Preface

Finding proper values of physical parameters in mathematical models is often quite a challenge. While many have gotten away with using just the mathematical symbols when doing science and engineering with pen and paper, the modern world of numerical computing requires each physical parameter to have a numerical value, otherwise one cannot get started with the computations. For example, in the simplest possible transient heat conduction simulation, a case relevant for a real physical material needs values for the heat capacity, the density, and the heat conduction coefficient of the material. In addition, relevant values must be chosen for initial and boundary temperatures as well as the size of the material. With a dimensionless mathematical model, as explained in Chapter 3.2, *no physical quantities* need to be assigned (!). Not only is this a simplification of great convenience, as one simulation is valid for any type of material, but it also actually increases the understanding of the physical problem.

Scaling of differential equations is basically a simple mathematical process, consisting of the chain rule for differentiation and some algebra. The *choice of scales*, however, is a non-trivial topic, which may cause confusion among practitioners without extensive experience with scaling. How to choose scales is unfortunately not well treated in the literature. Most of the times, authors just state scales without proper motivation. The choice of scales is highly problem-dependent and requires knowledge of the characteristic features of the solution or the physics of the problem. The present notes aim at explaining "all nuts and bolts" of the scaling technique, including choice of scales, the algebra, the interpretation of dimensionless parameters in scaled models, and how scaling impacts software for solving differential equations.

Traditionally, scaling was mainly used to identify small parameters in mathematical models, such that perturbation methods based on series expansions in terms of the small parameters could be used as an approximate solution method for differential equations. Nowadays, the greatest practical benefit of scaling is related to running numerical simulations, since scaling greatly simplifies the choice of values for the input data and makes the simulations results more widely applicable. The number of parameters in scaled models may be much less than the number of physical parameters in the original model. The parameters in scaled models are also dimensionless and express *ratios* of physical effects rather than levels of individual effects. Setting meaningful values of a few dimensionless numbers is much easier than determining physically relevant values for the original physical parameters.

Another great benefit of scaling is the physical insight that follows from dimensionless parameters. Since physical effects enter the problem through a few dimensionless groups, one can from these groups see how different effects compete in their impact on the solution. Ideally, a good physical understanding should provide the same insight, but it is not always easy to "think right" and realize how spatial and temporal scales interact with physical parameters. This interaction becomes clear through the dimensionless numbers, and such numbers are therefore a great help, especially for students, in developing a correct physical understanding.

Since we have a special focus on scaling related to numerical simulations, the notes contain a lot of examples on how to program with dimensionless differential equation models. Most numerical models feature quantities with dimension, so we show in particular how to utilize such existing models to solve the equations in the associated scaled model.

Scaling is not a universal mathematical technique as the details depend on the problem at hand. We therefore present scaling in a range of specific applications, starting with simple ODEs, progressing with basic PDEs, before attacking more complicated models, especially from fluid mechanics.

Chapter 1 discusses units and how to make programs that can automatically take care of unit conversion (the most frequent mathematical mistake in industry and science?). Section 2.1 introduces the mathematics of scaling and the thinking about scales in a simple ODE problem modeling exponential decay. The ideas are generalized to nonlinear ODEs and to systems of ODEs. Another ODE example, on mechanical vibrations, is treated in Section 2.2, where we cover many different physical contexts and different choices of scales. Scaling the standard, linear wave equation is the topic of Chapter 3.1, with discussion of how boundary and initial conditions influence the choice of scales. Another PDE example, the diffusion equation, appears in Chapter 3.2. Here we progress from a simple linear diffusion equation in 1D to a study of how scales are influenced by an oscillatory boundary condition. Nonlinear diffusion models, as well as convection-diffusion PDEs, are elaborated on. The final Chapter is devoted to many famous PDEs arising from continuum models: elasticity, viscous fluid flow, thermal convection, etc.

The mathematics is translated into complete computer codes for the ODE and simpler PDE problems.

Experimental fluid mechanics is a field full of relations involving dimensionless numbers such as the Grashof and Prandtl numbers, but none of the textbooks the authors have seen explain how these numbers actually relate to dimensionless forms of the governing equations. Consequently, this non-trivial topic is particularly highlighted in the fluid mechanics examples.

The mathematics in the first two chapters is very gentle and requires no more background than basic one-variable calculus and preferably some knowledge of differential equation models. The next chapter involves PDEs and assumes familiarity with basic models for wave phenomena, diffusion, and combined convection-diffusion. The final chapter is meant for readers with knowledge of the physics and mathematics of continuum mechanical models. The mathematical level of the text rises quickly after the first two chapters.

In the first two chapters, much of the mathematics is accompanied by complete (yet short) computer codes. The programming level requires familiarity with procedural programming in Python. As the mathematical level rises, the computer codes get much more comprehensive, and we refer to some files for computational examples in chapter three.

The pedagogy is to saturate the reader with lots of detailed examples to provide an understanding for the topic, primarily because the choice of scales depends on the problem at hand. One can also view the notes as a reference on how to scale many of the most important differential equation models in physics. For the simpler differential equations in Chapters 2 and 3, we present computer code for many computational examples, but the treatment of the advanced models in Chapter 4 is more superficial to limit the size of that chapter.

The exercises are named either Exercise or Problem. The latter is a stand-alone exercise without reference to the rest of the text, while the former typically extends a topic in the text or refers to sections or formulas in the text.

What this booklet is and is not

Books containing material on scaling and non-dimensionalization very often cover topics not treated in the present notes, e.g., the key topic of dimensional analysis and the famous Buckingham Pi Theorem [1, 8], which we discuss only briefly in section 1.1.3. Similarly, analytical solution methods like perturbation techniques and similarity solutions, which represent classical methods closely related to scaling and non-dimensionalization, are not addressed herein. There are numerous texts on perturbation techniques, and these methods build on an already scaled differential equations. Similarity solutions do not fit within the present scope since these involve non-dimensional *combinations* of the unscaled independent variables to derive new differential equations that are easier to solve.

Our scope is to scale differential equations to simplify the setting of parameters in numerical simulations, and at the same time understand more of the physics through interpretation of the dimensionless numbers that automatically arise from the scaling procedure.

With these notes, we hope to demystify the thinking involved in scale determination and encourage numerical simulations to be performed with dimensionless differential equation models.

All program and data files referred to in this book are available from the book's primary web site: URL: `http://hplgit.github.io/scaling-book/doc/web/`. This site also features a version of the book with exercises.

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