, \quad u_i = u_i^g + u_i^c, \quad i, j \in \{r, \vartheta, \varphi\}, \quad (2.2.17)$$

where the normal stresses of σ_{ij}^g and displacements u_r^g are given by Eqs. (2.1.26) and (2.1.24), respectively, and all other components vanish, i.e., $\sigma_{r\vartheta}^g = 0$, $\sigma_{r\varphi}^g = 0$, $\sigma_{\vartheta\varphi}^g = 0$, $u_\vartheta^g = 0$, $u_\varphi^g = 0$. In the same context Eq. (2.1.33) should be observed.

Moreover, the centrifugal parts read explicitly:

$$\begin{aligned} \sigma_{rr}^c &= \left(\frac{1}{10} - \frac{3+2\nu}{7+5\nu} P_2 \right) \frac{2}{3} \left(1 - \frac{r^2}{r_o^2} \right) \rho_0 \omega_o^2 r_o^2, \\ \sigma_{\vartheta\vartheta}^c &= \left(\frac{1}{15} \frac{3-\nu}{1-\nu} \left(1 - \frac{1+3\nu}{3-\nu} \frac{r^2}{r_o^2} \right) - \frac{3+3\nu}{3(7+5\nu)} \left[2P_2 \left(1 - \frac{1}{3+2\nu} \frac{r^2}{r_o^2} \right) \right. \right. \\ &\quad \left. \left. + \frac{d^2 P_2}{d\vartheta^2} \left(1 - \frac{2+\nu}{3+2\nu} \frac{r^2}{r_o^2} \right) \right] \right) \rho_0 \omega_o^2 r_o^2, \\ \sigma_{\varphi\varphi}^c &= \left(\frac{1}{15} \frac{3-\nu}{1-\nu} \left(1 - \frac{1+3\nu}{3-\nu} \frac{r^2}{r_o^2} \right) \right. \\ &\quad \left. - \frac{1}{3} \left[\left(\frac{2(1+\nu)}{7+5\nu} P_2 + \frac{2+\nu}{7+5\nu} \right) \frac{r^2}{r_o^2} - \frac{3+2\nu}{7+5\nu} \right] \right) \rho_0 \omega_o^2 r_o^2, \\ \sigma_{r\vartheta}^c &= -\frac{3+2\nu}{3(7+5\nu)} \frac{dP_2}{d\vartheta} \left(1 - \frac{r^2}{r_o^2} \right) \rho_0 \omega_o^2 r_o^2, \quad \sigma_{r\varphi}^c = 0, \quad \sigma_{\vartheta\varphi}^c = 0, \\ u_r^c &= \frac{\omega_o^2 m}{2\pi E} \frac{(1+\nu)(1-2\nu)}{1-\nu} \left[\frac{1}{10} \left(\frac{3-\nu}{1+\nu} - \frac{r^2}{r_o^2} \right) \right. \\ &\quad \left. - \frac{1-\nu}{1-2\nu} P_2 \left(\frac{3+2\nu}{7+5\nu} - \frac{1+\nu}{7+5\nu} \frac{r^2}{r_o^2} \right) \right] \frac{r}{r_o}, \\ u_\vartheta^c &= -\frac{\omega_o^2 m}{4\pi E} (1+\nu) \frac{dP_2}{d\vartheta} \left(\frac{3+2\nu}{7+5\nu} - \frac{2+\nu}{7+5\nu} \frac{r^2}{r_o^2} \right) \frac{r}{r_o}, \quad u_\varphi^c = 0. \end{aligned} \quad (2.2.18)$$

2.2.3 Evaluation and Discussion of the Results

It has already been indicated that the gravitational part of the linear-elastic solution is dominant in comparison to the centrifugal accelerations. Figure 2.12 illustrates the situation by showing the behavior of all **combined stresses** according to Eq. (2.2.14) as a function of radial position for various values of Poisson's ratio. In fact, the plots for the normal stresses were generated for the equatorial plane, i.e., by choosing $\vartheta = \pi/2$, and the one for the shear stress at $\vartheta = \pi/4$ in order to show the maximum values. For the numerical evaluation we have chosen the observed mean radius of the Earth, i.e., $r_o \equiv \bar{R}_E = 6.371 \times 10^6$ m, $\omega_{0,E} = 2\pi/86164$ s, and $m_E = 5.97 \times 10^{24}$ kg [28]. Thus we have $g = 9.81 \frac{\text{m}}{\text{s}^2}$, $a_c = 0.034 \frac{\text{m}}{\text{s}^2}$, and $\frac{g}{a_c} = 298.7$. These numbers already indicate the dominance of gravitation. In fact, in the case of the normal stresses it turns out that the gravitational parts in Eq. (2.2.14) are so strong that they conceal the dependence on the polar angle almost completely. All normal stresses are highly compressive. Note the striking similarity to the plots shown in Fig. 2.4 and the very slight difference between the two angular stresses. Both emphasizes our point that gravitation is dominant. Moreover, the shear stress depends hardly on Poisson's ratio.

Figure 2.13 illustrates the behavior of all stress components as given by Eq. (2.2.18), i.e., the action of centrifugal acceleration only. The radial as well as the shear stress show hardly any dependence on Poisson's ratio. Their behavior is depicted in Fig. 2.13_{1,4}. σ_{rr}^c was evaluated along the equator at $\vartheta = \pi/2$ and along the radius leading to the pole, i.e., $\vartheta = 0$, giving positive and negative values, respectively, as intuitively expected. $\sigma_{r\vartheta}^c$ was evaluated for $\vartheta = \pi/4$ at the location of maximum values. The angular normal stresses show a distinct dependence on ν . They

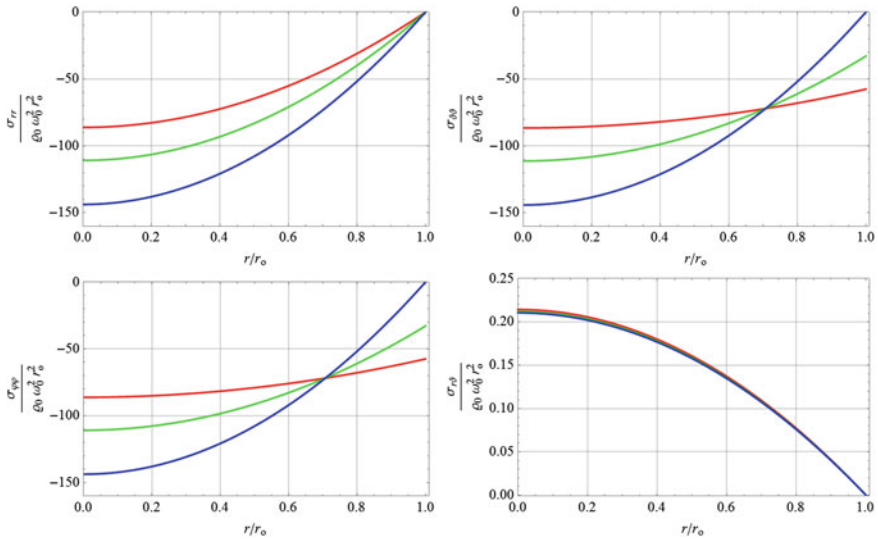


Fig. 2.12 Stress components for combined loading (color code as in Fig. 2.1)

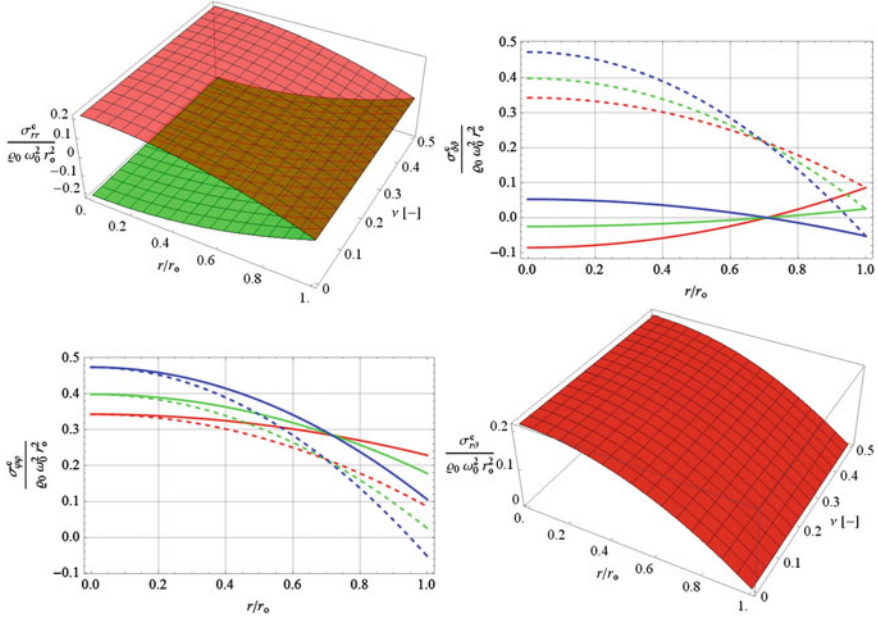


Fig. 2.13 Behavior of the stress components for centrifugal acceleration (color code as in Fig. 2.1)

were evaluated for three different choices of Poisson's ratio, $\nu = 0$ (red), $\nu = 0.3$ (green), and $\nu = 0.5$ (blue) at $\vartheta = \pi/2$ (solid lines) and $\vartheta = 0$ (dashed lines).

Turning now to the centrifugal displacement components the first two pictures in Fig. 2.14 show the non-vanishing dimensionless components of the centrifugal part of the displacement according to Eq. (2.2.18) for three different values of Poisson's ratio, namely $\nu = 0$ (red), $\nu = 0.3$ (green), and $\nu = 0.5$ (blue). u_r^c was evaluated at the equator, i.e., $\vartheta = \pi/2$, and for the pole, i.e., $\vartheta = 0$. This leads to positive and to negative values, respectively, which makes sense in view of the effect of the centrifugal acceleration on a deformable body (extension perpendicular to the axis of rotation accompanied by lateral contraction). u_ϑ^c was evaluated at the equator, i.e., $\vartheta = \pi/4$, where it assumes its extremum. We now concentrate upon the centrifugal displacement components when evaluated on the surface of the rotating sphere:

$$\left. \frac{u_r^c}{r_o} \right|_{r=r_o} = \frac{\omega_0^2 m}{2\pi E} \left[\frac{1}{5}(1 - 2\nu) - \frac{(1 + \nu)(2 + \nu)}{7 + 5\nu} P_2 \right], \quad (2.2.19)$$

$$\left. \frac{u_\vartheta^c}{r_o} \right|_{r=r_o} = -\frac{\omega_0^2 m}{2\pi E} \frac{(1 + \nu)^2}{7 + 5\nu} \frac{dP_2}{d\vartheta}.$$

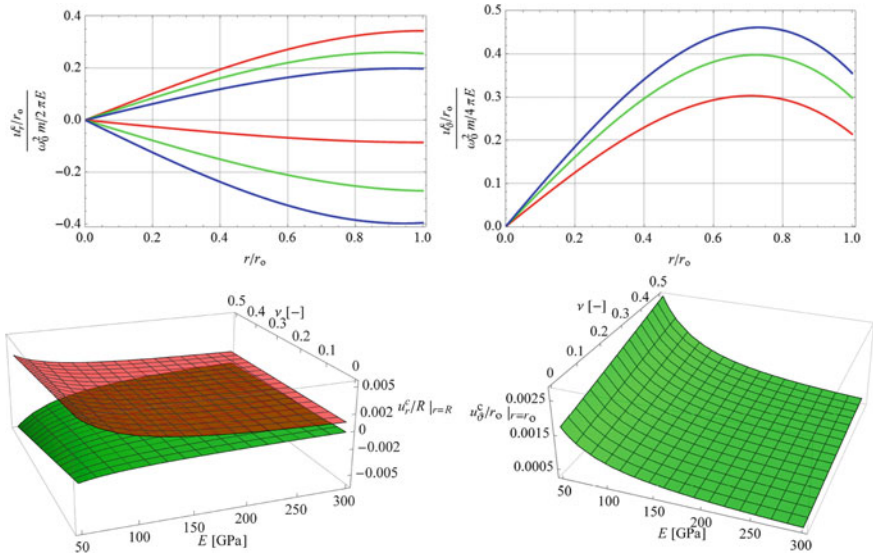


Fig. 2.14 Behavior of the displacement components for centrifugal acceleration (see text)

The third and fourth picture in Fig. 2.14 illustrate these relationships when evaluated in the equatorial plane of the Earth, $u_r^c/r_o|_{\vartheta=\pi/2}$, (positive values due to centrifugal acceleration), in the polar direction, $u_r^c/r_o|_{\vartheta=0}$, (negative values due to lateral contraction, i.e., the Poisson effect), and at 45°, $u_\vartheta^c/r_o|_{\vartheta=\pi/4}$, using Earth data from Table 2.1 (with $r_o = a$) for physically reasonable ranges of Young's moduli and Poisson's ratios. Obviously, all values stay below the 1% threshold and, hence, the message is that linear elasticity may be used to describe the centrifugal displacements and stresses even in the case of the Earth. The so-called flattening parameter, f , is defined by the ratio $f = \frac{a-c}{a}$, a and c being the radial distances from the Earth's center to the equator and to the pole, respectively. By using Eq. (2.2.19)₁ we compute the flattening as follows:

$$a = r_o + u_r(r = r_o, \vartheta = \pi/2), \quad c = r_o + u_r(r = r_o, \vartheta = 0) \quad \Rightarrow \quad (2.2.20)$$

$$f \approx \frac{2\rho_0 r_o^2 \omega_0^2}{3E} \frac{(1+\nu)(2+\nu)}{7+5\nu} [P(1) - P(0)] = \frac{\rho_0 r_o^2 \omega_0^2}{E} \frac{(1+\nu)(2+\nu)}{7+5\nu} \equiv \frac{\rho_0 r_o^2 \omega_0^2}{\mu} \frac{1+\nu/2}{7+5\nu},$$

if we neglect higher order terms in u_r as we should within the framework of a linear theory. Note that we may interpret r_o as the reference radius from which the displacement due to centrifugal actions is counted. And this reference position could be calculated within the framework of a linear-elastic theory (as in this section) or nonlinearly as in Chap. 3.

Indeed, Eq. (2.2.20) is the result originally presented by Thomson and Tait [23], p. 432. However, if one is interested in a *quantitative* prediction of the flattening, this relation has a serious drawback: For a particular terrestrial planet it is not evident which effective elastic constants, i.e., Young's modulus and Poisson's ratio, to use. On the other hand, if we believe that this simple Hookean model applies to terrestrial planets we may use it to determine their effective shear modulus or modulus of rigidity, $\mu \equiv G$, if we use the *experimentally observed* data for the flattening. The factor $\frac{1+\nu/2}{7+5\nu}$ is nearly constant for all possible values of ν . i.e., ca. 1/7:

$$\mu = \frac{3m\omega_0^2}{28\pi r_o f}. \quad (2.2.21)$$

If we evaluate this relation using Earth's data we obtain a value of 50 GPa, which is smaller than the value for iron or steel (roughly 70 GPa), which is often quoted in context with planet Earth.

References

1. Callister, W.D.: Materials Science and Engineering, 4th edn. John Wiley & Sons Inc, New York (1997)
2. Chakravorty, J.G.: Deformation and stresses in a non-homogeneous Earth model with a rigid core. *Pure Appl. Geophys.* **95**(1), 59–66 (1972)
3. Dziewonski, A.M., Anderson, D.L.: Preliminary reference Earth model. *Phys. Earth Planet. Inter.* **25**, 297–356 (1981)
4. Eringen, A.C.: Mechanics of continua. In: Robert, E., Krieger Publishing Co, USA (1980)
5. Fitzpatrick, R.: Fluid mechanics. <http://farside.ph.utexas.edu/teaching/336L/Fluid.pdf>. (2014)
6. Greve, R.: *Kontinuumsmechanik*. Springer, Berlin (2003)
7. Haupt, P.: *Continuum Mechanics and Theory of Materials*. Springer, Berlin (2002)
8. Hutter, K., Jöhnk, K.: *Continuum Methods of Physical Modeling: Continuum Mechanics, Dimensional Analysis, Turbulence*. Springer Science & Business Media (2013)
9. Jones, R.M.: *Deformation Theory of Plasticity*. Bull Ridge Publishing, Blacksburgh (2009)
10. Kienzler, R., Schröder, R.: *Einführung in die Höhere Festigkeitslehre*. Springer, Dordrecht (2009)
11. Lai, W.M., Rubin, D.H., Rubin, D., Krempf, E.: *Introduction to Continuum Mechanics*, 3rd edn. Butterworth-Heinemann (1993)
12. Landau, L.D., Lifshitz, E.M.: *Theory of Elasticity*, Volume 7 of Course of Theoretical Physics, 2nd edn. Pergamon Press Oxford, London (1970)
13. Liu, I.S.: *Continuum Mechanics*. Springer Science & Business Media (2013)
14. Love, A.E.H.: *A Treatise on the Mathematical Theory of Elasticity*, vol. 1. Cambridge University Press, Cambridge (1892)
15. Love, A.E.H.: *A Treatise on the Mathematical Theory of Elasticity*, 2nd edn. Cambridge University Press, Cambridge (1906)
16. Love, A.E.H.: *A Treatise on the Mathematical Theory of Elasticity*, 4th edn. Cambridge University Press, Cambridge (1927)
17. Müller, I.: *Thermodynamik—Grundlagen der Materialtheorie*. Bertelsmann-Universitätsverlag (1973)
18. Müller, W.H.: *An Expedition to Continuum Theory*. Springer, Dordrecht (2014)

19. Müller, W.H., Lofink, P.: The movement of the Earth: modeling of the flattening parameter. *Lect. Notes TICMI* **15**, 1–40 (2014)
20. Pan, S.K.: Deformation and stresses in different Earth models with a rigid core having varying elastic parameters. *Geofisica pura et applicata* **56**(1), 39–52 (1963)
21. Samanta, B.S.: Stresses in different rotating spherical Earth models with rigid core. *Pure Appl. Geophys.* **63**(1), 68–81 (1966)
22. Sokolnikoff, I.S.: *Mathematical Theory of Elasticity*, 4th edn. McGraw-Hill Book Company Inc, New York (1956)
23. Thomson, W., Tait, P.G.: *Treatise on Natural Philosophy, Part II*. Cambridge at the University Press, Cambridge (1912)
24. Timoshenko, S., Goodier, J.N.: *Theory of Elasticity*, 4th edn. McGraw-Hill Book Company Inc, New York (1951)
25. Truesdell, C., Noll, W.: *The Non-Linear Field Theories of Mechanics*. Springer, Heidelberg (1965)
26. Truesdell, C., Toupin, R.: *Handbook of Physics: The Classical Field Theories*, pp. 226–858. Springer-Verlag, Berlin (1960)
27. Wang, C.C., Truesdell, C.: *Introduction to Rational Elasticity*, vol. 1. Springer Science & Business Media (1973)
28. Wikipedia: Earth and Sidereal Time. <https://en.wikipedia.org/wiki/Earth>, https://en.wikipedia.org/wiki/Sidereal_time#Sidereal_time_and_solar_time (2015)
29. Wikipedia: Physical data of terrestrial planets and moons. <http://nssdc.gsfc.nasa.gov/planetary/factsheet>, <http://en.wikipedia.org/wiki/Moon>, [http://en.wikipedia.org/wiki/Io_\(moon\)](http://en.wikipedia.org/wiki/Io_(moon)), [http://en.wikipedia.org/wiki/Europa_\(moon\)](http://en.wikipedia.org/wiki/Europa_(moon)), [http://en.wikipedia.org/wiki/Ganymede_\(moon\)](http://en.wikipedia.org/wiki/Ganymede_(moon)), [http://en.wikipedia.org/wiki/Callisto_\(moon\)](http://en.wikipedia.org/wiki/Callisto_(moon)), [http://en.wikipedia.org/wiki/Titan_\(moon\)](http://en.wikipedia.org/wiki/Titan_(moon)) (2015)
30. WolframMathWorld: Laplace's Equation–Spherical Coordinates. <http://mathworld.wolfram.com/LaplacesEquationSphericalCoordinates.html> (2015)

The State of Deformation in Earthlike Self-Gravitating
Objects

Müller, W.H.; Weiss, W.

2016, XII, 111 p. 36 illus., 33 illus. in color., Softcover

ISBN: 978-3-319-32578-1