

Preface

I protest against the use of infinite magnitude as something completed, which in mathematics is never permissible. Infinity is merely a *façon de parler*, the real meaning being a limit which certain ratios approach indefinitely near, while others are permitted to increase without restriction.

C.F. Gauss, Brief an Schumacher (1831); Werke 8, 216 (1831)

In a letter to the astronomer H.C. Schumacher in 1831, Gauss was rebuking mathematicians for their use of the infinite as a number, and even for their use of the symbol for the infinite. It would be difficult to sustain that kind of finitism, regardless of any epistemological considerations: a good part of mathematics simply cannot survive with only the potential infinite.

The reaction against the infinite, as well as against complex or imaginary numbers, and against negative numbers before, are interesting examples of the difficulties faced, even by great minds, in accepting certain abstractions. Aristotle in Chaps. 4–8 of *Book III of Physics* argued against the actual infinite, advocating for the potential infinite. His idea was that natural numbers could never be conceived as a whole.

Euclid in a certain sense never proved that there exist infinitely many prime numbers. What was actually stated in Proposition 20 of *Book IX*, carefully avoiding the term infinite, was that “prime numbers are more than any previously thought (total) number of primes”, which agrees with his tradition.

It was only in the nineteenth century that G. Cantor dispelled all those accepted views by showing that an infinite set can be treated as a totality, as a full-fledged mathematical object with honorable properties, no less than the natural numbers.

Imaginary numbers were introduced to mathematics in the sixteenth century (through Girolamo Cardano, though others had already used them in different guises). These numbers caused an embarrassment among mathematicians for centuries, since they faced astonishing difficulties in accepting an extension of the concept of number, especially in light of the problem of computing the square root of -1 . Only after the fundamental works of L. Euler and Gauss did the complex

numbers rid themselves of the label “imaginary” given them by Descartes in 1637, and even then not without difficulty.

The mathematics of the infinite and of complex numbers, and all they represent in contemporary science, are triumphant cases of amplified concepts, but are not the only ones. A notable case of expansion of concepts, with deep implications for the development of contemporary logic, can be traced back to Frege and his famous article of 1891, *Funktion und Begriff* (see [1]).

In this seminal paper, Frege recalls how the meaning of the term ‘function’ has changed in the history of mathematics, and how the mathematical operations used to define functions have been extended by, as he says, ‘the progress of science’: basically, through passages (or transitions) to the limit, as in the process of defining a new function $y' = f'(x)$ from a function $y = f(x)$ (provided that the limits involved in the calculus exist), and through accepting complex numbers in domains and images of functions.

Starting from this point, Frege goes further into adding expressions that now we call predicates, such as ‘=’, ‘<’ and ‘>’. Leaving aside his philosophical motivations for seeing arithmetic as a “further development of logic”, what Frege started was a real revolution, that made possible the development of quantifiers and an unprecedented unification of propositional and predicate logic into a far more powerful system than any that preceded it.

Not only could the truth-values, *True* and *False*, be taken as outputs of a function, but any object whatsoever could be similarly treated. To rephrase an example from Frege himself, if we suppose ‘the capital of x ’ expresses a function, of which ‘the German Empire’ is the argument, Berlin is returned as the value of the function. In this way, Frege’s system could represent non-mathematical thoughts and predications, and founded the basis of the modern predicate calculus.

Frege’s idea of defining an independent notion of ‘concept’ as a function which maps every argument to one of the truth-values *True* or *False* was instrumental in the development of a strict understanding of the notions of ‘proof’, ‘derivation’, and ‘semantics’ as parts of the same logic mechanism. Regarding ‘concept’ as a wide and independent notion based on an amplification of the idea of function was an essential step for Frege’s fundamental break between the older Aristotelian tradition and the contemporary approach to logic.

Paraconsistency is the study of logical systems in which the presence of a contradiction does not imply triviality, that is, logical systems with a non-explosive negation \neg such that a pair of propositions A and $\neg A$ does not (always) trivialize the system. However, it is not only the syntactic and semantic properties of these systems that are worth studying. Some questions arise that are perennial philosophical problems. The question of the nature of the contradictions allowed by paraconsistent logics has been a focus of debate on the philosophical significance of paraconsistency. Although this book is primarily focused on the logico-mathematical development of paraconsistency, the technical results emphasized here aim to help, and hopefully to guide, the study of some of those philosophical problems.

Paraconsistent logics are able to deal with contradictory scenarios, avoiding triviality by means of the rejection of the Principle of Explosion, in the sense that these theories do not trivialize in the presence of (at least some) contradictory sentences. Different from traditional logic, in paraconsistent logics triviality is not necessarily connected to contradictoriness; in intuitive terms (a more formal account is given in Sect. 1.2) the situation could be described by the pictorial equation:

$$\text{contradictoriness} + \text{consistency} = \text{triviality}$$

The *Logics of Formal Inconsistency*, from now on **LFI**s, introduced in [2] and additionally developed in [3], are a family of paraconsistent logics that encompasses a great number of paraconsistent systems, including the majority of systems developed within the Brazilian tradition. An important characteristic of **LFI**s is that they are endowed with linguistic resources that permit to express the notion of consistency of sentences inside the object language by using a sentential unary connective referred to as ‘circle’: $\circ A$ meaning *A is consistent*. Explosion in the presence of contradictions does not hold in **LFI**s, as much as in any other paraconsistent logic. But **LFI**s are so designed that some contradictions will cause deductive explosion: *consistent* contradictions lead to triviality—intuitively, one can understand the notion of a ‘consistent contradiction’ as a contradiction involving well-established facts, or involving propositions that have conclusive favorable evidence. In this sense, **LFI**s are logics that permit one to separate the sentences for which explosion hold, from those for which explosion does not hold. It is not difficult to see that, in this way, reasoning under **LFI**s extend and expand the reasoning under classical logic: although **LFI**s are technically subsystems of classical logic, classical logic can be identified with that portion of **LFI**s that deals with ‘consistent contradictions’. Therefore **LFI**s subsume classical reasoning. This point will be explained in more detail in Sect. 1.2.

We may say that a first step in paraconsistency is the distinction between triviality and contradictoriness. But there is a second step, namely, the distinction between consistency and non-contradictoriness. In **LFI**s the consistency connective \circ is not only primitive, but it is also not necessarily equivalent to non-contradiction.

This is the most distinguishing feature of the logics of formal inconsistency. Once we break up the equivalence between $\circ A$ and $\neg(A \wedge \neg A)$, some quite interesting developments become available. Indeed, $\circ A$ may express notions of consistency independent from freedom from contradiction.

The most important conceptual distinction between **LFI**s and traditional logic is that **LFI**s start from the principle that assertions about the world can be divided into two categories: consistent sentences and non-consistent sentences. Consistent propositions are subjected to classical logic, and consequently a theory T that contains a pair of contradictory sentences $A, \neg A$ explodes only if A is taken to be a consistent sentence, linguistically marked as $\circ A$ (or $\circ \neg A$). This is the only distinction between **LFI**s and classical logic, albeit with far-reaching consequences:

classical logic in this form is augmented, in such a way that in most cases an **LFI** encodes classical logic.

The concept of **LFIs** generalizes and extends the famous hierarchy of C-systems introduced in [4] and popularized by hundreds of papers. At the same time, **LFIs** expand the classical logical stance, and consequently the majority of the traditional concepts and methods of classical logic, propositional or quantified (and even higher-order), can be adapted, with careful attention to detail.

Since, as much as intuitionistic logic, **LFIs** are more of an epistemic nature, rather than of an ontological, there is no point in advocating the replacement of classical logic with paraconsistent logic. Because **LFIs** extend the classical stance, the analogy with transfinite ordinal numbers and with complex numbers is compelling: in such cases, there is no rejection of what has come before, but a refinement of it.

It is not infrequent that an argument as of the skeptics, such as that given by Sextus Empiricus¹ against the sophists, is trumpeted against the need of paraconsistent logic, in science or reasoning in general:

[If an argument] leads to what is inadmissible, it is not we that ought to yield hasty assent to the absurdity because of its plausibility, but it is they that ought to abstain from the argument which constrains them to assent to absurdities, if they really choose to seek truth, as they profess, rather than drivel like children. Thus, suppose there were a road leading up to a chasm, we do not push ourselves into the chasm just because there is a road leading to it but we avoid the road because of the chasm; so, in the same way, if there should be an argument which leads us to a confessedly absurd conclusion, we shall not assent to the absurdity just because of the argument but avoid the argument because of the absurdity. So whenever such an argument is propounded to us we shall suspend judgement regarding each premiss, and when finally the whole argument is propounded we shall draw what conclusions we approve.

This argument, however, if it is not against the use of any logic, is indeed favorable to the kind of paraconsistency represented by **LFIs**. The notion of *consistency*—symbolized as \circ when applied to propositions—actually increases our wisdom: it does not stop one to jump into the chasm, but rather marks out the dangerous roads and, precisely, helps avoid such roads because of the chasm!

The idea that consistency can be taken as a primitive, independent notion, and be axiomatized for the good profit of logic is a new idea, which permits one to separate not only the notion of contradiction from the notion of deductive triviality, which is true of all paraconsistent logics, but also the notion of inconsistency from the notion of contradiction—as well as consistency from non-contradiction. This refined idea of consistency has great potential, as we shall see in detail in this book, as unanticipated as the possibilities that imaginary numbers, completed infinite, and Frege's idealization of a 'concept' as a function mapping arguments to one of the truth-values represented in mathematics, logic and philosophy. The rest of the book will speak for itself.

¹Sextus Empiricus, *Outlines of Pyrrhonism*, LCL 273: 318–319. http://www.loebclassics.com/view/sextus_empiricus-outlines_pyrrhonism/1933/pb_LCL273.3.xml

Book's Content: A Road Map

Chapter 1

Chapter 1 purports to clarify the whole project behind **LFI**s, making sense of its idealization and basic tenets. The paraconsistent viewpoint—materialized by means of **LFI**s—objects to classical logic, but only on the grounds that contradiction and triviality cease to coincide, and that contradiction ceases to coincide with inconsistency. But this requires no opposition to the classical stance, just the awareness that ‘classical’ logic involves some hidden assumptions, as discussed above in this chapter. In the light of this, Sect. 1.2 makes explicit some of the philosophical underpinning implicit in **LFI**s.

Section 1.3 will briefly retrace the motivations for the forerunners of **LFI**s and paraconsistency in general. No discussion of paraconsistency can avoid touching on, if only summarily, questions of the nature of logic, and Sect. 1.4 does this. Next challenges to be faced are questions about the nature of contradictions. Section 1.5 takes up this thorny philosophical topic from the times of the ancient Greece, cursorily discussing some remarks from Aristotle concerning three alleged versions of the *Principle of Non-Contradiction* that correspond to the three traditional aspects of logic, namely, ontological, epistemological, and linguistic.

This stance helps to give a justification for the rational acceptance of contradictory sentences, and to better appreciate the distinctions among contradiction, consistency, and negation, as characterized in Sect. 1.6. It will also help to make palatable the rationale behind the semantics of **LFI**s to be developed in all mathematical details in Chaps. 2 and 3, as well as to give support to alternative semantics for **LFI**s developed in Chap. 6.

There is a wide variety of reasons for repudiating (or at least to be cautioned against) classical logic, and many of them find an expression among paraconsistent logics. This chapter makes clear that **LFI**s are not coincidental with this spectrum of philosophical views, neither are they antagonistic, but can be combined with, and can complement, some of them. A summary of the main varieties of paraconsistency is given in Sect. 1.7, which attempts to clarify the position of **LFI**s with respect to other paraconsistent logics in the hope that this will justify some claims made in next chapters.

Chapter 2

Chapter 2 offers a careful survey of the basic logic of formal inconsistency, **mbC**: it is basic in the sense that, starting with positive classical logic \mathbf{CPL}^+ and adding a negation and a consistency operator, it is endowed with minimal properties in order to satisfy the definition of **LFI**s. The chapter also lays out the main notation, ongoing definitions and main ideas that will be used throughout the book. Positive classical logic is assumed as a natural starting point from which the **LFI**s will be defined, although in Chap. 5 some **LFI**s will be studied starting from other logics than \mathbf{CPL}^+ . A non-truth-functional valuation semantics for **mbC** is defined in Sect. 2.2, and its meaning and consequences explored in Sect. 2.3.

A remarkable feature of **LFI**s in general, and of **mbC** in particular, as mentioned above, is that classical logic (**CPL**) can be codified, or recovered, inside such logics, as shown, for instance, in Sect. 2.4.

One of the criteria proposed by da Costa in [5], p. 498, is that a paraconsistent calculus must contain as many of the schemata and rules of classical logic as can be endorsed without validating of the laws of explosion and non-contradiction. This vague criterion can be formalized in the sense that some **LFI**s can be proved to be maximal with respect to **CPL**, as in the case, for instance, of some three-valued **LFI**s treated in Chap. 4.

Moreover, in addition to being a subsystem of **CPL**, **mbC** is also an extension of **CPL**, obtained by adding to the latter a consistency operator \circ and a paraconsistent negation \neg (see Sect. 2.5). In this sense, **mbC** can be viewed, both, as a subsystem and as a conservative extension of **CPL**. A similar phenomenon holds for several other **LFI**s.

That section also sheds light on how **CPL** can be codified in **mbC**, showing that this can be achieved by way of a conservative translation, or by establishing a *Derivability Adjustment Theorem* (or DAT) between **CPL** and **mbC**. Section 2.5 also discusses an alternative formulation for **mbC** called \mathbf{mbC}^\perp , showing that by means of linguistic adaptations **mbC** can be directly introduced as an extension of **CPL**.

Chapter 3

Chapter 3 deals with extensions of **mbC**, which by its turn is a minimal extension of \mathbf{CPL}^+ with a consistency operator \circ and a paraconsistent negation \neg characterizable as an **LFI**. This chapter defines several extensions of **mbC**, strengthening or expanding different characteristics of this basic system.

In **mbC**, however, negation and consistency are totally separated concepts. The first extension of **mbC**, called **mbCciw**, is defined as the minimal extension guaranteeing that the truth-values of α and $\neg\alpha$ completely determine the truth-value of $\circ\alpha$.

Besides being a subsystem of classical logic, **mbC** is strong enough to contain the germ of classical negation, possessing a kind of hidden classical negation, as explained in Sect. 2.4 of Chap. 2. Section 3.2 of this chapter shows that here is another hidden operator in **mbC**: an alternative consistency operator \circ_β , one for each formula β . This operator establishes an important distinction, from a conceptual point of view, between **mbC** and \mathbf{mbC}^\perp as clarified in Sect. 3.4.

When he introduced his famous hierarchy C_n ($1 \leq n < \omega$) of paraconsistent systems, da Costa defined, for each system C_n , a kind of “well-behavedness” operator (later identified with consistency) in terms of the paraconsistent negation and conjunction (see Sect. 3.7). A special type of **LFI**s called **dC-systems**, characterized by the fact that the consistency operator can be defined in terms of the others, has been defined in [2]. The systems C_n of da Costa turn out to be examples of **dC-systems**. Section 3.3 of this chapter analyzes the formal notion of **dC-systems**, and investigates how to expand **mbC** in order to define the consistency

and/or the inconsistency operator in terms of the other connectives of the given signature.

In general terms, **LFI**s are concerned with the notion of consistency, expressed by the operator \circ . The notion of *inconsistency* of α is usually defined via the new operator $\neg\circ\alpha$, expressing the (formal) *inconsistency* of α . Section 3.5 studies the balance (or better, unbalance) between the formal concepts of consistency and inconsistency, defining a new **LFI** (**mbC**, which, in fact, is a **dc**-system) where inconsistency is a primitive notion and consistency is a defined one.

A natural requirement when characterizing consistency, as much as negation, is how consistency can be propagated through the remaining connectives. Sections 3.6 and 3.8 analyze extensions of **mbC** enjoying propagation of consistency in different forms, in the spirit of the historical systems of da Costa.

Chapter 4

Chapter 4 deals with matrices and algebraizability, and their consequences. In particular, the question of characterizability by finite matrices, as well as the algebraizability of (extensions of) **mbC** is tackled. Some negative results, in the style of the famous Dugundji's theorem for modal logics, are shown for several extensions of **mbC**. This results in new, compact proofs of previously established results, to the effect that a wide variety of **LFI**s extending **mbC** cannot be semantically characterized by finite matrices. Despite these general results, some three-valued extensions of **LFI**s can be characterized by finite matrices, and most of them are algebraizable in the well-known sense of Blok and Pigozzi. This is surprising, considering that several extensions of **mbC**, including the systems C_n of da Costa, cannot be algebraizable in Blok and Pigozzi's sense (and consequently, not in Lindenbaum and Tarski's sense).

On the topic of **LFI**s that can be defined matricially, the chapter also covers Halldén's logic of nonsense as well as Segerberg's variation, da Costa and D'Ottaviano's, logic **J3**, also known in its variants **LFI1** and **MPT**, Sette's logic **P1**, Priest's logic **LP**, the system **Ciore**, and several other related systems.

Chapter 5

Chapter 5 is devoted to giving an account of **LFI**s based on other logics, distinct from what was done in previous chapters, in which **LFI**s based exclusively on positive classical logic **CPL**⁺ were studied. Although several extensions of the basic system **mbC** have been proposed, including several three-valued logics (some of them even algebraizable in the sense of Blok and Pigozzi, which is not possible in the case of **mbC**) the underlying basis was always **CPL**⁺. This chapter, instead, analyzes **LFI**s defined over other logical basis, to wit: positive intuitionistic logic, the four-valued Belnap and Dunn's logic **BD**, some families of fuzzy logics, and some positive modal logics.

Section 5.1 starts by defining **LFI**s based on positive intuitionistic logic, instead of **CPL**⁺, beginning with paraconsistent logics based on **IPL**⁺ (taking as a basis Johansson's minimal logic and Nelson's logic). A weaker version of **mbC** called

imbC obtained from the former by changing the positive basis \mathbf{CPL}^+ to \mathbf{IPL}^+ is also investigated,

Section 5.2 is dedicated to the task of combining two paradigms of uncertainty: fuzziness and paraconsistency, with exciting possibilities. Taking as a basis the *monoidal t-norm based logic* MTL introduced in [6] as a generalization of the famous *basic fuzzy logic* BL due to P. Hájek (which, in turn, simultaneously generalizes three chief fuzzy logics, namely Łukasiewicz, Gödel-Dummett and Product logics) several new **LFI**s had been recently developed (see [7]).

Justified by the fact that MTL is the most general residuated fuzzy logic whose semantics is based on t-norms, the **LFI**s defined over MTL give a finely controlled combination of fuzzy and consistency (as well inconsistency) operators, giving rise to mathematical models for the novel notion of *fuzzy (in)consistency operators*, which formalizes the nice and natural idea of degrees of consistency and inconsistency.

Section 5.3 investigates a four-valued modal **LFI** based on N. Belnap and J.M. Dunn's logic \mathcal{BD} , a logic (based on their famous bilattice logic \mathcal{FOUR}) suitable for representing lack of information (a sentence is neither true nor false) or excess of information (a sentence is both true and false). The logic \mathcal{BD} was defined from the notion of *proposition surrogates* introduced by J.M. Dunn about five decades ago as a set-theoretic tool for representing De Morgan Lattices. The logic M_{4m} , a matrix logic expanding Belnap and Dunn's logic \mathcal{BD} by adding a modal operator, is then defined and proved to be an **LFI**. Moreover, it is a **dc**-system based on the logic preserving degrees of truth of the variety of bounded distributive lattices. The logic M_{4m} is based on the previous work by A. Monteiro on tetravalent modal algebras.

The chapter closes, in Sect. 5.4, with an overview of the notion of modal **LFI**s and their unfoldings.

Chapter 6

Chapter 6 studies alternative semantics for the **LFI**s presented in Chaps. 2 and 3, concentrating on the novel notion of swap structures. As much as modal logics, **LFI**s are in general non-truth-functional, and (as much as modal logics) have access to different kinds of semantics (like algebraic semantics, Kripke or relational semantics, topological semantics, and neighborhood semantics, among others) to better clarify their meaning, **LFI**s also naturally require a plurality of semantics. But unlike modal logics, **LFI**s in general do not have non-trivial logical congruences, and the question of defining other semantics for **LFI**s becomes more sensible. Standard tools, like categorial or algebraic semantics, will not work so easily for **LFI**s and the development of alternative semantical techniques for certain **LFI**s is an ongoing and relevant task.

The chapter clarifies the heritage of swap structures from M. Fidel's notion of twist structures (studied in Chap. 5), and also discusses the close relationship between the concept of Fidel structures, swap structures and non-deterministic matrices (or Nmatrices).

Section 6.8 surveys the possible-translations semantics (**PTS**s), a broad semantical concept introduced in 1990 that gives new philosophical interpretations

for some non-classical logics, and especially for paraconsistent logics. It happens that **PTS**s is a very general semantical notion, to the point that virtually any logic may have a **PTS** interpretation, under certain conditions. It also happens that several other semantical notions can be seen as particular cases of **PTS**s; those points are carefully explained in that section.

Chapter 7

Chapter 7 gives a full account of **LFI**s for first-order languages. The quantified versions of **LFI**s are essential for certain mathematical applications, such as set theory, and also for concrete applications in computer science, such as databases and logic programming. The combination of the consistency operator \circ with quantifiers \forall and \exists demands a careful treatment: now, the propagation of consistency through quantifiers has to be duly balanced, generalizing from the propagation of consistency for conjunction and disjunction. The intuitive idea, of course, is to regard the existential quantifiers as arbitrary conjunctions and disjunctions, but this has to be done taking a certain technical care.

The chapter is structured around a complete treatment of the system **QmbC**, a quantified extension of the system **mbC**, the basic **LFI** studied in Chap. 2. Other extensions of **QmbC**, such as **QCi** and **QmbC** \approx (the latter including an equality predicate), are also treated, keeping **QmbC** at the horizon. From the point of view of semantics, Tarskian first-order structures are now endowed with a paraconsistent bivaluation, and what results is a wide generalization of familiar model theory. An alternative approach to three-valued first-order **LFI**s is developed in detail in Sect. 7.9, based on the theory of quasi-truth. This treatment, of course, can be extended to other many-valued paraconsistent logics.

The paradigm of quasi-truth, which provides a way of accommodating the conceptual incompleteness inherent in scientific theories as studied in [8], views scientific theories from the perspective of paraconsistent logic. This paradigm offers a rational account for the dynamics of theory change, allowing for theories involving contradictions without triviality, with deep implications for the foundations of science and for the understanding of the scientific method. A generalization of the logical aspects of the theory of quasi-truth has been undertaken in [9], by means of a three-valued model theory for an **LFI** called **LPT1**, which in turn coincides (setting aside some details of language) with the quantified version of the three-valued paraconsistent logic **LFI1** introduced in Definition 4.4.41. An additional discussion on quasi-truth can be found in Sect. 9.3 of Chap. 9.

One of the aims of this chapter is to endorse the claim that basically the same results of classical model theory hold for **QmbC**, and for first-order **LFI**s in general, with certain provisos. Well-established results in traditional model theory such as the Completeness, Compactness and Lowenheim–Skolem Theorems can be proved for first-order **LFI**s along the same lines as the classical case. In this way, the chapter makes clear that first-order **LFI**s expand traditional logic, and allows for a revision of the uses of logic in mathematics and computer science from the vantage point of richer logics.

Chapter 8

The confusion between the concept of set on the one hand, and of class, or species, on the other hand, has plagued the foundations of set theory since its birth. The Principle of Comprehension (also referred to as the Principle of Naïve Comprehension, or Abstraction) was proposed in the nineteenth century, fruit of the somewhat romantic ideas of Dedekind, Cantor, and Frege, and states that for every property, expressed as a predicate, there exists a set consisting of exactly those objects that satisfy the predicate. This principle lurks behind certain tough paradoxes, such as Russell's paradox, and the history of contemporary set theory has much to do with efforts to rescue Cantor's naïve set theory from triviality, an inevitable consequence, in traditional logic, of the contradictions entailed by those paradoxes. Paraconsistent set theory has been an endeavor to save set theory from certain (it not all) paradoxes for at least three decades. Chapter 8 aims to offer a new approach to this question by means of employing **LFI**s and their powerful consistency operator. By assuming that not only sentences, but sets themselves can be classified as consistent or inconsistent objects, the basis for new paraconsistent set-theories that can resist certain paradoxes without falling into trivialism is established. One of the main motivations of this chapter, as stated in Sect. 8.1, is to rescue, together with Cantor's naïve set theory, the proper Cantor's intuition towards 'inconsistent sets'. Indeed, the chapter attempts to show that Cantor's treatment of inconsistent collections can be related to the one provided by means of **LFI**s.

Section 8.2 defines **ZFmbC**, a basic system of paraconsistent set theory whose underlying logic is **QmbC** \approx , and which contains two non-logical predicates (besides the equality predicate \approx): the binary predicate " \in " (for membership), and the unary predicate C (for consistency of sets). Section 8.3 proposes some extensions of **ZFmbC** by means of employing stronger **LFI**s as underlying logics and setting appropriate axioms for the consistency operator C for sets. Section 8.4 discusses the relationship between the notions of 'to be a consistent object in set theory' (as formalized in the chapter) and 'to be a set'. It shows that consistent objects can be (without risk of trivialism) regarded as sets, by means of an appropriate axiom. In the same spirit, proper classes can be regarded as inconsistent objects. Such affinities between consistent objects in set theory and sets, and between proper classes and inconsistent objects, though it cannot be strengthened into equivalence, testify to the richness of this approach.

Section 8.5, the last in the chapter, starts the discussion of models of paraconsistent set theory. If the construction of models for standard set theory is a fraught task, the analogue for paraconsistent set theory is adventurous, to say the least. One might consider standard models of paraconsistent set theory, where the ε relation of that model corresponds exactly to the membership relation \in of the universe of **ZFmbC** and its extensions, and the same for the consistency operator \circ , but it is also reasonable to make room for non-standard models. Only in this way could one venture into deeper waters, such as extending forcing machinery to paraconsistent

set theory. Although this is not done in this book, and it may be an ambitious project, it is not unrealistic.

Chapter 9

Chapter 9 attempts to clarify the close connections between paraconsistency and philosophy of science: in a nutshell, there are so many cases of contradictions, even if temporary, arising between scientific theories, as well as between facts and theories, that a paraconsistent approach to the foundations of science seem to be almost inevitable. Section 9.1 advocates an epistemological understanding of paraconsistency based upon the notion of evidence, and questions its significance for science supported by some examples of real situations, examined in Sect. 9.2. Consistency and contradiction in scientific theories can be understood by an epistemic approach to paraconsistency, we claim, inspired by some Kantian insights about the limits of human reason. Some historical examples of cases where scientists have held contradictory positions, and where science as a whole has gained from holding them, are reviewed in this section. The controversy surrounding the movement of the luminiferous aether of the nineteenth century, the controversies in the early development of quantum theory, the case of Mercury's orbit and the failure in hypothesizing Vulcan, a planet that only existed in the heads of certain astronomers, and the contradictions arising from the 'imponderable' phlogiston in the beginnings of the chemistry of the eighteenth century are illuminating cases. The provisional contradictions faced by Einstein just before he formulated the special theory of relativity in 1905 is another typical example of what we call epistemic contradictions, which arise between two non-contradictory theories that, when put together, yield contradictory results. The phenomenon is not restricted to natural sciences: the imaginary numbers, which baffled mathematicians and philosophers until the beginning of the twentieth century, is another piece of epistemic contradiction.

Section 9.3 reviews—from a more philosophical perspective—the concept of pragmatic truth, also referred to as quasi-truth, or partial truth, already analyzed from the formal point of view in Chap. 7. Quasi-truth, developed as part of efforts to expand the bounds of the traditional Tarskian account of formalized truth, proposes a partial (or pragmatic) notion of truth, intending to capture the meaning of wider, more flexible, theories of truth held by anti-realist thinkers in philosophy of science.

Section 9.4 emphasizes the evidence-based approach to paraconsistency, in the sense of understanding a pair of contradictory sentences as representing, and allowing us to reason about, conflicting evidence, defending this view as particularly promising for philosophical interpretations of paraconsistent logics.

The last section, Sect. 9.5, succinctly wraps up one of the chief points behind **LFIs**: they are concerned with truth, since classical logic can be fully recovered inside most of the **LFIs**, but they are also concerned with the notion of evidence, a notion weaker than truth that allows for an intuitive and plausible understanding of the acceptance of contradictions in some reasoning contexts. In this regard, both intuitionistic and paraconsistent logics may be conceived as normative theories of logical consequence endowed with an epistemic character. This view not only

stresses the brotherhood between the intuitionistic and the paraconsistent paradigms, but explains the adequacy of **LFI**s for wider accounts in the philosophy of science, and also their applicability in the fields of linguistics, theoretical computer science, inferential probability, and confirmation theory.

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