Chapter 2
A Basic Logic of Formal Inconsistency: mbC

In [1], the study of **LFI**s starts with **mbC**, which is basic in the following sense: it starts with positive classical logic (which is assumed as a reasonable basis—in Chap. 5, some **LFI**s will be studied which are based on logics other than positive classical logic) and has a negation and a consistency operator which are added with minimal properties to satisfy the definition of **LFI**s.

2.1 Introducing mbC

Since the first chapters of the book are exclusively devoted to propositional logics, some notation which will be used throughout the rest of the book will now be defined. From this chapter on, and in the rest of the book, the symbol ■ is used to mean the end of a definition, while □ is used to indicate the end of a proof, being omitted when a proof is not given explicitly.

Definition 2.1.1 (*Propositional signatures*) A *propositional signature* is a set Θ of symbols called *connectives*, together with the information concerning the arity of each connective. ■

Notation 2.1.2 For the entirety of the book the following symbols will be used for logical connectives (the intended meaning and the arity of each connective are included in the list below): $\wedge$ (conjunction, binary); $\vee$ (disjunction, binary); $\rightarrow$ (implication, binary); $\neg$ (weak negation, unary); $\circ$ (consistency operator, unary); $\bullet$ (inconsistency operator, unary); $\sim$ (strong negation, unary); $\bot$ (bottom formula, 0-ary, i.e., a propositional constant).

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Definition 2.1.3 Consider the following propositional signatures that will be used thereafter:

- $\Sigma = \{\wedge, \vee, \rightarrow, \neg, \circ\}$;
- $\Sigma_\bullet = \{\wedge, \vee, \rightarrow, \neg, \bullet\}$;
- $\Sigma_0 = \{\wedge, \vee, \rightarrow, \neg\}$;
- $\Sigma_+ = \{\wedge, \vee, \rightarrow\}$;
- $\Sigma_c = \{\wedge, \vee, \rightarrow, \sim\}$;
- $\Sigma_1 = \{\wedge, \vee, \rightarrow, \neg, \sim\}$;
- $\Sigma_2 = \{\bot, \wedge, \vee, \rightarrow, \neg\}$;
- $\Sigma_\bot = \{\bot, \rightarrow, \neg, \circ\}$.

Let $Var = \{p_1, p_2, \ldots\}$ be a denumerable set of propositional variables (will be fixed henceforth), and let Θ be any propositional signature. The propositional language generated by Θ from Var will be denoted by $\mathcal{L}_\Theta$. ■

In this book we will deal exclusively with so-called *Tarskian logics* (see, for instance, [2]):

Definition 2.1.4 (*Tarskian Logic*) A logic $\mathscr{L}$ defined over a language $\mathcal{L}$, which has a consequence relation $\vdash$, is *Tarskian* if it satisfies the following properties, for every $\Gamma \cup \Delta \cup \{\alpha\} \subseteq \mathcal{L}$:

(i) if $\alpha \in \Gamma$ then $\Gamma \vdash \alpha$;
(ii) if $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash \alpha$;
(iii) if $\Delta \vdash \alpha$ and $\Gamma \vdash \beta$ for every $\beta \in \Delta$ then $\Gamma \vdash \alpha$.

A logic satisfying item (ii) above is called *monotonic*. A logic $\mathscr{L}$ is said to be *finitary* if it satisfies the following:

(iv) if $\Gamma \vdash \alpha$ then there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \vdash \alpha$.

Finally, a $\mathscr{L}$ defined over a propositional language $\mathcal{L}$ generated by a signature from a set of propositional variables is called *structural* if it satisfies the following property:

(v) if $\Gamma \vdash \alpha$ then $\sigma[\Gamma] \vdash \sigma(\alpha)$, for every substitution σ of formulas for variables.[1]

A propositional logic is *standard* if it is Tarskian, finitary and structural (see [2]). ■

From now on, a logic $\mathscr{L}$ will be represented by a pair $\mathscr{L} = \langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ and $\vdash$ denote the language and the consequence relation of $\mathscr{L}$, respectively. If $\mathcal{L}$ is generated by a propositional signature Θ from Var, that is, $\mathcal{L} = \mathcal{L}_\Theta$ then we will write $\mathscr{L} = \langle \Theta, \vdash \rangle$.

Notation 2.1.5 Let $\mathscr{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic. Let α be a formula in $\mathcal{L}$ and let $X_1 \ldots X_n$ (for $n \geq 1$) be a finite sequence such that each X_i is either a set of formulas in $\mathcal{L}$ or a formula in $\mathcal{L}$. Then, as usual, $X_1, \cdots, X_n \vdash \alpha$ will stand for $X'_1 \cup \ldots \cup X'_n \vdash \alpha$

[1] In this book the following standard notation will be adopted: given a function f and a subset A of its domain, $f[A]$ will denote the set $\{f(a) \ : \ a \in A\}$.

where, for each i, X_i' is X_i, if X_i is a set of formulas, or X_i' is $\{X_i\}$, if X_i is a formula. Thus, for instance, if Γ and Δ are sets of formulas and $\{\alpha, \beta, \alpha_1, \ldots, \alpha_n\}$ is a set of formulas then

$$\Gamma, \alpha \vdash \beta; \quad \alpha \vdash \beta; \quad \Gamma, \Delta \vdash \beta; \quad \alpha_1, \ldots, \alpha_n \vdash \beta$$

will stand for

$$\Gamma \cup \{\alpha\} \vdash \beta; \quad \{\alpha\} \vdash \beta; \quad \Gamma \cup \Delta \vdash \beta; \quad \{\alpha_1, \ldots, \alpha_n\} \vdash \beta$$

respectively.

The main notion of this book can now be defined rigorously: the Logics of Formal Inconsistency.

Definition 2.1.6 A Tarskian logic $\mathscr{L}$ is *paraconsistent* if it has a (primitive or defined) negation $\neg$ such that $\alpha, \neg\alpha \nvdash_{\mathscr{L}} \beta$ for some formulas α and β in the language of $\mathscr{L}$. ■

If $\mathscr{L}$ has a *deductive implication* $\to$, in the sense that it satisfies the Deduction meta-theorem DMT (see Proposition 2.1.14(i) below), then $\mathscr{L}$ is paraconsistent iff the schema formula $\alpha \to (\neg\alpha \to \beta)$ is not valid. That is, the *explosion law* is not valid in $\mathscr{L}$ with respect to the negation $\neg$. In other words, the negation $\neg$ is not *explosive*.

The idea behind **LFI**s is to have a paraconsistent logic in which the explosion law is allowed in a *local* or *controlled* way. This is attained by the existence of a set of formulas $\bigcirc(p)$, depending exclusively on a single propositional variable p, such that the set $\bigcirc(\alpha)$, together with a contradiction $\{\alpha, \neg\alpha\}$ is explosive or logically trivial, that is: $\bigcirc(\alpha), \alpha, \neg\alpha \vdash_{\mathscr{L}} \beta$ for every α and β. A logic satisfying this property is called *gently explosive* in [1, 3]. Of course, it must be also required that $\bigcirc(\alpha)$ together with α not be trivial, as well as the combination of $\bigcirc(\alpha)$ with $\neg\alpha$ (otherwise, the principle of gently explosiveness will be redundant). As we shall see, there exists three ways to introduce **LFI**s. The original one (proposed in [3]) is the following:

Definition 2.1.7 Let $\mathscr{L} = \langle \Theta, \vdash \rangle$ be a standard logic. Assume that the signature Θ of $\mathscr{L}$ contains a negation $\neg$, and let $\bigcirc(p)$ be a nonempty set of formulas depending exactly on the propositional variable p. Accordingly, $\mathscr{L}$ is a *Logic of Formal Inconsistency* (an **LFI**, for short) (with respect to $\neg$ and $\bigcirc(p)$) if the following holds (here, $\bigcirc(\varphi) = \{\psi(\varphi) \ : \ \psi(p) \in \bigcirc(p)\}$):

(i) $\varphi, \neg\varphi \nvdash \psi$ for some φ and ψ, i.e., $\mathscr{L}$ is not explosive w.r.t. $\neg$;
(ii) there are two formulas α and β such that

   (ii.a) $\bigcirc(\alpha), \alpha \nvdash \beta$;
   (ii.b) $\bigcirc(\alpha), \neg\alpha \nvdash \beta$;

(iii) $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ.

■

When $\bigcirc(p)$ is a singleton, its element will be denoted by $\circ p$, where $\circ$ is the *consistency* operator. Item (i) of the above definition states that an **LFI** is, by definition, non-explosive (w.r.t. $\neg$). Because of item (iii), $\mathscr{L}$ is said to be *gently explosive* w.r.t. $\neg$ and $\bigcirc(p)$.

Observe that clauses (i) and (ii) are existential, while (iii) is universal. The pair of witnesses required to satisfy (i) and (ii) are possibly different, but (ii) is composed by two clauses, both of them being satisfied by the same pair (α, β). This could be weakened, obtaining the following weaker notion of **LFI**s:

Definition 2.1.8 Let $\mathscr{L} = \langle \Theta, \vdash \rangle$ be a standard logic. Assume that the signature Θ of $\mathscr{L}$ contains a negation $\neg$, and let $\bigcirc(p)$ be a nonempty set of formulas depending exactly on the propositional variable p. Then $\mathscr{L}$ is a *weak* **LFI** (with respect to $\neg$ and $\bigcirc(p)$) if the following holds:

(i) $\varphi, \neg\varphi \nvdash \psi$ for some φ and ψ;
(ii) $\bigcirc(\varphi), \varphi \nvdash \psi$ for some φ and ψ;
(iii) $\bigcirc(\varphi), \neg\varphi \nvdash \psi$ for some φ and ψ; and
(iv) $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ.

■

Observe that any **LFI** is a weak **LFI**, but the converse is not necessarily true. The notion of weak **LFI** was proposed as an alternative to the original one in [4–6], since it is more uniform: the sets $\{\alpha, \neg\alpha\}$, $\{\alpha, \circ\alpha\}$ and $\{\neg\alpha, \circ\alpha\}$ are not always deductively trivial, but the set $\{\alpha, \neg\alpha, \circ\alpha\}$ is always deductively trivial.[2] Finally, a stronger notion of **LFI**s (which is also more uniform than the original definition) could be proposed:

Definition 2.1.9 Let $\mathscr{L} = \langle \Theta, \vdash \rangle$ be a standard logic. Assume that the signature Θ of $\mathscr{L}$ contains a negation $\neg$, and let $\bigcirc(p)$ be a nonempty set of formulas depending exactly on the propositional variable p. Then $\mathscr{L}$ is a *strong* **LFI** (with respect to $\neg$ and $\bigcirc(p)$) if the following holds:

(i) there are two formulas α and β such that

 (i.a) $\alpha, \neg\alpha \nvdash \beta$;
 (i.b) $\bigcirc(\alpha), \alpha \nvdash \beta$;
 (i.c) $\bigcirc(\alpha), \neg\alpha \nvdash \beta$; and

(ii) $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ.

■

[2] It should be observed that the weak **LFI**s investigated in the three references mentioned above are also strong **LFI**s in the sense of Definition 2.1.9.

Remark 2.1.10 Clearly, any strong **LFI** is an **LFI**, but the converse is not necessarily true. It should be observed that all the **LFI**s introduced in the previous literature are strong **LFI**s. Moreover, if $\mathscr{L}$ is a propositional logic then $\mathscr{L}$ is a strong **LFI** whenever the following holds:

(i) if p and q are two different propositional variables then

  (i.a) $p, \neg p \nvdash q$;
  (i.b) $\bigcirc(p), p \nvdash q$;
  (i.c) $\bigcirc(p), \neg p \nvdash q$; and

(ii) $\bigcirc(\varphi), \varphi, \neg\varphi \vdash \psi$ for every φ and ψ.

Because of its simplicity, conditions (i) and (ii) above will be used along the rest of the book in order to prove that a given logic is a strong **LFI**.

The general definition of **LFI**s encompasses a wide range of paraconsistent logics. In order to formally express the properties of consistency, any logic featuring a consistency connective must present a set of logical axiom schemas and semantic rules governing this connective. Along these lines, in [1], a fundamental propositional **LFI** known as **mbC** was first introduced. Starting from positive classical logic plus *tertium non datur* ($\alpha \vee \neg\alpha$), **mbC** is intended to comply with the above definition in a minimal way: an axiom schema called (**bc1**) is added solely to describe the expected behavior of the consistency operator $\circ$, namely, the gentle explosion law (see Definition 2.1.12). In what follows, this logic will be described in its original language along with the statement of soundness and completeness theorems with respect to paraconsistent valuations.

Definition 2.1.11 (*Formula Complexity*) The complexity of a given formula $\varphi \in \mathcal{L}_\Sigma$, denoted by $l(\varphi)$, is recursively defined as follows:

1. If $\varphi = p$, where $p \in Var$, then $l(\varphi) = 1$;
2. If $\varphi = \neg\alpha$, then $l(\varphi) = l(\alpha) + 1$;
3. If $\varphi = \circ\alpha$, then $l(\varphi) = l(\alpha) + 2$;
4. If $\varphi = \alpha\#\beta$, where $\# \in \{\wedge, \vee, \rightarrow\}$, then $l(\varphi) = l(\alpha) + l(\beta) + 1$.

■

Definition 2.1.12 (**mbC**) The logic **mbC** is defined over the language $\mathcal{L}_\Sigma$ by the Hilbert calculus:

Axiom schemas:

$$\alpha \rightarrow (\beta \rightarrow \alpha) \tag{\textbf{Ax1}}$$

$$\Big(\alpha \rightarrow (\beta \rightarrow \gamma)\Big) \rightarrow \Big((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)\Big) \tag{\textbf{Ax2}}$$

$$\alpha \rightarrow \Big(\beta \rightarrow (\alpha \wedge \beta)\Big) \tag{\textbf{Ax3}}$$

$$(\alpha \wedge \beta) \rightarrow \alpha \tag{\textbf{Ax4}}$$

$$\big(\alpha \wedge \beta\big) \to \beta \tag{Ax5}$$

$$\alpha \to \big(\alpha \vee \beta\big) \tag{Ax6}$$

$$\beta \to \big(\alpha \vee \beta\big) \tag{Ax7}$$

$$\Big(\alpha \to \gamma\Big) \to \Big((\beta \to \gamma) \to \big((\alpha \vee \beta) \to \gamma\big)\Big) \tag{Ax8}$$

$$\big(\alpha \to \beta\big) \vee \alpha \tag{Ax9}$$

$$\alpha \vee \neg\alpha \tag{Ax10}$$

$$\circ\alpha \to \Big(\alpha \to \big(\neg\alpha \to \beta\big)\Big) \tag{bc1}$$

Inference rule:

$$\frac{\alpha \quad \alpha \to \beta}{\beta} \tag{MP}$$

■

Axiom (**bc1**) is called the *gentle explosion law*. In Theorem 2.3.2, we shall prove that, as expected, the logic **mbC** is an **LFI**.

Definition 2.1.13 Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$ be a set of formulas. A *derivation of* φ *from* Γ *in* ***mbC*** is a finite sequence $\varphi_1 \cdots \varphi_n$ of formulas in $\mathcal{L}_\Sigma$ such that φ_n is φ and, for every $1 \leq i \leq n$, the following holds:

1. φ_i is an instance of an axiom schema of **mbC**, or
2. $\varphi_i \in \Gamma$, or
3. there exist $j, k < i$ such that $\varphi_k = \varphi_j \to \varphi_i$ (and so φ_i follows from φ_j and φ_k by **MP**).

We say that φ *is derivable from* Γ *in* **mbC**, denoted by $\Gamma \vdash_{\mathbf{mbC}} \varphi$, if there exists a derivation of φ from Γ in **mbC**. ■

Observe that (**Ax1**)–(**Ax9**) plus **MP** constitute a Hilbert calculus over the signature $\Sigma_+ = \{\wedge, \vee, \to\}$ for positive classical propositional logic **CPL**$^+$ (the negation-free fragment of classical propositional logic **CPL**, see Definition 2.4.3), which is in fact the basis for **mbC** and its extensions.

The following meta-theorems of **mbC** will prove to be quite useful throughout the entirety of the book.

Proposition 2.1.14 *The calculus* **mbC** *satisfies the following properties:*
(i) $\Gamma, \alpha \vdash_{\mathbf{mbC}} \beta$ *iff* $\Gamma \vdash_{\mathbf{mbC}} \alpha \to \beta$ *(Deduction meta-theorem, DMT).*
(ii) If $\Gamma, \alpha \vdash_{\mathbf{mbC}} \varphi$ *and* $\Gamma, \beta \vdash_{\mathbf{mbC}} \varphi$ *then* $\Gamma, \alpha \vee \beta \vdash_{\mathbf{mbC}} \varphi$.
(iii) If $\Gamma, \alpha \vdash_{\mathbf{mbC}} \varphi$ *and* $\Gamma, \neg\alpha \vdash_{\mathbf{mbC}} \varphi$ *then* $\Gamma \vdash_{\mathbf{mbC}} \varphi$ *(Proof-by-cases).*

Proof (i) It is well known that, in order to prove that a Hilbert calculus satisfies DMT, it suffices to derive axioms (**Ax1**) and (**Ax2**), while **MP** must be the unique inference rule (see, for instance, [7]).
(ii) Suppose $\Gamma, \alpha \vdash_{\mathbf{mbC}} \varphi$ and $\Gamma, \beta \vdash_{\mathbf{mbC}} \varphi$. By DMT, both $\Gamma \vdash_{\mathbf{mbC}} \alpha \to \varphi$ and $\Gamma \vdash_{\mathbf{mbC}} \beta \to \varphi$. By axiom (**Ax8**), $\vdash_{\mathbf{mbC}} (\alpha \to \varphi) \to ((\beta \to \varphi) \to ((\alpha \vee \beta) \to \varphi))$ and so $\Gamma \vdash_{\mathbf{mbC}} (\alpha \vee \beta) \to \varphi$ by **MP** twice. Therefore $\Gamma, \alpha \vee \beta \vdash_{\mathbf{mbC}} \varphi$, by **MP**.
(iii) This is a consequence of item (ii) and the fact that $\alpha \vee \neg\alpha$ is a theorem of **mbC**. □

2.2 A Valuation Semantics for mbC

In [8, 9], N.C.A. da Costa and E. H. Alves proposed an original valuation semantics for C_1 over $\{0, 1\}$. A key feature of these valuations is that, as expected, they are defined as 2-valued $\mathbf{CPL}^+$-valuations with respect to the binary connectives (conjunction, disjunction and implication). However, the paraconsistent negation $\neg$ has a non-deterministic behavior w.r.t. this semantics: in general, if one of such valuations assigns the value 1 to a formula α, then the formula $\neg\alpha$ can receive either the value 0 or the value 1 (but not both) under the same valuation. That is: the truth-value of α does not uniquely determine the truth-value of $\neg\alpha$.

This kind of semantics (sometimes called *bivaluations*) were generalized to several **LFI**s in [1]. Based on that approach, in this section the logic **mbC** will be semantically characterized by a suitable valuation semantics over $\{0, 1\}$. The same will be done for several other **LFI**s in the next chapters of the book.

Definition 2.2.1 (*Valuations for* **mbC**) A function $v : \mathcal{L}_\Sigma \to \{0, 1\}$ is a *valuation for* **mbC**, or an **mbC**-*valuation*, if it satisfies the following clauses:

(***vAnd***) $v(\alpha \wedge \beta) = 1 \iff v(\alpha) = 1$ and $v(\beta) = 1$
(***vOr***) $v(\alpha \vee \beta) = 1 \iff v(\alpha) = 1$ or $v(\beta) = 1$
(***vImp***) $v(\alpha \to \beta) = 1 \iff v(\alpha) = 0$ or $v(\beta) = 1$
(***vNeg***) $v(\neg\alpha) = 0 \implies v(\alpha) = 1$
(***vCon***) $v(\circ\alpha) = 1 \implies v(\alpha) = 0$ or $v(\neg\alpha) = 0$.

The set of all such valuations will be designated by $V^{\mathbf{mbC}}$. ■

It should be observed that each **mbC**-valuation v assigns an unique truth-value (0 or 1) to each formula of **mbC**. However, because of clauses (*vNeg*) and (*vCon*), the value $v(\#\alpha)$ is not necessarily determined by the value $v(\alpha)$ of the immediate subformula α, for $\# \in \{\neg, \circ\}$.

For every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$, the following semantical consequence relation w.r.t. the set $V^{\mathbf{mbC}}$ of **mbC**-valuations can be naturally defined: $\Gamma \vDash_{\mathbf{mbC}} \varphi$ iff, for every $v \in V^{\mathbf{mbC}}$, if $v(\gamma) = 1$ for every $\gamma \in \Gamma$ then $v(\varphi) = 1$. The set $V^{\mathbf{mbC}}$ constitutes a sound and complete semantics for the logic **mbC**, as it will be proved below.

Theorem 2.2.2 (Soundness) *For every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$*:*

$$\Gamma \vdash_{\mathbf{mbC}} \varphi \quad \Longrightarrow \quad \Gamma \vDash_{\mathbf{mbC}} \varphi.$$

Proof The first step is to show the following:

Facts: Let v be an **mbC**-valuation.
(1) If γ is an instance of a **mbC** axiom schema, then $v(\gamma) = 1$.

(2) If α and β are formulas such that $v(\alpha) = v(\alpha \to \beta) = 1$, then $v(\beta) = 1$.

In order to prove (1), each **mbC** axiom schema must be checked. Observe that the clauses for the connectives $\wedge$, $\vee$ and $\to$ are the usual ones which characterize the truth-tables for these connectives in classical logic. Thus, axioms **Ax1**–**Ax9** (corresponding to positive classical logic) are sound w.r.t. this valuation semantics. Concerning axiom **Ax10**, let α be a formula. If $v(\neg\alpha) = 1$ then $v(\alpha \vee \neg\alpha) = 1$, by (vOr). Otherwise, if $v(\neg\alpha) = 0$ then $v(\alpha) = 1$, by $(vNeg)$, whence $v(\alpha \vee \neg\alpha) = 1$, by (vOr). Finally, let $\gamma = \circ\alpha \to \Big(\alpha \to \big(\neg\alpha \to \beta\big)\Big)$ be an instance of axiom **bc1**. If $v(\circ\alpha) = 0$ then $v(\gamma) = 1$, by $(vImp)$. Otherwise, if $v(\circ\alpha) = 1$ then either $v(\alpha) = 0$ or $v(\neg\alpha) = 0$. In both cases it is easy to see that $v(\gamma) = 1$, by $(vImp)$.
Item (2) is an easy consequence of the clause $(vImp)$ from Definition 2.2.1.
Once these facts are proved, the rest of the proof follows by induction on the length n of a derivation $\varphi_1 \ldots \varphi_n = \varphi$ in **mbC** of φ from Γ. Indeed, it will be proven by induction on n that, for any **mbC**-valuation v such that $v[\Gamma] \subseteq \{1\}$, $v(\varphi_i) = 1$ for every $1 \leq i \leq n$. In particular, $v(\varphi) = 1$, showing that $\Gamma \vDash_{\mathbf{mbC}} \varphi$ as required.

Thus, if $n = 1$, then φ_1 is either an instance of an axiom schema of **mbC** (and so the result follows by **Facts**(1)), or $\varphi_1 \in \Gamma$ (and so the result follows by hypothesis). Suppose now that the result holds for every formula ψ admitting a derivation in **mbC** from Γ of length $k \leq n$ (induction hypothesis), and suppose that $\varphi_1 \ldots \varphi_{n+1} = \varphi$ is a derivation in **mbC** of φ from Γ with length $n + 1$. By induction hypothesis, $v(\varphi_i) = 1$ for $1 \leq i \leq n$ (as each φ_i is derived from Γ in **mbC** with a derivation of length i). If $\varphi_{n+1} \in \Gamma$ or φ_{n+1} is an instance of an **mbC** axiom schema, the proof is as above. Otherwise, there exist $j, k \leq n$ such that $\varphi_k = \varphi_j \to \varphi_{n+1}$ and φ_{n+1} is obtained from φ_j and φ_k by **MP**. By **Facts**(2), $v(\varphi_{n+1}) = 1$. This completes the proof. □

The proof of completeness needs some definitions and results. Recall the notion of Tarskian Logic from Definition 2.1.4.

Definition 2.2.3 For a given Tarskian logic $\mathscr{L}$ over the language $\mathcal{L}$, let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$. The set Γ is *maximal* non-trivial with respect to φ in $\mathscr{L}$ if $\Gamma \nvdash_{\mathscr{L}} \varphi$ but $\Gamma, \psi \vdash_{\mathscr{L}} \varphi$ for any $\psi \notin \Gamma$. ■

Definition 2.2.4 Let $\mathscr{L}$ be a Tarskian logic. A set of formulas Γ is *closed in* $\mathscr{L}$, or a *closed theory of* $\mathscr{L}$, if the following holds for every formula ψ: $\Gamma \vdash_{\mathscr{L}} \psi$ iff $\psi \in \Gamma$.

Lemma 2.2.5 *Any set of formulas maximal non-trivial with respect to* φ *in* $\mathscr{L}$ *is closed, provided that* $\mathscr{L}$ *is Tarskian.*

Proof Let Γ be a set of formulas maximal non-trivial with respect to φ in $\mathscr{L}$. If $\psi \in \Gamma$ then $\Gamma \vdash_{\mathscr{L}} \psi$, as $\mathscr{L}$ is Tarskian. Conversely, if $\Gamma \vdash_{\mathscr{L}} \psi$, suppose that $\psi \notin \Gamma$. Then, by Definition 2.2.3, $\Gamma, \psi \vdash_{\mathscr{L}} \varphi$. However, given that $\mathscr{L}$ is Tarskian, it follows that $\Gamma \vdash_{\mathscr{L}} \varphi$, contradicting the hypothesis that Γ is maximal non-trivial with respect to φ in $\mathscr{L}$. Then $\psi \in \Gamma$ and so Γ is a closed theory. □

Now consider the following classical result:

Theorem 2.2.6 (Lindenbaum-Łos) *Let $\mathscr{L}$ be a Tarskian and finitary logic over the language $\mathcal{L}$. Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$ be such that $\Gamma \nvdash_{\mathscr{L}} \varphi$. There exists then a set Δ such that $\Gamma \subseteq \Delta \subseteq \mathcal{L}$ with Δ maximal non-trivial with respect to φ in $\mathscr{L}$.*

Proof We will adapt the proof in [2] (Theorem 22.2). By the well-ordering principle,[3] assume that the set $\mathcal{L}$ is well-ordered as a transfinite sequence $(\varphi_\lambda)_{\lambda<\theta}$, where θ is an ordinal. By transfinite recursion, a transfinite and increasing sequence of theories $(\Gamma_\lambda)_{\lambda<\theta}$ is defined as follows: $\Gamma_0 = \Gamma$, and for every $\lambda < \theta$,

$$\Gamma_\lambda = \begin{cases} \Gamma_\mu & \text{if } \lambda = \mu + 1 \text{ and } \Gamma_\mu, \varphi_\mu \vdash_{\mathscr{L}} \varphi \\ \Gamma_\mu \cup \{\varphi_\mu\} & \text{if } \lambda = \mu + 1 \text{ and } \Gamma_\mu, \varphi_\mu \nvdash_{\mathscr{L}} \varphi \\ \bigcup_{\mu<\lambda} \Gamma_\mu & \text{if } \lambda \text{ is a limit ordinal.} \end{cases}$$

Then $\Delta = \bigcup_{\lambda<\theta} \Gamma_\lambda$ satisfies the requirements. Indeed, observe firstly that $\Gamma \subseteq \Delta$. By transfinite induction, it is easy to prove that $\Gamma_\lambda \nvdash_{\mathscr{L}} \varphi$, for every $\lambda < \theta$: if $\lambda = 0$ then $\Gamma_\lambda \nvdash_{\mathscr{L}} \varphi$, by hypothesis. Assuming that $\Gamma_\mu \nvdash_{\mathscr{L}} \varphi$ for every $\mu < \lambda < \theta$, suppose that $\lambda = \mu + 1 < \theta$. Then $\Gamma_\mu \nvdash_{\mathscr{L}} \varphi$, by induction hypothesis, and so $\Gamma_\lambda \nvdash_{\mathscr{L}} \varphi$, by definition of Γ_λ. If λ is a limit ordinal, suppose that $\Gamma_\lambda \vdash_{\mathscr{L}} \varphi$. By finitariness of $\mathscr{L}$, there exists a finite subset Γ^{fin} of Γ_λ such that $\Gamma^{fin} \vdash_{\mathscr{L}} \varphi$. But $\Gamma_\mu \subseteq \Gamma_\kappa$ if $\mu < \kappa$ and so $\Gamma^{fin} \subseteq \Gamma_\mu$ for some $\mu < \lambda$. This means that $\Gamma_\mu \vdash_{\mathscr{L}} \varphi$ for some $\mu < \lambda$, contradicting the induction hypothesis.

By a similar argument, $\Delta \nvdash_{\mathscr{L}} \varphi$ is proved. Suppose now that $\psi \notin \Delta$. Then $\psi = \varphi_\mu$ for some $\mu < \theta$ and so $\varphi_\mu \notin \Gamma_{\mu+1}$, by definition of Δ. By construction of $\Gamma_{\mu+1}$, it follows that $\Gamma_\mu, \varphi_\mu \vdash_{\mathscr{L}} \varphi$ and so, by monotonicity of $\mathscr{L}$, it follows that $\Delta, \psi \vdash_{\mathscr{L}} \varphi$. This shows that Δ is maximal non-trivial with respect to φ in $\mathscr{L}$. □

Every logic $\mathscr{L}$ defined by a Hilbert calculus, where the inference rules are finitary, is Tarskian and finitary, and so Theorem 2.2.6 holds for $\mathscr{L}$. In particular, Theorem 2.2.6 holds for **mbC**.

Theorem 2.2.7 *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$, with Γ maximal non-trivial with respect to φ in* **mbC**. *The mapping $v : \mathcal{L}_\Sigma \to \{0, 1\}$ defined by:*

$$v(\psi) = 1 \quad \iff \quad \psi \in \Gamma$$

for all $\psi \in \mathcal{L}_\Sigma$ is a valuation for **mbC**.

[3] Since it is well-known, it is equivalent to the Axiom of Choice.

Proof It will be proved that v satisfies all the clauses of Definition 2.2.1.
1. Assume that $v(\alpha \vee \beta) = 1$. Then $\alpha \vee \beta \in \Gamma$. Suppose that neither $\alpha \in \Gamma$ nor $\beta \in \Gamma$. Then $\Gamma, \alpha \vdash_{\mathbf{mbC}} \varphi$ and $\Gamma, \beta \vdash_{\mathbf{mbC}} \varphi$. So $\Gamma, \alpha \vee \beta \vdash_{\mathbf{mbC}} \varphi$, by Proposition 2.1.14(ii). But then $\Gamma \vdash_{\mathbf{mbC}} \varphi$, which is a contradiction. Thus, either $\alpha \in \Gamma$ or $\beta \in \Gamma$ and so either $v(\alpha) = 1$ or $v(\beta) = 1$. Conversely, suppose that either $v(\alpha) = 1$ or $v(\beta) = 1$. Thus, either $\alpha \in \Gamma$ or $\beta \in \Gamma$. Suppose that $\alpha \in \Gamma$. As $\alpha \to (\alpha \vee \beta) \in \Gamma$, by axiom (**Ax6**) and by Lemma 2.2.5, it follows that $\alpha \vee \beta \in \Gamma$, by **MP**. From this, $v(\alpha \vee \beta) = 1$. Analogously, if $\beta \in \Gamma$ then $v(\alpha \vee \beta) = 1$ (now by using axiom (**Ax7**)). This shows that v satisfies clause (*vOr*) of Definition 2.2.1.
2. Assume that $v(\alpha \wedge \beta) = 1$. Then $\alpha \wedge \beta \in \Gamma$. As $(\alpha \wedge \beta) \to \alpha \in \Gamma$, by axiom (**Ax4**) and Lemma 2.2.5, it follows that $\alpha \in \Gamma$ by **MP**. From this, $v(\alpha) = 1$. Analogously, $v(\beta) = 1$ is proved by axiom (**Ax5**). Conversely, suppose that $v(\alpha) = 1$ and $v(\beta) = 1$. Then $\alpha \in \Gamma$ and $\beta \in \Gamma$. But $\alpha \to \Big(\beta \to \big(\alpha \wedge \beta\big)\Big) \in \Gamma$, by axiom (**Ax3**) and by Lemma 2.2.5. Then, by applying **MP** twice, it follows that $\alpha \wedge \beta \in \Gamma$ and so $v(\alpha \wedge \beta) = 1$. This shows that v satisfies clause (*vAnd*) of Definition 2.2.1.
3. Suppose that $v(\alpha \to \beta) = 1$. Then $\alpha \to \beta \in \Gamma$. If $\alpha \in \Gamma$, then $\beta \in \Gamma$ by **MP** and Lemma 2.2.5. Then $v(\alpha) = 1$ implies that $v(\beta) = 1$. This shows that either $v(\alpha) = 0$ or $v(\beta) = 1$. Conversely, suppose that either $v(\alpha) = 0$ or $v(\beta) = 1$. Then either $\alpha \notin \Gamma$ or $\beta \in \Gamma$. If $\beta \in \Gamma$, then $\alpha \to \beta \in \Gamma$ by axiom (**Ax1**), **MP** and Lemma 2.2.5. So $v(\alpha \to \beta) = 1$. Now, if $\alpha \notin \Gamma$ then, by the maximality of Γ, it follows that $\Gamma, \alpha \vdash_{\mathbf{mbC}} \varphi$. Suppose, by contradiction, that $\alpha \to \beta \notin \Gamma$. Then, again by the maximality of Γ, it follows that $\Gamma, \alpha \to \beta \vdash_{\mathbf{mbC}} \varphi$. Hence $\Gamma, (\alpha \to \beta) \vee \alpha \vdash_{\mathbf{mbC}} \varphi$, by Proposition 2.1.14(ii). But then $\Gamma \vdash_{\mathbf{mbC}} \varphi$ by axiom (**Ax9**), which leads to a contradiction. Therefore $\alpha \to \beta \in \Gamma$ and so $v(\alpha \to \beta) = 1$, showing that v satisfies clause (*vImp*) of Definition 2.2.1.
4. Suppose that $v(\neg\alpha) = 0$ and, by contradiction, that also $v(\alpha) = 0$. Then $\neg\alpha \notin \Gamma$ and $\alpha \notin \Gamma$. As Γ is maximal, it follows that $\Gamma, \neg\alpha \vdash_{\mathbf{mbC}} \varphi$ and $\Gamma, \alpha \vdash_{\mathbf{mbC}} \varphi$. By Proposition 2.1.14(iii), $\Gamma \vdash_{\mathbf{mbC}} \varphi$, which is a contradiction. Therefore $v(\neg\alpha) = 0$ implies that $v(\alpha) = 1$, and so v satisfies clause (*vNeg*) of Definition 2.2.1.
5. Suppose that $v(\circ\alpha) = 1$ and, by contradiction, that both $v(\alpha) = 1$ and $v(\neg\alpha) = 1$. Then $\circ\alpha \in \Gamma$ and both $\alpha \in \Gamma$ and $\neg\alpha \in \Gamma$. Thus, by axiom (**bc1**), **MP** twice and Lemma 2.2.5, $\beta \in \Gamma$ for every β. In particular, $\varphi \in \Gamma$, which is a contradiction. Therefore: $v(\circ\alpha) = 1$ implies that either $v(\alpha) = 0$ or $v(\neg\alpha) = 0$. This means that v satisfies clause (*vCon*) of Definition 2.2.1. □

Theorem 2.2.8 (Completeness of **mbC** w.r.t. valuations) *For every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$*:*

$$\Gamma \vDash_{\mathbf{mbC}} \varphi \quad \Longrightarrow \quad \Gamma \vdash_{\mathbf{mbC}} \varphi$$

Proof Suppose $\Gamma \nvdash_{\mathbf{mbC}} \varphi$ and let Δ be a maximal non-trivial set with respect to φ in **mbC** extending Γ (using Theorem 2.2.6). By Theorem 2.2.7, there is an **mbC**-valuation v, such that $v[\Gamma] \subseteq \{1\}$ (as $\Gamma \subseteq \Delta$) but $v(\varphi) = 0$ (as $\varphi \notin \Delta$). Therefore, $\Gamma \nvDash_{\mathbf{mbC}} \varphi$ and the theorem follows by contraposition. □

2.3 Applications of mbC-Valuations

From the soundness and completeness theorems for **mbC** w.r.t. valuations proved in the previous section, some properties of **mbC** can now be stated semantically, instead of using the associated Hilbert calculus.

We begin by observing that **mbC**-valuations can be used to construct truth-tables which exhibit a non-deterministic character.[4] Indeed, the clauses for **mbC**-valuations corresponding to the binary connectives $\to$, $\vee$ and $\wedge$ (see Definition 2.2.1) define the usual truth-tables for these connectives over $\{0, 1\}$. On the other hand, the paraconsistent negation $\neg$ defines the following diagram:

α	$\neg\alpha$	
1	1	v_1
	0	v_2
0	1	v_3

This means that there are three **mbC**-valuations (or scenarios) concerning a proposition α and its paraconsistent negation $\neg\alpha$, namely v_1, v_2 and v_3. According to v_1, $v_1(\alpha) = v_1(\neg\alpha) = 1$. According to v_2, $v_2(\alpha) = 1$ but $v_2(\neg\alpha) = 0$. Finally, in the third scenario v_3, $v_3(\alpha) = 0$ and $v_3(\neg\alpha) = 1$. Observe that the fourth scenario, namely $v_4(\alpha) = v_4(\neg\alpha) = 0$, is not allowed for **mbC**-valuations, in virtue of the clause (*vNeg*): α and $\neg\alpha$ can be both simultaneously true, but they cannot be simultaneously false. This means that **mbC** is paraconsistent but not paracomplete.[5]

Concerning the other non-classical connective, the consistency operator $\circ$, it can be better understood in terms of the formulas α and $\neg\alpha$ instead of analyzing α or $\neg\alpha$ alone:

Table 2.1

α	$\neg\alpha$	$\circ\alpha$	
1	1	0	v_1
	0	1	v_2
		0	v_3
0	1	1	v_4
		0	v_5

[4]However, such tables do not correspond to non-deterministic matrices in the sense of Avron and Lev (see [10, 11]). The relationship between valuations for **LFI**s and non-deterministic matrices will be analyzed in Chap. 6.

[5]There are logics which are simultaneously paraconsistent and paracomplete, that is, that allow the fourth scenario in which α and $\neg\alpha$ are both false. Logics of this kind are frequently called *paranormal* by the literature. One example of paranormality is a tetravalent modal logic that can be associated with Monteiro's tetravalent modal algebras, see [12]. This example will be analyzed in Chap. 5.

Observe that, if $v(\alpha) = v(\neg\alpha) = 1$, then $v(\circ\alpha)$ is forced to be 0. Otherwise, if $v(\alpha) \neq v(\neg\alpha)$ then the truth-value $v(\circ\alpha)$ of $\circ\alpha$ is arbitrary in **mbC**.

Remark 2.3.1 Diagrams as the one displayed in Table 2.1 can be naturally associated to the non-deterministic valuation semantic over $\{0, 1\}$ for **mbC** introduced in Definition 2.2.1. This idea was originally proposed by da Costa and Alves in [9], associated to the non-deterministic valuation semantic over $\{0, 1\}$ for C_1 defined there. It was proved by Fidel (see Theorem 7, p. 627 in [9]) that such diagrams, called *quasi-matrices* by Alves in [13],[6] provide a decision procedure for testing tautologies in C_1. It is easy to see that the same holds for **mbC** and for most of the **LFI**s to be analized in this book.

By using the soundness and completeness theorems for **mbC** w.r.t. valuations, it is easy to prove the following:

Theorem 2.3.2 *Let $\bigcirc(p) = \{\circ p\}$, for a propositional variable p. Then the logic* **mbC** *is a strong* **LFI** *(w.r.t. $\neg$ and $\bigcirc(p)$), according to Definition 2.1.9.*

Proof Assume that p and q are two different propositional variables. By considering the valuation v_1 of Table 2.1 and taking $v_1(q) = 0$, it follows that $p, \neg p \nvdash_{\mathbf{mbC}} q$ and clause (i.a) of Remark 2.1.10 thusly is satisfied. Now considering valuation v_2 and taking $v_2(q) = 0$, we show that $\circ p, p \nvdash_{\mathbf{mbC}} q$ and clause (i.b) of Remark 2.1.10 is satisfied. By considering valuation v_4 such that $v_4(q) = 0$, it follows that $\circ p, \neg p \nvdash_{\mathbf{mbC}} q$ and clause (i.c) of Remark 2.1.10 is satisfied. Finally, by means of the same table, it is clear that for no valuation it is the case that α, $\neg\alpha$ and $\circ\alpha$ are simultaneously true. Thus, clause (ii) of Remark 2.1.10 is satisfied. This shows that **mbC** is a strong **LFI** w.r.t. $\neg$ and $\circ$. □

The dependency of $\circ\alpha$ on $\{\alpha, \neg\alpha\}$, which was highlighted above, explains why the complexity $l(\circ\alpha)$ of $\circ\alpha$ is defined as $l(\alpha) + 2$. That is, the complexity of $\circ\alpha$ is strictly greater than the complexity of α and $\neg\alpha$ (recall Definition 2.1.11). By combining these non-deterministic truth-tables with the (deterministic) truth-table of conjunction, we can compare the consistency $\circ\alpha$ with the non-contradiction $\neg(\alpha \wedge \neg\alpha)$ on the one hand, and the inconsistency (or non-consistency) $\neg\circ\alpha$ with the contradiction $\alpha \wedge \neg\alpha$ on the other. This produces eight possible-scenarios (or **mbC**-valuations), which are depicted below.

[6]It should be observed that Ivlev (see [14]) and other authors use the term 'quasi-matrices' to refer to non-deterministic matrices in the sense of of A. Avron and I. Lev.

α	$\neg\alpha$	$\circ\alpha$	$\neg\circ\alpha$	$\alpha \wedge \neg\alpha$	$\neg(\alpha \wedge \neg\alpha)$	
1	1	0	1	1	1	v_1
					0	v_2
	0	1	1	0	1	v_3
			0	0	1	v_4
		0	1	0	1	v_5
0	1	1	1	0	1	v_6
			0	0	1	v_7
		0	1	0	1	v_8

From the table above, and by the adequacy of **mbC** w.r.t. valuations, the following becomes clear:

Proposition 2.3.3 *In* **mbC** *the following holds:*
(1) $\alpha \wedge \neg\alpha \vdash_{\mathbf{mbC}} \neg\circ\alpha$ *but* $\neg\circ\alpha \nvdash_{\mathbf{mbC}} \alpha \wedge \neg\alpha$*;*
(2) $\circ\alpha \vdash_{\mathbf{mbC}} \neg(\alpha \wedge \neg\alpha)$ *but* $\neg(\alpha \wedge \neg\alpha) \nvdash_{\mathbf{mbC}} \circ\alpha$*.*

Proof (1) By inspection of the table above, it follows that $\alpha \wedge \neg\alpha \vDash_{\mathbf{mbC}} \neg\circ\alpha$ but $\neg\circ\alpha \nvDash_{\mathbf{mbC}} \alpha \wedge \neg\alpha$ (because of v_3, v_5, v_6 and v_8).
(2) Looking at the table above, it follows that $\circ\alpha \vDash_{\mathbf{mbC}} \neg(\alpha \wedge \neg\alpha)$ but $\neg(\alpha \wedge \neg\alpha) \nvDash_{\mathbf{mbC}} \circ\alpha$ (because of v_1, v_5 and v_8). □

Item (1) of Proposition 2.3.3 shows that, in **mbC**, the notion of contradiction is strictly stronger than the notion of inconsistency (or non-consistency). In other words, in **mbC** every contradictory formula is inconsistent, but the converse is not always the case. By its turn, item (2) establishes that the notion of consistency is strictly stronger than the notion of non-contradiction in **mbC**. That is, every consistent formula is non-contradictory, but the converse does not hold in general.

Another simple example of application of **mbC**-valuations is the following:

Proposition 2.3.4 *In* **mbC** *the following holds:*
(1) $\neg\alpha \to \beta \vdash_{\mathbf{mbC}} \alpha \vee \beta$ *but* $\alpha \vee \beta \nvdash_{\mathbf{mbC}} \neg\alpha \to \beta$*;*
(2) $\circ\alpha, \alpha \vee \beta \vdash_{\mathbf{mbC}} \neg\alpha \to \beta$*.*

Proof Consider the following diagram:

α	β	$\neg\alpha$	$\alpha \vee \beta$	$\neg\alpha \to \beta$	
1	1	1	1	1	v_1
		0	1	1	v_2
1	0	1	1	0	v_3
		0	1	1	v_4
0	1	1	1	1	v_5
0	0	1	0	0	v_6

Clearly, $\neg\alpha \to \beta \vDash_{\mathbf{mbC}} \alpha \vee \beta$ but $\alpha \vee \beta \nvDash_{\mathbf{mbC}} \neg\alpha \to \beta$, because of valuation v_3. However, $v_3(\alpha) = v_3(\neg\alpha) = 1$, so $v_3(\circ\alpha) = 0$. Accordingly, $\circ\alpha, \alpha \vee \beta \vDash_{\mathbf{mbC}} \neg\alpha \to \beta$ given that v_3, the unique countermodel for the inference $\alpha \vee \beta \vDash_{\mathbf{mbC}} \neg\alpha \to \beta$, does not satisfy the premises: $v_3[\{\circ\alpha, \alpha \vee \beta\}] \nsubseteq \{1\}$. □

Finally, the following example, which comes from the same vein, shows that several contraposition rules for implication do not hold when the paraconsistent negation is taken into account:

Proposition 2.3.5 *In* **mbC** *the following holds:*
(1) $\alpha \to \beta \nvdash_{\mathbf{mbC}} \neg\beta \to \neg\alpha$ *but* $\circ\beta, \alpha \to \beta \vdash_{\mathbf{mbC}} \neg\beta \to \neg\alpha$*;*
(2) $\alpha \to \neg\beta \nvdash_{\mathbf{mbC}} \beta \to \neg\alpha$ *but* $\circ\beta, \alpha \to \neg\beta \vdash_{\mathbf{mbC}} \beta \to \neg\alpha$*;*
(3) $\neg\alpha \to \beta \nvdash_{\mathbf{mbC}} \neg\beta \to \alpha$ *but* $\circ\beta, \neg\alpha \to \beta \vdash_{\mathbf{mbC}} \neg\beta \to \alpha$*;*
(4) $\neg\alpha \to \neg\beta \nvdash_{\mathbf{mbC}} \beta \to \alpha$ *but* $\circ\beta, \neg\alpha \to \neg\beta \vdash_{\mathbf{mbC}} \beta \to \alpha$*.*

Proof Items (1) and (2). Consider the following diagram:

α	β	$\neg\alpha$	$\neg\beta$	$\alpha \to \beta$	$\neg\beta \to \neg\alpha$	$\alpha \to \neg\beta$	$\beta \to \neg\alpha$	
1	1	1	1	1	1	1	1	v_1
			0	1	1	0	1	v_2
		0	1	1	0	1	0	v_3
			0	1	1	0	0	v_4
1	0	1	1	0	1	1	1	v_5
		0	1	0	0	1	1	v_6
0	1	1	1	1	1	1	1	v_7
			0	1	1	1	1	v_8
0	0	1	1	1	1	1	1	v_9

Because of valuation v_3, $\alpha \to \beta \nvDash_{\mathbf{mbC}} \neg\beta \to \neg\alpha$ and $\alpha \to \neg\beta \nvDash_{\mathbf{mbC}} \beta \to \neg\alpha$. But $v_3(\beta) = v_3(\neg\beta) = 1$, so $v_3(\circ\beta) = 0$. Using an argument similar to the proof found in Proposition 2.3.4, it follows that $\circ\beta, \alpha \to \beta \vDash_{\mathbf{mbC}} \neg\beta \to \neg\alpha$ and $\circ\beta$, $\alpha \to \neg\beta \vDash_{\mathbf{mbC}} \beta \to \neg\alpha$.
Items (3) and (4). Consider the following diagram:

α	β	$\neg\alpha$	$\neg\beta$	$\neg\alpha \to \beta$	$\neg\beta \to \alpha$	$\neg\alpha \to \neg\beta$	$\beta \to \alpha$	
1	1	1	1	1	1	1	1	v_1
			0	1	1	0	1	v_2
		0	1	0	1	1	1	v_3
			0	0	1	1	1	v_4
1	0	1	1	0	1	1	1	v_5
		0	1	1	1	1	1	v_6
0	1	1	1	1	0	1	0	v_7
			0	1	1	0	0	v_8
0	0	1	1	0	0	1	0	v_9

Because of valuation v_7, $\neg\alpha \to \beta \nvDash_{\mathbf{mbC}} \neg\beta \to \alpha$ and $\neg\alpha \to \neg\beta \nvDash_{\mathbf{mbC}} \beta \to \alpha$. As $v_7(\beta) = v_7(\neg\beta) = 1$, $v_7(\circ\beta) = 0$. Therefore, $\circ\beta, \neg\alpha \to \beta \vDash_{\mathbf{mbC}} \neg\beta \to \alpha$ and $\circ\beta, \neg\alpha \to \neg\beta \vDash_{\mathbf{mbC}} \beta \to \alpha$. $\square$

Finally, valuation semantics allows one to clearly see that the logic **mbC** is not self-extensional in Wójcicki's sense (see [15]). We say that a propositional (Tarskian) logic $\mathscr{L}$ satisfies *weak replacement* if the following holds: given formulas α_i and β_i (for $1 \leq i \leq n$) such that $\alpha_1 \equiv \beta_1, \ldots, \alpha_n \equiv \beta_n$, then $\varphi(\alpha_1, \ldots, \alpha_n) \equiv \varphi(\beta_1, \ldots, \beta_n)$ for every formula $\varphi(p_1, \ldots, p_n)$. Here, $\alpha \equiv \beta$ is an abbreviation for $\alpha \vdash_{\mathscr{L}} \beta$ and $\beta \vdash_{\mathscr{L}} \alpha$. A logic is said to be *self-extensional* if it satisfies weak replacement.

It is easy to comprehend, by using valuations, that **mbC** does not satisfy weak replacement and so it is not self-extensional. Indeed, from $\alpha \equiv \beta$, it does not follow in general that $\#\alpha \equiv \#\beta$, for $\# \in \{\neg, \circ\}$. For instance, $(p_1 \wedge p_2) \equiv (p_2 \wedge p_1)$. However, neither $\neg(p_1 \wedge p_2) \equiv \neg(p_2 \wedge p_1)$ nor $\circ(p_1 \wedge p_2) \equiv \circ(p_2 \wedge p_1)$: it is enough to consider a **mbC**-valuation v such that $v(p_1 \wedge p_2) = 1$, $v(\neg(p_1 \wedge p_2)) = 1$ (and so $v(\circ(p_1 \wedge p_2)) = 0$), but $v(\neg(p_2 \wedge p_1)) = 0$ and $v(\circ(p_2 \wedge p_1)) = 1$.

Moreover, since **mbC** and several of its extensions do not admit a non-trivial logical consequence, they are not algebraizable even in the wide framework of Blok and Pigozzi (see [16–20]). The question of algebraizability of extensions of **mbC** will be analyzed in Chaps. 4 and 6.

2.4 Recovering Classical Logic Inside mbC

The laws governing the operator $\neg$ of **mbC** define a paraconsistent negation, making it weaker than the classical negation (which will be represented from now on by the symbol $\sim$). In order to clarify the relationship between **mbC** and classical propositional logic (**CPL**), the notion of *translations* and *conservative translations* between logics, introduced in [21], will be used. In what follows, if * is a mapping defined on formulas and Γ is a set of formulas, then $\Gamma^* \stackrel{\text{def}}{=} \{\gamma^* : \gamma \in \Gamma\}$.

Definition 2.4.1 (*Translation between Logics* ([21])) Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be logics with sets of formulas $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. A mapping $^*\colon \mathcal{L}_1 \to \mathcal{L}_2$ is said to be a *translation from* $\mathscr{L}_1$ *to* $\mathscr{L}_2$ if, for every $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_1$:

$$\Gamma \vdash_{\mathscr{L}_1} \alpha \quad \Longrightarrow \quad \Gamma^* \vdash_{\mathscr{L}_2} \alpha^*.$$

And it is said to be a *conservative translation* if it satisfies the stronger property:

$$\Gamma \vdash_{\mathscr{L}_1} \alpha \quad \Longleftrightarrow \quad \Gamma^* \vdash_{\mathscr{L}_2} \alpha^*. \qquad \blacksquare$$

Recall the notion of standard logic (Definition 2.1.4). A logic satisfying item (ii) of that definition is called *monotonic*, while a logic satisfying item (iv) is said to be *finitary*. The following result can be stated (see [6]):

Theorem 2.4.2 *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two logics, where $\mathscr{L}_1$ is finitary and $\mathscr{L}_2$ is monotonic, such that both logics have implications $\to$ and $\to'$ respectively, satisfying the Deduction meta-theorem DMT (see Proposition 2.1.14). Suppose that *: $\mathcal{L}_1 \to \mathcal{L}_2$ is a mapping for which:*

$$\vdash_{\mathscr{L}_1} \alpha \quad \Longrightarrow \quad \vdash_{\mathscr{L}_2} \alpha^* ,$$

and this mapping is such that $(\alpha \to \beta)^* = \alpha^* \to' \beta^*$. *Then* * *is a translation from* $\mathscr{L}_1$ *to* $\mathscr{L}_2$. *If, additionally, both* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *are finitary and monotonic, and* * *satisfies the stronger property:*

$$\vdash_{\mathscr{L}_1} \alpha \quad \Longleftrightarrow \quad \vdash_{\mathscr{L}_2} \alpha^* ,$$

then the mapping * *is a conservative translation.*

Proof Suppose that $\Gamma \vdash_{\mathscr{L}_1} \alpha$. By the finitariness of $\mathscr{L}_1$, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathscr{L}_1} \alpha$. Suppose that $\Gamma_0 = \{\gamma_1, \ldots, \gamma_n\}$ is nonempty. Then, given the fact that $\to$ satisfies DMT, $\vdash_{\mathscr{L}_1} \gamma_1 \to \big(\ldots \to (\gamma_n \to \alpha) \ldots\big)$. From the hypothesis on *, it is the case that:

$$\vdash_{\mathscr{L}_2} \Big(\gamma_1 \to \big(\ldots \to (\gamma_n \to \alpha) \ldots\big)\Big)^*$$

thus

$$\vdash_{\mathscr{L}_2} \gamma_1^* \to' \big(\ldots \to' (\gamma_n^* \to' \alpha^*) \ldots\big).$$

Given the fact that $\to'$ satisfies DMT,

$$\gamma_1^*, \ldots, \gamma_n^* \vdash_{\mathscr{L}_2} \alpha^*$$

and, given the monotonicity of $\mathscr{L}_2$, $\Gamma^* \vdash_{\mathscr{L}_2} \alpha^*$. The remaining is proved similarly. The case when Γ_0 is empty is even simpler. □

Definition 2.4.3 Let Σ_c be the signature $\{\wedge, \vee, \to, \sim\}$ (recall Definition 2.1.3). Let **CPL** be the Hilbert calculus for *Classical Propositional Logic* which is defined over the language $\mathcal{L}_{\Sigma_c}$, obtained from the Hilbert calculus for $\mathbf{CPL}^+$ (see comment after Definition 2.1.13) by adding the following axiom schemas:

$$\alpha \vee {\sim}\alpha \qquad \textbf{(TND)}$$

$$\alpha \to \big({\sim}\alpha \to \beta\big) \qquad \textbf{(exp)}$$

The consequence relation generated by **CPL** will be denoted by $\vdash_{\mathbf{CPL}}$. ■

Notice that the two axioms above, (**TND**) and (**exp**), are *tertium non datur* and the classical *explosion law*, both w.r.t. the negation $\sim$.

It will be useful to consider the expansion of **CPL** to the language generated by the signature $\Sigma_c^\circ = \Sigma_c \cup \{\circ\}$, obtained by adding a trivial consistency operator $\circ$ such that $\circ\alpha$ is always a top formula:

Definition 2.4.4 Let **CPL**$^\circ$ be the calculus over Σ_c° obtained by adding to **CPL** the axiom schema $\circ\alpha$. ■

It is clear that **CPL**$^\circ$ is sound and complete w.r.t. the classical 2-valued truth-tables for the connectives in Σ_c displayed below, with the addition of the following truth-table for $\circ$:

$\wedge$	1	0
1	1	0
0	0	0

$\vee$	1	0
1	1	1
0	1	0

$\to$	1	0
1	1	0
0	1	1

α	$\sim\alpha$
1	0
0	1

α	$\circ\alpha$
1	1
0	1

where 1 is the unique designated truth-value. That is: for every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma_c^\circ}$, $\Gamma \vdash_{\mathbf{CPL}^\circ} \varphi$ iff $\Gamma \models_{\mathbf{CPL}^\circ} \varphi$, where $\models_{\mathbf{CPL}^\circ}$ is the semantical consequence relation defined by the logical matrix above (recall the notion of logical matrix in Definition 4.1.2).

Proposition 2.4.5 *Let $t : \mathcal{L}_\Sigma \to \mathcal{L}_{\Sigma_c^\circ}$ be the mapping which replaces $\neg$ by $\sim$.*[7] *Then t is a translation from* **mbC** *to* **CPL**$^\circ$. *That is: for every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$, $\Gamma \vdash_{\mathbf{mbC}} \varphi$ implies $t[\Gamma] \vdash_{\mathbf{CPL}^\circ} t(\varphi)$. The mapping t is not a conservative translation.*

Proof It is clear that **mbC** is sound for the truth-tables above which caracterize the logic **CPL**$^\circ$, whenever $\neg$ is interpreted by the truth-table for $\sim$. Indeed, it is enough to see that every instance of an axiom of **mbC** is a tautology w.r.t. the truth-tables for **CPL**$^\circ$ when $\neg$ is interpreted as $\sim$. This means that, for every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$, $\Gamma \vdash_{\mathbf{mbC}} \varphi$ implies $t[\Gamma] \models_{\mathbf{CPL}^\circ} t(\varphi)$. As **CPL**$^\circ$ is complete for such semantics, it follows that, for every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$, $\Gamma \vdash_{\mathbf{mbC}} \varphi$ implies $t[\Gamma] \vdash_{\mathbf{CPL}^\circ} t(\varphi)$. This shows that t is a translation from **mbC** to **CPL**$^\circ$. The mapping t is not a conservative translation: for instance, if p and q are two distinct propositional variables, then $t(p), t(\neg p) \vdash_{\mathbf{CPL}^\circ} t(q)$ (i.e., $p, \sim p \vdash_{\mathbf{CPL}^\circ} q$) but $p, \neg p \nvdash_{\mathbf{mbC}} q$.

Recall that Σ_0 denotes the signature $\{\wedge, \vee, \to, \neg\}$ (see Definition 2.1.3).

Proposition 2.4.6 *Let $t_0 : \mathcal{L}_{\Sigma_0} \to \mathcal{L}_{\Sigma_c}$ be the mapping which replaces $\neg$ by $\sim$.*[8] *Then t_0 is a translation from the Σ_0-fragment of* **mbC** *to* **CPL**. *More precisely: for every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma_0}$, $\Gamma \vdash_{\mathbf{mbC}} \varphi$ implies $t_0[\Gamma] \vdash_{\mathbf{CPL}} t_0(\varphi)$. The mapping t_0 is not a conservative translation.*

Proof Observe that t_0 is the restriction to $\mathcal{L}_{\Sigma_0}$ of the mapping t of Proposition 2.4.5. Thus, for every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma_0}$, $\Gamma \vdash_{\mathbf{mbC}} \varphi$ implies $t_0[\Gamma] \vdash_{\mathbf{CPL}^\circ} t_0(\varphi)$. Combining

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[7] In formal terms, t is recursively defined as follows: $t(p) = p$ if $p \in Var$; $t(\neg\alpha) = \sim t(\alpha)$; $t(\circ\alpha) = \circ t(\alpha)$; and $t(\alpha \# \beta) = t(\alpha) \# t(\beta)$ if $\# \in \{\vee, \wedge, \to\}$.

[8] In formal terms, t_0 is recursively defined as follows: $t_0(p) = p$ if $p \in Var$; $t_0(\neg\alpha) = \sim t_0(\alpha)$; and $t_0(\alpha \# \beta) = t_0(\alpha) \# t_0(\beta)$ if $\# \in \{\vee, \wedge, \to\}$.

this with the clear fact that **CPL**$^\circ$ is a conservative extension of **CPL** it follows that, for every $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma_0}$, $\Gamma \vdash_{\mathbf{mbC}} \varphi$ implies $t_0[\Gamma] \vdash_{\mathbf{CPL}} t_0(\varphi)$. Thus, t_0 is a translation. The proof that t_0 is not a conservative translation is identical to the one for t given above, by changing **CPL**$^\circ$ by **CPL**. □

The last result simply shows that the fragment of **mbC** without $\circ$ is a proper sublogic of **CPL** (once we interpret $\neg$ as $\sim$). Moreover, Proposition 2.4.5 shows that the full logic **mbC** is a proper sublogic of **CPL**$^\circ$, the (inessential) expansion of **CPL** obtained by adding $\circ\alpha$ as a top formula. Indeed, some interactions valid in **CPL** between the negation and the other connectives are missing in **mbC**, for instance the ones described in Propositions 2.3.4(1) and 2.3.5. Of course, the fact that $\sim$ is explosive while $\neg$ is not is the first evidence that the negation $\neg$ of **mbC** is weaker than the negation $\sim$ of **CPL**.

However, **mbC** is not as weak as it seems: actually, **CPL** can be fully interpreted inside **mbC**, as we shall see. Moreover, **mbC** can be regarded as an expansion of **CPL** obtained by adding a consistency operator $\circ$ and a paraconsistent negation $\neg$ (see Sect. 2.5). In this sense, **mbC** can be seen both as a subsystem of **CPL**$^\circ$ and as a conservative extension of **CPL**.

There are two natural ways to reproduce **CPL** inside **mbC**: one is to consider conservative translations, while the other is to state a *Derivability Adjustment Theorem* (or DAT) between **CPL** and **mbC** as follows:

Theorem 2.4.7 *Let* $t' : \mathcal{L}_{\Sigma_c} \to \mathcal{L}_{\Sigma_0}$ *be the mapping which replaces* $\sim$ *with* $\neg$.[9] *Then the following holds: for every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma_c}$,

$$\Gamma \vdash_{\mathbf{CPL}} \varphi \;\; \textit{iff} \;\; \circ\Delta, t'[\Gamma] \vdash_{\mathbf{mbC}} t'(\varphi) \;\; \textit{for some } \Delta \subseteq \mathcal{L}_{\Sigma_0},$$

where $\circ\Delta = \{\circ\alpha \; : \; \alpha \in \Delta\}$.[10]

Proof For the ‘only if’ part, suppose that $\Gamma \vdash_{\mathbf{CPL}} \varphi$ and let π be a derivation $\varphi_1 \cdots \varphi_n$ of φ from Γ in **CPL**. The sequence $t'(\pi)$ provided by $t'(\varphi_1) \ldots t'(\varphi_n)$ is not, in general, a derivation of $t'(\varphi)$ from $t'[\Gamma]$ in **mbC** because some instances of the explosion law (**exp**) could occur in π. But the sequence $t'(\pi)$ can be transformed into a derivation in **mbC** as follows: for each $\varphi_i = \alpha_i \to (\sim\alpha_i \to \beta_i)$ occurring in π as an instance of axiom (**exp**) (not occurring, therefore, as an hypothesis or as the consequence of an application of **MP**), replace the formula $t'(\varphi_i) = t'(\alpha_i) \to (\neg t'(\alpha_i) \to t'(\beta_i))$ in $t'(\pi)$ by the following sequence: $\circ t'(\alpha_i)(\circ t'(\alpha_i) \to t'(\varphi_i))t'(\varphi_i)$. Observe that the latter sequence is a derivation of $t'(\varphi_i)$ from $\{\circ t'(\alpha_i)\}$ in **mbC**, as $\circ t'(\alpha_i) \to t'(\varphi_i)$ is an instance of axiom (**bc1**). After completing this procedure, a new sequence π' of formulas will be obtained from $t'(\pi)$. Let $\Delta \subseteq \mathcal{L}_{\Sigma_0}$ be the set of all the formulas $t'(\alpha_i)$ used in the procedure described above. In this way, π' is clearly a derivation in **mbC** of $t'(\varphi)$ from $\circ\Delta \cup t'[\Gamma]$, and consequently $\circ\Delta, t'[\Gamma] \vdash_{\mathbf{mbC}} t'(\varphi)$.

[9] In formal terms, t' is defined recursively as follows: $t'(p) = p$ if $p \in Var$; $t'(\sim\alpha) = \neg t'(\alpha)$; and $t'(\alpha \# \beta) = t'(\alpha) \# t'(\beta)$ if $\# \in \{\vee, \wedge, \to\}$.

[10] Notice that the mapping t' is necessary only because we consider different signatures for **mbC** and **CPL**.

For the 'if' part, suppose that $\circ\Delta, t'[\Gamma] \vdash_{\mathbf{mbC}} t'(\varphi)$ for some $\Delta \subseteq \mathcal{L}_{\Sigma_0}$. By Proposition 2.4.5 and the soundness of **CPL**$^\circ$ w.r.t. its truth-tables, it follows that $\circ t[\Delta], \Gamma \models_{\mathbf{CPL}^\circ} \varphi$, given that $t(t'(\beta)) = \beta$ for every $\beta \in \mathcal{L}_{\Sigma_c}$. However, every formula in $\circ t[\Delta]$ is a tautology in **CPL**$^\circ$. So, $\Gamma \models_{\mathbf{CPL}} \varphi$, by definition of the semantics of **CPL**$^\circ$. Given the completeness of **CPL** w.r.t. its truth-tables, $\Gamma \vdash_{\mathbf{CPL}} \varphi$. $\square$

By employing a conservative translation, another form to reproduce **CPL** inside **mbC** can be obtained. Before defining the translation mapping, it is necessary to observe that, as a consequence of axiom (**bc1**), a bottom formula $\bot_\beta$ is always definable in **mbC** from a given formula β as $\bot_\beta \stackrel{\text{def}}{=} \beta \wedge (\neg\beta \wedge \circ\beta)$:

β	$\neg\beta$	$\circ\beta$	$\bot_\beta$	
1	1	0	0	v_1
	0	1	0	v_2
		0	0	v_3
0	1	1	0	v_4
		0	0	v_5

which is represented here in a more compact and natural way:

β	$\bot_\beta$
1	0
0	0

Now, by combining $\bot_\beta$ with the implication connective $\to$ (as is done for instance, in intuitionistic logic or even in classical logic, see Sect. 2.5), a new unary operator (namely, a negation) can be defined as follows. Let $\beta(p)$ be a formula which depends exclusively on the propositional variable p, and let $\sim_{\beta(p)}$ be the unary operator defined as follows: $\sim_{\beta(p)} p \stackrel{\text{def}}{=} p \to \bot_{\beta(p)}$. By substitution, it is clear that $\sim_{\beta(\alpha)}\alpha = \alpha \to \bot_{\beta(\alpha)}$, for every formula α. To simplify the notation, we will write $\bot_\beta$ and $\sim_\beta\alpha$ instead of $\bot_{\beta(\alpha)}$ and $\sim_{\beta(\alpha)}\alpha$, respectively, for every formula α. Observe that the new operator produces the following table:

α	β	$\bot_\beta$	$\sim_\beta\alpha$	
1	1	0	0	v_1
1	0	0	0	v_2
0	1	0	1	v_3
0	0	0	1	v_4

or, in a more compact way,

α	$\sim_\beta\alpha$
1	0
0	1

Observe that $\sim_\beta$ satisfies the basic properties of a Boolean negation:

α	$\sim_\beta\alpha$	$\alpha \wedge \sim_\beta\alpha$	$\alpha \vee \sim_\beta\alpha$
1	0	0	1
0	1	0	1

and α is semantically equivalent to $\sim_\beta\sim_\beta\alpha$ (observe the abuse of notation in the last formula: indeed, it should be written as $\sim_{\beta(\sim_{\beta(\alpha)}\alpha)}\sim_{\beta(\alpha)}\alpha$). Because of these properties, the operator $\sim_\beta$ defined in **mbC** as above for every formula $\beta(p)$, is called a *strong* negation. In order to simplify the notation a bit more, from now on (and when there is no risk of confusion), we will write $\perp$ and $\sim$ instead of $\perp_\beta$ and $\sim_\beta$, respectively.

The bottom $\perp$ and the strong negation $\sim$ satisfy the following properties:

Proposition 2.4.8 *The following holds in* **mbC**, *for every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$*:*
(i) $\perp \vdash_{\mathbf{mbC}} \varphi$.
(ii) If $\Gamma, \varphi \vdash_{\mathbf{mbC}} \perp$ *then* $\Gamma \vdash_{\mathbf{mbC}} \neg\varphi$.
(iii) $\sim\varphi \vdash_{\mathbf{mbC}} \neg\varphi$, *and so* $\vdash_{\mathbf{mbC}} \sim\varphi \to \neg\varphi$.

Proof (i) It is a consequence of the definition of $\perp$ and axiom (**bc1**).
(ii) Assume that $\Gamma, \varphi \vdash_{\mathbf{mbC}} \perp$. By item (i), $\perp \vdash_{\mathbf{mbC}} \neg\varphi$ and then $\Gamma, \varphi \vdash_{\mathbf{mbC}} \neg\varphi$. Since is always the case that $\Gamma, \neg\varphi \vdash_{\mathbf{mbC}} \neg\varphi$, it follows that $\Gamma \vdash_{\mathbf{mbC}} \neg\varphi$, by Proposition 2.1.14(iii).
(iii) By definition of $\sim$ and by **MP**, $\sim\varphi, \varphi \vdash_{\mathbf{mbC}} \perp$. Then $\sim\varphi \vdash_{\mathbf{mbC}} \neg\varphi$, by item (ii). Finally, $\vdash_{\mathbf{mbC}} \sim\varphi \to \neg\varphi$ by DMT. □

More properties of the strong negation will be proved in Chap. 7, Proposition 7.2.2.

As was done in da Costa's logic C_1, another negation operator in **mbC** could be defined as follows: $\ominus\alpha \stackrel{\text{def}}{=} \neg\alpha \wedge \circ\alpha$. Despite $\ominus$ being explosive, it does not behave exactly as expected, as *tertium non datur* is not satisfied (see valuation v_5 below):

α	$\neg\alpha$	$\circ\alpha$	$\ominus\alpha$	$\alpha \wedge \ominus\alpha$	$\alpha \vee \ominus\alpha$	
1	1	0	0	0	1	v_1
	0	1	0	0	1	v_2
		0	0	0	1	v_3
0	1	1	1	0	1	v_4
		0	0	0	0	v_5

Additionally, $\ominus$ does not satisfy the law of double negation: take an **mbC**-valuation v such that $v(p) = 1$ for a given propositional variable. From this, it follows that $v(\ominus p) = 0$ and so $v(\neg\ominus p) = 1$. Suppose that $v(\circ\ominus p) = 0$. Then $v(\ominus\ominus p) = 0$, showing that $\nvdash_{\mathbf{mbC}} (p \to \ominus\ominus p)$. Now, consider an **mbC**-valuation v' such that $v'(p) = 0$, $v'(\ominus p) = 0$ (and so $v'(\neg\ominus p) = 1$), and $v'(\circ\ominus p) = 1$. From this, $v'(\ominus\ominus p) = 1$ and therefore $\nvdash_{\mathbf{mbC}} (\ominus\ominus p \to p)$.

It is possible to correct all the failures of the operator $\ominus$ by requiring the following additional property to the valuations:

$$v(\alpha) = 0 \implies v(\circ\alpha) = 1.$$

It is easy to prove that this strategy corresponds to adding to **mbC** the following axiom schema:

$$\alpha \vee \circ\alpha.$$

The following proposition shows that *tertium non datur* is indeed equivalent to the law above, and it shows also that the law of double negation will be automatically satisfied by the alternative negation $\ominus$:

Proposition 2.4.9 *Let $\mathscr{L}$ be an extension of* **mbC**.
(i) $\vdash_{\mathscr{L}} \alpha \vee \ominus\alpha$ *iff* $\vdash_{\mathscr{L}} \alpha \vee \circ\alpha$.
(ii) If $\vdash_{\mathscr{L}} \alpha \vee \ominus\alpha$ *then* $\vdash_{\mathscr{L}} \alpha \to \ominus\ominus\alpha$ *and* $\vdash_{\mathscr{L}} \ominus\ominus\alpha \to \alpha$.

Proof (i) Observe that $\alpha \vee \ominus\alpha = \alpha \vee (\neg\alpha \wedge \circ\alpha)$ is equivalent to $(\alpha \vee \neg\alpha) \wedge (\alpha \vee \circ\alpha)$ (by distributivity), which is in turn equivalent to $\alpha \vee \circ\alpha$ (given that $\alpha \vee \neg\alpha$ is a theorem of $\mathscr{L}$). This proves the result.
(ii) As observed above, an extension $\mathscr{L}$ proving the theorem $\alpha \vee \ominus\alpha$, is characterized by **mbC**-valuations which must satisfy: for every α, if $v(\alpha) = 0$ then $v(\circ\alpha) = 1$. It is easy to see, then, that the unique counterexamples of the law of double negation in both directions (see comment above) are now forbidden, as α is now logically equivalent to $\ominus\ominus\alpha$ in $\mathscr{L}$. □

This point will be resumed in Sect. 3.

Now, profiting from the strong negation $\sim_\beta$ definable in **mbC** (as discussed above), we show that the mapping that replaces the classical negation $\sim$, with the defined operator $\sim_\beta$, is a conservative translation from **CPL** to **mbC**. This constitutes an alternative way to recover classical logic inside **mbC**.

Proposition 2.4.10 *Fix a given formula $\beta(p_1) \in \mathcal{L}_\Sigma$ which depends exclusively on the propositional variable p_1 (for instance, $\beta = p_1$). The mapping $^\odot : \mathcal{L}_{\Sigma_c} \to \mathcal{L}_\Sigma$ is recursively defined as follows:*

$$\begin{aligned} p^\odot &= p, \text{ if } p \in Var; \\ (\sim\alpha)^\odot &= \sim_\beta(\alpha^\odot); \\ (\alpha \mathbin{\#} \beta)^\odot &= \alpha^\odot \mathbin{\#} \beta^\odot \ \text{ if } \# \in \{\vee, \wedge, \to\}. \end{aligned}$$

In this way $^{\odot}$ *is a conservative translation from* **CPL** *to* **mbC**. *That is: for every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma_c}$,

$$\Gamma \vdash_{\mathbf{CPL}} \varphi \;\; \textit{iff} \;\; \Gamma^{\odot} \vdash_{\mathbf{mbC}} \varphi^{\odot}.$$

Proof By Theorem 2.4.2, it suffices to prove that $\vdash_{\mathbf{CPL}} \varphi$ iff $\vdash_{\mathbf{mbC}} \varphi^{\odot}$, for every formula $\varphi \in \mathcal{L}_{\Sigma_c}$.

For the 'only if' part, recall that **CPL** is obtained from $\mathbf{CPL}^+$ by adding axiom schemas (**TND**) and (**exp**). As **mbC** is an extension of $\mathbf{CPL}^+$, and the translation of all the instances of axiom schemas and rules of $\mathbf{CPL}^+$ produces instances of axiom schemas and rules of $\mathbf{CPL}^+$ (and so of **mbC**), the only cases to be analyzed are the two axiom schemas (**TND**) and (**exp**). Thus, let $\varphi \vee {\sim}\varphi$ and $\varphi \to ({\sim}\varphi \to \psi)$ be instances of (**TND**) and (**exp**), respectively. They are translated by $^{\odot}$ as the formulas $\varphi^{\odot} \vee {\sim}_{\beta}(\varphi^{\odot})$ and $\varphi^{\odot} \to ({\sim}_{\beta}(\varphi^{\odot}) \to \psi^{\odot})$, respectively. But, as observed above, $\alpha \vee {\sim}_{\beta}\alpha$ is always true and $\alpha \wedge {\sim}_{\beta}\alpha$ is always false for the **mbC**-valuations, for every α. Being so, $\varphi^{\odot} \vee {\sim}_{\beta}(\varphi^{\odot})$ and $\varphi^{\odot} \to ({\sim}_{\beta}(\varphi^{\odot}) \to \psi^{\odot})$ are always true for **mbC**-valuations. Thus, they are derivable in **mbC**, by completeness. This shows that if $\vdash_{\mathbf{CPL}} \varphi$ then $\vdash_{\mathbf{mbC}} \varphi^{\odot}$.

For the 'if' part, suppose that $\vdash_{\mathbf{mbC}} \varphi^{\odot}$. By Proposition 2.4.5 it follows that $\vdash_{\mathbf{CPL}^{\circ}} t(\varphi^{\odot})$. However, in $\mathbf{CPL}^{\circ}$, the formula $t({\sim}_{\beta}\alpha)$ is equivalent to ${\sim}t(\alpha)$, for every α. As a consequence, it is easy to prove by induction on the complexity of φ that $t(\varphi^{\odot})$ is equivalent to φ in $\mathbf{CPL}^{\circ}$. Hence $\vdash_{\mathbf{CPL}^{\circ}} \varphi$ and so, by the fact that $\mathbf{CPL}^{\circ}$ is a conservative extension of **CPL**, it follows that $\vdash_{\mathbf{CPL}} \varphi$, which completes the proof. □

2.5 Reintroducing mbC as an Expansion of CPL

The results of the previous section, in particular Proposition 2.4.10, suggest that a different language based on the signature $\Sigma_{\perp} = \{\perp, \to, \neg, \circ\}$ could be used for **mbC** (recall Definition 2.1.3). This is the proposal found in [6], on which this section is based.

The notion of complexity of a formula in $\mathcal{L}_{\Sigma_{\perp}}$ is defined analogously to Definition 2.1.11:

Definition 2.5.1 *(Formula Complexity in* $\mathcal{L}_{\Sigma_{\perp}}$*)* The complexity of a given formula $\varphi \in \mathcal{L}_{\Sigma_{\perp}}$, denoted by $l(\varphi)$, is recursively defined as follows:

1. If $\varphi = p$, where $p \in Var \cup \{\perp\}$, then $l(\varphi) = 1$;
2. If $\varphi = \neg\alpha$, then $l(\varphi) = l(\alpha) + 1$;
3. If $\varphi = \circ\alpha$, then $l(\varphi) = l(\alpha) + 2$;
4. If $\varphi = \alpha \to \beta$, then $l(\varphi) = l(\alpha) + l(\beta) + 1$.

■

As observed in the previous section, any formula β defines a bottom constant in **mbC** as follows: $\perp_{\beta} \stackrel{\text{def}}{=} \beta \wedge (\neg\beta \wedge \circ\beta)$ and then **CPL** can be recovered within

mbC by means of a conservative translation (see Proposition 2.4.10). This suggests that it is possible to consider from the start a 0-ary connective $\bot$ and the axiom schemas for **CPL** in the signature $\Sigma_\bot$, as well as the corresponding axiom schemas for the paraconsistent negation $\neg$ and the consistency operator $\circ$, to obtain a new axiomatization of **mbC** in the signature $\Sigma_\bot$.

Besides the simplification it achieves (for instance, when doing proofs by induction on the complexity of a formula), a justification for the use of this new language is that $\bot$, being so important in the context of **LFI**s, is usually defined with respect to a formula $\beta(\alpha)$ as $\bot_{\beta(\alpha)}$, and so there is an infinitude of such bottom formulas. The same observation applies to the classical negation ($\sim$), which is defined as $\sim_{\beta(\alpha)}\alpha \stackrel{\text{def}}{=} \alpha \to \bot_{\beta(\alpha)}$, and so there are infinitely many classical negations inside **mbC**, with the index $\beta(\alpha)$ varying with α. For that reason, the inclusion of bottom $\bot$ to the signature allows to define a *distinguished* classical negation as expected:

Definition 2.5.2 *(Classical Negation in $\mathcal{L}_{\Sigma_\bot}$)* Let p be a propositional variable. The classical negation is defined in signature $\Sigma_\bot$ by means of the following formula:

$$\sim p \stackrel{\text{def}}{=} p \to \bot.$$

■

Hence, if α is any formula in $\mathcal{L}_{\Sigma_\bot}$ then $\sim\alpha \stackrel{\text{def}}{=} \alpha \to \bot$, by substitution. This constitutes a significant simplification with respect to the signature Σ, where the index of the negation varies with the formula being negated. From this, $\bot$ and $\sim$ can be considered the *canonical choices* for bottom and the classical negation inside **mbC**. Moreover, the new presentation of **mbC** in this signature is equivalent to consider this logic as an *expansion* of classical propositional logic **CPL** (this time defined in the signature $\{\to, \bot\}$) by adding a paraconsistent negation and a consistency operator. This allows **mbC** to be seen as a kind of bimodal logic based on **CPL**.

2.5.1 The New Presentation mbC$^\bot$ *of* mbC

Definition 2.5.3 (**mbC**$^\bot$) The calculus **mbC**$^\bot$ is defined over the language $\mathcal{L}_{\Sigma_\bot}$ by the following Hilbert calculus:

Axiom schemas:

$$\alpha \to (\beta \to \alpha) \qquad \textbf{(Ax1)}$$

$$\Big(\alpha \to \big(\beta \to \gamma\big)\Big) \to \Big(\big(\alpha \to \beta\big) \to \big(\alpha \to \gamma\big)\Big) \qquad \textbf{(Ax2)}$$

$$\sim\sim\alpha \to \alpha \qquad (\sim\sim)$$

$$\sim\alpha \to \neg\alpha \qquad (\sim\neg)$$

$$\circ\alpha \to \big(\neg\alpha \to \sim\alpha\big) \qquad (\mathbf{bc1}^\bot)$$

Inference rule:

$$\frac{\alpha \quad \alpha \to \beta}{\beta} \qquad \textbf{(MP)}$$

■

As was argued for **mbC** in Proposition 2.1.14, the calculus $\mathbf{mbC}^{\perp}$ satisfies the Deduction meta-theorem DMT:

Theorem 2.5.4 (Deduction meta-theorem) *For every* $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}_{\Sigma_\perp}$*:*

$$\Gamma \cup \{\varphi\} \vdash_{\mathbf{mbC}^\perp} \psi \quad \Longleftrightarrow \quad \Gamma \vdash_{\mathbf{mbC}^\perp} \varphi \to \psi.$$

The following result is well-known in the literature. See a proof, for instance, in Chap. 1 of Church's book [22] (where the system $\mathbf{CPL}_W$ below is called P_1).[11]

Proposition 2.5.5 *Let* Σ_W *be the signature* $\{\to, \perp\}$*. Consider the logic* $\mathbf{CPL}_W$ *given by the Hilbert calculus over the signature* Σ_W *which is formed by the axiom schemas* (**Ax1**)*,* (**Ax2**) *and* ($\sim\sim$) *plus the rule* (**MP**)*. In this way* $\mathbf{CPL}_W$ *constitutes a sound and complete axiomatization of* **CPL** *in the signature* Σ_W *(by taking* $\sim$ *as in Definition 2.5.2).*

Remark 2.5.6 The axiomatization $\mathbf{CPL}_W$ of **CPL** in the signature $\{\to, \perp\}$ was firstly proposed in 1939 by Wajsberg (see [23]).

The next technical lemma is required for establishing the completeness theorem in the following section.

Lemma 2.5.7 *All the following formulas are theorems of* $\mathbf{mbC}^{\perp}$*:*

1. $\perp \to \alpha$

2. $\big((\alpha \to \beta) \to \alpha\big) \to \alpha$

3. $(\alpha \to \gamma) \to \Big((\beta \to \gamma) \to \Big(\big((\alpha \to \beta) \to \beta\big) \to \gamma\Big)\Big)$

4. $(\alpha \to \gamma) \to \Big((\beta \to \gamma) \to \Big(\big((\alpha \to \perp) \to \beta\big) \to \gamma\Big)\Big)$

Proof It follows from the fact that all these formulas are classic tautologies in the signature $\{\to, \perp\}$ and they can therefore be derived in $\mathbf{mbC}^{\perp}$ by Proposition 2.5.5. □

[11] Not to be confused with Sette's **P1**, see Sect. 4.4.4. To be more precise P_1 contains, besides **MP**, the inference rule of Uniform Substitution, since the axioms are presented by using propositional variables instead of schema formulas.

2.5.2 Valuation Semantics for mbC

Now a valuation semantic, adapted from that for **mbC**, will be proposed for $\mathbf{mbC}^{\perp}$.

Definition 2.5.8 *(Valuations for* $\mathbf{mbC}^{\perp}$*)* A function $v : \mathcal{L}_{\Sigma_{\perp}} \to \{0, 1\}$ is a *valuation for* $\mathbf{mbC}^{\perp}$, or an $\mathbf{mbC}^{\perp}$*-valuation*, if it satisfies the following clauses:

(*vBot*) $v(\perp) = 0$
(*vNeg*) $v(\neg\alpha) = 0 \implies v(\alpha) = 1$
(*vCon*) $v(\circ\alpha) = 1 \implies v(\alpha) = 0$ or $v(\neg\alpha) = 0.$
(*vImp*) $v(\alpha \to \beta) = 1 \iff v(\alpha) = 0$ or $v(\beta) = 1$

The set of all such valuations will be designated by $V^{\mathbf{mbC}^{\perp}}$. ■

A technical result is given below, whose demonstration will be used latter on, more specifically in the proof of Theorem 2.5.20.

Lemma 2.5.9 *Let* $v_0 : Var \to \{0, 1\}$ *be a mapping. Then there exists a valuation* $v \in V^{\mathbf{mbC}^{\perp}}$ *extending* v_0.

Proof For $\psi \in \mathcal{L}_{\Sigma_{\perp}}$, the truth-values of $v(\psi)$ are defined by induction on $l(\psi)$. To begin, if ψ is such that $l(\psi) = 1$, then either $\psi \in Var$ or $\psi = \perp$. In the first case, the valuation is defined as $v(\psi) = v_0(\psi)$. In the second case, we define $v(\psi) = 0$.

Now, suppose that the valuation v is already defined for all ψ' such that $l(\psi') < n$, where $n > 1$ (induction hypothesis), and let ψ such that $l(\psi) = n$. According to the main connective of ψ, the definition goes as follows:

1. If $\psi = \alpha \to \beta$, then:

$$v(\alpha \to \beta) = 1 \iff v(\alpha) = 0 \text{ or } v(\beta) = 1$$

2. If $\psi = \neg\gamma$, then:

$$v(\neg\gamma) = \begin{cases} 1 & \text{if } v(\gamma) = 0, \text{ or} \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

3. If $\psi = \circ\gamma$, then:

$$v(\circ\gamma) = \begin{cases} 0 & \text{if } v(\gamma) = v(\neg\gamma) = 1, \textit{or} \\ \text{arbitrary} & \text{otherwise} \end{cases}.$$

It is worth noting that v is well-defined, by Definition 2.5.1 of complexity. Clearly, v is an $\mathbf{mbC}^{\perp}$-valuation extending v_0. The easy details are left to the reader. □

Remark 2.5.10 The reader should notice that, different to the case of valuations over logical matrices, which are homomorphisms between algebras (see Sect. 4.1 in Chap. 4), any mapping $v_0 : Var \to \{0, 1\}$ can be extended to more than one $\mathbf{mbC}^{\perp}$-valuation (in fact, there are infinite $\mathbf{mbC}^{\perp}$-valuations extending v_0). This is a consequence of the non-determinism inherent to valuations of this kind, in contrast with the determinism imposed by homomorphic valuations. The question of non-determinism versus determinism in the context of semantic approaches to **LFI**s will be analyzed with more details in Chap. 6.

Now we will prove that the new logic $\mathbf{mbC}^{\perp}$ is sound and complete for its semantics of valuations.

Theorem 2.5.11 (Soundness for $\mathbf{mbC}^{\perp}$) *For every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma_\perp}$*:*

$$\Gamma \vdash_{\mathbf{mbC}^{\perp}} \varphi \quad \Longrightarrow \quad \Gamma \models_{\mathbf{mbC}^{\perp}} \varphi.$$

Proof The proof of this theorem is entirely analogous than that of Theorem 2.2.2 for **mbC**. The only axioms to be analyzed are the new ones. Firstly observe that, for every $\mathbf{mbC}^{\perp}$-valuation v and for every formula α, $v({\sim}\alpha) = 1$ iff $v(\alpha) = 0$. Hence, axiom $({\sim}{\sim})$ is clearly valid. Let $\gamma = {\sim}\alpha \to \neg\alpha$ be an instance of axiom $({\sim}\neg)$, and let v be an $\mathbf{mbC}^{\perp}$-valuation. If $v({\sim}\alpha) = 0$ then $v(\gamma) = 1$, by $(vImp)$. Otherwise, if $v({\sim}\alpha) = 0$ then $v(\alpha) = 0$ and so $v(\neg\alpha) = 1$, by clause $(vNeg)$. From this, $v(\gamma) = 1$ by $(vImp)$. Finally, let δ be an instance $\circ\alpha \to \big(\neg\alpha \to {\sim}\alpha\big)$ of axiom $(\mathbf{bc1}^{\perp})$, and let v be an $\mathbf{mbC}^{\perp}$-valuation. If $v(\circ\alpha) = 0$ then $v(\delta) = 1$, by clause $(vImp)$. Otherwise, if $v(\circ\alpha) = 1$ then either $v(\alpha) = 0$ or $v(\neg\alpha) = 0$, by $(vCon)$. If $v(\alpha) = 0$ then $v(\neg\alpha) = v({\sim}\alpha) = 1$ and so $v(\neg\alpha \to {\sim}\alpha) = 1$. If $v(\neg\alpha) = 0$ then clearly $v(\neg\alpha \to {\sim}\alpha) = 1$ too. In both cases $v(\delta) = 1$. □

The proof of completeness will be also analogous to that of **mbC**.

Theorem 2.5.12 *Let* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma_\perp}$*, with* Γ *maximal non-trivial with respect to* φ *in* $\mathbf{mbC}^{\perp}$ *(recall Definition 2.2.3). The mapping* $v : \mathcal{L}_{\Sigma_\perp} \to \{0, 1\}$ *defined by:*

$$v(\psi) = 1 \quad \Longleftrightarrow \quad \psi \in \Gamma$$

for all $\psi \in \mathcal{L}_{\Sigma_\perp}$ *is a valuation for* $\mathbf{mbC}^{\perp}$.

Proof Let $\psi \in \mathcal{L}_{\Sigma_\perp}$ be an arbitrary formula.
1. $\psi = \perp$. Suppose, by contradiction, that $\perp \in \Gamma$. As $\vdash_{\mathbf{mbC}^{\perp}} \perp \to \varphi$ (by Lemma 2.5.7(1)) then $\perp \to \varphi \in \Gamma$, by Lemma 2.2.5. By **MP**, and using Lemma 2.2.5 again, it follows that $\varphi \in \Gamma$, which is a contradiction. Therefore $\perp \notin \Gamma$ and so $v(\perp) = 0$.
2. $\psi = \neg\alpha$. Suppose $\neg\alpha \notin \Gamma$ and, by contradiction, also $\alpha \notin \Gamma$. As Γ is maximal, it follows that $\Gamma, \neg\alpha \vdash_{\mathbf{mbC}^{\perp}} \varphi$ and $\Gamma, \alpha \vdash_{\mathbf{mbC}^{\perp}} \varphi$. By the deduction meta-theorem, $\Gamma \vdash_{\mathbf{mbC}^{\perp}} \alpha \to \varphi$ and $\Gamma \vdash_{\mathbf{mbC}^{\perp}} \neg\alpha \to \varphi$. Now, by Lemma 2.5.7(4), $\Gamma \vdash_{\mathbf{mbC}^{\perp}} \big((\alpha \to \perp) \to \neg\alpha\big) \to \varphi$. However, $(\alpha \to \perp) \to \neg\alpha$ is an instance of Axiom $({\sim}\neg)$, and then (by **MP**) $\Gamma \vdash_{\mathbf{mbC}^{\perp}} \varphi$, which is a contradiction. Therefore:

$$v(\neg\alpha) = 0 \quad \Longrightarrow \quad v(\alpha) = 1.$$

3. $\psi = \circ\alpha$. Suppose $\circ\alpha \in \Gamma$ and, by contradiction, that both $\alpha \in \Gamma$ and $\neg\alpha \in \Gamma$. Then, by Axiom (**bc1**$^\perp$) and Lemma 2.2.5, $\sim\alpha \in \Gamma$. By the definition of $\sim$ and by **MP**, it follows that $\perp \in \Gamma$. But then, as a consequence of Lemma 2.5.7(1), it can be inferred that $\varphi \in \Gamma$, which is a contradiction. Therefore, either $\alpha \notin \Gamma$ or $\neg\alpha \notin \Gamma$. That is:

$$v(\circ\alpha) = 1 \quad \Longrightarrow \quad v(\alpha) = 0 \text{ or } v(\neg\alpha) = 0.$$

4. $\psi = \alpha \to \beta$. Suppose $\alpha \to \beta \in \Gamma$. If $\alpha \in \Gamma$ then $\beta \in \Gamma$ by **MP** and Lemma 2.2.5. Therefore:

$$v(\alpha \to \beta) = 1 \quad \Longrightarrow \quad v(\alpha) = 0 \text{ or } v(\beta) = 1.$$

Now, suppose $\alpha \notin \Gamma$ or $\beta \in \Gamma$. If $\beta \in \Gamma$ then $\alpha \to \beta \in \Gamma$ by Axiom (**Ax1**), **MP** and Lemma 2.2.5. If $\alpha \notin \Gamma$ then, by the maximality of Γ, it follows that $\Gamma, \alpha \vdash_{\mathbf{mbC}^\perp} \varphi$. Now, suppose by contradiction that $\alpha \to \beta \notin \Gamma$. Then $\Gamma, \alpha \to \beta \vdash_{\mathbf{mbC}^\perp} \varphi$. By the deduction meta-theorem, both $\Gamma \vdash_{\mathbf{mbC}^\perp} (\alpha \to \beta) \to \varphi$ and $\Gamma \vdash_{\mathbf{mbC}^\perp} \alpha \to \varphi$. By Lemma 2.5.7(3), $\Gamma \vdash_{\mathbf{mbC}^\perp} \Big(\big((\alpha \to \beta) \to \alpha\big) \to \alpha\Big) \to \varphi$ and, by item 2 of that lemma, $\Gamma \vdash_{\mathbf{mbC}^\perp} \varphi$, which is a contradiction. Therefore:

$$v(\alpha) = 0 \text{ or } v(\beta) = 1 \quad \Longrightarrow \quad v(\alpha \to \beta) = 1.$$

□

The completeness of **mbC**$^\perp$ is then an immediate consequence of Theorems 2.5.12 and 2.2.6 (which clearly holds for **mbC**$^\perp$):

Corollary 2.5.13 (Completeness of **mbC**$^\perp$) *For every* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_{\Sigma_\perp}$*:*

$$\Gamma \vDash_{\mathbf{mbC}^\perp} \varphi \quad \Longrightarrow \quad \Gamma \vdash_{\mathbf{mbC}^\perp} \varphi$$

Proof Suppose $\Gamma \nvdash_{\mathbf{mbC}^\perp} \varphi$ and let Δ be a maximal non-trivial set with respect to φ in **mbC**$^\perp$ extending Γ (see Theorem 2.2.6). By Theorem 2.5.12, there is a valuation for **mbC**$^\perp$ satisfying Γ (as $\Gamma \subseteq \Delta$) but not φ (as $\varphi \notin \Delta$). Therefore, $\Gamma \nvDash_{\mathbf{mbC}^\perp} \varphi$ and the theorem follows by contraposition. □

2.5.3 Equivalence Between mbC *and* mbC$^\perp$

In this section, **mbC**$^\perp$ will be shown to be equivalent to its counterpart **mbC**.

By induction on the formula complexity, two mappings will now be defined which later on will be shown to be conservative translations.

Definition 2.5.14 Fix an arbitrary propositional variable in *Var*, for instance p_1. The mapping $^{\circledast} : \mathcal{L}_{\Sigma_\bot} \to \mathcal{L}_\Sigma$ is defined inductively for all $\varphi \in \mathcal{L}_{\Sigma_\bot}$ as follows:

$$\begin{aligned} q^{\circledast} &= q, \quad \text{if } q \in Var; \\ \bot^{\circledast} &= p_1 \wedge (\neg p_1 \wedge \circ p_1); \\ (\#\alpha)^{\circledast} &= \#(\alpha^{\circledast}) \quad \text{for } \# \in \{\neg, \circ\}; \\ (\alpha \to \beta)^{\circledast} &= \alpha^{\circledast} \to \beta^{\circledast}. \end{aligned}$$

■

Definition 2.5.15 The mapping $^* : \mathcal{L}_\Sigma \to \mathcal{L}_{\Sigma_\bot}$ is defined by induction on $l(\varphi)$, for all $\varphi \in \mathcal{L}_\Sigma$ as follows:

$$\begin{aligned} q^* &= q, \quad \text{if } q \in Var; \\ (\#\alpha)^* &= \#(\alpha^*) \quad \text{for } \# \in \{\neg, \circ\}; \\ (\alpha \to \beta)^* &= \alpha^* \to \beta^*; \\ (\alpha \vee \beta)^* &= \big(\alpha^* \to \bot\big) \to \beta^*; \\ (\alpha \wedge \beta)^* &= \big(\alpha^* \to (\beta^* \to \bot)\big) \to \bot. \end{aligned}$$

■

The injectivity of these mappings needs to be established, in order to be possible to properly define the valuations of Theorem 2.5.20. But, first, an intermediary result shall be given:

Lemma 2.5.16 *There are no formulas $\varphi, \psi \in \mathcal{L}_\Sigma$ which satisfy the following equation:*

$$\varphi^* \to \bot = \psi^*$$

Proof Suppose, by contradiction, there is a solution in $\mathcal{L}_\Sigma$ for the above identity and let φ and ψ be such a solution with a minimum value of $l(\varphi) + l(\psi)$. Observe now that $\bot$ is not in the image of *, and so $\psi \neq \alpha \to \beta$ and $\psi \neq \alpha \vee \beta$ for any of these would imply $\beta^* = \bot$. Therefore, the only way to get the image of ψ to be $\varphi^* \to \bot$ is with $\psi = \psi_1 \wedge \psi_2$. Therefore $\varphi^* \to \bot = \big(\psi_1^* \to (\psi_2^* \to \bot)\big) \to \bot$, and so $\varphi^* = \psi_1^* \to (\psi_2^* \to \bot)$.

There are two cases to be analyzed:

1. $\varphi = \varphi_1 \to \varphi_2$. Therefore $\varphi^* = \varphi_1^* \to \varphi_2^*$, $\varphi_1^* = \psi_1^*$ and $\varphi_2^* = \psi_2^* \to \bot$.
2. $\varphi = \varphi_1 \vee \varphi_2$. Therefore $\varphi^* = (\varphi_1^* \to \bot) \to \varphi_2^*$, $\varphi_1^* \to \bot = \psi_1^*$ and $\varphi_2^* = \psi_2^* \to \bot$.

In both cases (ψ_2, φ_2) is a solution to the equation in question with $l(\psi_2) + l(\varphi_2) < l(\varphi) + l(\psi)$, a contradiction. □

Theorem 2.5.17 *The mappings $^{\circledast} : \mathcal{L}_{\Sigma_\bot} \to \mathcal{L}_\Sigma$ and $^* : \mathcal{L}_\Sigma \to \mathcal{L}_{\Sigma_\bot}$ from Definitions 2.5.14 and 2.5.15 are injective.*

Proof Let $\varphi, \psi \in \mathcal{L}_{\Sigma_\perp}$ be such that $\varphi^{\circledast} = \psi^{\circledast}$. By induction on $l(\varphi) + l(\psi)$ it is easy to prove that $\varphi = \psi$. This is a consequence of the fact that, by Definition 2.5.14, there are not two different equations having, to the right, formulas with the same main connective.

Now, let $\varphi, \psi \in \mathcal{L}_\Sigma$ be such that $\varphi^* = \psi^*$. The proof is given by induction, analogous to $^{\circledast}$. However, the induction step for which the main connective of both sides of the above equation is $\to$ proves to be a bit more complicated. In fact, there are three equations in Definition 2.5.15 having, to the right, a formula with $\to$ as the main connective. So, let $\varphi^* = \alpha' \to \beta' = \psi^*$. Hence, there are the following possibilities:

(a) $\varphi = \alpha \to \beta$ and $\psi = \gamma \to \delta$. Therefore, $\varphi^* = \alpha^* \to \beta^* = \gamma^* \to \delta^* = \psi^*$. By unique readability, it follows that $\alpha^* = \gamma^*$ and $\beta^* = \delta^*$. The result is then obtained by the induction hypothesis: $\alpha = \gamma$ and $\beta = \delta$, which implies $\varphi = \psi$.
(b) $\varphi = \alpha \to \beta$ and $\psi = \gamma \vee \delta$. Therefore, $\varphi^* = \alpha^* \to \beta^* = (\gamma^* \to \perp) \to \delta^* = \psi^*$. By unique readability, $\alpha^* = \gamma^* \to \perp$, which is impossible by Lemma 2.5.16.
(c) $\varphi = \alpha \to \beta$ and $\psi = \gamma \wedge \delta$. This is impossible, for it would imply $\beta^* = \perp$.
(d) $\varphi = \alpha \vee \beta$ and $\psi = \gamma \vee \delta$. Then, as in item (a), $\alpha = \gamma$ and $\beta = \delta$, given the fact that $\alpha^* \to \perp = \gamma^* \to \perp$ and $\beta^* = \delta^*$. Then $\varphi = \psi$.
(e) $\varphi = \alpha \vee \beta$ and $\psi = \gamma \wedge \delta$. This is impossible, for it would imply $\beta^* = \perp$.
(f) $\varphi = \alpha \wedge \beta$ and $\psi = \gamma \wedge \delta$. Then $\alpha^* \to (\beta^* \to \perp) = \gamma^* \to (\delta^* \to \perp)$, which implies $\alpha = \gamma$ and $\beta = \delta$. Therefore $\varphi = \psi$. □

Corollary 2.5.18
1. Let $\varphi = \#\gamma \in \mathcal{L}_{\Sigma_\perp}$, with $\# \in \{\neg, \circ\}$. If $\varphi \in Im(^*) = \{\psi^* : \psi \in \mathcal{L}_\Sigma\}$, there exists a unique formula $\delta \in \mathcal{L}_\Sigma$ such that $\varphi = (\#\delta)^*$.
2. Let $\varphi = \#\gamma \in \mathcal{L}_\Sigma$, with $\# \in \{\neg, \circ\}$. If $\varphi \in Im(^{\circledast}) = \{\psi^{\circledast} : \psi \in \mathcal{L}_{\Sigma_\perp}\}$, there exists a unique formula $\delta \in \mathcal{L}_{\Sigma_\perp}$ such that $\varphi = (\#\delta)^{\circledast}$.

Proof This result is a direct consequence of the injectivity and the very definition of the mappings * and $^{\circledast}$. □

Lemma 2.5.19
1. Let $v \in V^{\mathbf{mbC}}$. Then the mapping $v' : \mathcal{L}_{\Sigma_\perp} \to \{0, 1\}$ defined by $v'(\varphi) \stackrel{def}{=} v(\varphi^{\circledast})$ is such that $v' \in V^{\mathbf{mbC}^\perp}$.
2. Let $v \in V^{\mathbf{mbC}^\perp}$. Then the mapping $v' : \mathcal{L}_\Sigma \to \{0, 1\}$ defined by $v'(\varphi) \stackrel{def}{=} v(\varphi^*)$ is such that $v' \in V^{\mathbf{mbC}}$.

Proof 1. Let $\varphi \in \mathcal{L}_{\Sigma_\perp}$ be an arbitrary formula. We will prove that v' satisfies the clauses from Definition 2.5.8.
(a) $\varphi = \perp$. Then $\varphi^{\circledast} = p_1 \wedge (\neg p_1 \wedge \circ p_1)$ and so $v(\varphi^{\circledast}) = 0$ for any valuation for **mbC**. Therefore $v'(\perp) = v(\varphi^{\circledast}) = 0$.
(b) $\varphi = \neg\alpha$. Then $\varphi^{\circledast} = \neg(\alpha^{\circledast})$. Therefore, if $v'(\neg\alpha) = 0$, then $v(\neg(\alpha^{\circledast})) = 0$ (by definition of v'). Now, as v is a valuation for **mbC**, it follows that $v(\alpha^{\circledast}) = 1$, and so $v'(\alpha) = 1$.
(c) $\varphi = \circ\alpha$. Then $\varphi^{\circledast} = \circ(\alpha^{\circledast})$. Therefore, if $v'(\circ\alpha) = 1$, then $v(\circ(\alpha^{\circledast})) = 1$. Now, as v is a valuation for **mbC**, $v(\alpha^{\circledast}) = v'(\alpha) = 0$ or $v(\neg(\alpha^{\circledast})) = v'(\neg\alpha) = 0$.

(d) $\varphi = \alpha \to \beta$. Then $\varphi^{\circledast} = \alpha^{\circledast} \to \beta^{\circledast}$. Therefore, $v'(\alpha \to \beta) = 1$ if, and only if, $v(\alpha^{\circledast} \to \beta^{\circledast}) = 1$. But the last occurs exactly when $v(\alpha^{\circledast}) = 0$ or $v(\beta^{\circledast}) = 1$, that is, exactly when $v'(\alpha) = 0$ or $v'(\beta) = 1$.
2. Let $\varphi \in \mathcal{L}_\Sigma$ be an arbitrary formula. We will prove that v' is an **mbC**-valuation. If φ is of the form $\neg\alpha$, $\circ\alpha$, or $\alpha \to \beta$, the proof is similar to that of item 1 above. Now, for the remaining cases:

(a) $\varphi = \alpha \vee \beta$. Then $\varphi^* = (\alpha^* \to \bot) \to \beta^*$ and therefore, $v'(\alpha \vee \beta) = v\big((\alpha^* \to \bot) \to \beta^*\big)$. Since $\to$ and $\bot$ are interpreted as in propositional classical logic, it follows that $v'(\alpha \vee \beta) = v(\varphi^*) = 1$ iff $v'(\alpha) = v(\alpha^*) = 1$ or $v'(\beta) = v(\beta^*) = 1$.
(b) $\varphi = \alpha \wedge \beta$. Then $\varphi^* = \big(\alpha^* \to (\beta^* \to \bot)\big) \to \bot$ and therefore, $v'(\alpha \wedge \beta) = v\Big(\big(\alpha^* \to (\beta^* \to \bot)\big) \to \bot\Big)$. In line with the argument in the previous item, it follows that $v'(\alpha \wedge \beta) = v(\varphi^*) = 1$ iff $v'(\alpha) = v(\alpha^*) = 1$ and $v'(\beta) = v(\beta^*) = 1$. $\square$

The next lemma establishes that when there is a model (or counter-model) for a formula φ in $\mathbf{mbC}^\bot$, there also exists a model (or counter-model) for $\varphi^{\circledast}$ in **mbC**. Similarly, given a model (or counter-model) for a formula φ in **mbC**, there also exists a model (or counter-model) for φ^* in $\mathbf{mbC}^\bot$. As it will become clear later on, this result suffices to prove that the translations in question are conservative ones.

Lemma 2.5.20
1. Let $v \in V^{\mathbf{mbC}}$. Therefore, there exists $v' \in V^{\mathbf{mbC}^\bot}$ such that $v'(\varphi^*) = v(\varphi)$, for every $\varphi \in \mathcal{L}_\Sigma$.
2. Let $v \in V^{\mathbf{mbC}^\bot}$. Therefore, there exists $v' \in V^{\mathbf{mbC}}$ such that $v'(\varphi^{\circledast}) = v(\varphi)$, for every $\varphi \in \mathcal{L}_{\Sigma_\bot}$.

Proof 1. Let v be a valuation for **mbC**. Define a mapping $v' : \mathcal{L}_{\Sigma_\bot} \to \big\{0, 1\big\}$ by induction on the complexity of the formula $\psi \in \mathcal{L}_{\Sigma_\bot}$ as follows:

– If $\psi = q \in Var$, then $v'(q) = v(q)$.
– If $\psi = \bot$, then $v'(\bot) = 0$.
– If $\psi = \delta \to \gamma$, then $v'(\delta \to \gamma) = 1$ iff $v'(\delta) = 0$ or $v'(\gamma) = 1$.
– If $\psi = \neg\gamma$, then

$$v'(\neg\gamma) = \begin{cases} 1 & \text{if } v'(\gamma) = 0 \\ v(\neg\delta) & \text{if } \neg\gamma = (\neg\delta)^* \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

– If $\psi = \circ\gamma$, then

$$v'(\circ\gamma) = \begin{cases} 0 & \text{if } v'(\gamma) = v'(\neg\gamma) = 1 \\ v(\circ\delta) & \text{if } \circ\gamma = (\circ\delta)^* \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Using Corollary 2.5.18, it is easy to prove by induction on the complexity of formulas that v' is well-defined and $v'(\varphi^*) = v(\varphi)$ for every $\varphi \in \mathcal{L}_\Sigma$. Additionally, $v' \in V^{\mathbf{mbC}^\bot}$ by the proof of Lemma 2.5.9.

2. Let v be a valuation for $\mathbf{mbC}^{\perp}$. Consider a mapping $v' : \mathcal{L}_{\Sigma} \to \{0, 1\}$ defined by induction as follows:

– If $\psi = q \in Var$, then $v'(q) = v(q)$.
– If $\psi = \delta \wedge \gamma$, then $v'(\delta \wedge \gamma) = 1$ iff $v'(\delta) = v'(\gamma) = 1$.
– If $\psi = \delta \vee \gamma$, then $v'(\delta \vee \gamma) = 0$ iff $v'(\delta) = v'(\gamma) = 0$.
– If $\psi = \delta \to \gamma$, then $v'(\delta \to \gamma) = 1$ iff $v'(\delta) = 0$ or $v'(\gamma) = 1$.
– If $\psi = \neg\gamma$, then

$$v'(\neg\gamma) = \begin{cases} 1 & \text{if } v'(\gamma) = 0 \\ v(\neg\delta) & \text{if } \neg\gamma = (\neg\delta)^{\circledast} \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

– If $\psi = \circ\gamma$, then

$$v'(\circ\gamma) = \begin{cases} 0 & \text{if } v'(\gamma) = v'(\neg\gamma) = 1 \\ v(\circ\delta) & \text{if } \circ\gamma = (\circ\delta)^{\circledast} \\ \text{arbitrary} & \text{otherwise.} \end{cases}$$

Using Corollary 2.5.18, one can simply prove by induction on the complexity of formulas that v' is well-defined and $v'(\varphi^{\circledast}) = v(\varphi)$ for every $\varphi \in \mathcal{L}_{\Sigma_{\perp}}$. Moreover, $v' \in V^{\mathbf{mbC}}$ by the proof of Lemma 2.5.9. □

The equivalence between these logics in the different languages can then be established by the following theorem:

Lemma 2.5.21 *The functions* $^{*} : \mathcal{L}_{\Sigma} \to \mathcal{L}_{\Sigma_{\perp}}$ *and* $^{\circledast} : \mathcal{L}_{\Sigma_{\perp}} \to \mathcal{L}_{\Sigma}$ *satisfy the following:*

$$\vdash_{\mathbf{mbC}^{\perp}} \varphi \iff \vdash_{\mathbf{mbC}} \varphi^{\circledast}$$
$$\vdash_{\mathbf{mbC}} \varphi \iff \vdash_{\mathbf{mbC}^{\perp}} \varphi^{*}.$$

Proof As a consequence of the completeness of the logics in both languages, the lemma can be proved by using valuation semantics, namely:

$$\models_{V^{\mathbf{mbC}^{\perp}}} \varphi \iff \models_{V^{\mathbf{mbC}}} \varphi^{\circledast}$$

$$\models_{V^{\mathbf{mbC}}} \varphi \iff \models_{V^{\mathbf{mbC}^{\perp}}} \varphi^{*}$$

or equivalently by contraposition:

$$\exists v \in V^{\mathbf{mbC}^{\perp}} : v(\varphi) = 0 \iff \exists v \in V^{\mathbf{mbC}} : v(\varphi^{\circledast}) = 0$$

$$\exists v \in V^{\mathbf{mbC}} : v(\varphi) = 0 \iff \exists v \in V^{\mathbf{mbC}^{\perp}} : v(\varphi^{*}) = 0.$$

For the first equivalence, suppose that there exists $v \in V^{\mathbf{mbC}^{\perp}}$ such that $v(\varphi) = 0$. By item 2 of Lemma 2.5.20, there exists $v' \in V^{\mathbf{mbC}}$ such that $v'(\varphi^{\circledast}) = v(\varphi) = 0$. Conversely, if $v(\varphi^{\circledast}) = 0$ for some $v \in V^{\mathbf{mbC}}$ then, by item 1 of Lemma 2.5.19, there exists $v' \in V^{\mathbf{mbC}^{\perp}}$ such that $v'(\varphi) = v(\varphi^{\circledast}) = 0$.

Now, suppose that there exists $v \in V^{\mathbf{mbC}}$ such that $v(\varphi) = 0$. Because of item 1 of Lemma 2.5.20, there exists $v' \in V^{\mathbf{mbC}^{\perp}}$ such that $v'(\varphi^*) = v(\varphi) = 0$. Conversely, if $v(\varphi^*) = 0$ for some $v \in V^{\mathbf{mbC}^{\perp}}$ then, by Lemma 2.5.19, item 2, there exists $v' \in V^{\mathbf{mbC}}$ such that $v'(\varphi) = v(\varphi^*) = 0$. □

Theorem 2.5.22 *The mapping* $^{\circledast} : \mathcal{L}_{\Sigma_{\perp}} \to \mathcal{L}_{\Sigma}$ *is a conservative translation from* $\mathbf{mbC}^{\perp}$ *to* **mbC**. *The mapping* $^{*} : \mathcal{L}_{\Sigma} \to \mathcal{L}_{\Sigma_{\perp}}$ *is a conservative translation from* **mbC** *to* $\mathbf{mbC}^{\perp}$.

Proof This result is a direct consequence of Theorem 2.4.2 and Lemma 2.5.21. □

It is worth noting that the disjunction $\vee$ inside $\mathbf{mbC}^{\perp}$ must be defined exactly as it is proposed here: if disjunction is interpreted as it usually is, in terms of the implication, the resulting mapping is no longer a conservative translation:

Proposition 2.5.23 *Let* $^{*} : \mathcal{L}_{\Sigma} \to \mathcal{L}_{\Sigma_{\perp}}$ *be the translation mapping of Definition 2.5.15 except in the case of the clause for* $\vee$*, which is replaced by the following:* $(\alpha \vee \beta)^* = \big(\alpha^* \to \beta^*\big) \to \beta^*$. *Then the mapping* **, thus defined, it is not a conservative translation even though it is a translation from* **mbC** *to* $\mathbf{mbC}^{\perp}$.

Proof First observe that both formulas $\alpha \vee \beta$ and $(\alpha \to \beta) \to \beta$ are translated into the same formula:

$$(\alpha \vee \beta)^* = \big(\alpha^* \to \beta^*\big) \to \beta^* = \big((\alpha \to \beta) \to \beta\big)^*,$$

and thus the translation is not injective. Moreover, there is a way to choose a formula whose translation under * is a theorem, while there is some other formula translated in the same theorem which is not a theorem of the source logic. Consider, for instance, the formula $\perp_{(\alpha\vee\beta)}$ in $\mathcal{L}_{\Sigma}$ and let φ be the formula $\neg\perp_{(\alpha\vee\beta)}$, that is:

$$\varphi = \neg\Big(\circ(\alpha \vee \beta) \wedge \neg(\alpha \vee \beta) \wedge (\alpha \vee \beta)\Big).$$

It is easy to see that φ^* is a theorem of $\mathbf{mbC}^{\perp}$, and φ is also a theorem of **mbC**. But now consider the following formula in $\mathcal{L}_{\Sigma}$:

$$\psi = \neg\Big(\circ\big((\alpha \to \beta) \to \beta\big) \wedge \neg(\alpha \vee \beta) \wedge (\alpha \vee \beta)\Big).$$

It is a straightforward task to prove that $\psi^* = \varphi^*$, however, ψ is not a theorem in the source logic. This shows that

$$\vdash_{\mathbf{mbC}} \psi \quad \not\Longleftarrow \quad \vdash_{\mathbf{mbC}^{\perp}} \psi^*.$$

□

This illustrates the consequences of a logic not being self-extensional (recall the end of Sect. 2.3) and draws our attention to the care required when dealing with this kind of logic. The last proposition shows that the right translation of disjunctions inside $\mathbf{mbC}^{\perp}$ is done through a schema formula that uses $\to$ and $\perp$. As the example above shows, the other formulation of disjunction (just using $\to$) simply does not work, despite both formulations being equivalent in $\mathbf{mbC}^{\perp}$.

Although **mbC** and $\mathbf{mbC}^{\perp}$ are equivalent in the sense of being inter-translatable, they are not the same logic: there are some subtle distinctions between them, that will be explored in Sect. 3.4.

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