Chapter 2
Dynamic String-Averaging Methods in Hilbert Spaces

In this chapter we study the convergence of dynamic string-averaging methods for solving common fixed point problems in a Hilbert space. Our main goal is to obtain an approximate solution of the problem in the presence of computational errors. We show that our dynamic string-averaging algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Moreover, for a known computational error, we find out what an approximate solution can be obtained and how many iterates one needs for this.

2.1 Preliminaries and the Main Result

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{\|x - y\| : \ y \in E\}.$$

For every point $x \in X$ and every positive number $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \leq r\}.$$

Suppose that m is a natural number, $\bar{c} \in (0, 1)$, $P_i : X \to X$, $i = 1, \dots, m$, for every integer $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \neq \emptyset \tag{2.1}$$

and that the inequality

$$\|z - x\|^2 \geq \|z - P_i(x)\|^2 + \bar{c}\|x - P_i(x)\|^2 \tag{2.2}$$

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holds for every integer $i \in \{1, \dots, m\}$, every point $x \in X$ and every point $z \in \mathrm{Fix}(P_i)$. Set

$$F = \cap_{i=1}^{m} \mathrm{Fix}(P_i). \tag{2.3}$$

For every positive number ϵ and every integer $i \in \{1, \dots, m\}$ set

$$F_\epsilon(P_i) = \{x \in X : \ \|x - P_i(x)\| \le \epsilon\}, \tag{2.4}$$

$$\tilde{F}_\epsilon(P_i) = F_\epsilon(P_i) + B(0, \epsilon), \tag{2.5}$$

$$F_\epsilon = \cap_{i=1}^{m} F_\epsilon(P_i) \tag{2.6}$$

and

$$\tilde{F}_\epsilon = \cap_{i=1}^{m} \tilde{F}_\epsilon(P_i) \tag{2.7}$$

A point belonging to the set F is a solution of our common fixed point problem while a point which belongs to the set $\tilde{F}_\epsilon$ is its ϵ-approximate solution.

We apply a dynamic string-averaging method with variable strings and weights in order to obtain a good approximative solution of the common fixed point problem.

Next we describe the dynamic string-averaging method with variable strings and weights.

By an index vector, we mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q, \ P[t] = P_{t_q} \cdots P_{t_1}. \tag{2.8}$$

It is not difficult to see that for each index vector t

$$P[t](x) = x \text{ for all } x \in F, \tag{2.9}$$

$$\|P[t](x) - P[t](y)\| = \|x - P[t](y)\| \le \|x - y\| \tag{2.10}$$

for every point $x \in F$ and every point $y \in X$.

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ satisfies } \sum_{t \in \Omega} w(t) = 1. \tag{2.11}$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega, w}(x) = \sum_{t \in \Omega} w(t) P[t](x), \ x \in X. \tag{2.12}$$

It is easy to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F, \tag{2.13}$$

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| = \|x - P_{\Omega,w}(y)\| \leq \|x - y\| \tag{2.14}$$

for every point $x \in F$ and every point $y \in X$.

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm.

Initialization: select an arbitrary point $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1},w_{k+1}}(x_k).$$

Fix a number

$$\Delta \in (0, m^{-1}] \tag{2.15}$$

and an integer

$$\bar{q} \geq m. \tag{2.16}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \leq \bar{q} \ \text{ for all } t \in \Omega, \tag{2.17}$$

$$w(t) \geq \Delta \text{ for all } t \in \Omega. \tag{2.18}$$

Fix a natural number $\bar{N}$.

In the studies of the common fixed point problem the goal is to find a point $x \in F$. In order to meet this goal we apply an algorithm generated by

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

such that for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}).$$

This algorithm generates, for any starting point $x_0 \in X$, a sequence $\{x_k\}_{k=0}^{\infty} \subset X$, where

$$x_{k+1} = P_{\Omega_{k+1},w_{k+1}}(x_k).$$

According to the results known in the literature, this sequence should converge to an element of F. In this chapter, we study the behavior of the sequences generated by $\{(\Omega_i, w_i)\}_{i=1}^{\infty}$ taking into account computational errors which always present in practice. These computational errors are bounded from above by a small constant depending only on our computer system which is denoted by δ. This computational error δ presents in all calculations which we do using our computer system. For example, if $x \in X$ and $i \in \{1, \dots, m\}$ and we need to calculate $P_i(x)$, then using our computer system we obtain a point $y \in X$ satisfying

$$\|y - P_i(x)\| \leq \delta.$$

If k is a natural number, $y_i \in X$, $i = 1, \dots, k$, $\alpha_i > 0$, $i = 1, \dots, k$ satisfying $\sum_{i=1}^{k} \alpha_i = 1$ and if need to calculate $\sum_{i=1}^{k} \alpha_i y_i$, then by using our computer system we obtain a point $y \in X$ satisfying

$$\|y - \sum_{i=1}^{k} \alpha_i y_i\| \leq \delta.$$

Surely, in this situation one cannot expect that the sequence of iterates generated by our algorithm converges to the set F. Our goal is to understand what approximate solutions of the common fixed point problem can be obtained.

We prove the following result (Theorem 2.1), which shows that in the presence of computational errors bounded from above by a constant δ, an ϵ_1-approximate solution can be obtained after $(n_0 - 1)\bar{N}$ iterations of the algorithm, where ϵ_1 and n_0 are constants depending on δ (see (2.23) and (2.24)).

In order to state Theorem 2.1 we need the following definitions.

Let $\delta \geq 0$, $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector. Define

$$\begin{aligned} A_0(x, t, \delta) = \{(y, \lambda) \in X \times R^1 : \ &\text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that} \\ &y_0 = x \text{ and for all } i = 1, \dots, p(t), \\ &\|y_i - P_{t_i}(y_{i-1})\| \leq \delta, \\ &y = y_{p(t)}, \\ &\lambda = \max\{\|y_i - y_{i-1}\| : \ i = 1, \dots, p(t)\}\}. \end{aligned} \tag{2.19}$$

Let $\delta \geq 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$\begin{aligned} A(x, (\Omega, w), \delta) = \{(y, \lambda) \in X \times R^1 : \ &\text{there exist} \\ &(y_t, \lambda_t) \in A_0(x, t, \delta), \ t \in \Omega \text{ such that} \\ &\|y - \sum_{t \in \Omega} w(t) y_t\| \leq \delta, \ \lambda = \max\{\lambda_t : \ t \in \Omega\}\}. \end{aligned} \tag{2.20}$$

Denote by $\mathrm{Card}(A)$ the cardinality of a set A. Suppose that the sum over empty set is zero.

Theorem 2.1. *Let $M > 0$ satisfy*

$$B(0, M) \cap F \neq \emptyset, \tag{2.21}$$

$\delta > 0$ satisfy

$$\delta \leq (2\bar{q}\bar{N})^{-1}, \tag{2.22}$$

a natural number n_0 satisfy

$$n_0 \geq 1 + 4M^2\delta^{-1}(\bar{q}+1)^{-1}(2M+4)^{-1}(4\bar{N})^{-1} \tag{2.23}$$

and let

$$\epsilon_1 = \bar{c}^{-1/2}(\bar{q}+1)(\bar{N}+2)(64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N})^{1/2}. \tag{2.24}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}), \tag{2.25}$$

$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{2.26}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta). \tag{2.27}$$

Then there exists an integer $q \in [0, n_0 - 1]$ such that

$$\|x_i\| \leq 3M + 1,\ i = 0, \dots, q\bar{N}, \tag{2.28}$$

$$\lambda_i \leq (64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N}\bar{c}^{-1})^{1/2}, \tag{2.29}$$
$$i = q\bar{N}+1, \dots, (q+1)\bar{N}.$$

Moreover, if an integer $q \in [0, n_0 - 1]$ satisfies (2.29), then for each $i = q\bar{N}, \dots, (q+1)\bar{N}$,

$$x_i \in \tilde{F}_{\epsilon_1}$$

and

$$\|x_i - x_j\| \leq (\bar{q}+1)\bar{N}(64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N}\bar{c}^{-1})^{1/2} \tag{2.30}$$

for each $i, j \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$.

Theorem 2.1 is proved in Sect. 2.2. It provides the estimations for the constants ϵ_1 and n_0, which follow from (2.23) and (2.24). Note that $\epsilon_1 = c_1\delta^{1/2}$ and $n_0 = \lfloor c_2\delta^{-1}\rfloor + 1$, where c_1 and c_2 are positive constants depending on M and $\lfloor u\rfloor$ denotes the integer part of u.

Let $\delta > 0$ satisfy (2.22) and a natural number n_0 satisfy (2.23). Assume that we apply an algorithm associated with

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

which satisfies (2.25) for each natural number j, under the presence of computational errors bounded from above by a constant δ and that our goal is to find an ϵ_1-approximate solution with ϵ_1 defined by (2.24). Theorem 2.1 also answers an important question: how we can find an iteration number k for which x_k is an ϵ_1-approximate solution of the common fixed point problem. By Theorem 2.1 we need just to find the smallest integer $q \in [0, \dots, n_0 - 1]$ satisfying (2.29).

Note that Theorem 2.1 is a generalization of the main result of [98] obtained for the convex feasibility problem.

2.2 Proof of Theorem 2.1

By (2.21) there exists a point

$$z \in B(0, M) \cap F. \tag{2.31}$$

Fix a positive number

$$\epsilon_0 = (64\Delta^{-1}\delta(\bar{q} + 1)(2M + 4)4\bar{N}\bar{c}^{-1})^{1/2}. \tag{2.32}$$

Assume that a nonnegative integer s satisfies for each integer $k \in [0, s]$,

$$\max\{\lambda_i :\ i = k\bar{N} + 1, \dots, (k + 1)\bar{N}\} > \epsilon_0. \tag{2.33}$$

By (2.26) and (2.31),

$$\|x_0 - z\| \le 2M. \tag{2.34}$$

Assume that an integer $k \in [0, s]$ satisfies

$$\|x_{k\bar{N}} - z\| \le 2M. \tag{2.35}$$

We prove the following auxiliary result.

Lemma 2.2. *Assume that an integer*

$$i \in [0, \bar{N} - 1] \tag{2.36}$$

satisfies

$$\|x_{k\bar{N}+i} - z\| \le 2M + i\delta(\bar{q} + 1). \tag{2.37}$$

Then

$$\|x_{k\bar{N}+i+1} - z\| \le \delta(\bar{q} + 1) + \|x_{k\bar{N}+i} - z\| \tag{2.38}$$

and

$$\|x_{k\bar{N}+i+1} - z\|^2 \le \|x_{k\bar{N}+i} - z\|^2 + \delta(\bar{q} + 1)(4M + 3). \tag{2.39}$$

If $\lambda_{k\bar{N}+i+1} > \epsilon_0$, *then*

$$\|x_{k\bar{N}+i+1} - z\|^2 - \|x_{k\bar{N}+i} - z\|^2 \le -32^{-1}\Delta\epsilon_0^2\bar{c}. \tag{2.40}$$

Proof. In view of (2.37),

$$(x_{k\bar{N}+i+1}, \lambda_{k\bar{N}+i+1}) \in A(x_{k\bar{N}+i}, (\Omega_{k\bar{N}+i+1}, w_{k\bar{N}+i+1}), \delta). \tag{2.41}$$

By (2.20) and (2.41) there exists vectors

$$(y_t, \alpha_t) \in A_0(x_{k\bar{N}+i}, t, \delta),\ \ t \in \Omega_{k\bar{N}+i+1} \tag{2.42}$$

such that

$$\|x_{k\bar{N}+i+1} - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t\| \le \delta, \tag{2.43}$$

$$\lambda_{k\bar{N}+i+1} = \max\{\alpha_t :\ t \in \Omega_{k\bar{N}+i+1}\}. \tag{2.44}$$

It follows from (2.19) and (2.42) that for each index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ there exists a finite sequence $\{y_i^{(t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_{k\bar{N}+i},\ y_{p(t)}^{(t)} = y_t, \tag{2.45}$$

$$\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \le \delta \text{ for each integer } j = 1, \dots, p(t), \tag{2.46}$$

$$\alpha_t = \max\{\|y_{i+1}^{(t)} - y_i^{(t)}\| :\ i = 0, \dots, p(t) - 1\}. \tag{2.47}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$$

be an index vector and let

$$j \in \{1, \dots, p(t)\}. \tag{2.48}$$

By (2.2), (2.3), and (2.31),

$$\|z - P_{t_j}(y_{j-1}^{(t)})\|^2 + \bar{c}\|P_{t_j}(y_{j-1}^{(t)}) - y_{j-1}^{(t)}\|^2 \le \|z - y_{j-1}^{(t)}\|^2. \tag{2.49}$$

It follows from (2.1), (2.3), (2.31), (2.46), and (2.48) that

$$\begin{aligned}\|z - y_j^{(t)}\|^2 &= \|z - P_{t_j}(y_{j-1}^{(t)}) + P_{t_j}(y_{j-1}^{(t)}) - y_j^{(t)}\|^2 \\ &\le \|z - P_{t_j}(y_{j-1}^{(t)})\|^2 + \|P_{t_j}(y_{j-1}^{(t)}) - y_j^{(t)}\|^2 \\ &\quad +2\|z - P_{t_j}(y_{j-1}^{(t)})\|\|P_{t_j}(y_{j-1}^{(t)}) - y_j^{(t)}\| \\ &\le \|z - P_{t_j}(y_{j-1}^{(t)})\|^2 + \delta^2 + 2\delta\|z - P_{t_j}(y_{j-1}^{(t)})\| \\ &\le \|z - P_{t_j}(y_{j-1}^{(t)})\|^2 + \delta^2 + 2\delta\|z - y_{j-1}^{(t)}\|.\end{aligned} \tag{2.50}$$

By (2.49) and (2.50),

$$\begin{aligned}\|z - y_j^{(t)}\|^2 \le \|z - y_{j-1}^{(t)}\|^2 - \|P_{t_j}(y_{j-1}^{(t)}) - y_{j-1}^{(t)}\|^2\bar{c} \\ +\delta^2 + 2\delta\|z - y_{j-1}^{(t)}\|.\end{aligned} \tag{2.51}$$

In view of (2.51),

$$\|z - y_j^{(t)}\| \le \|z - y_{j-1}^{(t)}\| + \delta. \tag{2.52}$$

Thus we have shown that the following property holds:

(P1) for each index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ and each integer $j \in \{1, \dots, p(t)\}$ relations (2.51) and (2.52) hold.

By property (P1), (2.17), (2.45), and (2.52), for each index vector $t \in \Omega_{k\bar{N}+i+1}$ and each integer $j \in \{1, \dots, p(t)\}$,

$$\begin{aligned}\|z - y_j^{(t)}\| \le \|z - y_0^{(t)}\| + \delta j = \|z - x_{k\bar{N}+i}\| + \delta j \\ \le \|z - x_{k\bar{N}+i}\| + \delta\bar{q}.\end{aligned} \tag{2.53}$$

It follows from (2.37) and (2.53) that for every index vector $t \in \Omega_{k\bar{N}+i+1}$ and every integer $j \in \{1, \dots, p(t)\}$,

$$\|z - y_j^{(t)}\| \le 2M + (1 + \bar{q})i\delta + \delta\bar{q} \le 2M + \delta(\bar{q}(i + 1) + i). \tag{2.54}$$

By (2.22), (2.36), (2.37), (2.45), and (2.54) the following property holds:

(P2) for every index vector $t \in \Omega_{k\bar{N}+i+1}$ and every $j \in \{0, 1, \dots, p(t)\}$,

$$\|z - y_j^{(t)}\| \le 2M + 2\delta\bar{q}\bar{N} \le 2M + 1. \tag{2.55}$$

In view of (2.45) and (2.53) for every index vector $t \in \Omega_{k\bar{N}+i+1}$,

$$\|z - y_t\| = \|z - y^{(t)}_{p(t)}\| \le \|z - x_{k\bar{N}+i}\| + \delta\bar{q}. \tag{2.56}$$

By (2.11), (2.43), and (2.56),

$$\begin{aligned}
\|x_{k\bar{N}+i+1} - z\| &\le \|x_{k\bar{N}+i+1} - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t\| \\
&+ \|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\| \\
&\le \delta + \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)\|y_t - z\| \le \delta + \|x_{k\bar{N}+i} - z\| + \delta\bar{q}, \\
\|x_{k\bar{N}+i+1} - z\| &\le \delta(\bar{q}+1) + \|x_{k\bar{N}+i} - z\|
\end{aligned}$$

and (2.38) is true.

It follows from (2.22), (2.36), (2.37), and (2.38) that

$$\begin{aligned}
\|x_{k\bar{N}+i+1} - z\|^2 &\le \|x_{k\bar{N}+i} - z\|^2 + \delta^2(\bar{q}+1)^2 + 2\delta(\bar{q}+1)\|x_{k\bar{N}+i} - z\| \\
&\le \|x_{k\bar{N}+i} - z\|^2 + \delta^2(\bar{q}+1)^2 + 2\delta(\bar{q}+1)(2M+1) \\
&\le \|x_{k\bar{N}+i} - z\|^2 + \delta(\bar{q}+1)(4M+3).
\end{aligned}$$

Thus (2.39) is true.

Assume that

$$\lambda_{k\bar{N}+i+1} > \epsilon_0. \tag{2.57}$$

In view of (2.44) there exists an index vector

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k\bar{N}+i+1} \tag{2.58}$$

such that

$$\alpha_s = \lambda_{k\bar{N}+i+1} > \epsilon_0. \tag{2.59}$$

By (2.47), (2.58), and (2.59), there exists an integer

$$j_0 \in \{1, \dots, p(s)\} \tag{2.60}$$

such that

$$\|y^{(s)}_{j_0} - y^{(s)}_{j_0-1}\| = \alpha_s > \epsilon_0. \tag{2.61}$$

By properties (P1), (P2), (2.36), (2.37), and (2.51) applied with $t = s, j = j_0$ we have

$$\begin{aligned}\|z - y_{j_0}^{(s)}\|^2 &\le \|z - y_{j_0-1}^{(s)}\|^2 - \|P_{s_{j_0}}(y_{j_0-1}^{(s)}) - y_{j_0-1}^{(s)}\|^2\bar{c} \\ &\quad +\delta^2 + 2\delta(2M+1) \\ &\le \|z - y_{j_0-1}^{(s)}\|^2 - \|P_{s_{j_0}}(y_{j_0-1}^{(s)}) - y_{j_0-1}^{(s)}\|^2\bar{c} + 2\delta(2M+2). \qquad (2.62)\end{aligned}$$

In view of (2.46) and (2.61),

$$\|P_{s_{j_0}}(y_{j_0-1}^{(s)}) - y_{j_0-1}^{(s)}\| \ge \|y_{j_0}^{(s)} - y_{j_0-1}^{(s)}\| - \|y_{j_0}^{(s)} - P_{s_{j_0}}(y_{j_0-1}^{(s)})\| > \epsilon_0 - \delta. \qquad (2.63)$$

By (2.62) and (2.63),

$$\|z - y_{j_0}^{(s)}\|^2 \le \|z - y_{j_0-1}^{(s)}\|^2 - \bar{c}(\epsilon_0 - \delta)^2 + 2\delta(2M+2). \qquad (2.64)$$

In view of property (P2), applied with $t = s$ for all integers $j \in \{0, 1, \ldots, p(s)\}$ we have

$$\|z - y_j^{(s)}\| \le 2M + 1. \qquad (2.65)$$

It follows from property (P1), (2.52) with $t = s$ and (2.65) that for all integers $j \in \{1, \ldots, p(s)\}$ we have

$$\begin{aligned}\|z - y_j^{(s)}\|^2 &\le \|z - y_{j-1}^{(s)}\|^2 + \delta^2 + 2\delta\|z - y_{j-1}^{(s)}\| \\ &\le \|z - y_{j-1}^{(s)}\|^2 + 2\delta(2M+2). \qquad (2.66)\end{aligned}$$

By (2.17), (2.45), (2.60), (2.64), and (2.66),

$$\begin{aligned}&\|z - x_{k\bar{N}+i}\|^2 - \|z - y_s\|^2 \\ &= \sum_{i=1}^{p(s)} [\|z - y_{i-1}^{(s)}\|^2 - \|z - y_i^{(s)}\|^2] \\ &\ge \bar{c}(\epsilon_0 - \delta)^2 - 2\delta(2M+2) - 2\delta(2M+2)\bar{q} \\ &\ge \bar{c}(\epsilon_0 - \delta)^2 - 2\delta(2M+2)(\bar{q}+1). \qquad (2.67)\end{aligned}$$

By properties (P1), (P2), and (2.52), for every index vector $t \in \Omega_{k\bar{N}+i+1}$ and every integer $j \in \{1, \ldots, p(t)\}$ we have

$$\begin{aligned}\|z - y_j^{(t)}\|^2 &\le \|z - y_{j-1}^{(t)}\|^2 + \delta^2 + 2\delta\|z - y_{j-1}^{(t)}\| \\ &\le \|z - y_{j-1}^{(t)}\|^2 + 2\delta(2M+2), \\ \|z - y_{j-1}^{(t)}\|^2 - \|z - y_j^{(t)}\|^2 &\ge -2\delta(2M+2). \qquad (2.68)\end{aligned}$$

In view of (2.17), (2.45), and (2.68), for every index vector $t \in \Omega_{k\bar{N}+i+1}$,

$$\|z - x_{k\bar{N}+i}\|^2 - \|z - y_t\|^2$$
$$= \sum_{i=1}^{p(t)} [\|z - y_{i-1}^{(t)}\|^2 - \|z - y_i^{(t)}\|^2] \geq -2\bar{q}\delta(2M+2). \tag{2.69}$$

Since the function $u \to \|u - z\|^2$, $u \in X$ is convex it follows from (2.11) and (2.58) that

$$\|\sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t - z\|^2$$
$$\leq \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)\|y_t - z\|^2$$
$$= \|z - x_{k\bar{N}+i}\|^2 + \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)[\|y_t - z\|^2 - \|z - x_{k\bar{N}+i}\|^2]$$
$$\leq \|z - x_{k\bar{N}+i}\|^2 + w_{k\bar{N}+i+1}(s)[\|y_s - z\|^2 - \|z - x_{k\bar{N}+i}\|^2]$$
$$+ \sum \{w_{k\bar{N}+i+1}(t)[\|y_t - z\|^2 - \|z - x_{k\bar{N}+i}\|^2] : \ t \in \Omega_{k\bar{N}+i+1} \setminus \{s\}\}. \tag{2.70}$$

It follows from (2.11), (2.18), (2.32), (2.67), (2.69), and (2.70) that

$$\|\sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t - z\|^2$$
$$\leq w_{k\bar{N}+i+1}(s)[-(\epsilon_0 - \delta)^2 \bar{c} + 2\delta(2M+2)(\bar{q}+1)]$$
$$+2\bar{q}\delta(2M+2) + \|z - x_{k\bar{N}+i}\|^2$$
$$\leq \|z - x_{k\bar{N}+i}\|^2 + 2\bar{q}\delta(2M+2)$$
$$-w_{k\bar{N}+i+1}(s)[4^{-1}\epsilon_0^2 \bar{c} - 2\delta(2M+2)(\bar{q}+1)]$$
$$\leq \|z - x_{k\bar{N}+i}\|^2 + 2\delta\bar{q}(2M+2) - \Delta(4^{-1}\epsilon_0^2 \bar{c} - 2\delta(M+2)(\bar{q}+1)). \tag{2.71}$$

In view of (2.32) and (2.71) we have

$$\|\sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t - z\|^2 - \|x_{k\bar{N}+i} - z\|^2$$
$$\leq 2\delta(2M+2)\bar{q} - 8^{-1}\Delta\epsilon_0^2 \bar{c} \leq -16^{-1}\Delta\epsilon_0^2 \bar{c}. \tag{2.72}$$

In view of (2.11), (2.43), and (2.72),

$$\begin{aligned}
&\|x_{k\bar{N}+i+1} - z\|^2 \\
&= \|x_{k\bar{N}+i+1} - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t + \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\|^2 \\
&\le \|x_{k\bar{N}+i+1} - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t\|^2 + \|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\|^2 \\
&+2\|x_{k\bar{N}+i+1} - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t\|\|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\| \\
&\le \delta^2 - 16^{-1}\bar{c}\Delta\epsilon_0^2 + \|x_{k\bar{N}+i} - z\|^2 \\
&+2\delta\|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\| \\
&\le \delta^2 - 16^{-1}\Delta\bar{c}\epsilon_0^2 + \|x_{k\bar{N}+i} - z\|^2 \\
&+2\delta\max\{\|y_t - z\| :\ t\in\Omega_{k\bar{N}+i+1}\}. \qquad (2.73)
\end{aligned}$$

In view of (2.32), (2.45), (2.73), and property (P2) we have

$$\begin{aligned}
&\|x_{k\bar{N}+i+1} - z\|^2 - \|x_{k\bar{N}+i} - z\|^2 \\
&\le \delta^2 - 16^{-1}\Delta\epsilon_0^2\bar{c} + 2\delta(2M+1) \\
&\le -16\Delta\epsilon_0^2\bar{c} + 2\delta(2M+2) \le -32^{-1}\Delta\epsilon_0^2\bar{c}.
\end{aligned}$$

This completes the proof of Lemma 2.2. □

It follows from (2.35), Lemma 2.2 applied by induction and (2.22) that for all integers $i = 0,\dots,\bar{N}-1$,

$$\|x_{k\bar{N}+i+1} - z\| \le \|x_{k\bar{N}+i} - z\| + \delta(\bar{q}+1),$$

$$\|x_{k\bar{N}+i+1} - z\| \le 2M + \delta(\bar{q}+1)(i+1) \le 2M + \delta(\bar{q}+1)\bar{N} \le 2M+1, \qquad (2.74)$$

$$\|x_{k\bar{N}+i} - z\| \le 2M+1,\ i = 0,\dots,\bar{N}. \qquad (2.75)$$

By (2.32), (2.33), (2.35), (2.74), and Lemma 2.2 we have

$$\begin{aligned}
&\|x_{(k+1)\bar{N}} - z\|^2 - \|x_{k\bar{N}} - z\|^2 \\
&= \sum_{i=0}^{\bar{N}-1}[\|x_{k\bar{N}+i+1} - z\|^2 - \|x_{k\bar{N}+i} - z\|^2] \\
&\le -32^{-1}\Delta\epsilon_0^2\bar{c} + \bar{N}\delta(\bar{q}+1)(4M+3) \le -64^{-1}\Delta\epsilon_0^2\bar{c}.
\end{aligned}$$

Thus we have shown that the following property holds:

(P3) if an integer $k \in [0, s]$ satisfies $\|x_{k\bar{N}} - z\| \le 2M$, then

$$\|x_j - z\| \le 2M + 1,\ j = k\bar{N}, \dots, (k+1)\bar{N},$$
$$\|x_{(k+1)\bar{N}} - z\|^2 - \|x_{k\bar{N}} - z\|^2 \le -64^{-1}\Delta\epsilon_0^2\bar{c}. \tag{2.76}$$

In view of (2.34) and property (P3) we have

$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N} \tag{2.77}$$

and (2.76) is true for every integer $k = 0, \dots, s$.

By (2.34) and (2.76),

$$64^{-1}\bar{c}\Delta\epsilon_0^2(s+1) \le \sum_{k=0}^{s}[\|x_{k\bar{N}} - z\|^2 - \|x_{(k+1)\bar{N}} - z\|^2]$$
$$= \|x_0 - z\|^2 - \|x_{(s+1)\bar{N}} - z\|^2 \le \|x_0 - z\|^2 \le 4M^2,$$
$$s + 1 \le 256M^2\Delta^{-1}\epsilon_0^{-2}\bar{c}^{-1}.$$

Thus we have shown that the following property holds:

(P4) If an integer $s \ge 0$ and for every integer $k \in [0, s]$ relation (2.33) holds, then

$$s \le 256M^2\Delta^{-1}\epsilon_0^{-2}\bar{c}^{-1} - 1,$$
$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N},$$
$$\|x_{k\bar{N}} - z\| \le 2M,\ k = 0, \dots, s+1.$$

By property (P4), (2.23), and (2.32), there exists an integer $q \in [0, n_0 - 1]$ such that for every integer k satisfying $0 \le k < q$,

$$\max\{\lambda_i :\ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_0,$$
$$\max\{\lambda_i :\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}\} \le \epsilon_0.$$

In view of (2.31), (2.34), property (P4), and the choice of q we have

$$\|x_{q\bar{N}} - z\| \le 2M,$$
$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, q\bar{N},$$
$$\|x_j\| \le 3M + 1,\ j = 0, \dots, q\bar{N}.$$

Assume that an integer $q \in [0, n_0 - 1]$ satisfies

$$\lambda_i \le \epsilon_0,\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{2.78}$$

Let

$$j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}. \tag{2.79}$$

It follows from (2.27) and (2.79),

$$(x_{j+1}, \lambda_{j+1}) \in A(x_j, (\Omega_{j+1}, w_{j+1}), \delta). \tag{2.80}$$

By (2.20), (2.78), and (2.80), there exist vectors

$$(y_t^{(j)}, \alpha_t^{(j)}) \in A_0(x_j, t, \delta),\ t \in \Omega_{j+1} \tag{2.81}$$

such that

$$\|x_{j+1} - \sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)}\| \le \delta, \tag{2.82}$$

$$\max\{\alpha_t^{(j)} : t \in \Omega_{j+1}\} \le \epsilon_0. \tag{2.83}$$

It follows from (2.19), (2.81), and (2.83) that for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ there exists a finite sequence $\{y_i^{(t,j)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t,j)} = x_j, \tag{2.84}$$

for every integer $i = 1, \dots, p(t)$,

$$\|y_i^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \le \delta, \tag{2.85}$$

$$y_{p(t)}^{(t,j)} = y_t^{(j)}, \tag{2.86}$$

$$\epsilon_0 \ge \alpha_t^{(j)} = \max\{\|y_i^{(t,j)} - y_{i-1}^{(t,j)}\| : i = 1, \dots, p(t)\}. \tag{2.87}$$

By (2.17), (2.84), (2.86), and (2.87), for every index vector $t \in \Omega_{j+1}$ and every integer $i = 1, \dots, p(t)$ we have

$$\|x_j - y_i^{(t,j)}\| \le i\epsilon_0 \le \epsilon_0 \bar{q}, \tag{2.88}$$

$$\|x_j - y_t^{(j)}\| \le \epsilon_0 \bar{q}. \tag{2.89}$$

In view of (2.85) and (2.88) for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ and every integer $i = 1, \dots, p(t)$,

$$\|x_j - P_{t_i}(y_{i-1}^{(t,j)})\|$$

$$\le \|x_j - y_i^{(t,j)}\| + \|y_i^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \le \epsilon_0 \bar{q} + \delta. \tag{2.90}$$

It follows from (2.11), (2.82), and (2.89) that

$$\|x_{j+1} - x_j\| \le \|x_{j+1} - \sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)}\|$$
$$+\|\sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)} - x_j\|$$
$$\le \delta + \sum_{t \in \Omega_{j+1}} w_{j+1}(t)\|y_t^{(j)} - x_j\| \le \delta + \epsilon_0 \bar{q}.$$

Combined with (2.32) this implies that

$$\|x_{j+1} - x_j\| \le \epsilon_0(\bar{q} + 1). \tag{2.91}$$

By (2.32), (2.84), (2.88), and (2.90), for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ and every integer $i = 1, \dots, p(t)$ we have

$$\|y_{i-1}^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \ge \|y_{i-1}^{(t,j)} - x_j\| + \|x_j - P_{t_i}(y_{i-1}^{(t,j)})\| \le 2\epsilon_0 \bar{q} + \delta \le \epsilon_0(2\bar{q} + 1). \tag{2.92}$$

In view of (2.84), (2.88), and (2.92), for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ and every integer $i = 1, \dots, p(t)$,

$$x_j \in \tilde{F}_{\epsilon_0(2\bar{q}+1)}(P_{t_i}).$$

Therefore

$$x_j \in \cap\{\cap_{i=1}^{p(t)} \tilde{F}_{\epsilon_0(2\bar{q}+1)}(P_{t_i}) : \ t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}\}. \tag{2.93}$$

It is clear that (2.91) and (2.93) are true for all integers $j = q\bar{N}, \dots, (q+1)\bar{N} - 1$. In view of (2.91), for every pair of integers $j_1, j_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{j_1} - x_{j_2}\| \le \epsilon_0(\bar{q} + 1)\bar{N}. \tag{2.94}$$

Let $s \in \{1, \dots, m\}$. By (2.25), there exist an integer $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$ and an index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ such that

$$s \in \{t_1, \dots, t_{p(t)}\}.$$

Together with (2.93) this implies that

$$x_j \in \tilde{F}_{\epsilon_0(2\bar{q}+1)}(P_s). \tag{2.95}$$

It follows from (2.94) and (2.95) that for every integer $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$ we have

$$x_i \in \tilde{F}_{\epsilon_0(\bar{q}+1)(\bar{N}+2)}(P_s).$$

Since the inclusion above holds for every integer $s \in \{1, \dots, m\}$ we conclude that for each $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$x_i \in \tilde{F}_{\epsilon_0(\bar{q}+1)(\bar{N}+2)} = \tilde{F}_{\epsilon_1}.$$

This completes the proof of Theorem 2.1. □

2.3 Asymptotic Behavior of Inexact Iterates

We use all the notation, definitions, and assumptions introduced in Sect. 2.1. It is not difficult to see that the following result holds.

Proposition 2.3. *Assume that for every $i \in \{1, \dots, m\}$,*

$$P_i(X) = Fix(P_i).$$

Then for every $i \in \{1, \dots, m\}$ and every $\epsilon > 0$,

$$F_\epsilon(P_i) \subset Fix(P_i) + B(0, \epsilon),$$

$$\tilde{F}_\epsilon(P_i) \subset F_{2\epsilon}(P_i),$$

$$\tilde{F}_\epsilon \subset F_{2\epsilon}.$$

Remark 2.4. If $P_i(X) = \mathrm{Fix}(P_i)$ for every $i \in \{1, \dots, m\}$, then in view of Proposition 2.3, we can easily obtain a version of Theorem 2.1, where in its conclusion the relation $x_i \in \tilde{F}_{\epsilon_1}$ is replaced by the inclusion $x_i \in F_{2\epsilon_1}$.

Proposition 2.5. *Assume that for every $i \in \{1, \dots, m\}$, every $x \in X$ and every $y \in X$,*

$$\|P_i(x) - P_i(y)\| \le \|x - y\|. \tag{2.96}$$

Then for every $i \in \{1, \dots, m\}$ and every $\epsilon > 0$,

$$\tilde{F}_\epsilon(P_i) \subset F_{3\epsilon}(P_i),$$

$$\tilde{F}_\epsilon \subset F_{3\epsilon}.$$

Proof. Let $i \in \{1, \dots, m\}$, $\epsilon > 0$ and $x \in \tilde{F}_\epsilon(P_i)$. Then there exists

$$y \in F_\epsilon(P_i) \tag{2.97}$$

such that

$$\|y - x\| \le \epsilon. \tag{2.98}$$

By (2.96)–(2.98),

$$\|x - P_i(x)\| \le \|x - y\| + \|y - P_i(y)\| + \|P_i(y) - P_i(x)\|$$
$$\le \epsilon + \epsilon + \|y - x\| \le 3\epsilon$$

and $x \in F_{3\epsilon}(P_i)$. Proposition 2.5 is proved. □

Remark 2.6. If (2.96) holds for all $x, y \in X$ and all $i \in \{1, \dots, m\}$, then we can easily obtain a version of Theorem 2.1, where in its conclusion the relation $x_i \in \tilde{F}_{\epsilon_1}$ is replaced by the inclusion $x_i \in F_{3\epsilon_1}$.

For each $z \in R^1$ set

$$\lfloor z \rfloor = \max\{i : \ i \text{ is an integer and } i \le z\}.$$

For each $M, \delta > 0$ set

$$\epsilon(\delta, M) = \bar{c}^{-1/2}(\bar{q} + 1)(\bar{N} + 2)(64\Delta^{-1}\delta(\bar{q} + 1)(2M + 4)4\bar{N})^{1/2}, \tag{2.99}$$

$$n(\delta, M) = \lfloor 2 + 4M^2\delta^{-1}(\bar{q} + 1)^{-1}(2M + 4)^{-1}(4\bar{N})^{-1} \rfloor. \tag{2.100}$$

Theorem 2.7. *Suppose that $\bar{\epsilon} \in (0, 1)$, $\bar{M} > 0$,*

$$\tilde{F}_{\bar{\epsilon}} \subset B(0, \bar{M}) \text{ and } F \neq \emptyset. \tag{2.101}$$

Let $M > \bar{M}$ and $\delta > 0$ satisfy

$$\delta \le (2\bar{q}\bar{N})^{-1} \text{ and } \epsilon(\delta, M) < \bar{\epsilon}. \tag{2.102}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}),$$
$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta).$$

Then

$$\|x_i\| \le 3M + 1 \text{ for all integers } i \ge 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ *such that*

$$0 \le q_0 \le n(\delta, M) - 1,$$

$$1 \le q_{p+1} - q_p \le n(\delta, M) \textit{ for all integers } p \ge 0$$

and that for each integer $p \ge 0$ *and each* $i = q_p\bar{N}, \dots, (q_p + 1)\bar{N}$,

$$x_i \in \tilde{F}_{\epsilon(\delta,M)}.$$

We can prove Theorem 2.7 applying by induction Theorem 2.1 and using (2.101) and (2.102).

Remark 2.8. Note that the set $\tilde{F}_{\bar{\epsilon}}$ is bounded if there exists an integer $j \in \{1, \dots, m\}$ such that the set $F_j(X)$ is bounded.

Assume that $C_1, \dots, C_m \subset X$ and $\cap_{i=1}^m C_i \neq \emptyset$. We say that the family of sets $\{C_1, \dots, C_m\}$ has a bounded regularity property [7] if for each $\epsilon > 0$ and each $M > 0$ there exists $\delta > 0$ such that if $x \in B(0, M)$ satisfies $d(x, C_i) \le \delta$ for all $i = 1, \dots, m$, then $d(x, \cap_{i=1}^m C_i) \le \epsilon$.

Theorem 2.9. *Suppose that*

$$P_i(X) = Fix(P_i), \; i = 1, \dots, m,$$

the family of sets $\{Fix(P_i), \; i = 1, \dots, m\}$ *has the bounded regularity property,* $M > 0$ *satisfies*

$$B(0, M) \cap F \neq \emptyset$$

and that $\epsilon_0 \in (0, 1)$. *Let* $\epsilon_1 \in (0, \epsilon_0)$ *be such that the following property holds:*

(i) if $z \in B(0, 3M + 2)$ *satisfies* $d(x, Fix(P_i)) \le 2\epsilon_1$ *for all* $i = 1, \dots, m$, *then* $d(z, F) \le \epsilon_0$.

Let $\delta > 0$ *satisfy*

$$\delta \le (2\bar{q}\bar{N})^{-1} \textit{ and } \epsilon(\delta, M) \le \epsilon_1.$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}),$$

$$x_0 \in B(0, M) \textit{ and } \{x_i\}_{i=1}^{\infty} \subset X, \; \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta).$$

Then there exists an integer $q \in [0, n(\delta, M) - 1]$ *such that*

$$\|x_i\| \leq 3M + 1, \; i = 0, \dots, q\bar{N}, \tag{2.103}$$

$$\lambda_i \leq (64\Delta^{-1}\delta(\bar{q} + 1)(2M + 4)4\bar{N}\bar{c}^{-1})^{1/2}, \tag{2.104}$$

$$i = q\bar{N} + 1, \dots, (q + 1)\bar{N}.$$

Moreover, if an integer $q \in [0, n(\delta, M) - 1]$ *satisfies (2.103) and (2.104), then for each* $i = q\bar{N}, \dots, (q + 1)\bar{N}$,

$$d(x_i, F) \leq \epsilon_0 \tag{2.105}$$

and

$$\|x_i - x_j\| \leq \epsilon(\delta, M) \text{ for each } i, j \in \{q\bar{N}, \dots, (q + 1)\bar{N}\}. \tag{2.106}$$

Proof. By Theorem 2.1, there exists an integer $q \in [0, n(\delta, M) - 1]$ such that (2.103) and (2.104) hold.

Assume that an integer $q \in [0, n(\delta, M) - 1]$ satisfies (2.103) and (2.104). By Theorem 2.1, (2.106) holds and

$$x_i \in \tilde{F}_{\epsilon_1}, \; i = q\bar{N}, \dots, (q + 1)\bar{N}.$$

Together with Proposition 2.3 this implies that for all $i = q\bar{N}, \dots, (q + 1)\bar{N}$,

$$x_i \in \tilde{F}_{\epsilon_1} \subset F_{2\epsilon_1} \subset \cap_{j=1}^{m}(\mathrm{Fix}(P_i) + B(0, 2\epsilon_1)). \tag{2.107}$$

In view of (2.103) and (2.106), for all $i = q\bar{N}, \dots, (q + 1)\bar{N}$,

$$\|x_i\| \leq \epsilon(\delta, M) + \|x_{q\bar{N}}\| \leq 3M + 1 + \epsilon(\delta, M) \leq 3M + 2. \tag{2.108}$$

By (2.107), (2.108), property (i), and the choice of ϵ_1, for all $i = q\bar{N}, \dots, (q + 1)\bar{N}$,

$$d(x_i, F) \leq \epsilon_0.$$

Theorem 2.9 is proved. □

Applying by induction Theorem 2.9 we obtain the following result.

Theorem 2.10. *Suppose that*

$$P_i(X) = Fix(P_i), \; i = 1, \dots, m,$$

the family of sets $\{Fix(P_i),\ i = 1, \dots, m\}$ *has the bounded regularity property,* $\bar{M} > 0$ *satisfies*

$$F \subset B(0, \bar{M}),$$

$M > \bar{M} + 1$ *and that* $\epsilon_0 \in (0, 1)$. *Let* $\epsilon_1 \in (0, \epsilon_0)$ *be such that the following property holds:*

if $z \in B(0, 3M + 2)$ *satisfies* $d(z, Fix(P_i)) \le 2\epsilon_1$ *for all* $i = 1, \dots, m$*, then* $d(z, F) \le \epsilon_0$.

Let $\delta > 0$ *satisfy*

$$\delta \le (2\bar{q}\bar{N})^{-1} \text{ and } \epsilon(\delta, M) \le \epsilon_1.$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}),$$

$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta).$$

Then

$$\|x_i\| \le 3M + 1 \text{ for all integers } i \ge 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ *such that*

$$0 \le q_0 \le n(\delta, M) - 1,$$

$$1 \le q_{p+1} - q_p \le n(\delta, M) \text{ for all integers } p \ge 0$$

and that for each integer $p \ge 0$ *and each* $i = q_p\bar{N}, \dots, (q_p + 1)\bar{N}$,

$$d(x_i, F) \le \epsilon_0.$$

The following result is proved in Sect. 2.4.

Theorem 2.11. *Suppose that*

$$P_i(X) = Fix(P_i),\ i = 1, \dots, m,$$

the family of sets $\{Fix(P_i),\ i = 1,\dots,m\}$ *has the bounded regularity property,* $F \neq \emptyset$, $\bar{M} > 0$ *satisfies*

$$F \subset B(0, \bar{M}),$$

$M > \bar{M} + 1$ *and that* $\epsilon_0 \in (0, 1)$. *Let* $\epsilon_1 \in (0, \epsilon_0/2)$ *be such that the following property holds:*

(ii) if $z \in B(0, 3M + 2)$ *satisfies* $d(z, Fix(P_i)) \leq 2\epsilon_1$ *for all* $i = 1,\dots,m$, *then* $d(z, F) \leq \epsilon_0/2$.

Let $\delta_0 > 0$ *satisfy*

$$\delta_0 \leq (2\bar{q}\bar{N})^{-1} \text{ and } \epsilon(\delta_0, M) \leq \epsilon_1 \tag{2.109}$$

and let a positive number δ *satisfy*

$$\delta < \delta_0 \text{ and } \delta n(\delta_0, M)\bar{N}(\bar{q} + 1) < \epsilon_0/2. \tag{2.110}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1,\dots,m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\}),$$

$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta).$$

Then

$$\|x_i\| \leq 3M + 1 \text{ for all integers } i \geq 0$$

and for each integer $i \geq (n(\delta_0, M) - 1)\bar{N}$,

$$d(x_i, F) \leq \epsilon_0.$$

2.4 Proof of Theorem 2.11

By Theorem 2.10,

$$\|x_i\| \leq 3M + 1 \text{ for all integers } i \geq 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ such that

$$0 \le q_0 \le n(\delta_0, M) - 1, \tag{2.111}$$

$$1 \le q_{p+1} - q_p \le n(\delta_0, M) \text{ for all integers } p \ge 0 \tag{2.112}$$

and that for each integer $p \ge 0$ and each $i = q_p\bar{N}, \dots, (q_p + 1)\bar{N}$,

$$d(x_i, F) \le \epsilon_0/2. \tag{2.113}$$

Assume that an integer $p \ge 0$ and that an integer i satisfies

$$(q_p + 1)\bar{N} \le i < q_{p+1}\bar{N} \tag{2.114}$$

and

$$d(x_i, F) \le \epsilon_0/2 + (i - (q_p + 1)\bar{N})\delta(\bar{q} + 1). \tag{2.115}$$

(Note that in view of (2.113), inequality (2.115) is true for $i = (q_p + 1)\bar{N}$.)

Let $\gamma > 0$. By (2.115), there exists $z \in X$ such that

$$z \in F,$$

$$\|x_i - z\| < \epsilon_0/2 + (i - (q_p + 1)\bar{N})\delta(\bar{q} + 1) + \gamma. \tag{2.116}$$

Set

$$\gamma_1 = \epsilon_0/2 + (i - (q_p + 1)\bar{N})\delta(\bar{q} + 1) + \gamma. \tag{2.117}$$

The inclusion

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), \delta) \tag{2.118}$$

is true. By (2.20) and (2.118) there exists

$$(y_t, \alpha_t) \in A_0(x_i, t, \delta),\ t \in \Omega_{i+1} \tag{2.119}$$

such that

$$\|x_{i+1} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t\| \le \delta, \tag{2.120}$$

$$\lambda_{i+1} = \max\{\alpha_t :\ t \in \Omega_{i+1}\}. \tag{2.121}$$

It follows from (2.19) and (2.119) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_i, \; y_{p(t)}^{(t)} = y_t, \tag{2.122}$$

$$\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \le \delta \text{ for each integer } j = 1, \dots, p(t), \tag{2.123}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| : \; j = 0, \dots, p(t) - 1\}. \tag{2.124}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}.$$

By (2.116), (2.117), and (2.122),

$$\|z - y_0^{(t)}\| = \|z - x_i\| \le \gamma_1. \tag{2.125}$$

Assume that an integer i satisfies $0 \le i < p(t)$ and

$$\|z - y_i^{(t)}\| \le \gamma_1 + i\delta. \tag{2.126}$$

(Note that in view of (2.125), inequality (2.126) is true for $i = 0$.) By (2.2), (2.116), (2.123), and (2.126),

$$\begin{aligned}\|z - y_{i+1}^{(t)}\| &\le \|z - P_{t_{i+1}}(y_i^{(t)})\| + \|P_{t_{i+1}}(y_i^{(t)}) - y_{i+1}^{(t)}\| \\ &\le \|z - y_i^{(t)}\| + \delta \le \gamma_1 + (i+1)\delta.\end{aligned}$$

Thus we have shown by induction that (2.126) holds for all $i = 0, \dots, p(t)$. Combined with (2.17) and (2.122) this implies that

$$\|z - y_t\| = \|z - y_{p(t)}^{(t)}\| \le \gamma_1 + p(t)\delta \le \gamma_1 + \bar{q}\delta \tag{2.127}$$

for every $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$.

It follows from (2.11), (2.117), (2.120), and (2.127) that

$$\begin{aligned}\|z - x_{i+1}\| &\le \|z - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t\| + \|\sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t - x_{i+1}\| \\ &\le \sum_{t \in \Omega_{i+1}} w_{i+1}(t) \|z - y_t\| + \delta \le \gamma_1 + \bar{q}\delta + \delta \\ &\le \epsilon_0/2 + (i - (q_p + 1)\bar{N})\delta(\bar{q} + 1) + (\bar{q} + 1)\delta + \gamma.\end{aligned}$$

Since γ is any positive number we conclude that

$$\|z - x_{i+1}\| \leq \epsilon_0/2 + (i + 1 - (q_p + 1)\bar{N})\delta(\bar{q} + 1).$$

Thus we have shown by induction that (2.115) holds for all $i = (q_p + 1)\bar{N}, \dots, q_{p+1}\bar{N}$. Combined with (2.110), (2.111), (2.112), and (2.113) this implies that for each integer $i \geq (n(\delta_0, M) - 1)\bar{N}$,

$$d(x_i, F) \leq \epsilon_0/2 + n(\delta_0, M)\bar{N}\delta(\bar{q} + 1) < \epsilon_0.$$

Theorem 2.11 is proved. □

2.5 Auxiliary Results

We use the notation, definitions, and assumptions introduced in Sects. 2.1 and 2.3.

Proposition 2.12. *Let*

$$P_i(X) = Fix(P_i),\ i = 1, \dots, m,$$

$M > 0$ *satisfy*

$$F \cap B(0, M) \neq \emptyset,$$

$r > 0$ *and* k *be a natural number. Assume that*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}),$$

$$x_0 \in B(0, M) \tag{2.128}$$

and

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), r).$$

Then for all integers $i = 0, \dots, k$,

$$\|x_i\| \leq 3M + k(\bar{q} + 1)r.$$

Proof. Fix

$$z \in F \cap B(0, M). \tag{2.129}$$

By (2.128) and (2.129),

$$\|z - x_0\| \le 2M. \tag{2.130}$$

We show that for all $i = 0, \ldots, k$,

$$\|z - x_i\| \le 2M + i(\bar{q} + 1)r. \tag{2.131}$$

In view of (2.130), inequality (2.131) holds for $i = 0$. Assume that an integer i satisfies $0 \le i < k$ and that (2.131) holds.

The inclusion

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), r) \tag{2.132}$$

is true. By (2.20) and (2.132) there exist

$$(y_t, \alpha_t) \in A_0(x_i, t, r),\ t \in \Omega_{i+1} \tag{2.133}$$

such that

$$\|x_{i+1} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t\| \le r, \tag{2.134}$$

$$\lambda_{i+1} = \max\{\alpha_t :\ t \in \Omega_{i+1}\}. \tag{2.135}$$

It follows from (2.19) and (2.133) that for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_i,\ y_{p(t)}^{(t)} = y_t, \tag{2.136}$$

$$\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \le r \text{ for each integer } j = 1, \ldots, p(t), \tag{2.137}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| :\ j = 0, \ldots, p(t) - 1\}. \tag{2.138}$$

Let

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}.$$

We show that for all $j = 0, \ldots, p(t)$,

$$\|z - y_j^{(t)}\| \le \|z - x_i\| + jr. \tag{2.139}$$

Note that in view of (2.136), inequality (2.139) is true for $j = 0$.

Assume that an integer j satisfies $0 \leq j < p(t)$ and that (2.139) holds. By (2.2), (2.129), (2.137), and (2.139),

$$\|z - y_{j+1}^{(t)}\| \leq \|z - P_{t_{j+1}}(y_j^{(t)})\| + \|P_{t_{j+1}}(y_j^{(t)}) - y_{j+1}^{(t)}\|$$
$$\leq \|z - y_j^{(t)}\| + r \leq \|z - x_i\| + (j+1)r.$$

Thus we have shown by induction that (2.139) holds for all $j = 0, \dots, p(t)$. Combined with (2.17) and (2.136) this implies that

$$\|z - y_t\| = \|z - x_i\| + p(t)r \leq \|z - x_i\| + \bar{q}r \tag{2.140}$$

for all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$. It follows from (2.11), (2.134), and (2.140) that

$$\|z - x_{i+1}\| \leq \|z - \sum_{t \in \Omega_{i+1}} w_{i+1}(t)y_t\| + \|\sum_{t \in \Omega_{i+1}} w_{i+1}(t)y_t - x_{i+1}\|$$
$$\leq \sum_{t \in \Omega_{i+1}} w_{i+1}(t)\|z - y_t\| + r \leq \|z - x_i\| + \bar{q}r + r.$$

Combined with (2.31) this implies that

$$\|z - x_{i+1}\| \leq \|z - x_i\| + \bar{q}r + r \leq 2M + r(\bar{q}+1)(i+1).$$

Thus we have shown by induction that (2.131) holds for all $i = 0, \dots, k$. Together with (2.129) this implies that for all $i = 0, \dots, k$,

$$\|x_i\| \leq 3M + (\bar{q}+1)kr.$$

Proposition 2.12 is proved. □

Proposition 2.13. *Let*

$$P_i(X) = Fix(P_i), \ i = 1, \dots, m,$$

$F \neq \emptyset$, $r > 0$ *and* $\bar{M} > 0$ *satisfy*

$$F \subset B(0, \bar{M}).$$

Suppose that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j the following properties:

(a)

$$\{1, \ldots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \ldots, t_{p(t)}\});$$

(b) there exists $i(j) \in \{j, \ldots, j + \bar{N} - 1\}$ *such that for each*

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i(j)}$$

there exists $s \in \{t_1, \ldots, t_{p(t)}\}$ *for which*

$$P_s(X) \subset B(0, \bar{M}).$$

Assume that

$$\{x_i\}_{i=0}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), r).$$

Then for all integers $i \geq \bar{N}$,

$$\|x_i\| \leq 6\bar{M} + r(\bar{q} + 1)(\bar{N} + 3) + 3r.$$

Proof. Fix

$$z \in F \cap B(0, \bar{M}). \tag{2.141}$$

Assume that $p \geq 0$ is an integer. By property (b) there is $\tilde{i} \in \{p + 1, \ldots, p + \bar{N}\}$ such that the following property holds:

(c) for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{\tilde{i}}$ there exists $s \in \{t_1, \ldots, t_{p(t)}\}$ for which

$$P_s(X) \subset B(0, \bar{M}).$$

The inclusion

$$(x_{\tilde{i}}, \lambda_{\tilde{i}}) \in A(x_{\tilde{i}-1}, (\Omega_{\tilde{i}}, w_{\tilde{i}}), r). \tag{2.142}$$

is true. By (2.20) and (2.142) there exist

$$(y_t, \alpha_t) \in A_0(x_{\tilde{i}-1}, t, r), \ t \in \Omega_{\tilde{i}} \tag{2.143}$$

such that

$$\|x_{\tilde{i}} - \sum_{t \in \Omega_{\tilde{i}}} w_{\tilde{i}}(t) y_t\| \le r, \tag{2.144}$$

$$\lambda_{\tilde{i}} = \max\{\alpha_t : \ t \in \Omega_{\tilde{i}}\}. \tag{2.145}$$

It follows from (2.19) and (2.143) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{\tilde{i}}$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_{\tilde{i}-1}, \ y_{p(t)}^{(t)} = y_t, \tag{2.146}$$

$$\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \le r \text{ for each integer } j = 1, \dots, p(t), \tag{2.147}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| : \ j = 0, \dots, p(t) - 1\}. \tag{2.148}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{\tilde{i}}.$$

In view of property (c), there exists $\tilde{j} \in \{t_1, \dots, t_{p(t)}\}$ such that

$$P_{t_{\tilde{j}}}(X) \subset B(0, \bar{M}). \tag{2.149}$$

By (2.147),

$$\|y_{\tilde{j}}^{(t)} - P_{t_{\tilde{j}}}(y_{\tilde{j}-1}^{(t)})\| \le r. \tag{2.150}$$

Relations (2.149) and (2.150) imply that

$$\|y_{\tilde{j}}^{(t)}\| \le \bar{M} + r. \tag{2.151}$$

It follows from (2.141) and (2.51) that

$$\|z - y_{\tilde{j}}^{(t)}\| \le 2\bar{M} + r. \tag{2.152}$$

Assume that an integer j satisfies

$$\tilde{j} \le j < p(t)$$

and

$$\|z - y_j^{(t)}\| \le 2\bar{M} + r + (j - \tilde{j})r. \tag{2.153}$$

(Note that in view of (2.152), inequality (2.153) is true for $j = \tilde{j}$.) By (2.2), (2.14), (2.147), and (2.153),

$$\|z - y^{(t)}_{j+1}\| \le \|z - P_{t_{j+1}}(y^{(t)}_j)\| + \|P_{t_{j+1}}(y^{(t)}_j) - y^{(t)}_{j+1}\|$$
$$\le \|z - y^{(t)}_j\| + r \le 2\bar{M} + r + (j + 1 - \tilde{j})r.$$

Thus we have shown by induction that (2.153) holds for all $j = \tilde{j}, \dots, p(t)$. Together with (2.17) and (2.146) this implies that

$$\|z - y_t\| = \|z - y^{(t)}_{p(t)}\| \le 2\bar{M} + r + (p(t) - \tilde{j})r \le 2\bar{M} + r + r\bar{q},$$
$$\|z - y_t\| \le 2\bar{M} + r(\bar{q} + 1) \text{ for all } t \in \Omega_{\tilde{i}}. \tag{2.154}$$

By (2.11), (2.144), and (2.154),

$$\|x_{\tilde{i}} - z\| \le \|x_{\tilde{i}} - \sum_{t\in\Omega_{\tilde{i}}} w_{\tilde{i}}(t)y_t\| + \|\sum_{t\in\Omega_{\tilde{i}}} w_{\tilde{i}}(t)y_t - z\|$$
$$\le r + \sum_{t\in\Omega_{\tilde{i}}} w_{\tilde{i}}(t)\|y_t - z\| \le r + 2\bar{M} + r(\bar{q} + 1).$$

Thus we have shown that the following property holds:
for each integer $p \ge 0$ there exists $\tilde{i} \in \{p + 1, \dots, p + \bar{N}\}$ such that

$$\|x_{\tilde{i}} - z\| \le 2\bar{M} + r(\bar{q} + 2).$$

This property implies that there exists a strictly increasing sequence of natural numbers $\{p_i\}_{i=1}^{\infty}$ such that

$$1 \le p_1 \le \bar{N},$$
$$1 \le p_{i+1} - p_i \le \bar{N} \text{ for all integers } i \ge 1$$

and that

$$\|x_{p_i} - z\| \le 2\bar{M} + r(\bar{q} + 2) \text{ for all integers } i \ge 1.$$

Applying Proposition 2.12 we obtain that for all integers $i \ge \bar{N}$,

$$\|x_i\| \le 6\bar{M} + r(\bar{q} + 1)(\bar{N} + 3)r + 3r.$$

Proposition 2.13 is proved. □

2.6 A Convergence Result

We use the notation, definitions, and assumptions introduced in Sects. 2.1 and 2.3. We prove the following convergence result under the assumption that the computation errors tend to zero.

Theorem 2.14. *Suppose that*

$$P_i(X) = Fix(P_i),\ i = 1, \dots, m,$$

$F \neq \emptyset$, *the family* $\{Fix(P_i) : \ i = 1, \dots, m\}$ *has the bounded regularity property,* $\epsilon > 0$, $\bar{M} > 0$ *satisfy*

$$F \subset B(0, \bar{M})$$

and that a sequence $\{\delta_i\}_{i=1}^{\infty} \subset (0, \infty)$ *satisfies*

$$\lim_{i \to \infty} \delta_i = 0. \tag{2.155}$$

Then there exist a natural number k_1 *such that the following assertion holds.*

Let

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfy for each natural number j *the following properties:*

(P5)

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\});$$

(P6) there exists $i(j) \in \{j, \dots, j + \bar{N} - 1\}$ *such that for each* $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i(j)}$ *there exists* $s \in \{t_1, \dots, t_{p(t)}\}$ *for which*

$$P_s(X) \subset B(0, \bar{M}).$$

Assume that

$$\{x_i\}_{i=0}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta_i).$$

Then for all integers $i \geq k_1$,

$$d(x_i, F) \leq \epsilon.$$

Proof. Set

$$r = \max\{\delta_i : \ i = 1, 2, \dots\}. \tag{2.156}$$

By Theorem 2.11, there exists $\bar{\delta} \in (0, 1)$ such that the following property holds:

(P7) for each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

which satisfies for each natural number j

$$\{1, \ldots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \ldots, t_{p(t)}\}),$$

each

$$x_0 \in B(0, 6\bar{M} + r(\bar{q} + 1)(\bar{N} + 3) + 3r)$$

and each pair of sequences $\{x_i\}_{i=1}^{\infty} \subset X$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ which satisfies for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \bar{\delta})$$

we have

$$d(x_i, F) \leq \epsilon$$

for each integer $i \geq \bar{N}n(\bar{\delta}, 6\bar{M} + r(\bar{q} + 1)(\bar{N} + 3) + 3r)$. By (2.155), there is an integer $k_0 \geq 1$ such that

$$\delta_i \leq \bar{\delta} \text{ for all integers } i \geq k_0. \tag{2.157}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j properties (P5) and (P6) and that

$$\{x_i\}_{i=0}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta_i). \tag{2.158}$$

By (2.156), (2.158), properties (P5) and (P6), and Proposition 2.13,

$$\|x_i\| \leq 6\bar{M} + r(\bar{q} + 1)(\bar{N} + 3) + 3r \text{ for all integers } i \geq \bar{N}. \tag{2.159}$$

It follows from (2.157), (2.159), and property (P7) that

$$d(x_i, F) \leq \epsilon$$

for all integers $i \geq \bar{N} + k_0 + \bar{N}n(\bar{\delta}, 6\bar{M} + r(\bar{q} + 1)(\bar{N} + 3) + 3r)$. Theorem 2.14 is proved. $\square$

2.7 Asymptotic Behavior of Exact Iterates

We use the notation, definitions, and assumptions introduced in Sects. 2.1 and 2.3.

Theorem 2.15. *Let* $M > 0$ *satisfy*

$$B(0, M) \cap F \neq \emptyset$$

and let $\epsilon > 0$. *Assume that*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{2.160}$$

satisfies for every natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}), \tag{2.161}$$

$$x_0 \in B(0, M) \tag{2.162}$$

and $\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfy for every natural number i,*

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), 0).$$

Then

$$Card(\{i \in \{0, 1, \dots\} :\ x_i \notin \tilde{F}_\epsilon\}) \leq \bar{N}(4M^2\bar{c}^{-1}\Delta^{-1}\epsilon^{-2}(\bar{N}+1)^2\bar{q}^2 + 1) + 1.$$

Proof. Fix a point

$$z \in B(0, M) \cap F. \tag{2.163}$$

Set

$$\gamma_0 = \epsilon(\bar{N}+1)^{-1}\bar{q}^{-1}. \tag{2.164}$$

Let $i \geq 0$ be an integer. The inclusion

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), 0) \tag{2.165}$$

is true. By (2.20) and (2.165) there exist vectors

$$(y_t, \alpha_t) \in A_0(x_i, t, 0),\ t \in \Omega_{i+1} \tag{2.166}$$

such that

$$x_{i+1} = \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t, \tag{2.167}$$

$$\lambda_{i+1} = \max\{\alpha_t :\ t \in \Omega_{i+1}\}. \tag{2.168}$$

It follows from (2.19) and (2.166) that for every $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_i,\ y_{p(t)}^{(t)} = y_t, \tag{2.169}$$

$$y_j^{(t)} = P_{t_j}(y_{j-1}^{(t)}) \text{ for each integer } j = 1, \dots, p(t), \tag{2.170}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| : j = 0, \dots, p(t) - 1\}. \tag{2.171}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}.$$

By (2.2), (2.163) and (2.170), for every integer j satisfying $0 \le j < p(t)$, we have

$$\begin{aligned} \|z - y_j^{(t)}\|^2 - \|z - y_{j+1}^{(t)}\|^2 &= \|z - y_j^{(t)}\|^2 - \|z - P_{t_{j+1}}(y_j^{(t)})\|^2 \\ &\ge \bar{c}\|y_j^{(t)} - y_{j+1}^{(t)}\|^2. \end{aligned} \tag{2.172}$$

In view of (2.17), (2.169) and (2.172),

$$\begin{aligned} \|z - x_i\|^2 - \|z - y_t\|^2 &= \|z - y_0^{(t)}\|^2 - \|z - y_{p(t)}^{(t)}\|^2 \\ &= \sum_{j=0}^{p(t)-1} (\|z - y_j^{(t)}\|^2 - \|z - y_{j+1}^{(t)}\|^2) \\ &\ge \bar{c} \sum_{j=0}^{p(t)-1} \|y_j^{(t)} - y_{j+1}^{(t)}\|^2 \ge \bar{c}\alpha_t^2. \end{aligned} \tag{2.173}$$

It follows from (2.11), (2.18), (2.167), (2.168), and (2.173) that

$$\begin{aligned} \|z - x_{i+1}\|^2 &= \|z - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t\|^2 \le \sum_{t \in \Omega_{i+1}} \|z - y_t\|^2 w_{i+1}(t) \\ &\le \sum_{t \in \Omega_{i+1}} w_{i+1}(t)(\|z - x_i\|^2 - \bar{c}\alpha_t^2) \\ &\le \|z - x_i\|^2 - \bar{c}\Delta \sum_{t \in \Omega_{i+1}} \alpha_t^2 \le \|z - x_i\|^2 - \bar{c}\Delta\lambda_{i+1}^2. \end{aligned}$$

Thus

$$\|z - x_{i+1}\|^2 \le \|z - x_i\|^2 - \bar{c}\Delta\lambda_{i+1}^2 \text{ for all integers } i \ge 0. \tag{2.174}$$

By (2.162), (2.163), and (2.174), for each natural number n,

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_n\|^2 = \sum_{i=0}^{n-1}(\|z - x_i\|^2 - \|z - x_{i+1}\|^2)$$

$$\geq \sum_{i=0}^{n-1} \bar{c}\Delta\lambda_{i+1}^2 \geq \bar{c}\Delta\gamma_0^2 \mathrm{Card}(\{i \in \{1, \dots, n\} : \lambda_i \geq \gamma_0\}).$$

Since the relation above holds for every natural number n we conclude that

$$\mathrm{Card}(\{i \in \{1, 2, \dots, \} : \lambda_i \geq \gamma_0\}) \leq 4M^2\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2} + 1. \tag{2.175}$$

Assume that an integer $i \geq 1$ and $\lambda_i < \gamma_0$. The inclusion

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), 0) \tag{2.176}$$

is true. By (2.20) and (2.176) there exist vectors

$$(y_t, \alpha_t) \in A_0(x_{i-1}, t, 0), \ t \in \Omega_i \tag{2.177}$$

such that

$$x_i = \sum_{t \in \Omega_i} w_i(t) y_t, \tag{2.178}$$

$$\lambda_i = \max\{\alpha_t : t \in \Omega_i\}. \tag{2.179}$$

It follows from (2.19) and (2.177) that for every $t = (t_1, \dots, t_{p(t)}) \in \Omega_i$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_{i-1}, \ y_{p(t)}^{(t)} = y_t, \tag{2.180}$$

$$y_j^{(t)} = P_{t_j}(y_{j-1}^{(t)}) \text{ for each integer } j = 1, \dots, p(t), \tag{2.181}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| : j = 0, \dots, p(t) - 1\}. \tag{2.182}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_i$$

be an index vector. By (2.179), (2.181), (2.182) and the inequality $\lambda_i < \gamma_0$, for every $j = 0, \dots, p(t) - 1$,

$$y_j^{(t)} \in F_{\gamma_0}(P_{t_{j+1}}). \tag{2.183}$$

It follows from (2.17), (2.179), (2.180), (2.182), (2.183), and the inequality $\lambda_i < \gamma_0$ that for every integer $j = 0, \dots, p(t)$ we have

$$\|x_{i-1} - y_j^{(t)}\| \le j\lambda_i \le \bar{q}\gamma_0$$

and if $j < p(t)$, then

$$x_{i-1} \in \tilde{F}_{\bar{q}\gamma_0}(P_{t_j+1}).$$

Therefore

$$x_{i-1} \in \tilde{F}_{\bar{q}\gamma_0}(P_s) \text{ for all } s = 1, \dots, p(t) \tag{2.184}$$

and

$$\|x_{i-1} - y_t\| \le \bar{q}\gamma_0 \tag{2.185}$$

for all $t \in \Omega_i$. In view of (2.185),

$$x_{i-1} \in \cap\{\tilde{F}_{\bar{q}\gamma_0}(P_s) : \ s \in \cup_{t\in\Omega_i}\{t_1, \dots, t_{p(t)}\}\}. \tag{2.186}$$

It follows from (2.11), (2.178), and (2.185) that

$$\begin{aligned}\|x_{i-1} - x_i\| &= \|x_{i-1} - \sum_{t\in\Omega_i} w_i(t)y_t\| \\ &\le \sum_{t\in\Omega_i} w_i(t)\|x_{i-1} - y_t\| \le \gamma_0\bar{q}. \end{aligned} \tag{2.187}$$

Set

$$E_0 = \{i \in \{1, 2, \dots\} : \ \lambda_i \ge \gamma_0\}. \tag{2.188}$$

By (2.175), (2.186), (2.187), and (2.188),

$$\mathrm{Card}(E_0) \le 4M^2\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2} + 1 \tag{2.189}$$

and the following property holds:

(P8) if an integer $i \ge 1$ satisfies the inequality $\lambda_i < \gamma_0$, then

$$x_{i-1} \in \tilde{F}_{\bar{q}\gamma_0}(P_s), \ s \in \cup_{t\in\Omega_i}\{t_1, \dots, t_{p(t)}\}, \tag{2.190}$$

$$\|x_{i-1} - x_i\| \le \gamma_0\bar{q}. \tag{2.191}$$

Set

$$E_1 = \{i \in \{1, 2, \dots\} : \ [i, i + \bar{N} - 1] \cap E_0 \ne \emptyset\}. \tag{2.192}$$

By (2.164), (2.189), and (2.191) we have

$$\begin{aligned}\mathrm{Card}(E_1) &\le \bar{N}(4M^2\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}+1)\\ &\le \bar{N}(4M^2\bar{c}^{-1}\Delta^{-1}\bar{q}^2(\bar{N}+1)^2\epsilon^{-2}+1).\end{aligned} \tag{2.193}$$

Assume that a natural number $j \notin E_1$. In view of (2.192),

$$\{j,\dots,j+\bar{N}-1\}\cap E_0=\emptyset.$$

Together with (2.188) this implies that for every integer $i \in \{j,\dots,j+\bar{N}-1\}$, the inequality $\lambda_i < \gamma_0$ is true and (2.190) and (2.191) hold. In view of (2.191) which holds for every integer $i \in \{j,\dots,j+\bar{N}-1\}$ and for every pair of integers $i_1, i_2 \in \{j-1,\dots,j+\bar{N}-1\}$ we have

$$\|x_{i_1}-x_{i_2}\| \le \gamma_0\bar{N}\bar{q}. \tag{2.194}$$

By (2.161), (2.194), and (2.190) which holds for each $i \in \{j,\dots,j+\bar{N}-1\}$,

$$x_j \in \tilde{F}_{\bar{q}\gamma_0(\bar{N}+1)}(P_s),\ s \in \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\})=\{1,\dots,m\}.$$

Together with (2.164) this implies that

$$x_j \in \tilde{F}_\epsilon$$

for all $j \in \{1,2,\dots,\}\setminus E_1$. Theorem 2.15 is proved. □

Note that Theorem 2.15 is a generalization of the main result of [93] obtained for the convex feasibility problem.

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