Analytical-Numerical Solutions for First-Order Periodic Boundary Value Problems Using the Reproducing Kernel Algorithm

Ayed Al e'damat, Mohammed Al-Smadi, Iryna Komashynska, Ali Ateiwi, and Ala' Alrawajfi

Abstract This paper proposes an efficient numerical algorithm to obtain an approximate solution of first-order periodic boundary value problems. This new algorithm is based on a reproducing kernel Hilbert space method. Its exact solution is calculated in the form of series in reproducing kernel space with easily computable components. In addition, convergence analysis for this method is discussed. In this sense, some numerical examples are given to show the effectiveness and performance of the proposed method. The results reveal that the method is quite accurate, simple, straightforward, and convenient to handle a various range of differential equations.

Keywords Analytical-numerical solutions • Reproducing kernel algorithm • Periodic boundary condition • Boundary value problems

AMS subject classifications: 34K28, 34K13, 34B15, 47B32.

A. Al e'damat
Department of Mathematics, Faculty of Science, Al-Hussein Bin Talal University, P.O. Box 20, Ma'an, Jordan

M. Al-Smadi
Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan

I. Komashynska
Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan

A. Ateiwi (✉)
Department of Mathematics, Faculty of Science, Al-Hussein Bin Talal University, P.O. Box 20, Ma'an, Jordan

Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan
e-mail: ateiwi@hotmail.com

A. Alrawajfi
Department of Financial and Administrative Sciences, Ma'an College, Al-Balqa Applied University, P.O. Box 194, Ma'an, Jordan

S. Pinelas et al. (eds.), *Differential and Difference Equations with Applications*, Springer Proceedings in Mathematics & Statistics 164,
DOI 10.1007/978-3-319-32857-7_2

1 Introduction

Boundary value problems (BVPs) with periodic boundary conditions have become a focus of research in many fields of physics, engineering, and mathematics, including molecular dynamics, mechanical systems, computer simulations, and composite materials with a periodic microstructure and so on. When such problems are solved numerically, the periodicity condition is often imposed strongly; in other words, the values on periodic edges are required to match exactly. For typical examples, see [18, 19].

The purpose of this paper is to extend the application of the reproducing kernel Hilbert space method (RKHM) to provide approximate solution of a class of first-order periodic BVPs of the following form:

$$u'(x) + g(x)u(x) = f(x, u(x))\,;\quad 0 \le x \le 1, \tag{1}$$

subject to the periodic boundary condition

$$u(0) - u(1) = 0, \tag{2}$$

where $g(x)$ is continuous function, $f(x,u) \in W_2^1[0,1]$, $u = u(x) \in W_2^2[0,1]$ is an unknown function to be determined, $\|f(x,u(x)) - f(x,\overline{u}(x))\|_{W_2^1} \le M\|u(x) - \overline{u}(x)\|_{W_2^1}$ for $x \in [0,1]$, $M \in \mathbb{R}$, $f(x,u)$ is linear or nonlinear function of u depending on the problem discussed, and $W_2^2[0,1]$ and $W_2^1[0,1]$ are reproducing kernel spaces defined in the next section. Throughout this paper, we assume that the BVP models (1) and (2) have a unique smooth solution on the given interval $[0,1]$.

The numerical solvability of BVPs with periodic boundary conditions of different orders has been pursued in literature. To mention a few, Peng [22] has discussed the existence and multiplicity of the positive solutions for first-order periodic BVPs. Al-Smadi et al. [4] have developed an iterative method for systems of first-order periodic BVPs based on the RKHM. Lia [20] has presented the existence of positive solution for fourth-order periodic BVPs. On the other hand, this method has been implemented in several operator, differential, integral, and integrodifferential equations side by side with their theories for instance, singular BVPs [12], singularly perturbed multipantograph delay equations (Geng and Qian, 2014), partial differential equations [17], Fredholm-Volterra integrodifferential equation [2, 5, 6], Fredholm integrodifferential equation ([1, 3, 14]), Volterra integrodifferential equation [7, 8], Fredholm-Volterra integral equation [11], operator equations [21], Fuzzy differential equations [9], and others [10, 15, 16]. The basic motivation of this paper is to apply the RKHM to develop an approach for obtaining the representation of exact and approximate solutions for a class of periodic BVPs (1) and (2), whereas the condition for determining solutions can be imposed in reproducing kernel space. However, this approach is simple, needs less effort to achieve the results, and is effective.

The paper is organized as follows. In Sect. 2, reproducing kernel spaces are presented in order to construct their reproducing kernel functions. In Sect. 3, representations of exact solution for BVPs (1) and (2) together with some essential results are introduced. Meanwhile, an iterative method for solving first-order periodic BVPs is described based on these reproducing kernel spaces. Subsequently, the analysis of the method is discussed in Sect. 4. In Sect. 5, numerical examples are simulated to show the reasonableness of our theory and to demonstrate the high performance of the proposed method. Finally, some conclusions are summarized in the last section.

2 Preliminaries and Materials

In this section, we utilize the reproducing kernel concept to construct the space $W_2^2[0,1]$ in which every function satisfies the periodic boundary condition (2) and formulate its reproducing kernel function. Besides, we present some basic results and remarks in the reproducing kernel theory and its applications.

Definition 1. Let E be a nonempty abstract set. A function $K : E \times E \to \mathbb{R}$ is a reproducing kernel of the Hilbert space $\mathscr{H}$ if:

1. For each $x \in E, K(\cdot,x) \in \mathscr{H}$.
2. For each $x \in E$ and $\varphi \in \mathscr{H}$, $\langle \varphi, K(\cdot,x)\rangle = \varphi(x)$.

The last condition is called *the reproducing property*: the value of the function φ at the point x is reproducing by the inner product of φ with $K(\cdot,x)$.

Remark 1. A Hilbert space $\mathscr{H}$ of functions on a set E is called a reproducing kernel Hilbert space (RKHS) if there exists a reproducing kernel K of $\mathscr{H}$. That is, a Hilbert space which possesses a reproducing kernel is called the RKHS.

Definition 2. The Hilbert space $W_2^m[0,1]$, $m \in \mathbb{N}$, is called a reproducing kernel if for each fixed x in $[0,1]$, there exist $K(x,y) \in W_2^m[0,1]$ such that $\langle u(y), K(x,y)\rangle_{W_2^m} = u(x)$ for any $u(y) \in W_2^m[0,1]$ and $y \in [0,1]$.

Definition 3. The reproducing kernel space $W_2^2[0,1]$ defined as $W_2^2[0,1] = \Big\{u(x) :$ $u'(x)$ is absolutely continuous real-valued function, $u''(x) \in L^2[0,1]$, and $u(0) = u(1)\Big\}$. The inner product and norm in $W_2^2[0,1]$ are given, respectively, by

$$\langle u(x), v(x)\rangle_{W_2^2} = u(0)v(0) + u'(0)v'(0) + \int_0^1 u''(t)v''(t)dt, \tag{3}$$

and $\|u\| = \langle u,u\rangle^{\frac{1}{2}}$, where $u,\ v \in W_2^2[0,1]$.

Remark 2. The space $W_2^2[0,1]$ is a complete reproducing kernel space, and its reproducing kernel function $K(x,y)$ can be written as

$$k(x,y) = \begin{cases} \sum_{i=1}^{4} c_i(x) y^{i-1}, \ y \le x, \\ \sum_{i=1}^{4} d_i(x) y^{i-1}, \ y > x, \end{cases} \tag{4}$$

where $c_i(x)$ and $d_i(x), \ i = 1,2,3,4$ will be given by the following assumptions:

Let's assume that $K(x,y) \in W_2^2[0,1]$ satisfies the generalized differential equations

$$\begin{cases} \frac{\partial^4 k(x,y)}{\partial y^4} = \delta(y-x), \frac{\partial^2 k(x,1)}{\partial y^2} = 0, k(x,0) + \frac{\partial^3 k(x,0)}{\partial y^3} + c_1 = 0, \\ \frac{\partial k(x,0)}{\partial y} - \frac{\partial^2 k(x,0)}{\partial y^2} = 0, \frac{\partial^3 k(x,1)}{\partial y^3} + c_1 = 0. \end{cases} \tag{5}$$

where δ is the Dirac delta function.

On the other hand, for $x \neq y$, $K(x,y)$ is the solution of the constant differential equation $\frac{\partial^4 k(x,y)}{\partial y^4} = 0$, subject to the boundary conditions (5). That is, the characteristic equation is given by $\lambda^4 = 0$ and the eigenvalues are $\lambda = 0$ with multiplicity 4. Hence, the general solution can be written as in Eq. (4).

In addition, assume that $K(x,y)$ satisfies the equations $\frac{\partial^m k(x,x+0)}{\partial y^m} = \frac{\partial^m k(x,x-0)}{\partial y^m}$ for $m = 0,1,2$, and $\frac{\partial^3 k(x,x+0)}{\partial y^3} - \frac{\partial^m k(x,x-0)}{\partial y^m} = -1$. Through the last descriptions together with the boundary conditions (5), the unknown coefficients $c_i(x)$ and $d_i(x), \ i = 1,2,3,4$ are uniquely obtained.

However, the representation of the reproducing kernel function $K(x,y)$ in $W_2^2[0,1]$, using Mathematica software package, is provided by

$$K(x,y) = \begin{cases} \frac{1}{48}\left[x^3 y\left(6+3y-y^2\right)+3x^2 y\left(-6-3y+y^2\right)+6xy\left(2+y+y^2\right)-8\left(-6+y^3\right)\right], y \le x, \\ \frac{1}{48}\left[48+6xy\left(2-3y+y^2\right)+3x^2 y\left(2-3y+y^2\right)-x^3\left(8-6y-3y^2+y^3\right)\right], y>x. \end{cases} \tag{6}$$

Here, it should be noted that the kernel function $K(x,y)$ is unique, symmetric, and nonnegative for any fixed $x \in [0,1]$. For detailed method for obtaining the reproducing kernel function, we refer to [12].

Theorem 1. An arbitrary bounded set of $W_2^2[0,1]$ is a compact set of $C[0,1]$.

Proof Let $\{u_n(x)\}_{n=1}^{\infty}$ be a bounded set of $W_2^2[0,1]$ such that $\|u_n(x)\| < M$, where M is positive constant. From representation of $K(x,y)$, we have $\left|u^{(i)}(x)\right| = \left|\left\langle u(x), \partial_x^i K(x,y)\right\rangle_{W_2^2}\right| \le \partial_x^i \|K(x,y)\|_{W_2^2} \|u(x)\|_{W_2^2}$. Since $\partial_x^i K(x,y), \ i = 1,2,\ldots$ is uniformly bounded about x and y, we have $\left|u^{(i)}(x)\right| \le M_i \|u(x)\|_{W_2^2}$. Accordingly, $\|u(x)\|_c \le M$.

Now, we need to prove that $\{u_n(x)\}_{n=1}^{\infty}$ is a compact set of $C[0,1]$, that is, $\{u_n(x)\}_{n=1}^{\infty}$ are equicontinuous functions. From the property of $K(x,y)$, we have

$$|u_n(x_1) - u_n(x_2)| = \left|\langle u(y), K(x_1,y) - K(x_2,y)\rangle_{W_2^2}\right|$$
$$\leq \|u(x)\|_{W_2^2}\|K(x_1,y) - K(x_2,y)\|_{W_2^2} \leq M\left\|K(x_1,y) - K(x_2,y)_{W_2^2}\right\|.$$

By "mean-value theorem of differentials" and the symmetry of $K(x,y)$, it follows that

$$|K(x_2,y) - K(x_1,y)| = |K(y,x_2) - K(y,x_1)| = \left|\frac{d}{dx}K(y,x)\right|_{x=\eta}|x_2 - x_1| \leq N|x_2 - x_1|.$$

Thus, if $\gamma \leq |x_2 - x_1| \leq \frac{\epsilon}{NM}$, then one can get $|u_n(x_1) - u_n(x_2)| < \epsilon$.

Definition 4. The reproducing kernel space $W_2^1[0,1]$ defined as $W_2^1[0,1] = \Big\{u(x) :$ $u'(x)$ is absolutely continuous real-valued function, $u'(x) \in L^2[0,1]\Big\}$. The inner product and norm in $W_2^1[0,1]$ are given, respectively, by

$$\langle u(x), v(x)\rangle_{W_2^1} = u(0)v(0) + \int_0^1 u'(t)v'(t)dt, \tag{7}$$

and $\|u\| = \langle u,u\rangle^{\frac{1}{2}}$, where $u,\ v \in W_2^1[0,1]$.

In 2006, Lin and Cui have proved that the space $W_2^1[0,1]$ is a complete reproducing kernel and its reproducing kernel is given by

$$G(x,y) = \begin{cases} (1+y), y \leq x, \\ (1+x), y > x. \end{cases} \tag{8}$$

3 Adaptation of Reproducing Kernel Algorithm

In this section, the formulation of a linear differential operator and the implementation method are presented in $W_2^2[0,1]$. After a while, the construction of orthogonal function systems is introduced based on the use of the Gram-Schmidt orthogonalization process in order to obtain exact and approximate solutions of periodic BVPs (1) and (2). To do this, we define a differential operator $L : W_2^2[0,1] \to W_2^1[0,1]$ such that $Lu(x) = u'(x) + g(x)u(x)$. Thus, the periodic BVPs (1) and (2) can be converted into the form

$$\begin{cases} Lu(x) = f\left(x, u(x)\right),\ 0 \leq x \leq 1, \\ u(0) - u(1) = 0, \end{cases} \tag{9}$$

where $u(x) \in W_2^2\left[0, 1\right]$ and $f\left(x, y\right) \in W_2^1\left[0, 1\right]$ as $y = y(x)\Big) \in W_2^2\left[0, 1\right],\ y \in (-\infty, \infty), x \in \left[0, 1\right].$

Corollary 1 *The operator L: $W_2^2\left[0, 1\right] \to W_2^1\left[0, 1\right]$ is a bounded linear operator.*

Proof It is so easy to see that L is a linear operator. Thus, it is enough to show that L is a bounded operator. From Definition 4, we have

$$\|Lu\|_{W_2^1}^2 = \langle Lu, Lu\rangle_{W_2^1} = [(Lu)(0)]^2 + \int_0^1 [(Lu)'(x)]^2 dx.$$

By reproducing property of $K(x, y)$, we have

$$.\begin{cases} u(x) = \langle u(y),\ K\left(x, y\right)\rangle_{W_2^2}, \\ (Lu)(x) = \Big\langle u,\ LK\left(x, y\right)\Big)\Big\rangle_{W_2^2}, \\ (Lu)'(x) = \Big\langle u,\ LK\left(x, y\right)\Big)'\Big\rangle_{W_2^2}. \end{cases}$$

By Schwarz inequality, we get

$$|(Lu)(x)| = \left|\Big\langle u,\ LK\left(x, y\right)\Big)\Big\rangle_{W_2^2}\right| \leq \|LK\left(x, y\right)\|_{W_2^2}\|u\|_{W_2^2} = M_1\|u\|_{W_2^2},$$

and

$$|(Lu)'(x)| = \left|\langle u, (LK\left(x, y\right))'\rangle_{W_2^2}\right| \leq \|(LK\left(x, y\right))'\|_{W_2^2}\|u\|_{W_2^2} = M_2\|u\|_{W_2^2},$$

where $M_1, M_2 > 0$ are positive constants.

Thus $[(Lu)(0)]^2 \leq M_1^2\|u\|_{W_2^2}^2$, $[(Lu)'(x)]^2 \leq M_2^2\|u\|_{W_2^2}^2$ and $\int_0^1 [(Lu)'(x)]^2 dx \leq M_2^2\|u\|_{W_2^2}^2$. That is,

$$\|(Lu)(x)\|_{W_2^1}^2 = [(Lu)(0)]^2 + \int_0^1 [(Lu)'(x)]^2 dx \leq \left(M_1^2 + M_2^2\right)\|u\|_{W_2^2}^2 = M\|u\|_{W_2^2}^2,$$

where $M = M_1^2 + M_2^2 > 0$.

Now, we construct an orthogonal system of functions $\{\psi_i(x)\}_{i=1}^{\infty}$ of $W_2^2[0, 1]$ by setting $\Phi_i(x) = G\left(x, x_i\right)$ and $\Psi_i(x) = L^*\Phi_i(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$

and L^* is the conjugate operator of L. Consequently, in terms of the properties of $G(x,y)$, one obtains $\langle u(x), \Psi_i(x)\rangle_{W_2^2} = \langle u(x), L^*\Phi_{\mathrm{i}}(x)\rangle_{W_2^2} = \langle Lu(x), \Phi_i(x)\rangle_{W_2^1} = Lu\,(x_i)\,,\ i = 1, 2, \ldots$

Lemma 1 The fact $\Psi_i(x) = \frac{d}{dy}K\,(x,y)\,|_{y=x_i}, i = 1, 2, \ldots$ holds.

Proof From reproducing property of, we can obtain that $\Psi_i(x) = \langle \Psi_i(y), K\,(x,y)\rangle_{W_2^2} = \langle L^*\Phi_{\mathrm{i}}(x), K\,(x,y)\rangle_{W_2^2} = \langle \Phi_{\mathrm{i}}(x), LK\,(x,y)\rangle_{W_2^1} = LK\,(x,x_i) = \frac{d}{dy}K\,(x,y)\,|_{y=x_i}$.

Lemma 2 If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$; then $\{\Psi_i(x)\}_{i=1}^{\infty}$ is a complete system of $W_2^2[0,1]$.

Proof For each fixed $u(x) \in W_2^2\,[0,1]$, let $\langle u(x), \Psi_i(x)\rangle_{W_2^2} = 0$. That is, $\langle u(x), \Psi_i(x)\rangle_{W_2^2} = \langle u(x), L^*\Phi_{\mathrm{i}}(x)\rangle_{W_2^2} = \langle Lu(x), \Phi_i(x)\rangle_{W_2^1} = Lu\,(x_i) = 0,\ i = 1,\ 2, \ldots$. Therefore, $Lu(x) = 0$ from the density of $\{x_i\}_{i=1}^{\infty}$ on $[0,1]$, as well as $u(x) = 0$ from the existence of L^{-1} and the continuity of $u(x)$.

The orthonormal system functions $\left\{\overline{\Psi}_i(x)\right\}_{i=1}^{\infty}$ of $W_2^2[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\Psi_i(x)\}_{i=1}^{\infty}$ as follows:

$$\overline{\Psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\Psi_k(x), \tag{10}$$

where β_{ik} are orthogonalization coefficients $\beta_{ii} > 0,\ i = 1, 2, \ldots, n\big)$ that are given by

$$\beta_{ij} = \frac{1}{\|\Psi_i\|}, \quad \text{for} \quad i = j = 1,$$

$$\beta_{ij} = \frac{1}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1}\left(\left\langle \Psi_i, \overline{\Psi}_k\right\rangle_{W_2^2}\right)^2}} \quad \text{for} \quad i = j \neq 1, \quad \text{and}$$

$$\beta_{ij} = \frac{-\sum_{j=k}^{i-1}\left\langle \Psi_i, \overline{\Psi}_k\right\rangle_{W_2^2}\beta_{jk}}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1}\left(\left\langle \Psi_i, \overline{\Psi}_k\right\rangle_{W_2^2}\right)^2}} \quad \text{for} \quad i > j.$$

Theorem 2. For each $u(x)$ in $W_2^2[0,1]$, the series $\sum_{i=1}^{\infty}\left\langle u(x), \overline{\Psi}_i(x)\right\rangle \overline{\Psi}_i(x)$ is convergent in the sense of the norm $\|\cdot\|_{W_2^2}$. On the other hand, if $\{x_i\}_{i=1}^{\infty}$ is dense on [0, 1] and $u(x) \in W_2^2\,[0,1]$ is the solution of problem model (9), then $u(x)$ satisfy the following form:

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} f\left(x_k, u\left(x_k\right)\right)\overline{\Psi}_i(x), \tag{11}$$

and the approximate solution can be obtained by

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} f\left(x_k, u\left(x_k\right)\right)\overline{\Psi}_i(x), \tag{12}$$

where $u_0(x) \in W_2^2\,[0,1]$ (u_0 fixed).

Proof Since $u(x) \in W_2^2\,[0,1]$, $u(x)$ can be expanded in the form of Fourier series about $\left\{\overline{\Psi}_i(x)\right\}_{i=1}^{\infty}$ as $u(x) = \sum_{i=1}^{\infty}\left\langle u(x), \overline{\Psi}_i(x)\right\rangle \overline{\Psi}_i(x)$, and since the space $W_2^2[0,1]$ is the Hilbert space, then the series $u(x) = \sum_{i=1}^{\infty}\left\langle u(x), \overline{\Psi}_i(x)\right\rangle \overline{\Psi}_i(x)$ is convergent in the norm $\|.\|_{W_2^2}$. From the Fourier series expansion and by Eq. (7), $u(x)$ can be written as

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}\left\langle u(x), \overline{\Psi}_i(x)\right\rangle_{W_2^2}\overline{\Psi}_i(x) = \sum_{i=1}^{\infty}\left\langle u(x), \sum_{k=1}^{i}\beta_{ik}\Psi_k(x)\right\rangle_{W_2^2}\overline{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x), \Psi_k(x)\right\rangle_{W_2^2}\overline{\Psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x), L^*\Phi_k(x)\right\rangle_{W_2^2}\overline{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\left\langle \beta_{ik} Lu(x), \Phi_k(x)\right\rangle_{W_2^1}\overline{\Psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} Lu\left(x_k\right)\overline{\Psi}_i(x) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} f\left(x_k, u\left(x_k\right)\right)\overline{\Psi}_i(x).
\end{aligned}$$

Therefore, the form in Eq. (11) is the exact solution of Eq. (9). By truncating the series in Eq. (11), we obtain the th-truncated series approximate solution as in Eq. (12). So, the proof of the theorem is complete.

Lemma 3. If $u(x) \in W_2^2\,[0,1]$, then there exists a positive constant M such that $\left\|u^{(i)}(x)\right\|_c \leq M\|u(x)\|_{W_2^2}$, $i = 0, 1$, where $\|u(x)\|_c = \max_{0<x<1}|u(x)|$.

Proof For any $x_1, x_2 \in [0,1]$, we have $u^{(i)}\left(x_1\right) = \left\langle u\left(x_2\right), \partial_{x_1}^{i} K\left(x_1, x_2\right)\right\rangle_{W_2^2}, i = 0, 1$. By the expression form of $K(x, y)$, it follows that $\left\|\partial_x^i K(x, y)\right\|_{W_2^2} \leq M_i, i = 0, 1$. Thus, $\left|u^{(i)}\left(x_1\right)\right| = \left|\left\langle u\left(x_2\right), \partial_{x_1}^{i} K\left(x_1, x_2\right)\right\rangle_{W_2^2}\right| \leq \left\|\partial_{x_1}^{i} K\left(x_1, x_2\right)\right\|_{W_2^2}\left\|u\left(x_2\right)\right\|_{W_2^2} \leq \cdots M_i\|u(x)\|_{W_2^2}, i = 0, 1$. Hence, $\left\|u^{(i)}(x)\right\|_c \leq \max_i\left\{M_i\right\}\|u(x)\|_{W_2^2}, i = 0, 1$. The proof is complete.

Corollary 2. *The approximate solution* $u_n(x)$ *and its derivative* $u'_n(x)$ *are converging uniformly to the exact solution* $u(x)$ *and its derivative* $u'(x)$ *as* $n \to \infty$*, respectively.*

Proof Form Lemma 3, for any $x \in [0,1]$, it easy to see that $\left|u_n^{(i)}(x)-u^{(i)}(x)\right| = \left|\left\langle u_n(x)-u(x), \partial_x^i K(x,x)\right\rangle_{W_2^2}\right| \leq \left\| \partial_x^i K(x,x)\right\|_{W_2^2} \|u_n(x)-u(x)\|_{W_2^2} \leq M_i \|u_n(x)-u(x)\|_{W_2^2}$, $i=0,1$.

Hence, if $\|u_n(x)-u(x)\|_{W_2^2} \to 0$ as $n \to \infty$, then the approximate solution $u_n(x)$ and its derivative $u'_n(x)$ are converging uniformly to the exact solution $u(x)$ and its derivative $u'(x)$ as $n \to \infty$, respectively. So, the proof of the theorem is complete.

Remark 3. In order to solve Eq. (1) numerically using the RKHS technique, we have the following two cases:

Case 1: If Eq. (1) is linear, then the exact and approximate solutions can be obtained directly from Eqs. (11) and (12), respectively.
Case 2: If Eq. (1) is nonlinear, then in this case the exact and approximate solutions can be obtained by using the following algorithm:

Algorithm 1 According to Eq. (11), the representation of the solution of problem (1) can be denoted by

$$u(x) = \sum_{i=1}^{\infty} B_i \overline{\Psi}_i(x), \tag{13}$$

where $B_i = \sum_{k=1}^{i} \beta_{ik} f\left(x_k, u_{k-1}\left(x_k\right)\right)$. In fact, $B_i, i = 1,2,\ldots,$ in Eq. (13) are unknown, so we will approximate them using the known A_i as follows: For a numerical computations, let the initial function $u_0\left(x_1\right) = 0$, set $u_0\left(x_1\right) = u\left(x_1\right)$, and define the n-term approximation to $y_s(x)$ by

$$u_n(x) = \sum_{i=1}^{n} A_i \overline{\Psi}_i(x), \tag{14}$$

where the coefficients A_i of $\overline{\Psi}_i(x), i = 1,2,\ldots,n$, are given by

$$\begin{cases} A_1 = \beta_{11} f\left(x_1, u_0\left(x_1\right)\right), u_1(x) = A_1 \overline{\Psi}_1(x), \\ A_2 = \sum_{k=1}^{2} \beta_{2k} f\left(x_1, u_{k-1}\left(x_k\right)\right), u_2(x) = \sum_{i=1}^{2} A_i \overline{\Psi}_i(x), \\ u_{n-1}(x) = \sum_{i=1}^{n-1} A_i \overline{\Psi}_i(x), A_n = \sum_{k=1}^{n} \beta_{nk} f\left(x_1, u_{k-1}\left(x_k\right)\right). \end{cases} \tag{15}$$

Consequently, the unknown coefficients $B_i, i = 1,2,\ldots,$ in Eq. (13) will be approximate using the known coefficients $A_i, i = 1,2,\ldots,$ given in Eq. (14).

However, in the iterative process of the series (14), we can guarantee that the approximation $u_n(x)$ satisfies the periodic boundary condition (2).

4 Convergence Analysis of the Method

In this section, we will prove that the iterative formula (14) is convergent to the exact solution of Eq. (9) in the sense of the norm of $W_2^2[0,1]$. In fact, this result is fundamental in the RKHS theory and its applications. The remaining lemmas are collected in order to prove the pre-recent theorem.

Lemma 4 If $\|u_n(x) - u(x)\|_{W_2^2} \to 0,\ x_n \to y,\ (n \to \infty)$, and $f(x,z)$ is continuous in $[0,1]$ with respect to x, z for $x \in [0,1]\,,\ z \in (-\infty, \infty)$, then the following are held in the sense of the norm of $W_2^2[0,1]$:

(a) $u_{n-1}(x_n) \to u(y)$ *as* $n \to \infty$.
(b) $f(x_n, u_{n-1}(x_n)) \to f(y, u(y))$, *as* $n \to \infty$.

Proof For part (a), note that

$$\begin{aligned}|u_{n-1}(x_n) - u(y)| &= |u_{n-1}(x_n) - u_{n-1}(y) + u_{n-1}(y) - u(y)| \\ &\le |u_{n-1}(x_n) - u_{n-1}(y)| + |u_{n-1}(y) - u(y)|\,.\end{aligned}$$

By reproducing property of $K(x,y)$, we have $u_{n-1}(x_n) = \langle u_{n-1}(x), K(x_n,x)\rangle_{W_2^2}$ and $u_{n-1}(y) = \langle u_{n-1}(x), K(y,x)\rangle_{W_2^2}$. Thus,

$$\begin{aligned}|u_{n-1}(x_n) - u_{n-1}(y)| &= \left|\langle u_{n-1}(x), K(x_n,x) - K(y,x)\rangle_{W_2^2}\right| \\ &\le \|u_{n-1}(x)\|_{W_2^2}\|K(x_n,x) - K(y,x)\|_{W_2^2}.\end{aligned}$$

From the symmetry of $K(x,y)$, it follows that $\left\|K(x_n,x) - K(y,x)_{W_2^2}\right\| \to 0$ as $x_n \to y, n \to \infty$. Hence, $|u_{n-1}(x_n) - u_{n-1}(y)| \to 0$ as soon as $x_n \to y,\ (n \to \infty)$. On the other hand, for any $x \in [0,1]$, by using Corollary 2, it holds that $|u_{n-1}(y) - u(y)| \to 0$ as $n \to \infty$. Therefore, $u_{n-1}(x_n) \to u(y)$ in the sense of $\|\cdot\|_{W_2^2}$ as $x_n \to y$ and $n \to \infty$. Thus, for part (b), by means of the continuation of $f(\cdot)$, it is obtained that $f(x_n, u_{n-1}(x_n)) \to f(y, u(y))$ as $x_n \to y$ and $n \to \infty$.

Lemma 5 For the approximate solution $u_n(x)$ in iterative formula (14), the following relations hold:

(a) $Lu_n(x_j) = f(x_j, u_{j-1}(x_j))\,, j \le n$,
(b) $Lu_n(x_j) = Lu(x_j)\,, j \le n$.

Proof For part (a), the proof will be obtained by mathematical induction. For $j \le n$, we have

$$Lu_n\left(x_j\right)=\sum_{i=1}^{n}A_i L\overline{\Psi}_i(x)=\sum_{i=1}^{n}A_i\left\langle L\overline{\Psi}_i(x),\Phi_{\mathrm{j}}(x)\right\rangle_{W_2^1}=\sum_{i=1}^{n}A_i\left\langle\overline{\Psi}_i(x),L^*\Phi_{\mathrm{j}}(x)\right\rangle_{W_2^2}$$
$$=\sum_{i=1}^{n}A_i\left\langle\overline{\Psi}_i(x),\Psi_j(x)\right\rangle_{W_2^2}.$$

That is,

$$Lu_n\left(x_j\right)=\sum\nolimits_{i=1}^{n}A_i\left\langle\overline{\Psi}_i(x),\Psi_j(x)\right\rangle_{W_2^2}. \tag{16}$$

Multiplying both sides of Eq. (15) by β_{jl}, summing for l from 1 to j, and using the orthogonality of $\left\{\overline{\Psi}_i(x)\right\}_{i=1}^{\infty}$ yields that

$$\sum_{l=1}^{j}\beta_{jl}Lu_n\left(x_l\right)=\sum_{i=1}^{n}A_i\left\langle\overline{\Psi}_i(x),\sum_{l=1}^{j}\beta_{jl}\Psi_l(x)\right\rangle_{W_2^2}=\sum_{i=1}^{\infty}A_i\left\langle\overline{\Psi}_i(x),\overline{\Psi}_j(x)\right\rangle_{W_2^2}$$
$$=A_j=\sum_{l=1}^{j}\beta_{jl}f\left(x_l,u_{l-1}\left(x_l\right)\right).$$

If $j=1$, then $Lu_n\left(x_1\right)=f\left(x_1,u_0\left(x_1\right)\right)$. Besides, if $j=2$, then $\beta_{21}Lu_n\left(x_1\right)+\beta_{22}Lu_n\left(x_2\right)=\beta_{21}f\left(x_1,u_0\left(x_1\right)\right)+\beta_{22}f\left(x_2,u_1\left(x_2\right)\right)$, that is, $Lu_n\left(x_2\right)=f\left(x_2,u_1\left(x_2\right)\right)$. Thus $Lu_n\left(x_j\right)=f\left(x_j,u_{j-1}\left(x_j\right)\right)$ for $j\leq n$.

For part (b), from Corollary 2 as well as by taking limits in Eq. (14), we have $u(x)=\sum\nolimits_{i=1}^{\infty}A_i\overline{\Psi}_i(x)$. Thus, $u_n(x)=P_n u(x)$, where P_n is an orthogonal projector from the space $W_2^2[0,1]$ to $\mathrm{Span}\{\Psi_1,\Psi_2,\ldots,\Psi_n\}$. Therefore, $Lu_n\left(x_j\right)=\left\langle Lu_n(x),\Phi_j(x)\right\rangle_{W_2^1}=\left\langle u_n(x),L^*\Phi_j(x)\right\rangle_{W_2^2}=\left\langle P_n u(x),\Psi_j(x)\right\rangle_{W_2^2}=\left\langle u(x),P_n\Psi_j(x)\right\rangle_{W_2^2}=\left\langle u(x),\Psi_j(x)\right\rangle_{W_2^2}=\left\langle u(x),L^*\Phi_j(x)\right\rangle_{W_2^2}=\left\langle Lu(x),\Phi_j(x)\right\rangle_{W_2^1}=Lu\left(x_j\right)$. So, the proof of the lemma is complete.

Lemma 6 The sequence $\{u_n(x)\}_{n=1}^{\infty}$ in the iterative formula (14) is monotone increasing in the sense of $\|\cdot\|_{W_2^2}$.

Theorem 3. Suppose that $\{x_i\}_{i=1}^{\infty}$ is dense on a compact interval $[0,1]$ and $\|u_n(x)\|_{W_2^2}$ is bounded in formula (14), then the n-term approximate solution $u_n(x)$ in the iterative formula (14) is convergent to the exact solution $u(x)$ of Eq. (9) in the space $W_2^2[0,1]$ and $u(x)=\sum\nolimits_{i=1}^{\infty}A_i\overline{\Psi}_i(x)$, where $A_i, i=1,2,\ldots$ are given by Eq. (14).

Proof First of all, we will prove the convergence of $u_n(x)$. From iterative formula (14), we infer that $u_{n+1}(x)=u_n(x)+A_{n+1}\overline{\Psi}_{n+1}(x)$. By the orthogonality of $\left\{\overline{\Psi}(x)\right\}_{i=1}^{\infty}$, it follows that $\|u_{n+1}\|_{W_2^2}^2=\|u_n\|_{W_2^2}^2+\left(A_{n+1}\right)^2=\|u_{n-1}\|_{W_2^2}^2+\left(A_n\right)^2$

$+\ (A_{n+1})^2 = \cdots = \|u_0\|^2_{W_2^2} + \sum_{i=1}^{n+1}(A_i)^2$. From Lemma 6, the sequence $\|u_n\|_{W_2^2}$ is monotone increasing, and from the boundedness of $\|u_n\|_{W_2^2}$, we have $\sum_{i=1}^{\infty}(A_i)^2 < \infty$, that is, $\{A_i\}_{i=1}^{\infty} \in l^2\ (i = 1, 2, \ldots)$. Hence, $\|u_n\|_{W_2^2}$ is convergent as $n \to \infty$.

Let $m > n$; for $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{n+1} - u_n)$, it follows that

$$\begin{aligned}\|u_m(x) - u_n(x)\|^2_{W_2^2} &= \|u_m(x) - u_{m-1}(x) + u_{m-1}(x) - \cdots + u_{n+1}(x) - u_n(x)\|^2_{W_2^2} \\ &\le \|u_m(x) - u_{m-1}(x)\|^2_{W_2^2} + \cdots + \|u_{n+1}(x) - u_n(x)\|^2_{W_2^2} \\ &= \sum_{i=n+1}^{m}(A_i)^2 \to 0,\ (n \to \infty).\end{aligned}$$

Considering the completeness of $W_2^2[0,1]$, there exists $u(x) \in W_2^2\,[0,1]$ such that $u_n(x) \to u(x)$ as $n \to \infty$ in sense of $\|.\|_{W_2^2}$.

Secondly, we will prove that $u(x)$ is the solution of Eq. (9). Since $\{x_i\}_{i=1}^{\infty}$ is dense on compact interval [0, 1], thus for any $x \in [0,1]$, there exists subsequence $\{x_{n_j}\}$ such that $x_{n_j} \to x$, as $j \to \infty$. From Lemma 5, $Lu_n\left(x_{n_j}\right) = f\left(x_{n_j}, u_{j-1}\left(x_{n_j}\right)\right)$. Hence, let $j \to \infty$; we have $Lu(x) = f\,(x, u(x))$. That is, $u(x)$ is solution of Eq. (9). The proof is complete.

Theorem 4. Assume that $u_n(x) \in W_2^2\,[0,1]$ is the solution of BVP (9) and $r_n(x) = \|u(x) - u_n(x)\|_{W_2^2}$ is an error function, where $u_n(x)$ is the approximate solution that is given by iterative formula (14). Then the sequence of number $\{r_n\}$ is monotone decreasing in the sense of $\|.\|_{W_2^2}$ and $r_n \to 0$ as $n \to \infty$.

Proof Based on the previous results, it is obvious that

$$\begin{aligned}\|r_n(x)\|^2_{W_2^2} = \|u(x) - u_n(x)\|^2_{W_2^2} &= \left\| \sum_{i=n+1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F\left(x_k, u_{k-1}\left(x_k\right), Tu_{k-1}\left(x_k\right)\right)\overline{\Psi}_i(x)\right\|^2_{W_2^2} \\ &= \left\|\sum_{i=n+1}^{\infty}A_i\overline{\Psi}_i(x)\right\|^2_{W_2^2} = \sum_{i=n+1}^{\infty}(A_i)^2,\end{aligned}$$

and $\|r_{n-1}(x)\|^2_{W_2^2} = \sum_{i=n}^{\infty}(A_i)^2$. Thus, $\|r_n(x)\|_{W_2^2} \le \|r_{n-1}(x)\|_{W_2^2}$. Consequently, the error r_n is monotone decreasing in the sense of $\|.\|_{W_2^2}$. The proof is complete.

5 Applications and Test Problems

In this section, some numerical examples are studied to demonstrate the performance, accuracy, and applicability of the present method for both linear and nonlinear problems. Results obtained are compared with the exact solution of each example and are found to be in good agreement with each other. In the process of computation, all the symbolic and numerical computations performed by using Mathematica software package.

Example 1 Consider the following linear equation

$$u'(x) + u(x) = x^2 + x - 1, 0 \leq x \leq 1, \tag{17}$$

subject to periodic boundary condition

$$u(0) - u(1) = 0 \tag{18}$$

The exact solution is $u(x) = x\,(x-1)$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, \ldots, n$. The numerical results at some selected grid points for $n = 51$ are given in Table 1.

To show the accuracy of the present method for our tested problems, we report two types of error. The first one is the absolute error, $\mathrm{Abs}_n(x)$, and the second one is the relative error, $\mathrm{Rel}_n(x)$, which are defined, respectively, by $\mathrm{Abs}_n(x) = |u(x) - u_n(x)|$, $\mathrm{Rel}_n(x) = \frac{\mathrm{Abs}_n(x)}{|u(x)|}$, where $x \in [0, 1]$, $u_n(x)$ is the n-term approximation of $u(x)$ obtained by the RKHS method, and $u(x) \in W_2^2\,[0, 1]$ is the exact solution.

Example 2 Consider the following nonlinear equation

$$u'(x) + u(x)e^{-u(x)} = \frac{2x - 1 + \ln\left(x^2 - x + 1\right)}{x^2 - x + 1}, 0 \leq x \leq 1, \tag{19}$$

subject to periodic boundary condition

$$u(0) - u(1) = 0 \tag{20}$$

Table 1 Numerical results for Example 1

x_i	$u(x)$	$u_{51}(x)$	$\mathrm{Abs}_{51}(x)$	$\mathrm{Rel}_{51}(x)$
0.16	−0.1344	−0.13440010447668405	1.04477×10^{-7}	7.77356×10^{-7}
0.32	−0.2176	−0.21760010031925470	1.00319×10^{-7}	4.61026×10^{-7}
0.48	−0.2496	−0.24960009873547984	9.87355×10^{-8}	3.95575×10^{-7}
0.64	−0.2304	−0.23040009968473152	9.96847×10^{-8}	3.95575×10^{-7}
0.80	−0.1600	−0.16000010319136138	1.03191×10^{-7}	6.44946×10^{-7}
0.96	−0.0384	−0.03840010934533122	1.09345×10^{-7}	2.84753×10^{-6}

Table 2 Numerical results for Example 2

x_i	$u(x)$	$u_{51}(x)$	$\text{Abs}_{51}(x)$	$\text{Rel}_{51}(x)$
0.16	−0.1443323708899199	−0.1443322995528097	7.13371×10^{-8}	4.94256×10^{-7}
0.32	−0.2453891602615295	−0.2453890219359414	1.38326×10^{-7}	5.63699×10^{-7}
0.48	−0.2871488812901222	−0.2871487102608685	1.71029×10^{-7}	5.95612×10^{-7}
0.64	−0.2618843796306403	−0.2618842287680727	1.50863×10^{-7}	5.76066×10^{-7}
0.80	−0.1743533871447777	−0.1743532975615985	8.95832×10^{-8}	5.13802×10^{-7}
0.96	−0.0391567152011939	−0.0391566982754902	1.69257×10^{-8}	4.32255×10^{-7}

The exact solution is $u(x) = \ln\left(x^2 - x + 1\right)$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, \ldots, n$. The numerical results at some selected grid points for $n = 51$ are given in Table 2.

6 Conclusion

The main concern of this work has been to propose an efficient algorithm for the solutions of first-order periodic BVPs. The goal has been achieved by introducing the RKHS method to solve this class of differential equations. We can conclude that the RKHS method is a powerful and efficient technique in finding approximate solution $u_n(x)$ for linear and nonlinear problems. In the proposed algorithm, the solution $u(x)$ and the approximate solution $u_n(x)$ are represented in the form of series in $W_2^2[0, 1]$. Moreover, the approximate solution and its derivative converge uniformly to the exact solution and its derivative, respectively.

References

1. Abu Arqub, O., Al-Smadi, M., Shawagfeh, N.: Solving Fredholm integro-differential equations using reproducing kernel Hilbert space method. Appl. Math. Comput. **219**, 8938–8948 (2013)
2. Abu Arqub, O., Al-Smadi, M.: Numerical algorithm for solving two-point, second-order periodic boundary value problems for mixed integro-differential equations. Appl. Math. Comput. **243**, 911–922 (2014)
3. Abu-Gdairi, R., Al-Smadi, M.: An efficient computational method for 4th-order boundary value problems of Fredholm IDEs. Appl. Math. Sci. **7**(96), 4761–4774 (2013)
4. Al-Smadi, M., Abu Arqub, O., El-Ajou, A.: A numerical iterative method for solving systems of first-order periodic boundary value problems. J. Appl. Math. **2014**, 135465 (2014). p. 10
5. Al-Smadi, M., Abu Arqub, O., Momani, S.: A computational method for two point boundary value problems of fourth-order mixed integrodifferential equations. Math. Probl. Eng. **2013**, 832074 (2013). p. 10
6. Abu Arqub, O., Al-Smadi, M., Momani, S.: Application of reproducing kernel method for solving nonlinear Fredholm-Volterra integro-differential equations. Abstr. Appl. Anal. **2012**, 839836 (2012). p. 16

7. Al-Smadi, M., Abu Arqub, O., Shawagfeh, N.: Approximate solution of BVPs for 4th-order IDEs by using RKHS method. Appl. Math. Sci. **6**, 2453–2464 (2012)
8. Altawallbeh, Z., Al-Smadi, M., Abu-Gdairi, R.: Approximate solution of second-order integrodifferential equation of Volterra type in RKHS method. Int. J. Math. Anal. **7**(44), 2145–2160 (2013)
9. Abu Arqub, O., Al-Smadi, M., Momani, S., Hayat, T.: Numerical solutions of fuzzy differential equations using reproducing kernel Hilbert space method. Soft Computing. **2015**, 1–20 (2015). doi:10.1007/s00500-015.1707-4
10. Al-Smadi, M., Altawallbeh, Z.: Solution of system of Fredholm integro-differential equations by RKHS method. Int. J. Contemp. Math. Sci. **8**(11), 531–540 (2013)
11. Cui, M., Du, H.: Representation of exact solution for the nonlinear Volterra-Fredholm integral equations. Appl. Math. Comput. **182**, 1795–1802 (2006)
12. Geng, F.: Solving singular second order three-point boundary value problems using reproducing kernel Hilbert space method. Appl. Math. Comput. **215**, 2095–2102 (2009)
13. Geng, F.Z., Qian, S.P.: Solving Singularly Perturbed Multipantograph Delay Equations Based on the Reproducing Kernel Method. Abstr. Appl. Anal. **2014**, 794716 (2014). p. 6
14. Komashynska, I., Al-Smadi, M.: Iterative reproducing kernel method for solving second-order integrodifferential equations of Fredholm type. J. Appl. Math. **2014**, 459509 (2014). p. 11
15. Komashynska, I., Al-Smadi, M., Abu Arqub, O., Momani, S.: An efficient analytical method for solving singular initial value problems of nonlinear systems. Appl. Math. Inform. Sci. **10**(2), 647–656. In press.
16. Komashynska, I., Al-Smadi, M., Ateiwi, A., Al e'damat, A.: An oscillation of the solution for a nonlinear second-order stochastic differential equation. J. Comput. Anal. Appl. **20**(5), 860–868 (2016)
17. Lin, Y., Cui, M., Yang, L.: Representation of the exact solution for a kind of nonlinear partial differential equations. Appl. Math. Lett. **19**, 808–813 (2006)
18. Lakshmikantham, V., Leela, S.: Remarks on first and second order periodic boundary value problems. Nonlinear Anal. **8**, 281–287 (1984)
19. Lakshmikantham, V.: Periodic boundary value problems of first and second order differential equations. J. Appl. Math. Simulat. **2**, 131–138 (1989)
20. Lia, Y.: Positive solutions of fourth-order periodic boundary value problems. Nonlinear Anal. **54**, 1069–1078 (2003)
21. Li, C., Cui, M.: The exact solution for solving a class nonlinear operator equations in the reproducing kernel space. Appl. Math. Comput. **143**, 393–399 (2003)
22. Peng, S.: Positive solutions for first order periodic boundary value problem. Appl. Math. Comput. **158**, 345–351 (2004)

http://www.springer.com/978-3-319-32855-3

Differential and Difference Equations with Applications
ICDDEA, Amadora, Portugal, May 2015, Selected
Contributions
Pinelas, S.; Došlá, Z.; Došlý, O.; Kloeden, P.E. (Eds.)
2016, XV, 451 p. 36 illus., 18 illus. in color., Hardcover
ISBN: 978-3-319-32855-3