Pure Strategy Nash Equilibrium and Simultaneous-Move Games with Complete Information

2

Introduction

This chapter analyzes behavior in relatively simple strategic settings: simultaneous-move games of complete information. Let us define the two building blocks of this chapter: best responses and Nash equilibrium.

Best response. A strategy s_i^* is a best response of player i to a strategy profile s_{-i} selected by other players if it provides player i with a weakly larger payoff than any of his available strategies $s_i \in S_i$. Formally, strategy s_i^* is a best response to s_{-i} if and only if

$$u_i\left(s_i^*, s_{-i}\right) \geq u_i(s_i, s_{-i}) \text{ for all } s_i \in S_i.$$

We then say that strategy s_i^* is a best response to s_{-i}, and denote it as $s_i^* \in BR(s_{-i})$.

For instance, in a two-player game, s_1^* is a best response for player 1 to strategy s_2 selected by player 2 if and only if $u_1\left(s_1^*, s_2\right) \geq u_1(s_1, s_2)$ for all $s_1 \in S_1$ thus implying that $s_1^* \in BR_1(s_2)$.

We next define a Nash equilibrium by requiring that every player uses best responses to his opponents' strategies, i.e., players use mutual best responses.

Nash equilibrium. Strategy profile $s^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a Nash equilibrium if every player i's strategy is a best response to his opponents' strategies; that is, if for every player i his strategy s_i^* satisfies

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i(s_i, s_{-i}^*) \text{ for all } s_i \in S_i$$

or, more compactly, strategy s_i^* is a best response to s_{-i}^*, i.e., $s_i^* \in BR_i(s_{-i}{}^*)$.

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In words, every player plays a best response to his opponents' strategies, and his conjectures about his opponents' behavior must be correct (otherwise, players could have incentives to modify their strategies and, thus, not be in equilibrium). As a consequence, players do not have incentives to deviate from their Nash equilibrium strategies; and we can understand such strategy profile as stable.

We initially focus on games where two players select between two possible strategies, such as the Prisoner's Dilemma game (where two prisoners decide to either cooperate or defect), and the Battle of the Sexes game (where a husband and a wife choose whether to attend the football game or the opera). Afterwards, we explain how to find best responses and equilibrium behavior in games where players choose among a continuum of strategies, such as in the Cournot game of quantity competition, and games where players' actions impose externalities on other players. Furthermore, we illustrate how to find best responses in games with more than two players, and how to identify Nash Equilibria in these contexts.

We finish this chapter with one application from law and economics about the incentives to commit crimes and to prosecute them through law enforcement, and a Cournot game in which the merging firms benefit from efficiency gains.[1]

Exercise 1—Prisoner's Dilemma[A]

Two individuals have been detained for a minor offense and confined in separate cells. The investigators suspect that these individuals are involved in a major crime, and separately offer each prisoner the following deal, as depicted in the matrix in Fig. 2.1.: If you confess while your partner doesn't, you will leave today without serving any time in jail; if you confess and your partner also confesses, you will serve 5 years in jail; if you don't confess and your partner does, you have to serve 15 years in jail (since you did not cooperate with the prosecutor but your partner provided us with evidence against you); finally, if none of you confess, you will serve one year in jail (since we only have limited evidence against you). If both players must simultaneously choose whether or not to confess, and they cannot coordinate their strategies, which is the Nash Equilibrium (NE) of the game?

Answer

Every player $i = \{1, 2\}$ has a strategy space of $S_i = \{C, NC\}$. In a NE, every player has complete information about all players' strategies and maximizes his own payoff, taking the strategy of his opponents as given. That is, every player selects his best response to his opponents' strategies. Let's start finding the best responses of player 1, for each of the possible strategies of player 2.

[1]While the Nash equilibrium solution concept allows for many applications in the area of industrial organization, we only explore some basic examples in this chapter, relegating many others to Chap. 5 (Applications to Industrial Organization).

		Player 2	
		confess	Not confess
Player 1	confess	-5, -5	0, -15
	Not confess	-15, 0	-1, -1

Fig. 2.1 Prisoner’s dilemma game (Normal-form)

Player 1

If player 2 confesses (in the left-hand column), player 1’s best response is to Confess, since his payoff from doing so, −5, is larger than that from not confessing, −15.[2] This is indicated in Fig. 2.2 by underlining the payoff that player 1 obtains from playing this best response, −5. If, instead, player 2 does Not confess (in the right-hand column), player 1’s best response is to confess, given that his payoff from doing so, 0, is larger than that from not confessing, −1.[3] This is also indicated in Fig. 2.2 with the underlined best-response payoff 0. Hence, we can compactly represent player 1’s best response ($BR_1(s_2)$) as $BR_1(C) = C$ to Confess, and $BR_1(NC) = C$ to not confess. Importantly, this implies that player 1 finds confess a strictly dominant strategy, as he chooses to confess regardless of what player 2 does.

Player 2

A similar argument applies for player 2. In particular, since the game is symmetric, we find that: (1) when player 1 confesses (in the top row), player 2’s best response is to Confess, since $-5 > -15$; and (2) when player 1 does Not confess (in the bottom row), player 2’s best response is to Confess, since $0 > -1$.[4] Hence, player 2’s best response can be expressed as $BR_2(C) = C$ and $BR_2(NC) = C$, also indicating that Confess is a strictly dominant strategy for player 2, since he selects this

[2] A common trick many students use in order to be able to focus on the fact that we are examining the case in which player 2 confesses (in the left-hand column) is to cover with their hand (or a piece of paper) the columns in which player 2 selects strategies different from Confess (in this case, that means covering Not confess, but in larger matrices it would imply covering all columns except for the one we are analyzing at that point.) Once we focus on the column corresponding to Confess, player 1’s best response becomes a straightforward comparison of his payoff from Confess, −5, and that from Not confess, −15, which helps us underline the largest of the two payoffs, i.e., −5.

[3] In this case, you can also focus on the column corresponding to Not confess by covering the column of Confess with your hand. This would allow you to easily compare player 1’s payoff from Confess, 0, and Not confess, −1, underlining the largest of the two, i.e., 0.

[4] Similarly as for player 2, you can now focus on the row selected by player 1 by covering with your hand the row he did not select. For instance, when player 1 chooses Confess, you can cover the row corresponding to Not confess, which allows for an immediate comparison of the payoff when player 2 responds with Confess, −5, and when he does not, −15, and underline the largest of the two, i.e., −5. An analogous argument applies to the case in which player 1 selects Not confess, where you can cover the row corresponding to Confess with your hand.

		Player 2	
		confess	Not confess
Player 1	confess	-5, -5	0, -15
	Not confess	-15, 0	-1, -1

Fig. 2.2 Prisoner's dilemma game (Normal-form)

strategy regardless of his opponent's strategy. Payoffs associated with player 2's best responses are underlined in Fig. 2.3. with red color.

We can now see that there is a single cell in which both players are playing a mutual best response, (C, C), as indicated by the fact that both players' payoffs are underlined (i.e., both players are playing best responses to each other's strategies). Intuitively, since we have been underling best response payoffs, a cell that has the payoffs of all players underlined entails that every player is selecting a best response to his opponent's strategies, as required by the definition of NE. Therefore, strategy profile (C, C) is the unique Nash Equilibrium (NE) of the game.

$$NE = (C, C)$$

Equilibrium vs. Efficiency. This outcome is, however, inefficient since it does not maximize social welfare (where social welfare is understood as the sum of both players' payoffs). In particular, if players could coordinate their actions, they would both select not to confess, giving rise to outcome (NC, NC), where both players' payoffs strictly improve relative to the payoff they obtain in the equilibrium outcome (C, C), i.e., they would only serve one year in jail rather than five years. This is a common feature in several games with intense competitive pressures, in which a conflict emerges between individual incentives (to confess in this example) and group/society incentives (not confess). Finally, notice that the NE is consistent with IDSDS. Indeed, since both players use strictly dominant strategies in the NE of the game, the equilibrium outcome according to NE coincides with that resulting from the application of IDSDS.

		Player 2	
		confess	Not confess
Player 1	confess	-5, -5	0, -15
	Not confess	-15, 0	-1, -1

Fig. 2.3 Prisoner's dilemma game (Normal-form)

Exercise 2—Battle of the Sexes[A]

A husband and a wife are leaving work, and do not remember which event they are attending to tonight. Both of them, however, remember that last night's argument was about either attending to the football game (the most preferred event for the husband) or the opera (the most preferred event for the wife). To make matters worse, their cell phones broke, so they cannot call each other to confirm which event they are attending to. As a consequence, they must simultaneously and independently decide whether to attend to the football game or the opera.

The payoff matrix in Fig. 2.4 describes the preference of the husband (wife) for the football game (opera, respectively), but also indicates that both players prefer to be together rather than being alone (even if they are alone at their most preferred event). Find the set of Nash Equilibria of this game.

Answer
Every player $i = \{H, W\}$ has strategy set $S_i = \{F, O\}$. In order to find the Nash equilibrium of this game, let us separately identify the best responses of each player to his opponent's strategies.

Husband
Let's first analyze the husband's best responses $BR_H(F)$ and $BR_H(O)$. If his wife goes to the football game (focusing our attention in the left-hand column), the husband prefers to also attend the football game since his payoff from doing so, 3, exceeds that from attending the opera by himself, 0. If, instead, his wife attends the opera (in the right-hand column), the husband prefers to attend the opera with her, given that his payoff from doing so, 1, while low (he dislikes opera!), is still larger than that from going to the football game alone, 0. Hence we can summarize the husband's best response as $BR_H(F) = F$ and $BR_H(O) = O$; as indicated in the underlined payoffs in the matrix of Fig. 2.5. Intuitively, the husband's best response is thus to attend the same event as his wife.

Wife
A similar argument applies to the wife, who also best responds by attending the same event as her husband, i.e., $BR_W(F) = F$ in the top row when her husband attends the football game, and $BR_W(O) = O$ in the bottom row when he goes to the opera; as illustrated in the payoffs underlined in red color in the matrix of Fig. 2.6.

		Wife	
		Football	Opera
Husband	Football	3, 1	0, 0
	Opera	0, 0	1, 3

Fig. 2.4 Battle of the sexes game (Normal-form representation)

		Wife	
		Football	Opera
Husband	Football	$\underline{3}$, 1	0, 0
	Opera	0, 0	$\underline{1}$, 3

Fig. 2.5 Battle of the sexes game—underlining best response payoffs for Husband

		Wife	
		Football	Opera
Husband	Football	$\underline{3}$, $\underline{1}$	0, 0
	Opera	0, 0	$\underline{1}$, $\underline{3}$

Fig. 2.6 Battle of the sexes game—underlining best response payoffs for Husband and Wife

Therefore, we found two strategy profiles in which both players are playing mutual best responses: (F, F) and (O, O), as indicated by the two cells in the matrix where both players' payoffs are underlined. Hence, this game has two pure-strategy Nash equilibria (*psNE*): (F, F), where both players attend the football game, and (O, O), where they both attend the opera. This can be represented formally as:

$$psNE = \{(F, F), (O, O)\}$$

Exercise 3—Pareto Coordination[A]

Consider the game in Fig. 2.7, played by two firms $i = \{1, 2\}$, each of them simultaneously and independently selecting to adopt either technology A or B. Technology A is regarded as superior by both firms, yielding a payoff of 2 to each firm if they both adopt it, while the adoption of technology B by both firms only entails a payoff of 1. Importantly, if firms do not adopt the same technology, both

		Firm 2	
		Technology A	Technology B
Firm 1	Technology A	2, 2	0, 0
	Technology B	0, 0	1, 1

Fig. 2.7 Pareto coordination game (Normal-form)

		Firm 2	
		Technology A	Technology B
Firm 1	Technology A	$\underline{2}$, 2	0, 0
	Technology B	0, 0	$\underline{1}$, 1

		Firm 2	
		Technology A	Technology B
Firm 1	Technology A	$\underline{2}$, $\underline{2}$	0, 0
	Technology B	0, 0	$\underline{1}$, $\underline{1}$

Fig. 2.8 Pareto coordination game—underlining best response payoffs for Firm 1 (left matrix) and for both firms (right matrix)

obtain a payoff of zero. This can be explained because, even if firm i adopts technology A, such a technology is worthless if firm i cannot exchange files, new products and practices with the other firm $j \neq i$. Find the set of Nash Equilibria (NE) in this game.

Answer
Firm 1. Let's first examine firm 1's best response. Similarly as in the battle of the sexes game, firm 1's best response is to adopt the same technology as firm 2, i.e., $BR_1(A) = A$ when firm 2 chooses technology A, and $BR_1(B) = B$ when firm 2 selects technology B; as indicated in the payoffs underlined in blue color in the left-hand matrix of Fig. 2.8.
Firm 2. A similar argument applies to firm 2, since firms' payoffs are symmetric, i.e., $BR_2(A) = A$ and $BR_2(B) = B$; as depicted in the payoffs underlined in red color in the right-hand matrix.

Hence, we found two pure strategy Nash equilibria: (A, A) and (B, B), which are depicted in the matrix as the two cells were both players' payoffs are underlined.[5]

$$psNE = \{(A, A), (B, B)\}$$

Finally, note that, while either of the two technologies could be adopted in equilibrium, only one of them is efficient, (A, A), while the other equilibrium, (B, B), is inefficient, i.e., both firms would be better off if they could coordinate their simultaneous adoption of technology A.[6]

Exercise 4—Cournot game of Quantity Competition[A]

Consider an industry with two firms competing in quantities, i.e., Cournot competition. For simplicity, assume that firms are symmetric in costs, $c > 0$, with no fixed costs and that they face a linear inverse demand $p(Q) = a - bQ$, where $a > c$,

[5]In both of these Nash equilibria, firms are playing mutual best responses, and thus no firm has incentives to unilaterally deviate.

[6]However, no firm has incentives to unilaterally move from technology B to A when its competitor is selecting technology B.

$b > 0$, and Q denotes aggregate output. Note that the assumption $a > c$ implies that the highest willingness to pay for the first unit is larger than the marginal cost that firms must incur in order to produce the first unit, thus indicating that a positive production level is profitable in this industry. If firms simultaneously and independently select their output level, q_1 and q_2, find the Nash Equilibrium (NE) of the Cournot game of quantity competition.

Answer

The profits of firm i are given by:

$$\pi_i = p(Q) \cdot q_i - cq_i$$

Given that $Q = q_1 + q_2$, every firm i chooses its production level q_i, taking the output level of its rival, q_j, as given. That is, every firm i solves

$$\max_{q_i} \left(a - bq_i - bq_j\right)q_i - cq_i$$

Taking first-order conditions with respect to q_i, we obtain

$$a - 2bq_i - bq_j - c = 0$$

and solving for q_i, we find firm i's best response function, $BR_i(q_j)$ that is,

$$q_i\left(q_j\right) = \frac{a-c}{2b} - \frac{1}{2}q_j$$

Figure 2.9 depicts the best response function of firm i, which originates at $\frac{a-c}{2b}$, indicating the production level that firm i sells when firm j is inactive, i.e., the monopoly output level; and decreases as its rival, firm j, produces a larger amount of output. Intuitively, firm i's and j's output are strategic substitutes, so that firm i is forced to sell fewer units when the market becomes flooded of firm j's products. When firm j's production is sufficiently large, i.e., firm j produces more than $\frac{a-c}{b}$ units, firm i is forced to remain inactive, i.e., $q_i^* = 0$.[7] This property is illustrated in the figure by the fact that firm i's best response function coincides with the horizontal axis (zero production) for all $q_j > \frac{a-c}{b}$.

A similar argument applies to firm j, obtaining best response function $q_j(q_i) = \frac{a-c}{2b} - \frac{1}{2}q_i$, as depicted in Fig. 2.10. (Note that we use the same axis, in order to be able to represent both best response functions in the same figure in our ensuing discussion.)

If we superimpose firm j's best response function on top of firm i's, we can visually see that the point where both functions cross each other represents the Nash Equilibrium of the Cournot game of quantity competition (Fig. 2.11).

[7] In order to obtain the output level of firm j that forces firm i to be inactive, set $q_i = 0$ on firm i's best response function, and solve for q_j. The output you obtain should coincide with the horizontal intercept of firm i's best response function in Fig. 2.9.

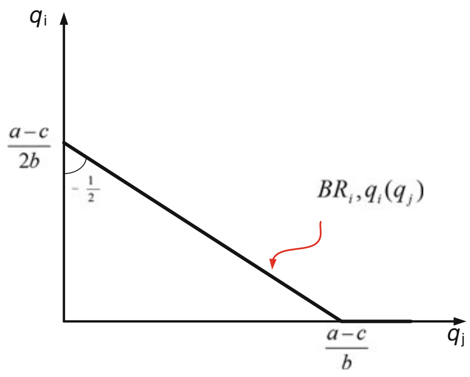

Fig. 2.9 Cournot game—Best response function of firm *i*

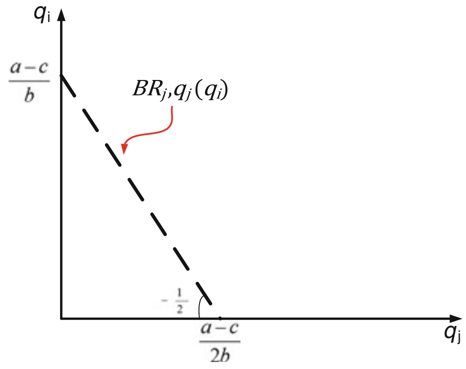

Fig. 2.10 Cournot game—Best response function of firm *j*

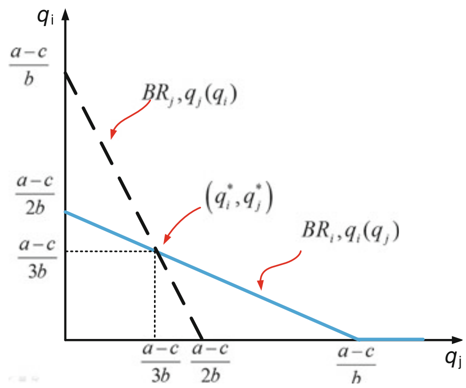

Fig. 2.11 Cournot game—Best response functions and Nash-equilibrium

In order to precisely find the point in which both best response functions cross each other, let us simultaneously solve for q_i^* and q_j^* by, for instance, plugging $q_j(q_i) = \frac{a-c}{2b} = \frac{1}{2}q_i$ into $q_i(q_j)$, as we do next,

$$q_i^* = \frac{a-c}{2b} - \frac{1}{2}\left(\frac{a-c}{2b} - \frac{1}{2}q_i^*\right)$$

which simplifies to

$$q_i^* = \frac{a-c+bq_i^*}{4b}$$

and solving for q_i^*, we find the equilibrium output level for firm i,

$$q_i^* = \frac{a-c}{3b}$$

and that of firm j,

$$q_j^* = \frac{a-c}{2b} - \frac{1}{2}\frac{a-c}{3b} = \frac{a-c}{3b}.$$

As we can see from the results, both firms produce exactly the same quantities, since they both have the same technology (they both face the same production costs). Hence, the pure strategy Nash Equilibrium is:

$$psNE = \left\{q_i^*, q_j^*\right\} = \left\{\frac{a-c}{3b}, \frac{a-c}{3b}\right\}$$

(Notice that this exercise assumes, for simplicity, that firms are symmetric in their production costs. In subsequent chapters we investigate how firms' equilibrium production is affected when one of them exhibits a cost advantage ($c_i < c_j$). We also examine how firms' competition is affected when more than two firms interact in the same industry. see Chap. 5 for more details.)

Exercise 5—Games with Positive Externalities[B]

Two neighboring countries, $i = 1, 2$, simultaneously choose how many resources (in hours) to spend in recycling activities, r_i. The average benefit (π_i) for every dollar spent on recycling is:

$$\pi_i\left(r_i, r_j\right) = 10 - r_i + \frac{r_j}{2},$$

and the (opportunity) cost per hour for each country is 4. Country *i's* average benefit is increasing in the resources that neighboring country *j* spends on his recycling because a clean environment produces positive external effects on other countries.

Part (a) Find each country's best-response function, and compute the Nash Equilibrium (NE), (r_1*, r_2*)
Part (b) Graph the best-response functions and indicate the pure strategy Nash Equilibrium on the graph.
Part (c) On your previous figure, show how the equilibrium would change if the intercept of one of the countries' average benefit functions fell from 10 to some smaller number.

Answer

Part (a) Since the gains of recycling are given by $(\pi_i \cdot r_i)$, and the costs of the activity are ($4r_i$), country 1's maximization problem consists of selecting the amount of hours devoted to recycling r_1 that solves:

$$\max_{r_1}\left(10 - r_1 + \frac{r_2}{2}\right)r_1 - 4r_1$$

Taking the first-order condition with respect to r_1

$$10 - 2r_1 + \frac{r_2}{2} - 4 = 0$$

Rearranging and solving for r_1 yields country 1's best-response function (BRF_1):

$$r_1(r_2) = 3 + \frac{r_2}{4}$$

Symmetrically, Country 2's best-response function is

$$r_2(r_1) = 3 + \frac{r_1}{4}$$

Inserting best-response function $r_2(r_1)$ into $r_1(r_2)$ yields

$$r_1 = 3 + \frac{3 + \frac{r_1}{4}}{4},$$

and, rearranging, we obtain an equilibrium level of recycling of r_1* = 4 for country 1. Hence, country 2's equilibrium recycling level is

$$r_2^* = 3 + \frac{4}{4} = 4$$

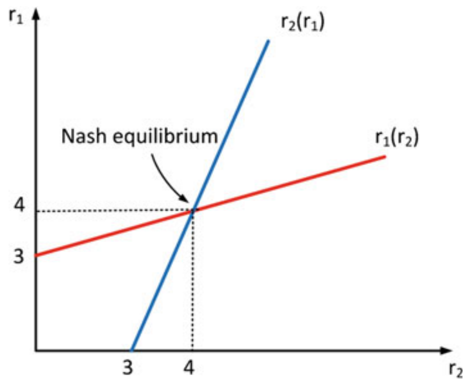

Fig. 2.12 Positive externalities—Best response functions and Nash-equilibrium

Note that, alternatively, countries' symmetry implies $r_1^* = r_2^*$. Hence, in BRF_1 we can eliminate the subscript (since both countries' recycling level coincides in equilibrium), and thus $r = 3 + \frac{r}{4}$, which, solving for r yields a symmetric equilibrium recycling of $r^* = 4$.

Hence, the psNE is given by:

$$psNE = \{r_1^* = 4,\ r_2^* = 4\}$$

Part (b) Both best response functions originate at 3 and increase with a positive slope of 1/4, as depicted in Fig. 2.12. Intuitively, countries' strategies are strategic complements, since an increase in r_2 induces Country 1 to strategically increase its own level of recycling, r_1, by 1/4.

Part (c) A reduction in the benefits from recycling produces a fall in the intercept of one of the countries' average benefit function, for example in Country 2. This change is indicated in Fig. 2.13 by the leftward shift (following the arrow) in

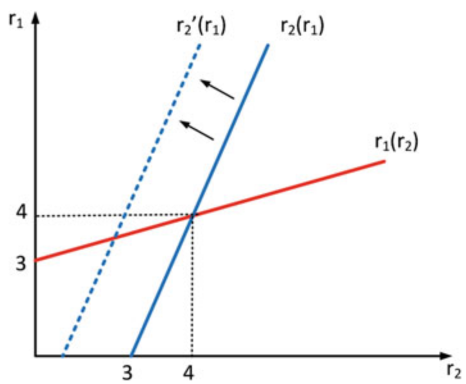

Fig. 2.13 Shift in best response functions, change in Nash-equilibrium

Country 2's best response function. In the new Nash Equilibrium, Country 2 recycles a lot less while Country 1 recycles a little less.

Exercise 6—Traveler's Dilemma[B]

Consider the following game, often referred to as the "traveler's dilemma." An airline loses two identical suitcases that belong to two different travelers. The airline is liable for up to \$100 per suitcase. The airline manager, in order to obtain an honest estimate of each suitcase, separates each traveler i in a different room and proposes the following game: "Please write an integer $x_i \in [2, 100]$ in this piece of paper. As a manager, I will use the following reimbursement rule:

- If both of you write the same estimate, $x_1 = x_2 = x$, each traveler gets x.
- If one of you writes a larger estimate, i.e., $x_i > x_j$ where $i \neq j$, then:
  - The traveler who wrote the *lowest* estimate (traveler j) receives $x_j + k$, where $k > 1$; and
  - The traveler who wrote the *largest* estimate (traveler i) only receives $\max\{0, x_j - k\}$."

Part (a) Show that asymmetric strategy profiles, in which travelers submit different estimates, cannot be sustained as Nash equilibria.
Part (b) Show that symmetric strategy profiles, in which both travelers submit the same estimate, and such estimate is strictly larger than 2, cannot be sustained as Nash equilibria.
Part (c) Show that the symmetric strategy profile in which both travelers submit the same estimate $(x_1, x_2) = (2, 2)$ is the unique pure strategy Nash equilibrium.
Part (d) Does the above result still hold when the traveler writing the largest amount receives $x_j - k$ rather than $\max\{0, x_j - k\}$? Intuitively, since $k > 1$ by definition, a traveler can now receive a negative payoff if he submits the lowest estimate and $x_j < k$.

Answer

Part (a) We first show that asymmetric strategy profiles, (x_1, x_2) with $x_1 \neq x_2$, cannot be sustained as a Nash equilibrium. Consider, without loss of generality, that player 1 submits a higher estimate than player 2, $x_1 > x_2$. In this setting, it is easy to see that player 1 has incentives to deviate: he now obtains a payoff of $\max\{0, x_2 - k\}$, and he could increase his payoff by submitting an estimate that matches that of player 2, i.e., $x_1 = x_2$, which guarantees him a payoff of x_2 (as now the estimates from both travelers coincide), where

$$\max\{0, x_2 - k\} < x_2 \text{ for all } x_2 \text{ given that } k > 1.$$

Part (b) Using a similar argument, we can show symmetric strategy profiles in which both travelers submit the same estimate (but higher than two), i.e., (x_1, x_2), with $x_1 = x_2 > 2$, cannot be supported as Nash equilibria either. To see this, note that in such strategy profile every player i obtains a payoff $x_i = x_j$, but he can increase his payoff by deviating towards a lower estimate, i.e., $x_i = x_j - 1$ since estimates must be integer numbers. With such a deviation, player i's estimate becomes the lowest, and he thus obtains a payoff of

$$x_i + k = \left(x_j - 1\right) + k,$$

where $\left(x_j - 1\right) + k > x_j$ since $k > 1$ by definition.

Part (c) Hence, the only remaining strategy profile is that in which both travelers submit an estimate of 2, $x_1 = x_2 = 2$. Let us now check if it can be sustained as a Nash equilibrium, by showing that every player i has no profitable deviation. Every traveler i obtains a payoff of 2 under the proposed strategy profile. If player i deviates towards a higher price, traveler i would be now submitting the highest estimate, and thus would obtain a payoff of

$$\max\{0, x_j - k\} = \max\{0, 2 - k\},$$

where $\max\{0, 2 - k\} < 2$ since $k > 1$ by definition. That is, submitting a higher estimate reduces player i's payoff. Finally, note that submitting a lower estimate is not feasible since estimates must satisfy $x_i \in [2, 100]$ by definition.

Alternative approach: While the above analysis tests whether a specific strategy profile can/cannot be sustained as Nash Equilibrium of the game, a more direct approach would identify each player's best response function, and then find the point where player 1's and 2's best response functions cross each other, which constitutes the NE of the game. For a given estimate from player j, x_j, if player i writes an estimate lower than x_j, $x_i < x_j$, player i obtains a payoff of $x_i + k$, which is larger than the payoff he obtains from matching player j's estimate, i.e., $x_i = x_j = x$, as long as $x_i + k > x_j$, that is, if $k > x_j - x_i$. Intuitively, player i profitably undercuts player j's estimate if x_i is not extremely lower than x_j.[8] If, instead, player i writes a larger estimate than player j, $x_i > x_j$, his payoff becomes $\max\left\{0, x_j - k\right\}$, which is lower than his payoff from matching player j's estimate, i.e., $x_i = x_j = x$, since $\max\left\{0, x_j - k\right\} < x_j$.

In summary, player i does not have incentives to submit a higher estimate than player j's, but rather an estimate that is k-units lower than player j's estimate. Hence, player i's best response function can be written as

$$x_i\left(x_j\right) = \max\left\{2, x_j - k\right\}$$

[8]For instance, if $x_j = 5$ and $k = 2$, then player i has incentives to write an estimate of $x_i = 5 - 2 = 3$, but not lower than 3 since his payoff, $x_i + k$, is increasing in his own estimate x_i.

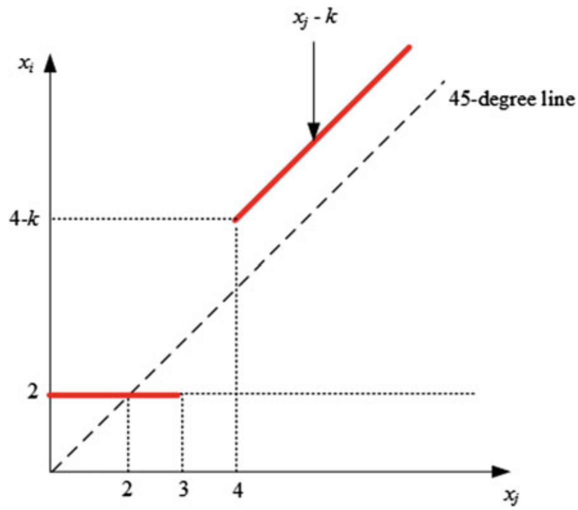

Fig. 2.14 Traveler's dilemma, best response functions

since the estimate that he writes must lie on the interval [2, 100]. This best response function is depicted in Fig. 2.14. Specifically, player i's best response function originates at $x_i = 2$ when player j submits $x_j = 2$; remains at $x_i = 2$ when $x_j = 3$ (since $k > 1$ entails that $3 - k < 2$); and becomes $x_i = \max\{2, 4 - k\}$ when $x_j = 4$, thus increasing in x_j. For instance, if $k = 2$, player i's best response function is $x_i = 2$ when player j's estimate is $x_j = 2, 3, 4$, but increases to $x_i = 3$ when player j's estimate is $x_j = 5$, and generally becomes $x_i(x_j) = x_j - 2$ for all $x_j > 4$. Graphically, this function has a flat segment for low values of x_j, but then increases in x_j in a straight line located k-units below the 45-degree line. A similar argument applies to player j's best response function. Hence, player 1's and 2's best response functions only cross at $x_i = x_j = 2$ (Fig. 2.14).

Part (d) Our above argument did not rely on the property of positive payoffs for the traveler submitting the highest estimate. Hence, all the previous proof applies to this reimbursement rule as well, implying that $x_1 = x_2 = 2$ is the unique pure strategy Nash equilibrium of the game.

Exercise 7—Nash Equilibria with Three Players[B]

Find all the Nash equilibria of the following three-player game (see Fig. 2.15), in which player 1 selects rows (a, b, or c), player 2 chooses columns (x, y, or z), and player 3 selects a matrix (either A in the left-hand matrix, or B in the right-hand matrix).

Player 3: matrix A

Player 2

		x	y	z
Player 1	a	2,0,4	1,1,1	1,2,3
	b	3,2,3	0,1,0	2,1,0
	c	1,0,2	0,0,3	3,1,1

Player 3: matrix B

Player 2

		x	y	z
Player 1	a	2,0,3	4,1,2	1,1,2
	b	1,3,2	2,2,2	0,4,3
	c	0,0,0	3,0,3	2,1,0

Fig. 2.15 Normal-form game with 3 players

Player 3: matrix A

Player 2

		x	y	z
Player 1	a	2,0,4	1,1,1	1,2,3
	b	3,2,3	0,1,0	2,1,0
	c	1,0,2	0,0,3	3,1,1

Player 3: matrix B

Player 2

		x	y	z
Player 1	a	2,0,3	4,1,2	1,1,2
	b	1,3,2	2,2,2	0,4,3
	c	0,0,0	3,0,3	2,1,0

Fig. 2.16 Normal-form game with 3 players. BR_3 when player 2 chooses x

Answer

Player 3. Let's start by evaluating the payoffs for player 3 when Player 2 selects x (first column). The arrows in Fig. 2.16 help us keep track of player 3's pairwise comparison. For instance, when player 1 chooses a and player 2 selects x (in the top left-hand corner of either matrix), player 3 prefers to respond with matrix A, which gives him a payoff of 4, rather than with B, which only yields a payoff of 3. This comparison is illustrated by the top arrow in Fig. 2.16. A similar argument applies for the second arrow, which fixes the other players' strategy profile at (b, x), and for the third arrow, which fixes their strategy profile at (c, x). The highest payoff that player 3 obtains in each of these three pairwise comparisons is circled in Fig. 2.16.

Hence, we obtain that player 3's best responses are $BR_3(x, a) = A$, $BR_3(x, b) = A$, and $BR_3(x, c) = A$.

If player 2 selects y (in the second column of each matrix), player 3's pairwise comparisons are given by the three arrows in Fig. 2.17. In terms of best responses, this implies that $BR_3(y, a) = B$, $BR_3(y, b) = B$, and $BR_3(y, c) = \{A, B\}$. The highest payoff that player 3 obtains in each pairwise comparison are also circled in Fig. 2.17.

Player 3: matrix *A*

Player 2

		x	y	z
Player 1	a	2,0,4	1,1,1	1,2,3
	b	3,2,3	0,1,0	2,1,0
	c	1,0,2	0,0,3	3,1,1

Player 3: matrix *B*

Player 2

		x	y	z
Player 1	a	2,0,3	4,1,2	1,1,2
	b	1,3,2	2,2,2	0,4,3
	c	0,0,0	3,0,3	2,1,0

Fig. 2.17 Normal-form game with 3 players. BR_3 when player 2 chooses y

Player 3: matrix *A*

Player 2

Player 1	x	y	z
a	2,0,$\underline{4}$	1,1,1	1,2,3
b	3,2,$\underline{3}$	0,1,0	2,1,0
c	1,0,$\underline{2}$	0,0,$\underline{3}$	3,1,1

Player 3: matrix *B*

Player 2

Player 1	x	y	z
a	2,0,3	4,1,$\underline{2}$	1,1,2
b	1,3,2	2,2,$\underline{2}$	0,4,3
c	0,0,0	3,0,$\underline{3}$	2,1,0

Fig. 2.18 Normal-form game with 3 players. BR_3 when player 2 chooses z

Player 3: matrix *A*

Player 2

Player 1	x	y	z
a	2,0,$\underline{4}$	1,1,1	1,2,$\underline{3}$
b	3,2,$\underline{3}$	0,1,0	2,1,0
c	1,0,$\underline{2}$	0,0,$\underline{3}$	3,1,$\underline{1}$

Player 3: matrix *B*

Player 2

Player 1	x	y	z
a	2,0,3	4,1,$\underline{2}$	1,1,2
b	1,3,2	2,2,$\underline{2}$	0,4,$\underline{3}$
c	0,0,0	3,0,$\underline{3}$	2,1,0

Fig. 2.19 Normal-form game with 3 players. Underlining player 3's response payoffs

If player 2 selects z (in the third column of each matrix), player 3's pairwise comparisons are depicted by the three arrows in Fig. 2.18 which in terms of best responses yields $BR_3(z, a) = A$, $BR_3(z, b) = B$, and $BR_3(z, c) = A$. Hence, the payoff matrix that arises after underlying (or circling) the payoff corresponding to the best responses of player 3 is the following (see Fig. 2.19)

Player 2. Let's now identify player 2's best responses as depicted in the circled payoffs of the matrices in Fig. 2.20. In particular, we take player 1's strategy as given (fixing the row) and player 3's as given (fixing the matrix) (where player 3 chooses matrix A). We obtain that player 2's best responses are $BR_2(a, A) = z$ when player 1 chooses a (in the top row), $BR_2(b, A) = x$ when player 1 selects b (in the middle row), and $BR_2(c, A) = z$ when player 1 chooses c (in the bottom row). Visually, notice that we are now fixing our attention on a matrix (strategy of player 3) and on a row (strategy of player 1), and horizontally comparing the payoff that player 2 obtains from selecting the left, middle or right-hand column. Similarly, when player 3 chooses matrix B, we obtain that player 2's best responses are $BR_2(a, B) = \{y, z\}$ when player 1 selects a (in the top row) since both y and z yield the same payoff, \$1, $BR_2(b, B) = x$ when player 1 chooses b (in the middle row), and $BR_2(c, B) = z$ when player 1 selects c (in the bottom row).[9]

Therefore, the matrices that arise after underlying the best response payoffs of player 2 are those in Fig. 2.21.

[9]Visually, this implies fixing your attention on the first row of the left-hand matrix, and horizontally search for which strategy of player 2 (column) provides this player with the highest payoff.

Player 3: matrix *A*

Player 2

Player 1		x	y	z
	a	$2,0,\underline{4}$	1,1,1	$1,2,\underline{3}$
	b	$3,2,\underline{3}$	0,1,0	2,1,0
	c	$1,0,\underline{2}$	$0,0,\underline{3}$	$3,1,\underline{1}$

Player 3: matrix *B*

Player 2

Player 1		x	y	z
	a	2,0,3	$4,1,\underline{2}$	1,1,2
	b	1,3,2	$2,2,\underline{2}$	$0,4,\underline{3}$
	c	0,0,0	$3,0,\underline{3}$	2,1,0

Fig. 2.20 Normal-form game with 3 players. Circling best responses of player 2

Player 3: matrix *A*

Player 2

Player 1		x	y	z
	a	$2,0,\underline{4}$	1,1,1	$1,\underline{2},\underline{3}$
	b	$3,\underline{2},\underline{3}$	0,1,0	2,1,0
	c	$1,0,\underline{2}$	$0,0,\underline{3}$	$3,\underline{1},\underline{1}$

Player 3: matrix *B*

Player 2

Player 1		x	y	z
	a	2,0,3	$4,\underline{1},\underline{2}$	$1,\underline{1},2$
	b	1,3,2	$2,2,\underline{2}$	$0,\underline{4},\underline{3}$
	c	0,0,0	$3,0,\underline{3}$	$2,\underline{1},0$

Fig. 2.21 Normal-form game with 3 players

Player 1. Let's finally identify player 1's best responses, given player's 2 strategy (fixing the column) and given player 3's strategy (fixing the matrix).

When player 3 chooses A (left matrix), player 1's best responses become $BR_1(x, A) = b$ when player 2 chooses x (left-hand column), $BR_1(y, A) = a$ when player 2 selects y (middle column), and $BR_1(z, A) = c$ when player 2 chooses z (right-hand column). Visually, notice that we are now fixing our attention on a matrix (strategy of player 3) and on a column (strategy of player 2), and vertically comparing the payoff that player 1 obtains from choosing the top, middle or bottom row[10]. Operating in an analogous fashion when player 3 chooses B (right-hand matrix), we obtain player 1's best responses: $BR_1(x, B) = a$ when player 2 chooses x (in the left-hand column), $BR_1(y, B) = a$ when player 2 selects y (middle column), and $BR_1(z, B) = c$ when player 2 chooses z (right-hand column).

We hence found three pure strategy Nash equilibria: (b, x, A), (c, z, A) and (a, y, B); as depicted in the cells where the payoffs of all players are underlined (Fig. 2.22), as these cells correspond to outcomes where players employ mutual best responses to each others' strategies (Fig. 2.23).

In summary, the Nash equilibria of this three player game are

$$psNE = \{(b, x, A), (c, z, A), (a, y, B)\}$$

[10]For instance, in finding $BR_1(x, A)$, we fix the matrix in which player 3 selects A (left matrix), and the column that player 2 selects x (left-hand column), and compare the payoffs that player 1 would obtain from responding with the first row (a), \$2, the second row (b), \$3, or with the third row (c), \$1. Hence, $BR_1(x, A) = b$. A similar argument applies to other best responses of player 1.

Player 3: matrix A

Player 1 \ Player 2	x	y	z
a	$2,0,\underline{4}$	$\underline{1},1,1$	$1,\underline{2},\underline{3}$
b	$\underline{3},\underline{2},\underline{3}$	$0,1,0$	$2,1,0$
c	$1,0,\underline{2}$	$0,0,\underline{3}$	$\underline{3},\underline{1},\underline{1}$

Player 3: matrix B

Player 1 \ Player 2	x	y	z
a	$\underline{2},0,3$	$\underline{4},\underline{1},\underline{2}$	$1,\underline{1},2$
b	$1,3,2$	$2,2,\underline{2}$	$0,\underline{4},\underline{3}$
c	$0,0,0$	$3,0,\underline{3}$	$\underline{2},\underline{1},0$

Fig. 2.22 Normal-form game with 3 players

Player 3: matrix A

Player 1 \ Player 2	x	y	z
a	$2,0,\underline{4}$	$\underline{1},1,1$	$1,\underline{2},\underline{3}$
b	$\boxed{\underline{3},\underline{2},\underline{3}}$	$0,1,0$	$2,1,0$
c	$1,0,\underline{2}$	$0,0,\underline{3}$	$\boxed{\underline{3},\underline{1},\underline{1}}$

Player 3: matrix B

Player 1 \ Player 2	x	y	z
a	$\underline{2},0,3$	$\boxed{\underline{4},\underline{1},\underline{2}}$	$1,\underline{1},2$
b	$1,3,2$	$2,2,\underline{2}$	$0,\underline{4},\underline{3}$
c	$0,0,0$	$3,0,\underline{3}$	$\underline{2},\underline{1},0$

Fig. 2.23 Normal-form game with 3 players. Nash equilibria

Exercise 8—Simultaneous-Move Games with $n \geq 2$ Players[B]

Consider a game with $n \geq 2$ players. Simultaneously and independently, the players choose between two options, X and Y. These options might represent, for instance, two available technologies for the n firms operating in an industry, e.g., selling smartphones with the Android operating system or, instead, opt for the newer Windows Phone operating system from Microsoft. That is, the strategy space for each player i is $S_i = \{X, Y\}$. The payoff of each player who selects X is:

$$2m_x - m_x^2 + 3$$

where m_x denotes the number of players who choose X. The payoff of each player who selects Y is

$$4 - m_y$$

where m_y is the number of players who choose Y. Note that $m_x + m_y = n$.

Part (a) For the case of only two players, $n = 2$, represent this game in its normal form, and find the pure-strategy Nash equilibria.
Part (b) Suppose now that $n = 3$. How many psNE does this game have?
Part (c) Consider now a game with $n > 3$ players. Identify an asymmetric psNE, i.e., an equilibrium in which a subset of players chooses X, while the remaining players choose Y.

Answer

Part (a) When both players choose X, m_x = 2 and m_y = 0, thus implying that every player's payoff is $2m_x - m_x^2 + 3$. Replacing $m_x = 2$, we obtain a payoff of

$$2(2) - (2)^2 + 3 = 3$$

for both players, as indicated in the cell corresponding to outcome (X, X) in the payoff matrix in Fig. 2.24. When, instead, both players choose Y, m_y = 2 and m_x = 0, and players' payoff becomes $4 - m_y = 4 - 2 = 2$; as depicted in outcome (Y, Y) of the payoff matrix. Finally, if only one player chooses X and another chooses Y, m_x = 1 and m_y = 1, this yield a payoff of

$$2m_x - m_x^2 + 3 = 2(1) - (1)^2 + 3 = 4$$

for the player who chose X, and $4 - m_y = 4 - 1 = 3$ for the player who chose Y; as represented in outcomes (X, Y) and (Y, X) in the payoff matrix (see cells away from the main diagonal in Fig. 2.24).

As usual, underlined payoffs represent a payoff corresponding to a player's best response, as we next separately describe for each player.

Player 1: In particular, player 1's best responses are $BR_1(X) = \{X, Y\}$ when player 2 chooses X (in the left-hand column) since player 1 is indifferent between responding with X (in the top row) or Y (in the bottom row) given that they both yield a payoff of $3. As a result, we underline both payoffs of $3 for player 1 in the column in which player 2 chooses X. If, instead, player 2 chooses Y (in the right-hand column), player 1's best response is $BR_1(Y) = \{X\}$, since player 1 obtains a higher payoff by selecting X ($4), than by choosing Y ($2). As a consequence, we underline the payoff of player 1 associated to his best response.

Player 2: Similarly, for player 2 we find that, when player 1 chooses X (in the top row), player 2 best responds with $BR_2(X) = \{X, Y\}$, since both X and Y yield a payoff of $3; while if player 1 selects Y (bottom row), player 2's best response is $BR_2(Y) = \{X\}$, since X yields a payoff of $4 while Y only entails a payoff of $2.

Therefore, since there are three cells where payoffs of all players have been underlined as the best responses, they represent strategy profiles where players' play mutual best responses, i.e. Nash equilibria of the game. There are, hence, three pure strategy Nash equilibrium in this game:

$$(\text{X, X}), (\text{X, Y}) \text{ and } (\text{Y, X}).$$

		Player 2	
		X	Y
Player 1	X	3,3	4,3
	Y	3,4	2,2

Fig. 2.24 Normal-form game with n = 2 players

Part (b) When introducing three players, the normal form representation of the game is depicted in the matrices of Figs. 2.25 and 2.26. (This is a standard three-player simultaneous-move game similar to that in the previous exercise).

In order to identify best responses, player 1 and 2 operate as in previous exercises, i.e., taking the action of player 3 (matrix player) as given, and comparing their payoffs across rows for player 1 and across columns for player 2. However, player 3 compares his payoffs across matrices, for a given strategy profile of player 1 and 2. In particular, this pairwise payoff comparison of player 3 is analogous to that depicted in Exercises 2.6 and 2.8 of this chapter. For instance, if player's 1 and 2 select (X, X), then player 3 obtains a payoff of only 0 if he were to select X as well (in the upper matrix), but a higher payoff of 3 if he, instead, selects Y (in the lower matrix). For this reason, we underline 3 in the third component of the cell corresponding to (X, X, Y), in the upper left-hand corner of the lower matrix. A similar argument applies for identifying other best responses of player 3, where we compare the third component of every cell across the two matrices. For instance, when player 1 and 2 select (X, Y), player 3 obtains a payoff of 3 if he chooses X (in the upper matrix), but only a payoff of 2 if he selects Y (in the lower matrix), which leads us to underline 3 in the third component of the payoff in the cell (X, Y) of the upper matrix.

Following a similar approach, we can see in Figs. 2.25 and 2.26 that there are three outcomes for which the payoffs of all players have been underlined (i.e., players are selecting mutual best responses). Specifically, the pure strategy Nash equilibria of the game with $n = 3$ players are:

$$psNE = \{(\mathrm{X}, \mathrm{Y}, \mathrm{X}), (\mathrm{Y}, \mathrm{X}, \mathrm{X}), (\mathrm{X}, \mathrm{X}, \mathrm{Y})\}$$

Part (c) When $n > 3$ players compete in this simultaneous-move game, the payoff from selecting strategy Y is

$$4 - m_y = 4 - (n - m_x)$$

		Player 2	
		X	Y
Player 1	X	0,0,0	$\underline{3},\underline{3},\underline{3}$
	Y	$\underline{3},\underline{3},\underline{3}$	2,2,$\underline{4}$

Fig. 2.25 Normal-form game when player 3 chooses s_3 = X

		Player 2	
		X	Y
Player 1	X	$\underline{3},\underline{3},\underline{3}$	$\underline{4}$,2,2
	Y	2,$\underline{4}$,2	1,1,1

Fig. 2.26 Normal-form game when player 3 chooses s_3 = Y

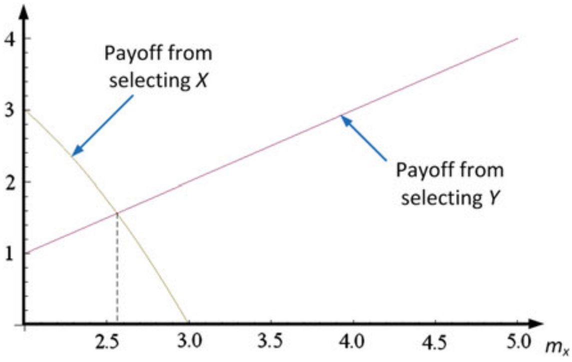

Fig. 2.27 Payoffs from selecting strategy X and Y

where the number of players selecting Y, m_y, is represented as those players who did not choose X, i.e., $m_y = n - m_x$. The payoff from selecting X is

$$2m_x - m_x^2 + 3$$

Hence, a player selects strategy X if and only if his payoff from selecting X is weakly higher than from choosing Y, that is

$$4 - (n - m_x) \geq 2m_x - m_x^2 + 3$$

Solving for m_x, yields that the number of players selecting strategy X is $m_x = \frac{1 \pm \sqrt{1-4(1-n)}}{2}$. For instance, in the case of $n = 5$ players, the above expression becomes $m_x = 2.56$ players, which implies that three players select X. (Note that the above result for m_x produces two roots, $m_x = 2.56$ and $m_x = -1.56$, but we only focus on the positive root.)

For illustration purposes, Fig. 2.27 depicts the payoff from selecting strategy Y when $n = 5$ interact, $4 - (5 - m_x) = m_x - 1$, and that from strategy X, $2m_x - m_x^2 + 3$. Intuitively, the payoff from strategy X is decreasing in the number of players choosing it, m_x (rightward movement in Fig. 2.27). Similarly, the payoff from selecting Y is also decreasing in the number of players choosing it, $n - m_x$; as depicted by leftward movements in Fig. 2.27. These incentives a negative network externality. For instance, settings in which a particular technology is very attractive when few other firms use it, but becomes less attractive as many other firms use it.

Exercise 9—Political Competition (Hoteling Model)[B]

Consider two candidates competing for office: Democrat (D) and Republican (R). While they can compete along several dimensions (such as their past policies, their endorsements from labor unions, their advertising, and even their looks!), we assume for simplicity that voters compare the two candidates according to only one dimension (e.g., the budget share that each candidate promises to spend on education). Voters' ideal policies are uniformly distributed along the interval [0, 1], and each votes for the candidate with a policy promise closest to the voter's ideal. Candidates simultaneously and independently announce their policy positions. A candidate's payoff from winning is 1, and from losing is −1. If both candidates receive the same number of votes, then a coin toss determines the winner of the election.

Part (a) Show that there exists a unique pure strategy Nash equilibrium, and that in involves both candidates proposals to promise a policy closest to the median voter.
Part (b) Show that with three candidates (democrat, republican, and independent), no pure strategy Nash equilibrium exists.

Answer

Part (a) Let $x_i \in [0, 1]$ denote candidate i's policy, where $i = \{D, R\}$. Hence, for a strategy profile (x_D, x_R) where, for instance, $x_D > x_R$, voters to the right-hand side of x_D vote democrat (since x_D is close to their ideal policy than x_R is), as well as half of the voters in the segment between x_D and x_R; as depicted in Fig. 2.28. In contrast, voters to the left-hand side of x_R and half of those in the segment between x_D and x_R vote republican. (The opposite argument applies for strategy profiles (x_D, x_R) satisfying $x_D < x_R$.)

We can now show that there exists a unique Nash in which both candidates announce $x_D = x_R = 0.5$. Our proof is similar to that in the Traveler's Dilemma game. First demonstrate that asymmetric strategy profiles where $x_D \neq x_R$ cannot be sustained as Nash equilibria of the game; second, to show that symmetric strategy profiles where $x_D = x_R = x$ but $x \neq 0.5$ cannot be supported as Nash equilibria either; and third, to demonstrate that symmetric strategy profile $x_D = x_R = 0.5$ can be sustained as Nash equilibrium of the game.

Let's first consider asymmetric strategy profiles where each candidate makes a different policy promise $x_i \neq x_j$, where $i = \{D, R\}$ and $j \neq i$:

Case 1 If $x_i < x_j < 0.5$, candidate i could increase his chances to win by positioning himself ε to the right of x_j (where $\varepsilon > 0$ is assumed to be small). Thus, any strategy profile where $x_i < x_j < 0.5$ cannot be supported as a Nash equilibrium.

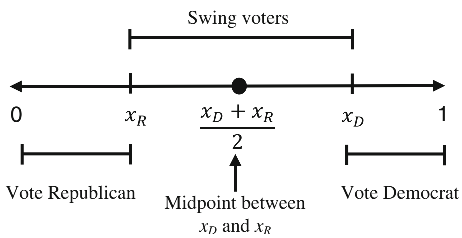

Fig. 2.28 Allocation of voters

Case 2 If $0.5 < x_i < x_j$, candidate j could increase his chances to win by positioning himself ε to the left of x_i. Thus, any strategy profile where $0.5 < x_i < x_j$ cannot be supported as a Nash equilibrium either.

Case 3 If $x_i < 0.5 < x_j$, candidate i could increase his chances to win by positioning him self ε to the left of x_j. Thus, any case where $x_i < 0.5 < x_j$ cannot be supported as a Nash equilibrium. (Note that the candidate j would also want to deviate to ε right to the candidate i). Thus, there cannot be *asymmetric* Nash equilibria.

Let us now consider *symmetric* strategy profiles where both candidates make the same policy promise, but their common policy differs from 0.5.

Case 1 If $x_D = x_R < 0.5$, a tie occurs, and each candidate wins the election with probability 1/2. However, candidate D could win the election with certainty by positioning himself ε to the right of x_R. (A similar argument applies to candidate R, who would also have incentives to position himself ε to the right of x_D.) Thus, any strategy profile where $x_D = x_R < 0.5$ cannot be supported as a Nash equilibrium.

Case 2 A similar argument applies if $0.5 < x_D = x_R$, where a tie occurs and every candidate wins the election with probability 1/2. However, candidate R could increase his chances of winning by positioning himself ε to the left of x_D. Thus, any strategy profile where $0.5 < x_D = x_R$ cannot be supported as a Nash equilibrium either.

Finally, if both candidates choose the same policy, $x_D = x_R = x$, and such common policy is $x = 1/2$, each candidate receives half of the votes, and wins the election with probability 0.5. In this setting, however, neither candidate has incentives to deviate; otherwise his votes would fall from half of the electorate, guaranteeing him to lose the election. Therefore, there exists only one Nash equilibrium, in which $x_D = x_R = 0.5$.

Part (b) Suppose that a Nash equilibrium exists with a triplet of policy proposals (x_D^*, x_R^*, x_I^*), where D denotes democrat, R republican, and I independent. We will next show that: (1) symmetric strategy profiles in which all candidates make the same that proposal, $x_D^* = x_R^* = x_I^*$, cannot be sustained as Nash equilibria; (2) asymmetric strategy profiles where two candidates choose the same proposal, but a third candidate differs, cannot be supported as equilibria either; and (3) asymmetric strategy profiles in which all three candidates choose different proposals cannot be sustained as equilibria; ultimately entailing that no pure strategy equilibrium exists.

First case. Consider, first, symmetric policy proposals $x_D^* = x_R^* = x_I^*$. All candidates, hence, receive the same number of votes (one third of the electorate), and each candidate wins with probability 1/3. While candidates didn't have incentives to alter their policy promises in a setting with two candidates, with three of them we can see that candidates have incentives to deviate from such strategy profile. In particular, any candidate can win the election by moving to the right (if their common policy satisfies $x_D^* = x_R^* = x_I^* < 2/3$) or moving to the left (if their common policy satisfies $x_D^* = x_R^* = x_I^* > 1/3$); as depicted in Fig. 2.29. Similarly,

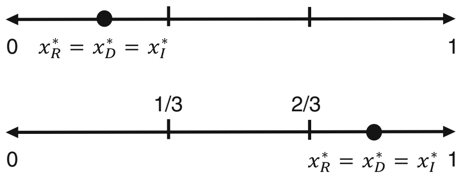

Fig. 2.29 Allocation of voters

when the location of the three candidates satisfies $x_D^* = x_R^* = x_I^* > 1/2$ each candidate has incentives to deviate towards the left, while if $x_D^* = x_R^* = x_I^* < 1/2$ each candidate has incentives to deviate to the right.

Second case. Consider now strategy profiles in which two candidates choose the same policy, $x_i^* = x_j^* = x^*$, but the third candidate differs, $x^* \neq x_k^*$, where $i \neq j \neq k$. If their policies satisfy $x^* < x_k^*$, then candidate k has incentives to approach x^*, i.e., $x^* + \varepsilon$, as such position increases his votes. Similarly, if $x^* > x_k^*$, candidate k has incentives to approach x^*, i.e., $x^* - \varepsilon$, which increases his votes. Since we found that at least one player has a profitable deviation, the above strategy profile cannot be sustained as a Nash equilibrium.

Third case. Finally, consider strategy profiles where all three candidates make different policy promises, $x_i^* \neq x_j^* \neq x_k^*$. The candidate that is located the farthest on the right will be able to win by moving ε to the right of its closest competitor; implying that the original strategy profile cannot be equilibrium. (A similar argument applies to the other candidates, such as that located the farthest to the left, who could win by moving to the left of its closest competitor.) Therefore, there exists no pure strategy Nash Equilibrium in this game.

Exercise 10—Tournaments[B]

Several strategic settings can be modeled as a tournament, whereby the probability of winning a certain prize not only depends on how much effort you exert, but also on how much effort other participants in the tournament exert. For instance, wars between countries, or R&D competitions between different firms in order to develop a new product, not only depend on a participant's own effort, but on the effort put by its competitors. Let's analyze equilibrium behavior in these settings. Consider that the benefit that firm 1 obtains from being the first company to launch a new drug is \$36 million. However, the probability of winning this R&D competition against its rival (i.e., being the first to launch the drug) is $\frac{x_1}{x_1+x_2}$, which increases with this firm's own expenditure on R&D, x_1, relative to total expenditure by both firms, $x_1 + x_2$. Intuitively, this suggests that, while spending more than its rival, i.e., $x_1 > x_2$, increases firm 1's chances of being the winner, the fact that $x_1 > x_2$ does not guarantee that firm 1 will be the winner. That is, there is still some randomness as to which firm will be the first to develop the new drug, e.g., a firm can spend more resources than its rival but be "unlucky" because its laboratory exploits a few weeks before being able to develop the drug. For simplicity, assume that firms' expenditure cannot exceed 25, i.e., $x_i \in [0, 25]$. The cost is simply x_i, so firm 1's profit function is

$$\pi_1(x_1, x_2) = 36\left(\frac{x_1}{x_1 + x_2}\right) - x_1$$

and there is an analogous profit function for firm 2:

$$\pi_2(x_1, x_2) = 36\left(\frac{x_2}{x_1 + x_2}\right) - x_2$$

You can easily check that these profit functions are increasing and concave in a firm's own expenditure. Intuitively, this indicates that, while profits increase in the firm's R&D, the first million dollar is more profitable than the 10th million dollar, e.g., the innovation process is more exhausted.

Part (a) Find each firm's best-response function.
Part (b) Find a symmetric Nash equilibrium, i.e., $x_1^* = x_2^* = x^*$.

Answer

Part (a) Firm 1's optimal expenditure is the value of x_1 for which the first derivative of its profit function equals zero. That is,

$$\frac{\partial \pi_1(x_1, x_2)}{\partial x_1} = 36\left[\frac{x_1 + x_2 - x_1}{(x_1 + x_2)^2}\right] - 1 = 0$$

Rearranging, we find

$$36\left[\frac{x_2}{(x_1 + x_2)^2}\right] - 1 = 0$$

which simplifies to

$$36x_2 = (x_1 + x_2)^2$$

and further rearranging

$$6\sqrt{x_2} = x_1 + x_2$$

Solving for x_1, we obtain firm 1's best response function

$$x_1(x_2) = 6\sqrt{x_2} - x_2$$

Figure 2.30 depicts firm 1's best response function, $x_1(x_2) = 6\sqrt{x_2} - x_2$ as a function of its rival's expenditure, x_2 in the horizontal axis for the admissible set $x_2 \in [0, 25]$.

It is straightforward to show that, for all values of $x_2 \in [0, 25]$, firm 1's best response also lies in the admissible set $x_1 \in [0, 25]$. In particular, the maximum of BR_1 occurs at $x_2 = 9$ since

$$\frac{\partial BR_1(x_2)}{\partial x_2} = \frac{\partial\left[6\sqrt{x_2} - x_2\right]}{\partial x_2} = 3(x_2)^{-\frac{1}{2}} - 1$$

Hence, the point at which this best response function reaches its maximum is that in which its derivative is zero, i.e., $3(x_2)^{-1/2} - 1 = 0$, which yields a value of $x_2 = 9$. At this point, firm 1's best response function informs us that firm 1 optimally spends $6\sqrt{9} - 9 = 9$. Finally, note that the best response function is concave in its rival expenditure, x_2, since

$$\frac{\partial^2 BR_1(x_2)}{\partial x_2^2} = -\frac{3}{2}(x_2)^{-\frac{3}{2}} < 0.$$

By symmetry, firm 2's best response function is $x_2(x_1) = 6\sqrt{x_1} - x_1$.

Part (b) In a symmetric Nash equilibrium $x_1^* = x_2^* = x^*$. Hence, using this property in the best-response functions found in part (c), yields

$$x^* = 6\sqrt{x^*} - x^*$$

Rearranging, we obtain $2x^* = 6\sqrt{x^*}$, and solving for x^*, we find $x^* = 9$. Hence, the unique symmetric Nash equilibrium has each firm spending 9. As Fig. 2.31 depicts, the points at which the best response function of player 1 and 2 cross each other occur at the 45-degree line (so the equilibrium is symmetric). In particular, those points are the origin, i.e., (0, 0), but this case is uninteresting since it implies that no firm spends money on R&D, and (9, 9).

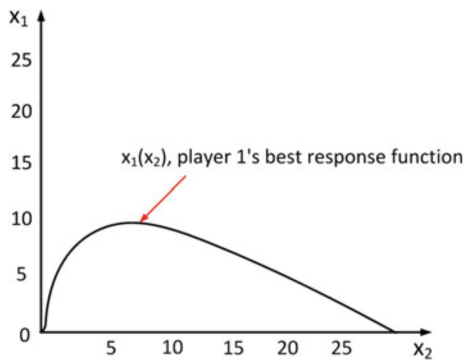

Fig. 2.30 Tournament—Firm 1's best response function

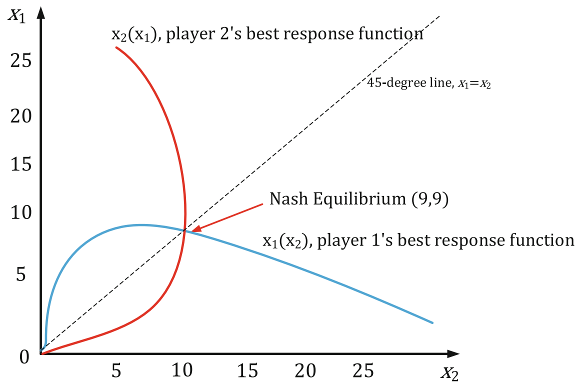

Fig. 2.31 Tournament–Best response functions and Nash-equilibrium

Exercise 11—Lobbying[A]

Consider two interest groups, A and B, seeking to influence a government policy, each with opposed interest (group A's most preferred policy is 0, while that of group B is 1). Each group simultaneously and independently chooses a monetary contribution to government officials, $s_i \in [0, 1]$, where $i = \{A, B\}$. The policy (x) that the government implements is a function of the contributions from both interest groups, as follows:

$$x(s_A, s_B) = \frac{1}{2} - s_A + s_B$$

Hence, if interest groups contribute zero (or if their contributions coincide, thus canceling each other), the government implements its ideal policy, $\frac{1}{2}$. [In this simplified setting, the government is not a strategic player acting in the second stage of the game, since its response to contributions is exogenously described by policy function $x(s_A, s_B)$.] Finally, assume that the interest groups have the following utility functions:

$$u_A(s_A, s_B) = -[x(s_A, s_B)]^2 - s_A$$

$$u_B(s_A, s_B) = -[1 - x(s_A, s_B)]^2 - s_B$$

which decrease in the contribution to the government, and in the squared distance between their ideal policy (0 for group A, and 1 for group B) and the implemented policy $x(s_A, s_B)$. Find the Nash equilibrium of this simultaneous-move game.

Answer

Substituting the policy function into the utility function of every group, we obtain

$$u_A(s_A, s_B) = -\left[\frac{1}{2} - s_A + s_B\right]^2 - s_A$$

$$u_B(s_A, s_B) = -\left[1 - \left(\frac{1}{2} - s_A + s_B\right)\right]^2 - s_B$$

Taking first order conditions with respect to s_A in the utility function of group A yields

$$2\left[\frac{1}{2} - s_A + s_B\right] - 1 = 0$$

Rearranging, we obtain $1 - 2s_A + 2s_B - 1 = 0$, which, solving for s_A, yields the best-response function for interest group A, $s_A(s_B) = s_B$.

Similarly, taking first order conditions with respect to s_B in the utility function of group B, we find

$$2\left[1 - \left(\frac{1}{2} - s_A + s_B\right)\right] - 1 = 0$$

which simplifies to $-s_A + s_B = 0$, thus yielding the best-response function for interest group B, $s_B(s_A) = s_A$. Graphically, both best response functions coincide with the 45-degree line, and completely overlap to one another. As a consequence, the set of pure strategy NEs is given by all the points in the 45-degree line, i.e., all points satisfying $s_A = s_B$, or, more formally, the set

$$\{(s_A, s_B) \in [0, 1]^2 : s_A = s_B\}.$$

Furthermore, since both interest groups are contributing the same amount to the government, their contributions cancel out, and the government implements its preferred policy, $\frac{1}{2}$. Finally, note that the strategic incentives in this game are similar to those in other Pareto Coordination games with symmetric NEs. While the game has multiple NEs in which both groups choose the same contribution level, the NE in which both groups choose a zero contribution Pareto dominates all other NEs with positive contributions.

Exercise 12—Incentives and Punishment[B]

Consider the following "law and economics" game, between a criminal and the government. The criminal selects a level of crime, $y \geq 0$, and the government chooses a level of law enforcement $x \geq 0$. Both choices are simultaneous and independent, and utility functions of the government (G) and the criminal (C) are, respectively,

$$u_G = -\frac{y^2}{x} - xc^4 \qquad \text{and} \qquad u_C = \frac{1}{1+xy}\sqrt{y}$$

Intuitively, the government takes into account that crime, y, is harmful for society (i.e., y enters negatively into the government's utility function u_G), and that each unit of law enforcement, x, is costly to implement, at a cost of c^4 per unit. In contrast, the criminal enjoys $\sqrt{y}$ if he is not caught, which definitely occurs when $x = 0$ (i.e., his utility becomes $u_c = \sqrt{y}$ when $x = 0$), while the probability of not being caught is $\frac{1}{1+xy}$.

Part (a) Find each player's best-response function. Depict these best-response functions, with x on the horizontal axis and y on the vertical axis.
Part (b) Compute the Nash equilibrium of this game.
Part (c) Explain how the equilibrium levels of law enforcement, x and crime, y, found in part (b) change as the cost of law enforcement, c, increases.

Answer

Part (a) First, note that the government, G, selects the level of law enforcement, x, that solves

$$\max_{x} \; -\frac{y^2}{x} - xc^4$$

Taking first-order conditions with respect to x yields

$$\frac{y^2}{x^2} - c^4 = 0$$

Rearranging and solving for x, we find the government's (G) best response function, BR_G, to be

$$x(y) = \frac{y}{c^2}.$$

Intuitively, the government's level of law enforcement, x, increases in the amount of criminal activity, y, and decreases in the cost of every unit of law enforcement, c.

Second, the criminal, C, selects the level of crime, y, that solves

$$\max_{y} \frac{1}{1+xy}\sqrt{y}$$

Taking first-order conditions with respect to y yields

$$\frac{1}{2y^{1/2}(1+xy)} - \frac{y^{1/2}x}{(1+xy)^2} = 0$$

Rearranging and solving for y, we find the criminal's best response function, BR_C, to be

$$y(x) = \frac{1}{x}.$$

which decreases in the level of law enforcement chosen by the government, x.

The government's best response function, BR_G, and the criminal's best response function, BR_C, are represented in Fig. 2.32. Note that $BR_C\left(\text{i.e.}, y(x) = \frac{1}{x}\right)$ is clearly decreasing in x but becomes flatter as x increases, i.e., $\frac{dy(x)}{dx} = -\frac{1}{x^2} < 0$ and $\frac{d^2y(x)}{dx^2} = \frac{2}{x^3} > 0$. In order to depict the government's best response function BR_G, $x(y) = \frac{y}{c^2}$, it is convenient to solve for y which yields $y = c^2x$. As depicted in Fig. 2.32, BR_G originates at (0, 0) and has a slope of c.

Part (b) We find the values of x and y that simultaneously solve both players' best response functions $x = \frac{y}{c^2}$ and $y = \frac{1}{x}$. For instance, you can plug the second expression into the first expression. This yields $x^* = \frac{1/x^*}{c^2}$, which, solving for x^*, entails an equilibrium level of law enforcement of $x^* = \frac{1}{c}$. Therefore, the equilibrium level of crime is $y\left(\frac{1}{c}\right) = \frac{1}{1/c} = c$. The Nash equilibrium is, hence, $x^* = \frac{1}{c}$ and $y^* = c$; as illustrated in the point where best response function BR_G crosses BR_C in Fig. 2.32.

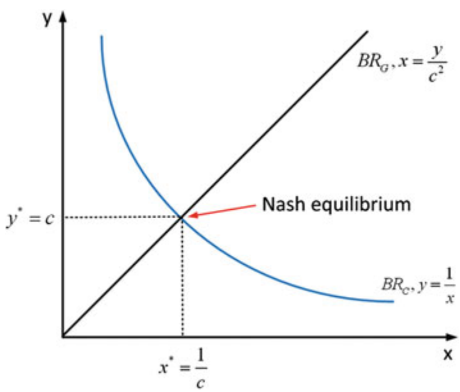

Fig. 2.32 Incentives and Punishment

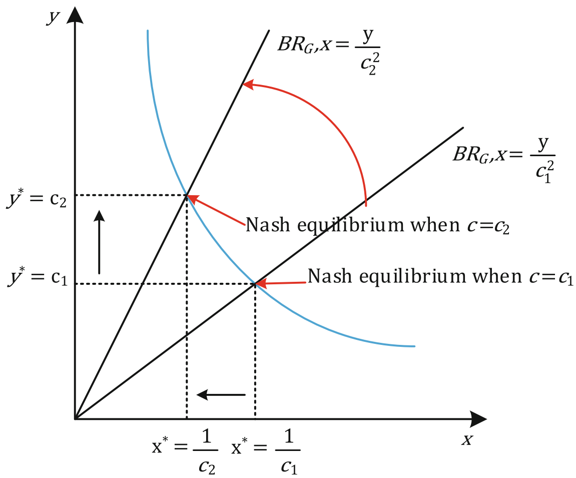

Fig. 2.33 Incentives and Punishment-Comparative statics

Part (c) As Fig. 2.33 illustrates, an increase in the cost of enforcement c pivots the government's best response function, BR_G leftward, with center at the origin (recall that c is the slope of BR_G). In contrast, the criminal's best response function is unaffected (since it is independent on c). This pivoting effect produces a new crossing point that lies to the northwest of the original Nash equilibrium, entailing a higher level of criminal activity (y) and a lower level of enforcement (x).

Exercise 13—Cournot mergers with Efficiency Gains[B]

Consider an industry with three identical firms each selling a homogenous good and producing at a constant cost per unit c with $1 > c > 0$. Industry demand is given by $p(Q) = 1 - Q$, where $Q = q_1 + q_2 + q_3$. Competition in the marketplace is in quantities.

Part (a) Find the equilibrium quantities, price and profits.
Part (b) Consider now a merger between two of the three firms, resulting in duopolistic structure of the market (since only two firms are left: the merged firm and the remaining firm). The merger might give rise to efficiency gains, in the sense that the firm resulting from the merger produces at a cost $e \cdot c$, with $e \leq 1$ (whereas the remaining firm still has a cost c):

i. Find the post-merger equilibrium quantities, price and profits.
ii. Under which conditions does the merger reduce prices?
iii. Under which conditions is the merger beneficial to the merging firms?

Answer

Part (a)

Each firm $i = \{1, 2, 3\}$ has a profit of

$$\pi_i = (1 - Q - c)q_i.$$

Hence, since $Q \equiv q_1 + q_2 + q_3$, profits can be rewritten as:

$$\pi_i = \big(1 - (q_i + q_j + q_k) - c\big)q_i,$$

The first-order conditions are given by

$$1 - 2q_i - q_j - q_k - c = 0$$

since firms are symmetric $q_i = q_j = q_k = q$ in equilibrium, that is

$$1 - 2q - q - q - c = o$$

or

$$1 - 4q - c = 0.$$

Solving for q at the symmetric equilibrium yields a Cournot output of,

$$q_c = \frac{1 - c}{4}$$

Hence, equilibrium prices are $p_C = 1 - \frac{1-c}{4} - \frac{1-c}{4} - \frac{1-c}{4} = 1 - 3\left(\frac{1-c}{4}\right) = \frac{1+3c}{4}$ and every firm i's equilibrium profits are $\pi_C = \left(\frac{1+3c}{4} - c\right)\frac{1-c}{4} = \frac{(1-c)^2}{16}$.

Part (b)

i. After the merger, two firms are left: firm 1, with cost $e \cdot c$, and firm 3, with cost c. Hence, the two profit functions are now given by:

$$\pi_1 = (1 - Q - ec)q_1.$$

$$\pi_3 = (1 - Q - c)q_3$$

Taking first order conditions of π_1 with respect to q_1 yields

$$1 - 2q_1 - q_3 - ec = 0$$

and, solving for q_1, we obtain firm 1's best response function

$$q_1(q_3) = \frac{1 - ec}{2} - \frac{1}{2}q_3$$

Similarly taking first-order conditions of firm 3’s profits, π_3, with respect to q_3 yields

$$1 - 2q_3 - q_1 - c = 0$$

which, solving for q_3, provides us with firm 3’s best response function

$$q_3(q_1) = \frac{1 - c}{2} - \frac{1}{2}q_1$$

Plugging $q_3(q_1)$ into $q_1(q_3)$, yields

$$q_1^* = \frac{1 - ec}{2} - \frac{1}{2}\left(\frac{1 - c}{2} - \frac{1}{2}q_1^*\right)$$

Rearranging and solving for q_1^*, we obtain firm 1’s equilibrium output

$$q_1^* = \frac{1 - c(2e - 1)}{3}$$

Plugging this output level into firm 3’s best response function yields an equilibrium output of

$$q_3^* = \frac{1 - c(2 - e)}{3}$$

Note that the outsider firm can sell a positive output at equilibrium only if the merger does not give rise to strong cost savings: that is $q_3 \geq 0$ if $e \geq \frac{2c-1}{c}$ (if $c < 1/2$, then the previous payoff becomes $\frac{2c-1}{c} < 0$, implying that $e \geq \frac{2c-1}{c}$ holds for all $e \geq 0$, ultimately entailing that the outsider firm will always sell at the equilibrium. We hence concentrate on values of c that satisfy $c > 1/2$.) Figure 2.34 illustrates cutoff $e > \frac{2c-1}{c}$, where $c > 1/2$. The region of (e, c)-combinations above this cutoff indicate parameters for which the merger is not sufficiently cost saving to induce the outside firm to produce positive output levels. The opposite occurs when the cost-saving parameter, c, is lower than $(2c - 1)/c$, thus indicating that the merger is so cost saving that the nonmerged firm cannot profitably compete against the merged firm.

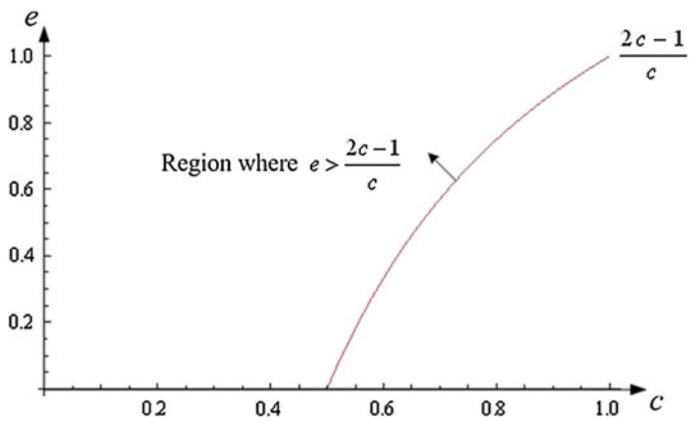

Fig. 2.34 Positive production after the merger

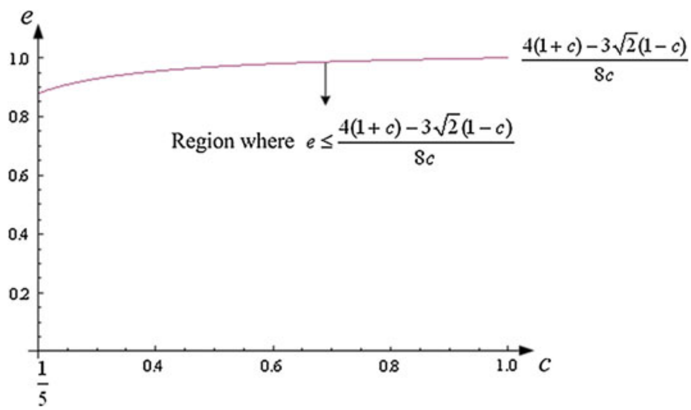

Fig. 2.35 Profitable mergers if e is low enough

- The equilibrium price is $p_m = 1 - \frac{1-c(2e-1)}{3} - \frac{1-c(2-e)}{3} = \frac{1+c(1+e)}{3}$, and equilibrium profits are given by $\pi_1 = \frac{(1-c(2e-1))^2}{9}$ and $\pi_3 = \frac{(1-c(2-e))^2}{9}$.

ii. Prices decrease after the merger only if there are sufficient efficiency gains: that is, $p_m \le p_c$ can be rewritten as $e \le \frac{5c-1}{4c}$. Note that if $c < 1/5$, then $\frac{5c-1}{4c} < 0$, implying that $e \le \frac{5c-1}{4c}$ cannot hold for any $e \ge 0$. As a consequence, $p_m > p_c$, and prices will never fall no matter how strong efficiency gains, e, are

iii. To see if the merger is profitable, we have to study the inequality $\pi_1 \ge 2\pi_c$, which, solving for e, yields

$$e \leq \frac{4(1+c) - 3\sqrt{2}(1-c)}{8\mathrm{c}}$$

In other words, the merger is profitable only if it gives rise to enough cost savings. Figure 2.35 depicts the cutoff of e where costs are restricted to $c \in \left[\frac{1}{5}, 1\right]$. Notice that if cost savings are sufficiently strong, i.e., parameter e is sufficiently small as depicted in the region below the cutoff, the merger is profitable.

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