Chapter 2
Symbolic Dynamics

A number system consists of a continuous value mapping whose domain is a symbolic space of infinite words and whose range is the extended real line. We say that the value mapping is a **symbolic extension** of $\overline{\mathbb{R}}$. Symbolic spaces and symbolic extensions are treated in symbolic dynamics which is based on the theory of compact metric spaces (see e.g., Hocking and Young [1]).

2.1 Metric Spaces

Definition 2.1 A **metric space** (X, d) consists of a set X and a metric $d : X \times X \to [0, \infty)$ which gives the distance $d(x, y)$ of points $x, y \in X$. The following properties are assumed:
1. $d(x, y) = 0 \Leftrightarrow x = y$,
2. $d(x, y) = d(y, x)$: symmetry,
3. $d(x, z) \leq d(x, y) + d(y, z)$: triangle inequality.

We refer to elements of X as points. A classical example of a metric space is the n-dimensional **Euclidean space** $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$ with the Euclidean metric given by

$$d_e(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

In particular, the set $\mathbb{R}$ of real numbers is a metric space with metric $d_e(x, y) = |x - y|$. The extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is a metric space with the angle metric (see Sect. 1.4)

$$d_a(x, y) = \frac{1}{\pi}\text{arccotg}\,\frac{|xy + 1|}{|y - x|},\ \ d_a(x, \infty) = \frac{1}{\pi}\text{arccotg}|x|.$$

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Given a metric space (X, d), the **ball** with center $x \in X$ and radius $r > 0$ is the set

$$B_r(x) = \{y \in X : \ d(y, x) < r\}.$$

In $(\mathbb{R}, d_e)$, balls are open intervals $B_r(x) = (x - r, x + r)$. The **interior** $\mathrm{int}(Y)$ and **closure** $\overline{Y}$ of a set $Y \subseteq X$ are defined by

$$\begin{aligned} \mathrm{int}(Y) &= \{x \in X : \ \exists r > 0, B_r(x) \subseteq Y\}, \\ \overline{Y} &= \{x \in X : \ \forall r > 0, B_r(x) \cap Y \neq \emptyset\}, \end{aligned}$$

so $\mathrm{int}(Y) \subseteq Y \subseteq \overline{Y}$, $\overline{X \setminus Y} = X \setminus \mathrm{int}(Y)$, and $\mathrm{int}(X \setminus Y) = X \setminus \overline{Y}$, where

$$X \setminus Y = \{x \in X : \ x \notin Y\}$$

is the **set difference** of Y from X. For example, if $Y = [0, 1) \subset \mathbb{R}$ is a semiclosed interval, then $\mathrm{int}(Y) = (0, 1)$ and $\overline{Y} = [0, 1]$. If $Y, Z \subseteq X$, then

$$\begin{aligned} \mathrm{int}(Y \cap Z) &= \mathrm{int}(Y) \cap \mathrm{int}(Z), \\ \mathrm{int}(Y \cup Z) &\supseteq \mathrm{int}(Y) \cup \mathrm{int}(Z), \\ \overline{Y \cap Z} &\subseteq \overline{Y} \cap \overline{Z}, \\ \overline{Y \cup Z} &= \overline{Y} \cup \overline{Z}. \end{aligned}$$

A set $Y \subseteq X$ is **open**, if $Y = \mathrm{int}(Y)$, and **closed** if $\overline{Y} = Y$. It follows that $Y \subseteq X$ is closed iff $X \setminus Y$ is open. By the triangle inequality, every ball $B_r(x)$ is an open set. A semi-open (or semi-closed) interval $[a, b) = \{x \in \mathbb{R} : \ a \leq x < b\}$ is neither closed nor open in $\mathbb{R}$. A set is **clopen** if it is both closed and open. The sets $\emptyset$ and X are clopen in any metric space. If they are the only clopen sets, then we say that X is a **connected space**. The Euclidean space $\mathbb{R}^n$ is connected. The union of two intervals $[0, 1] \cup [2, 3]$ with the Euclidean metric $d_e(x, y) = |x - y|$ is not a connected space, since $[0, 1]$ and $[2, 3]$ are its clopen sets.

A sequence $\{x_n \in X : \ n \geq 0\}$ of points of X converges to a point $x \in X$ if for every $\varepsilon > 0$ there exists n_0 such that $d(x_n, x) < \varepsilon$ for every $n \geq n_0$. In this case we say that $\{x_n : \ n \geq 0\}$ is a **convergent sequence**. A sequence cannot converge to two distinct points, so we write $\lim_{n \to \infty} x_n = x$ if x_n converge to x. A **subsequence** of $\{x_n : \ n \geq 0\}$ is any sequence $\{x_{n_i} : \ i \geq 0\}$, where $\{n_i : \ i \geq 0\}$ is an increasing sequence of indices. If (X, d) is a metric space and $Y \subseteq X$, then d restricted to $Y \times Y$ is a metric and we say that (Y, d) is a **subspace** of X.

Definition 2.2 A metric space is **compact** if any its sequence has a convergent subsequence. A subset of a metric space is compact, if it is compact as a subspace.

The real line $\mathbb{R}$ is not compact, since the sequence $\{x_n = n : \ n \geq 0\}$ has no convergent subsequence. The open interval $(0, 1)$ is not compact either since the sequence $\{x_n = 1/n : \ n > 0\}$ has in $(0, 1)$ no convergent subsequence: all its subsequences converge to zero, which is not in the space (0, 1). A closed bounded interval $[a, b]$ is compact in $\mathbb{R}$. We show that a set $Y \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded. We say that a set $Y \subseteq X$ is **bounded**, if $Y \subseteq B_r(x)$ for some $x \in X$ and $r > 0$. This happens iff the set has a finite **diameter**

$$\operatorname{diam}(Y) = \sup\{d(y, z) : \; y, z \in Y\}.$$

Proposition 2.3 *1. A compact subset of a metric space is closed and bounded.*
2. A closed subset of a compact space is compact.
3. A subset of an Euclidean space $\mathbb{R}^n$ is compact iff it is closed and bounded.

Proof 1. Let $Y \subseteq X$ be compact and assume by contradiction that it is not closed, so there exists $y \in \overline{Y} \setminus Y$. For each $n > 0$ there exists $y_n \in Y$ such that $d(y_n, y) < 1/n$, so $\lim_{n\to\infty} y_n = y \in X \setminus Y$. Each subsequence of $\{Y_n : \; n \geq 0\}$ has the same limit y. This means that no its subsequence has a limit in Y and this is a contradiction. Assume that Y is not bounded. Take any $y_0 \in Y$. There exist points $y_n \in Y$ such that $d(y_n, y_0) > n$, and the sequence $\{y_n : \; n \geq 0\}$ has no convergent subsequence. This is a contradiction.
2. Let X be compact and let $Y \subseteq X$ be closed. A sequence $\{y_n \in Y : \; n \geq 0\}$ has a subsequence which converges to some $y \in X$. Since X is closed, $y \in Y$, so Y is compact.
3. Let $Y \subseteq \mathbb{R}$ be closed and bounded and $x_n \in Y$. There exists an interval $[a_0, b_0] \supseteq Y$. Denote by $c_0 = \frac{a_0+b_0}{2}$. An infinite number of x_n belong either to $[a_0, c_0]$ or to $[c_0, b_0]$. In the former case set $[a_1, b_1] = [a_0, c_0]$ and in the latter case set $[a_1, b_1] = [c_0, b_0]$. Let n_1 be the first index with $x_{n_1} \in [a_1, b_1]$. We continue by induction. At each step k the interval $[a_k, b_k]$ is one half of the interval $[a_{k-1}, b_{k-1}]$ and contains an infinite number of x_n. Let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in [a_k, b_k]$. Then $\{x_{n_k} : \; k \geq 1\}$ converges to the common limit of a_k and b_k. Since Y is closed, this limit belongs to Y, so Y is compact. If $Y \subseteq \mathbb{R}^n$ is closed and bounded, and $\{x_m = (x_{m,1}, \ldots, x_{m,n}) : \; m \geq 0\}$ is a sequence in Y, then for each coordinate $i \leq n$, $\{x_{m,i} : \; m \geq 0\}$ is a bounded sequence. There exists a subsequence whose first coordinate converges, a subsequence of this subsequence whose second coordinate converges, etc. Thus there exists a subsequence of $\{x_m : \; m \geq 0\}$ which converges in each coordinate. Since Y is closed, the limit belongs to Y. □

A **cover** of a space X is any collection $\mathcal{U} = \{U_i : \; i \in I\}$ of sets $U_i \subseteq X$ whose union is X. The index set I may be finite or infinite with arbitrary cardinality. If all U_i are open, we say that $\mathcal{U}$ is an **open cover**. If $J \subseteq I$ and $\bigcup_{i\in J} U_i = X$, we say that $\{U_i : \; i \in J\}$ is a **subcover** of $\mathcal{U}$. The **diameter** of a cover is the supremum of the diameters of its elements.

Proposition 2.4 *Let X be a metric space. The following three conditions are equivalent.*
1. X is compact.
2. Every open cover of X has a finite subcover.

3. *If* $\{V_n \subseteq X : n \geq 0\}$ *is a sequence of nonempty closed sets such that* $V_{n+1} \subseteq V_n$, *then the intersection* $\bigcap_{n\geq 0} V_n$ *is nonempty.*

Proof $1 \Rightarrow 2$: Assume that $\mathcal{U} = \{U_n \subseteq X : n \geq 0\}$ is a countable cover which does not have a finite subcover. Then there exist points $x_n \in X \setminus (U_0 \cup \cdots \cup U_n)$. The sequence $\{x_n : n \geq 0\}$ has a converging subsequence $\lim_{k\to\infty} x_{n_k} = x$. Since $\mathcal{U}$ is a cover, $x \in U_n$ for some n. Since U_n is open, $x_{n_k} \in U_n$ for each sufficiently large n_k and this is a contradiction. If $\mathcal{U}$ is an uncountable cover, then its countable cover should be first found using the concept of countable open basis (see e.g., Hocking and Young [1]).
$2 \Rightarrow 3$: Let $\emptyset \neq V_{n+1} \subseteq V_n \subseteq X$ be nonempty closed sets and assume that their intersection is empty. Then $\{U_n = X \setminus V_n : n \geq 0\}$ is an open cover of X and has a finite subcover, so there exists n such that $X = U_0 \cup \cdots \cup U_n = X \setminus V_n$. This implies $V_n = \emptyset$ which is a contradiction.
$3 \Rightarrow 1$. For a sequence $\{x_n \in X : n \geq 0\}$, set $V_n = \overline{\{x_i : i \geq n\}}$. Then $V_{n+1} \subseteq V_n$ are nonempty and closed, so there exists $x \in \bigcap_n V_n$. Since V_1 is closed, $B_1(x) \cap V_1 \neq \emptyset$, so there exists n_1 such that $x_{n_0} \in B_1(x)$. In a similar way we show that there exists $n_2 > n_1$ such that $x_{n_2} \in B_{1/2}(x)$. By induction we get a subsequence $\{x_{n_k} : k \geq 0\}$ such that $x_{n_k} \in B_{1/k}(x)$, so $\lim_{k\to\infty} x_{n_k} = x$. □

A mapping $F : X \to Y$ from a set X to a set Y assigns to elements $x \in X$ elements $F(x) \in Y$. If $G : Y \to Z$ is another mapping, then the composition $G \circ F : X \to Z$ is defined by $(G \circ F)(x) = G(F(x))$. A mapping $F : X \to Y$ is **injective**, if $x \neq x' \in X$ implies $F(x) \neq F(x')$. It is **surjective**, if for each $y \in Y$ there exists $x \in X$ such that $y = F(x)$. A mapping is **bijective**, if it is both injective and surjective. A bijective mapping $F : X \to Y$ has the inverse mapping $F^{-1} : Y \to X$ such that $F^{-1}(F(x)) = x$ for every $x \in X$, so the compositions $F^{-1} \circ F = \mathrm{Id}_X$, $F \circ F^{-1} = \mathrm{Id}_Y$ are the identity mappings on X and Y. If (X, d_X) and (Y, d_Y) are metric spaces, then we say that a mapping $F : X \to Y$ is continuous at $x \in X$, if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x' \in X, (d_X(x, x') < \delta \Rightarrow d_Y(F(x), F(x')) < \varepsilon).$$

We say that F is a **continuous** mapping, if it is continuous at every point $x \in X$. We say that F is a **homeomorphism** if it is bijective and both F and F^{-1} are continuous. Metric spaces X, Y are **homeomorphic**, if there exists a homeomorphism from X to Y. For example, the function $F(x) = 1/x$ is a homeomorphism between the intervals $X = (0, 1)$ and $Y = (1, \infty)$.

Proposition 2.5 *A mapping* $F : X \to Y$ *between metric spaces is continuous iff for every open set* $U \subseteq Y$, *the preimage*

$$F^{-1}(U) = \{x \in X : F(x) \in U\}$$

is an open set in X. An equivalent condition is that the preimage $F^{-1}(V)$ of every closed set $V \subseteq Y$ is a closed set in X.

Proof Assume that F is continuous and let $U \subseteq Y$ be an open set. If $x \in F^{-1}(U)$, then $F(x) \in U$, so there exists $\varepsilon > 0$ such that $B_\varepsilon(F(x)) \subseteq U$. By the continuity of F in x there exists $\delta > 0$ such that if $y \in B_\delta(x)$ then $F(y) \in B_\varepsilon(F(x)) \subseteq U$. This means that $B_\delta(x) \subseteq F^{-1}(U)$, so $F^{-1}(U)$ is open in X. Conversely assume that the preimage of any open set is open. Given $x \in X$ and $\varepsilon > 0$, the ball $U = B_\varepsilon(F(x))$ is an open set, so its preimage $F^{-1}(U)$ is open in X. Since $x \in F^{-1}(U)$ there exists $\delta > 0$ such that $B_\delta(x) \subseteq F^{-1}(U)$ and this is just the condition of continuity. If $V \subseteq Y$ is a closed set, then $F^{-1}(Y \setminus V) = X \setminus F^{-1}(V)$ is an open set so $F^{-1}(V)$ is a closed set. □

Proposition 2.6 *If X is a compact space and $F : X \to Y$ is continuous and surjective, then Y is compact. If F is also injective (and therefore bijective), then $F^{-1} : Y \to X$ is continuous, so F is a homeomorphism.*

Proof Let $\{U_i : i \in I\}$ be an open cover of Y. Then $\{F^{-1}(U_i) : i \in I\}$ is an open cover of X so it has a finite subcover $\{F^{-1}(U_i) : i \in K\}$, and $\{U_i : i \in K\}$ is an open cover of Y. Thus Y is compact. Assume that F is bijective. If $V \subseteq X$ is a closed set then it is compact and $(F^{-1})^{-1}(V) = F(V)$ is a compact set and therefore closed. By the preceding proof, F^{-1} is continuous. □

The stereographic projection $\mathbf{d}(x) = \frac{2x+i(x^2-1)}{x^2+1}$ is a bijective mapping $\mathbf{d} : \overline{\mathbb{R}} \to \mathbb{S}$. With the angle metric on $\overline{\mathbb{R}}$ and the Euclidean metric on $\mathbb{S} \subset \mathbb{C}$, $\mathbf{d}$ is a homeomorphism. Since $\mathbb{S}$ is a closed and bounded subset of $\mathbb{C} \approx \mathbb{R}^2$, it is compact and $\overline{\mathbb{R}}$ is compact too.

Theorem 2.7 *Any open cover $\mathcal{U} = \{U_a : a \in A\}$ of a compact space X has a* **Lebesgue number** $L > 0$ *such that* $\forall x \in X, \exists a \in A, B_L(x) \subseteq U_a$.

Proof Let $\mathcal{U} = \{U_a : a \in A\}$ be an open cover of X. If $U_a = X$ for some $a \in A$, then any $L > 0$ is a Lebesgue number of $\mathcal{U}$. Assume therefore that $U_a \neq X$ for each $a \in A$. Define a function $f : X \to (0, \infty)$ by

$$f(x) = \sup\{r > 0 : \exists a \in A, B_r(x) \subseteq U_a\} < \infty.$$

We show that f is continuous: If $d(x, y) < \delta$ and $0 < r < f(x)$, then there exists $a \in A$ such that $B_r(x) \subseteq U_a$, $B_{r-\delta}(y) \subseteq U_a$, so $f(y) > r - \delta$. Since this holds for any $r < f(x)$, we get $f(y) \geq f(x) - \delta$. Interchanging x and y we get $f(x) \geq f(y) - \delta$, so $|f(x) - f(y)| \leq \delta$ and this proves the continuity of f (see Fig. 2.1). By Proposition 2.6, a continuous image of a compact space is compact, so $f(X) \subseteq (0, \infty)$ is compact and therefore closed. Since $f(X)$ does not contain zero, its minimum $L_0 = \min f(X)$ is positive. If $0 < L < L_0$, then L is a Lebesgue number of $\mathcal{U}$. □

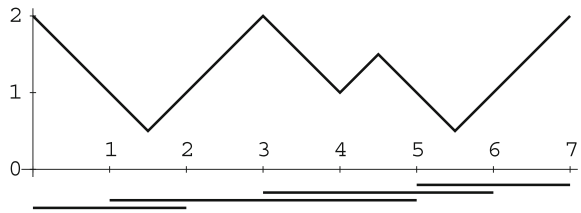

Fig. 2.1 The function $f(x) = \sup\{r > 0 : \exists a \in A, B_r(x) \subseteq U_a\}$ for the cover $\mathcal{U} = \{[0, 2), (1, 5), (3, 6), (5, 7]\}$ of $X = [0, 7]$

We say that a mapping $F : X \to Y$ is **uniformly continuous** if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, x' \in X (d(x, x') < \delta \Rightarrow d(F(x), F(x')) < \varepsilon).$$

A uniformly continuous map is continuous. The map $f : (0, 1) \to (1, \infty)$ defined by $f(x) = 1/x$ is continuous but not uniformly continuous.

Proposition 2.8 *If $F : X \to Y$ is a continuous map and X is compact, then F is uniformly continuous.*

Proof Pick $\varepsilon > 0$. For each $x \in X$ there exists $\delta_x > 0$ such that if $d_X(y, x) < \delta_x$, then $d_Y(F(x), F(y)) < \frac{\varepsilon}{2}$. Let $\delta > 0$ be a Lebesgue number of the open cover $\mathcal{U} = \{B_{\delta_x}(x) : x \in X\}$. If $y, z \in X$ and $d_X(y, z) < \delta$, then there exists $x \in X$ such that $B_\delta(y) \subseteq B_{\delta_x}(x)$, so both y, z belong to $B_{\delta_x}(x)$ and therefore $d_Y(F(y), F(z)) \le d_Y(F(y), F(x)) + d_Y(F(x), F(z)) < \varepsilon$. □

2.2 The Cantor Space

Recall that if A is an alphabet (a finite set with at least two elements), then the distance of words $u, v \in A^\omega$ is defined by

$$d(u, v) = 2^{-n}, \text{ where } n = \min\{k \ge 0 : u_k \ne v_k\},$$

so $d(u, v) < 2^{-n}$ iff $u_{[0,n]} = v_{[0,n]}$. Clearly $d(u, v) = 0$ iff $u = v$ and d is symmetric, $d(u, v) = d(v, u)$. To show that d satisfies the triangle inequality, let $d(u, v) = 2^{-n}$, $d(v, w) = 2^{-m}$ and $p = \min\{m, n\}$. Then $u_{[0,p)} = v_{[0,p)} = w_{[0,p)}$, so

$$d(u, w) \le 2^{-p} \le \max\{d(u, v), d(v, w)\} \le d(u, v) + d(v, w).$$

Thus (A^ω, d) is a metric space which is called a **power space**.

To get insight to the topology of the power spaces A^ω, we show that these spaces are homeomorphic to the **Cantor middle third set**

$$C = [0, 1] \setminus (\tfrac{1}{3}, \tfrac{2}{3}) \setminus (\tfrac{1}{9}, \tfrac{2}{9}) \setminus (\tfrac{7}{9}, \tfrac{8}{9}) \setminus (\tfrac{1}{27}, \tfrac{2}{27}) \setminus \cdots$$

The set C is obtained from the closed unit interval $[0, 1]$ by deleting the open middle third interval $(\frac{1}{3}, \frac{2}{3})$ and repeating this deleting procedure indefinitely with the remaining closed intervals (see Fig. 1.6). If we express the numbers $x \in [0, 1]$ in the ternary system $x = \sum_{n\geq 0} u_n 3^{-n-1}$, where $u_n \in \{0, 1, 2\}$, then the interval $(\frac{1}{3}, \frac{2}{3})$ consists of points whose first digit is $u_0 = 1$. The endpoints of this intervals have two expansions: $\frac{1}{3} = .10^\omega = .02^\omega$, $\frac{2}{3} = .20^\omega = .12^\omega$, so $[0, 1]\setminus(\frac{1}{3}, \frac{2}{3})$ consists of points which have ternary expansions with $u_0 \neq 1$. By induction, we show that C consists of points which have ternary expansions with digits $u_i \in \{0, 2\}$.

Proposition 2.9 *The Cantor middle third set C is homeomorphic to* $\{0, 1\}^\omega$.

Proof Define $\Phi_3 : \{0, 1\}^\omega \to C$ by $\Phi_3(u) = \sum_{i\geq 0} 2u_i \cdot 3^{-i-1}$. If $d(u, v) = 2^{-n}$, then $u_{[0,n)} = v_{[0,n)}$, $u_n \neq v_n$, so

$$|\Phi_3(u) - \Phi_3(v)| = \left|\sum_{i=n}^{\infty} 2(u_i - v_i)3^{-i-1}\right| \leq 2\sum_{i=n}^{\infty} 3^{-i-1} = \frac{2 \cdot 3^{-n-1}}{1 - \frac{1}{3}} = 3^{-n},$$

$$|\Phi_3(u) - \Phi_3(v)| \geq 2 \cdot 3^{-n-1} - 2\sum_{i=n+1}^{\infty} 3^{-i-1} = 3^{-n-1}.$$

This shows that Φ_3 is bijective. If $d(u, v) < 2^{-n+1}$ then $|\Phi_3(u) - \Phi_3(v)| \leq 3^{-n}$ and if $|x - y| < 3^{-n-1}$ then $d(\Phi_3^{-1}(x), \Phi_3^{-1}(y)) < 2^{-n}$. This means that Φ_3 is a homeomorphism. □

While the Cantor middle third set C is obtained from the closed unit interval by deleting the middle thirds, the unit interval is obtained from the Cantor middle third set by gluing the endpoints of its cylinders. This is done by the mapping $\Phi_2 \circ \Phi_3^{-1} : C \to [0, 1]$ (see Fig. 2.2 left), where $\Phi_3 : \{0, 1\}^\omega \to C$ is the homeomorphism from the proof of Proposition 2.9 and $\Phi_2 : \{0, 1\}^\omega \to [0, 1]$ is defined by $\Phi_2(u) = \sum_{i=0}^{\infty} u_i \cdot 2^{-i-1}$, The mapping $\Phi_2 \circ \Phi_3^{-1}$ defined on C can be extended to a continuous mapping $f : [0, 1] \to [0, 1]$ which is constant on the intervals deleted from the Cantor middle third set. This mapping is known as the Devil's staircase (see Fig. 2.2 right).

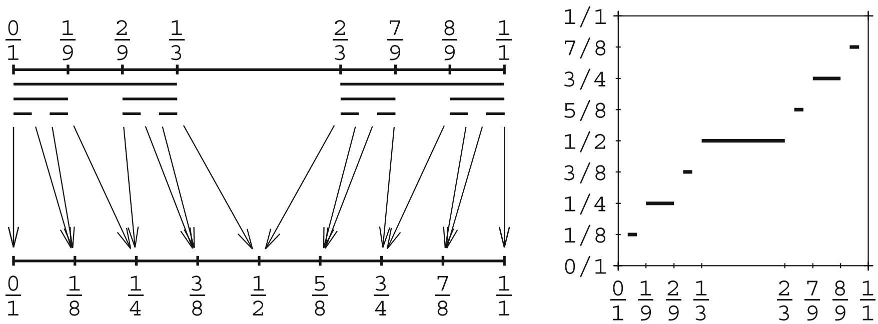

Fig. 2.2 The mapping $\Phi_2 \circ \Phi_3^{-1} : C \to [0, 1]$ (*left*) and the Devil's staircase (*right*)

Proposition 2.10 *If A is an alphabet, then A^ω is homeomorphic to* $\{0, 1\}^\omega$.

Proof For $A = \{0, 1, \ldots, k\}$, $k \geq 2$ define a mapping $\psi : A \to \{0, 1\}^+$ by

$$\psi(k) = 1^k,\ \ \psi(a) = 1^a 0 \text{ for } a < k.$$

Then ψ is a **prefix code**. This means that for every binary word $u \in A^+$ of length at least k there exists a unique $a \in A$ such that $f(a)$ is a prefix of u. This implies that every infinite binary word can be uniquely decomposed into the code words $f(u_i)$. Define $\Psi : A^\omega \to \{0, 1\}^\omega$ by $\Psi(u) = \psi(u_0)\psi(u_1)\ldots$ (concatenation). Then Ψ is bijective. If $d(x, y) \leq 2^{-n}$ then $d(\Psi(x), \Psi(y)) \leq 2^{-n}$ since the length of each $\psi(a)$ is at least 1. If $d(\Psi(x), \Psi(y)) \leq 2^{-kn}$ then $d(x, y) \leq 2^{-n}$ since the length if each $\psi(a)$ is at most k. Thus both Ψ and Ψ^{-1} are continuous, so Ψ is a homeomorphism. □

Proposition 2.11 *If A is an alphabet and $u \in A^*$, then the* **cylinder**

$$[u] = \{w \in A^\omega :\ w_{[0,|u|)} = u\}$$

of u is a clopen (closed and open) set.

Proof If $w \in [u]$ then $[u] = B_{2^{-n+1}}(w)$ is an open ball (whose center is any its element), so $[u]$ is an open set. The complement

$$A^\omega \setminus [u] = \bigcup \{[v] :\ v \in A^n \setminus \{u\}\}$$

is a union of open sets so it is open and therefore $[u]$ is closed. □

We characterize the power spaces A^ω by three topological properties.

Definition 2.12 1. A metric space X is **perfect** if it has no isolated points, i.e., if

$$\forall x \in X, \forall \varepsilon > 0, \exists y \in X, 0 < d(y, x) < \varepsilon.$$

2. A metric space X is **totally disconnected** if its points can be separated by clopen sets, i.e., if
$$x \neq y \ \Rightarrow\ \exists W \text{ clopen}, x \in W, y \in X \setminus W.$$

3. A metric space is a **Cantor space** if it is compact, perfect, and totally disconnected.

Theorem 2.13 *A metric space is a Cantor space iff it is homeomorphic to a power space A^ω.*

Proof 1. By Propositions 2.9 and 2.10, A^ω is homeomorphic to the Cantor middle third set which is compact by Proposition 2.3.

2. We show that A^ω is perfect: Since A has at least two elements, for each $w \in A^\omega$ there exists $z \in A^\omega$ such that $z_{[0,n)} = w_{[0,n)}$, $z_n \neq w_n$, so $d(w, z) = 2^{-n}$.
3. We show that A^ω is totally disconnected: For $w \neq z$ there exists n such that $w_n \neq z_n$, $w \in W = [w_{[0,n]}]$, $z \in A^\omega \setminus W$.
The converse proof that each Cantor space is homeomorphic to $\{0, 1\}^\omega$ can be found e.g., in Hocking and Young [1] or Kůrka [2].

We say that a metric space X is a **symbolic space** if it is homeomorphic to a closed subspace of A^ω. Symbolic spaces are compact and totally disconnected but not necessarily perfect. For example, every finite metric space is a symbolic space. Continuous mappings between symbolic spaces can be characterized combinatorially.

Proposition 2.14 *A mapping $F : A^\omega \to B^\omega$ between power spaces is continuous iff there exists a sequence of mappings $\{f_n : A^{k_n} \to B;\ n \geq 0\}$ such that $F(u)_n = f_n(u_{[0,k_n)})$.*

Proof Let F be continuous. By Proposition 2.8, F is uniformly continuous, so for every $\varepsilon = 2^{-n}$ there exists $\delta = 2^{-k_n+1}$ such that

$$\begin{aligned} d(u, v) < \delta \ &\Rightarrow\ d(F(u), F(v)) < \varepsilon, \\ u_{[0,k_n)} = v_{[0,k_n)} \ &\Rightarrow\ F(u)_{[0,n]} = F(v)_{[0,n]}. \end{aligned}$$

Thus $F(u)_n$ depends only on $u_{[0,k_n)}$ and this dependence defines f_n: For $v \in A^{k_n}$ we have $f(v) = a$ iff $F(u)_n = a$ for some $u \in [v]$. Conversely, if $F(u)_n = f_n(u_{[0,k_n)})$, then F is uniformly continuous. □

2.3 Redundant Symbolic Extensions

If we have a symbolic extension $\Phi : X \to \overline{\mathbb{R}}$, we want to perform arithmetical operations on symbolic representations of real numbers. A unary arithmetical operation like a linear function $g(x) = ax + b$ is a continuous mapping on $\overline{\mathbb{R}}$ (with $g(\infty) = \infty$). Its symbolic extension is a mapping $f : X \to X$ such that $g(\Phi(x)) = \Phi(f(x))$ for each $x \in X$. Symbolic extensions of continuous mappings exist provided Φ is redundant, i.e., provided the cylinder intervals $\Phi([u])$ overlap. The redundancy encountered in Sect. 1.2 is thus a topological concept.

Definition 2.15 We say that a continuous surjective mapping $\Phi : X \to Y$ is a **symbolic extension**, if X is a symbolic space. We say that a continuous mapping $\Phi : X \to Y$ is **redundant**, if for every continuous mapping $\Psi : X \to Y$ there exists a continuous mapping $F : X \to X$ such that $\Phi \circ F = \Psi$.

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr, "\Psi"'] & X \arrow[d, "\Phi"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\Phi"'] & X \arrow[d, "\Phi"] \\ Y \arrow[r, "G"] & Y \end{tikzcd} \qquad \begin{tikzcd} X^n \arrow[r, "F"] \arrow[d, "\Phi^n"'] & X \arrow[d, "\Phi"] \\ Y^n \arrow[r, "G"] & Y \end{tikzcd}$$

If $\Phi : X \to Y$ is a redundant symbolic extension, then continuous selfmappings of Y can be lifted to X. If $G : Y \to Y$ is a continuous mapping, then for $G \circ \Phi : X \to Y$ there exists a continuous mapping $F : X \to X$ such that $\Phi \circ F = G \circ \Phi$. We say that F is an **extension** of G by Φ. This can be generalized to mappings of several variables:

Proposition 2.16 *Let $\Phi : X \to Y$ be a redundant symbolic extension. Then for each continuous mapping $G : Y^n \to Y$ there exists a continuous mapping $F : X^n \to X$ such that $\Phi \circ F = G \circ \Phi^n$ (see the diagram).*

Proof If X is a Cantor space, then X^n is also a Cantor space and therefore it is homeomorphic to X. Let $H : X^n \to X$ be a homeomorphism. For a continuous mapping $G : Y^n \to Y$ we have a continuous mapping $g = G \circ \Phi^n \circ H^{-1} : X \to Y$, so there exists a continuous mapping $f : X \to X$ such that $\Phi \circ f = g$. For $F = f \circ H : X^n \to X$ we get $\Phi \circ F = \Phi \circ f \circ H = g \circ H = G \circ \Phi^n$. □

The redundancy implies surjectivity: If $\Phi : X \to Y$ is redundant and $y \in Y$, then for the constant mapping $\Psi : X \to Y$ given by $\Psi(x) = y$ there exists a mapping $F : X \to X$ with $\Phi \circ F = \Psi$, so for any $x \in X$, $\Phi(F(x)) = \Psi(x) = y$. Since the continuous image of a compact space is compact, only compact spaces can have symbolic extensions. In particular, the real line $\mathbb{R}$ has no symbolic extension.

Example 2.17 The binary value mapping $\Phi_{2,0,1} : \{0,1\}^\omega \to [0,1]$ defined by $\Phi_{2,0,1}(u) = \sum_{i \geq 0} u_i \cdot 2^{-i-1}$ is a symbolic extension which is not redundant.

Proof The mapping $\Phi = \Phi_{2,0,1}$ is clearly continuous and surjective. We show that it is not redundant. Let $c \in (0,1)$ be an irrational number and consider the mapping $g(x) = \frac{x}{2c}$ on $[0,1]$. Since c is irrational, there exists a unique $u \in \{0,1\}^\omega$ with $\Phi(u) = c$. Assume that $f : \{0,1\}^\omega \to \{0,1\}^\omega$ is an extension of g by Φ and denote by $a = f(u)_0 \in \{0,1\}$. Since f is continuous at u, there exists $n > 0$ such that $f([u_{[0,n)}]) \subseteq [a]$, so

$$g\Phi([u_{[0,n)}]) = \Phi(f([u_{[0,n)}])) \subseteq \Phi([a]).$$

However, c is an inner point of $\Phi([u_{[0,n)}])$ and $g(c) = \frac{1}{2}$, so $g\Phi([u_{[0,n)}])$ is included neither in $\Phi([0]) = [0, \frac{1}{2}]$ nor in $\Phi([1]) = [\frac{1}{2}, 1]$. This is a contradiction. □

If Y is a subspace of X then the closure and interior of a set $V \subseteq Y$ in Y usually differs from its closure and interior in X. The closure of V in Y is

$$\{y \in Y : \forall r > 0, B_r(y) \cap V \neq 0\} = \overline{V} \cap Y,$$

where $\overline{V}$ is the closure of V in X. For the interior of V in Y we get

$$\mathrm{int}_Y(V) = \{y \in Y : \exists r > 0, B_r(y) \cap Y \subseteq V\} = Y \setminus \overline{Y \setminus V}.$$

For example, $\mathrm{int}_{[0,2]}([0,1]) = [0,2] \setminus \overline{(1,2]} = [0,1)$: the point 0 is an inner point of $[0,1]$ regarded as a subspace of $[0,2]$.

Theorem 2.18 *Let $\Phi : A^\omega \to Y$ be a surjective and continuous mapping and assume that there exist integers $0 = n_0 < n_1 < n_2 < \cdots$ such that for every $k \geq 0$ and for every $u \in A^{n_k}$, $\{\mathrm{int}_{\Phi([u])}(\Phi([uv])) : uv \in A^{n_{k+1}}\}$ is a cover of $\Phi([u])$. Then Φ is redundant.*

Proof For each integer $k \geq 0$ there exists $\lambda_{k+1} > 0$ such that for each $u \in A^{n_k}$, the open cover $\{\mathrm{int}_{\Phi([u])}(\Phi([uv])) : uv \in A^{n_{k+1}}\}$ of $\Phi([u])$ has a Lebesgue number λ_{k+1}. If $\Psi : A^\omega \to Y$ is continuous, then it is uniformly continuous and for $k \geq 1$ there exists m_k such that

$$d(x, y) \leq 2^{-m_k} \Rightarrow d(\Psi(x), \Psi(y)) < \lambda_k.$$

We construct a sequence of mappings $\{f_k : A^{m_k} \to A^{n_k} : k \geq 0\}$ such that $\Psi([u]) \subseteq \Phi([f_k(u)])$ for $u \in A^{m_k}$. For $k = 0$ and $u \in A^{m_0}$ we have $u = \lambda$, so $\Phi([\lambda]) = \Phi(A^\omega) = Y$. Thus $\{\mathrm{int}(\Phi([u]) : u \in A^{n_1}\}$ is a cover of Y with a Lebesgue number λ_1. Given $u \in A^{m_1}$, choose a point $x \in [u]$. By the uniform continuity we have $\Psi([u]) \subseteq B_{\lambda_1}(\Psi(x))$. There exists $f_1(u) \in A^{n_1}$ such that $B_{\lambda_1}(\Psi(x)) \subseteq \mathrm{int}(\Phi([f_1(u)])) \subseteq \Phi([f_1(u)])$, so $\Psi([u]) \subseteq \Phi([f_1(u)])$. Assume by induction that we have constructed $f_k : A^{m_k} \to A^{n_k}$ such that $\Psi([u]) \subseteq \Phi([f_k(u)])$ for $u \in A^{m_k}$. Let $uv \in A^{m_{k+1}}$ and choose a point $x \in [uv]$, so $\Psi(x) \in \Psi([u]) \subseteq \Phi([f_k(u)])$. Since

$$\{\mathrm{int}_{\Phi([f_k(u)])}(\Phi([f_k(u)w])) : f_k(u)w \in A^{n_{k+1}}\}$$

is an open cover of $\Phi([f_k(u)])$, there exists $f_{k+1}(uv) \in A^{n_{k+1}}$ with prefix $f_k(u)$ such that

$$\Psi([uv]) \subseteq B_{\lambda_{k+1}}(\Psi(x)) \subseteq \Phi([f_{k+1}(uv)]).$$

For $u \in A^\omega$ we have a chain of prefixes $f_1(u_{[0,m_1)}) \sqsubseteq f_2(u_{[0,m_2)}) \sqsubseteq \cdots$, so there exists $F(u) \in A^\omega$ such that $F(u)_{[0,n_k)} = f_k(u_{[0,m_k)})$. Then $F : A^\omega \to A^\omega$ is continuous. For each $u \in A^\omega$ and for each k we have

$$\Psi(u) \in \Psi([u_{[0,m_k)}]) \subseteq \Phi([f_k(u_{[0,m_k)})]) = \Phi([F(u)_{[0,n_k)}]).$$

Since $\Phi(F(u)) \in \Phi([F(u)_{[0,n_k)}])$, we get $\Psi = \Phi \circ F$. □

Corollary 2.19 *If $\beta > 1$, and r, s are integers with $s - r > \beta - 1$, then the mapping $\Phi_{\beta,r,s} : [r, s]^\omega \to W_\lambda$ from Sect. 1.5 is redundant.*

Theorem 2.20 *If X is a Cantor space and Y is compact metric space, then there exists a symbolic redundant extension $\Phi : X \to Y$.*

Proof We can assume $X = \{0, 1\}^\omega$. There exists a finite open cover of Y of diameter at most 2^{-1}. Repeating some of the sets if necessary, we can assume that its number of elements is a power of 2. Thus there exists $n_1 > n_0 = 0$, and an open cover

$\mathcal{V}_1 = \{V_u : u \in \{0,1\}^{n_1}\}$ of Y of diameter at most 2^{-1}. We continue by induction. Assume that we have constructed an open cover

$$\mathcal{V}_k = \{V_u : u \in \{0,1\}^{n_k}\}$$

of Y of diameter at most 2^{-k}. There exists $n_{k+1} > n_k$, such that for each $u \in \{0,1\}^{n_k}$ there exists an open cover $\mathcal{W}(u)$ of $\overline{V_u}$ with diameter 2^{-k-1} which has $2^{n_{k+1}} - 2^{n_k}$ elements. We can index them so that $\mathcal{W}(u) = \{W_{uv} : uv \in \{0,1\}^{n_{k+1}}\}$. Set $V_{uv} = V_u \cap W_{uv}$. Then

$$\mathcal{V}_{k+1} = \{V_{uv} : uv \in \{0,1\}^{n_{k+1}}\}$$

is an open cover of Y with diameter at most 2^{-k-1}. If $u \in \{0,1\}^{n_k}$ and $v \in \{0,1\}^{n_{k+1}-n_k}$, then $V_{uv} \subseteq V_u$. For $u \in \{0,1\}^{\omega}$, $\bigcap_{k\ge 0} \overline{V_{u_{[0,n_k)}}} \neq \emptyset$ has zero diameter and therefore contains a unique element

$$\Phi(u) \in \bigcap_{k\ge 0} \overline{V_{u_{[0,n_k)}}}.$$

Then $\Phi : \{0,1\}^{\omega} \to Y$ is continuous and surjective. Clearly $\Phi([u]) \subseteq \overline{V_u}$ for each $u \in A^{n_k}$. To prove the opposite inclusion, let $x \in \overline{V_u}$. There exists $v \in \{0,1\}^{n_{k+1}-n_k}$ such that $x \in W_{uv}$. For each $r > 0$ we have $B_r(x) \cap V_v \cap W_{uv} \neq \emptyset$, so $x \in \overline{V_{uv}}$. Continuing in this way we construct an infinite word w with prefix u such that $x \in \overline{V_{w_{[0,n_l)}}}$ for each l, so $x = \Phi(w) \in \Phi([u])$. Thus $\Phi([u]) = \overline{V_u}$ for each $u \in \{0,1\}^{n_k}$. We show that for each $u \in \{0,1\}^{n_k}$,

$$\{\mathrm{int}_{\overline{V_u}}(\overline{V_{uv}}) : uv \in \{0,1\}^{n_{k+1}}\}$$

is a cover of $\overline{V_u}$. For $x \in \overline{V_u}$ there exists v such that $x \in W_{uv}$ and there exists $r > 0$ such that $B_r(x) \subseteq W_{uv}$. If $z \in B_r(x) \cap \overline{V_u}$, then for every $s > 0$ we have $B_s(z) \cap V_u \cap W_{uv} \neq \emptyset$, so $z \in \overline{V_u \cap W_{uv}} = \overline{V_{uv}}$. Thus $B_r(x) \cap \overline{V_u} \subseteq \overline{V_{uv}}$ and therefore $x \in \mathrm{int}_{\overline{V_u}}(\overline{V_{uv}})$. By Theorem 2.18, Φ is redundant. $\square$

Positional number systems for bounded intervals studied in Sect. 1.5 can be obtained from contractive iterative systems.

Definition 2.21 Let X be a metric space.

1. We say that a mapping $F : X \to X$ is **contracting** if there exists an increasing continuous function $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(0) = 0$, $\psi(t) < t$ for $t > 0$ and $\mathrm{diam}(F(V)) \le \psi(\mathrm{diam}(V))$ for every set $V \subseteq X$.
2. A **contracting iterative system** over an alphabet A is a pair (X, F), where X is a compact metric space and $F = \{F_a : X \to X : a \in A\}$ is a system of contractions indexed by the letters of A.

In particular, if there exists $0 < q < 1$ such that $d(F(x), F(y)) \le q \cdot d(x, y)$, then F is contracting. Any contracting mapping is continuous.

Theorem 2.22 *Let (X, F) be a contractive iterative system over A. There exists a continuous* **value mapping** $\Phi : A^\omega \to X$ *such that*
1. $\{\Phi(u)\} = \bigcap_{n>0} F_{u_{[0,n)}}(X)$ *for* $u \in A^\omega$.
2. $F_u(\Phi(v)) = \Phi(uv)$ *for* $u \in A^*$, $v \in A^\omega$.
3. *If* $u \in A^*$ *then* $\Phi([u]) \subseteq F_u(X)$.
4. $\Phi(u) = \lim_{n\to\infty} F_{u_{[0,n)}}(z)$ *for any* $z \in X$.
5. $\Phi : A^\omega \to X$ *is surjective iff* $\bigcup_{a\in A} F_a(X) = X$.
6. *If* $\Phi : A^\omega \to X$ *is surjective, then* $\Phi([u]) = F_u(X)$ *for each* $u \in A^*$.
7. *If every* F_a *is injective and* $X = \bigcup_{a\in A} \operatorname{int}(F_a(X))$*, then* $\Phi : A^\omega \to X$ *is redundant.*

Proof 1. Since $F_{u_{[0,n+1)}}(X) \subseteq F_{u_{[0,n)}}(X)$ are nonempty closed sets, their intersection is nonempty. By the assumption there exist real functions ψ_a such that $\operatorname{diam}(F_a(V)) \leq \psi_a(\operatorname{diam}(V)) < \operatorname{diam}(V)$ for every $V \subseteq X$. Set $\psi(t) = \max\{\psi_a(t) :\ a \in A\}$. Then

$$\begin{aligned}\operatorname{diam}(F_{u_{[0,n)}}(X)) &\leq \psi(\operatorname{diam}(F_{u_{[1,n)}}(X))) \leq \psi^2(\operatorname{diam}(F_{u_{[2,n)}}(X))) \leq \cdots \\ &\leq \psi^n(\operatorname{diam}(X)).\end{aligned}$$

Since $\lim_{n\to\infty} \psi^n(\operatorname{diam}(X)) = 0$, the intersection $\bigcap_{n>0} F_{u_{[0,n)}}(X)$ has zero diameter and contains a unique point which is by definition $\Phi(u)$.
2. Both $F_u(\Phi(v))$ and $\Phi(uv)$ belong to all $F_{uv_{[0,n)}}(X)$, so they are equal.
3. If $v \in A^\omega$ then $uv \in [u]$ and $\Phi(uv) = F_u(\Phi(v)) \in F_u(X)$, so $\Phi([u]) \subseteq F_u(X)$. Since $\operatorname{diam}(\Phi([u])) \leq \operatorname{diam}(F_u(X)) \leq \psi^{|u|}(\operatorname{diam}(X))$, $\Phi : A^\omega \to X$ is continuous.
4. Since $\Phi(u), F_{u_{[0,n)}}(z) \in F_{u_{[0,n)}}(X)$, we get $d(\Phi(u), F_{u_{[0,n)}}(z)) \leq \psi^n(\operatorname{diam}(X))$. It follows $\lim_{n\to\infty} F_{u_{[0,n)}}(z) = \Phi(u)$.
5. For each $u \in A^\omega$ we have $\Phi(u) \in F_{u_0}(X)$, so $\Phi(A^\omega) \subseteq \bigcup_{a\in A} F_a(X)$. If $\bigcup_{a\in A} F_a(X) \neq X$, then Φ is not surjective. Conversely, assume that $\bigcup_{a\in A} F_a(X) = X$. Then for every $u \in A^*$ we have

$$\bigcup_{a\in A} F_{ua}(X) = F_u\left(\bigcup_{a\in A} F_a(X)\right) = F_u(X).$$

Given $x \in X$, there exists u_0 such that $x \in F_{u_0}(X)$, there exists u_1 such that $x \in F_{u_{[0,1]}}(X)$ and by induction we construct $u \in A^\omega$ such that $x \in F_{u_{[0,n)}}(X)$ for each n, so $x = \Phi(u)$.
6. If $x \in F_u(X)$ then $x = F_u(y)$ for some $y \in X$ and there exists $v \in A^\omega$ with $y = \Phi(v)$, so $x = \Phi(uv)$ and $x \in \Phi([u])$.
7. Since $\Phi([u]) = F_u(X)$, by Theorem 2.18 it suffices to show that

$$\{\operatorname{int}_{F_u(X)}(F_{ua}(X)) :\ a \in A\}$$

is a cover of $F_u(X)$ for each $u \in A^*$. Let $x \in F_u(X)$, so $x = F_u(y)$ for some $y \in X$. By the assumption there exists $a \in A$ and $\varepsilon > 0$ such that $B_\varepsilon(y) \subseteq F_a(X)$. Since $F_u^{-1} : F_u(X) \to X$ is a homeomorphism, there exists $\delta > 0$ such that $F_u^{-1}(B_\delta(x) \cap F_u(X)) \subseteq B_\varepsilon(y) \subseteq F_a(X)$, so $B_\delta(x) \cap F_u(X) \subseteq F_{ua}(X)$ and $x \in \mathrm{int}_{F_u(X)}(F_{ua}(X))$. □

Thus the mapping $\Phi_{\beta,r,s} : [r,s]^\omega \to W_\lambda$ from Sect. 1.5 is the value mapping of the contractive iterative system with alphabet $A = [r,s]$ and transformations $F_a(x) = \frac{x+a}{\beta}$.

2.4 Subshifts

The value mappings of number systems need not be defined on the whole power space A^ω but only on some its subset. Subshifts are subsets of the power space defined by finite forbidden words (see e.g., Lind and Marcus [3] or Kůrka [2]).

Definition 2.23 For an alphabet A and a set $D \subseteq A^*$ of **forbidden words**, denote by

$$\Sigma_D = \{u \in A^\omega : \forall v \in D : v \not\sqsubseteq u\}.$$

We say that a nonempty set $\Sigma \subseteq A^\omega$ is a **subshift**, if $\Sigma = \Sigma_D$ for some $D \subseteq A^*$. If $D \subset A^*$ is a finite set then we say that Σ_D is a **subshift of finite type** (SFT). The **order** of a SFT Σ is the smallest $p \geq 2$ such that there exists $D \subseteq A^p$ with $\Sigma = \Sigma_D$.

To forbid a word $u \in A^*$ is equivalent to forbidding words ua for all $a \in A$. Thus any SFT has an order. For example the SFT $\Sigma_{\{00,111\}} = \Sigma_{\{000,001,111\}}$ in $A = \{0,1\}$ has order 3. Some examples of SFT of order 2 in the alphabet $A = \{0,1\}$ are

$$\begin{aligned}
\Sigma_{\{00,11\}} &= \{(01)^\omega, (10)^\omega\},\\
\Sigma_{\{10\}} &= \{0^n 1^\omega : n \geq 0\} \cup \{0^\omega\},\\
\Sigma_{\{11\}} &= \{0, 10\}^\omega.
\end{aligned}$$

The subshift $\Sigma_{\{00,11\}}$ is finite, $\Sigma_{\{10\}}$ is countable and $\Sigma_{\{11\}}$ is uncountable: any concatenation of 10 with 0 belongs to $\Sigma_{\{11\}}$. An example of a subshift which is not of finite type is the **soliton subshift** of words which contain at most one occurrence of 1. Its forbidden set is $D = \{10^n 1 : n \geq 0\}$.

The **shift map** $\sigma : A^\omega \to A^\omega$ is defined by $\sigma(u)_i = u_{i+1}$. Thus $\sigma(u) = u_1 u_2 \ldots$ is obtained from u by forgetting the first letter u_0. The shift map is continuous since $d(\sigma(u), \sigma(v)) \leq 2d(u,v)$.

Proposition 2.24 *A nonempty set $\Sigma \subseteq A^\omega$ is a subshift iff it is closed and shift-invariant, i.e., if $\sigma(w) \in \Sigma$ whenever $w \in \Sigma$.*

Proof If forbidden words do not occur in w then they do not occur in $\sigma(w)$, so Σ_D is shift-invariant. To show that Σ_D is closed, we show that its complement is open. If $u \in A^\omega \setminus \Sigma_D$, then for some $i < j$, $u_{[i,j)} \in D$, and no $w \in A^\omega$ with $w_{[0,j)} = u_{[0,j)}$

belongs to Σ_D, so $[u_{[0,j)}] \subseteq A^\omega \setminus \Sigma_D$. This means that $A^\omega \setminus \Sigma_D$ is open and therefore Σ_D is closed. Conversely assume that $\Sigma \subseteq A^\omega$ is closed and shift-invariant and set

$$D = \{v \in A^* : \forall u \in \Sigma, v \not\sqsubseteq u\}.$$

If $u \in \Sigma$ and $v \in D$ then $v \not\sqsubseteq u$, so $u \in \Sigma_D$. Thus we have proved $\Sigma \subseteq \Sigma_D$. If $u \in A^\omega \setminus \Sigma$, then, since $A^\omega \setminus \Sigma$ is open, there exists $v = u_{[0,n)}$ such that $[v] \subseteq A^\omega \setminus \Sigma$. Assume by contradiction that v occurs in some $w \in \Sigma$, so $v = w_{[i,i+n)}$. Then $\sigma^i(w) \in \Sigma$, but $\sigma^i(w) \in [v] \subseteq A^\omega \setminus \Sigma$ and this is a contradiction. It follows that $v \in D$ and therefore $u \in A^\omega \setminus \Sigma_D$. Thus we have shown $A^\omega \setminus \Sigma \subseteq A^\omega \setminus \Sigma_D$, so $\Sigma_D \subseteq \Sigma$. □

Definition 2.25 The **language** of a subshift $\Sigma \subseteq A^\omega$ is the set of finite words which occur as subwords of infinite words of Σ:

$$\mathcal{L}(\Sigma) = \{u \in A^* : \exists x \in \Sigma, u \sqsubseteq x\}.$$

We denote by $\mathcal{L}^n(\Sigma) = \mathcal{L}(\Sigma) \cap A^n$. If $\Sigma = \Sigma_D$ then we denote by $\mathcal{L}_D = \mathcal{L}(\Sigma_D)$, $\mathcal{L}_D^n = \mathcal{L}_D \cap A^n$.

Some examples are

$$\begin{aligned}
\mathcal{L}_{\{00,11\}} &= \{\lambda, 0, 1, 01, 10, 010, 101, 0101, 1010, \ldots\},\\
\mathcal{L}_{\{10\}} &= \{\lambda, 0, 1, 00, 01, 11, 000, 001, 011, 111, \ldots\},\\
\mathcal{L}_{\{11\}} &= \{\lambda, 0, 1, 00, 01, 10, 000, 001, 010, 100, 101, \ldots\}.
\end{aligned}$$

Definition 2.26 A nonempty set $L \subseteq A^*$ is an **extendable language**, if
1. for any $u \in L$ and for any $v \sqsubseteq u$ we have $v \in L$,
2. for any $u \in L$ there exists $a \in A$ such that $ua \in L$.

The subshift of an extendable language $L \subseteq A^*$ is

$$\mathcal{S}(L) = \{x \in A^\omega : \forall n \geq 0, x_{[0,n)} \in L\}.$$

Proposition 2.27 *1. If L is an extendable language then $\mathcal{S}(L)$ is a subshift and $\mathcal{L}(\mathcal{S}(L)) = L$.*
2. If Σ is a subshift then $\mathcal{L}(\Sigma)$ is an extendable language and $\mathcal{S}(\mathcal{L}(\Sigma)) = \Sigma$.

Proof 1. Let $L \subseteq A^*$ be an extendable language. For $n > 0$ set

$$X_n = \{x \in A^\omega : x_{[0,n)} \in L\},$$

so $\mathcal{S}(L) = \bigcap_{n>0} X_n$. Since L contains words of any length, X_n is nonempty. Since X_n is a finite union of cylinders, it is closed. Since $X_{n+1} \subseteq X_n$, their intersection $\mathcal{S}(L)$ is nonempty and closed. Clearly, $\mathcal{S}(L)$ is invariant, so it is a subshift. We show $\mathcal{L}(\mathcal{S}(L)) = L$. If $u \in L$, $|u| = n$, then there exists $u_n \in A$ such that $u_{[0,n]} \in L$.

Repeating this infinitely many times we extend u to a point $x \in A^\omega$ with prefix u such that for any m, $x_{[0,m)} \in L$. Thus $x \in \mathcal{S}(L)$ and $u \in \mathcal{L}(\mathcal{S}(L))$, so $L \subseteq \mathcal{L}(\mathcal{S}(L))$. Conversely, if $u \in \mathcal{L}(\mathcal{S}(L))$, then there exists $x \in \mathcal{S}(L)$ with $u = x_{[i,j)}$ for some $i < j$. Since $x_{[0,j)} \in L$ and u is its subword, $u \in L$. Thus $\mathcal{L}(\mathcal{S}(L)) \subseteq L$.
2. Let $\Sigma \subseteq A^\omega$ be a subshift. If $v \sqsubseteq u \in \mathcal{L}(\Sigma)$, then $u \sqsubseteq x$ for some $x \in \Sigma$ and therefore $v \sqsubseteq x$, so $v \in \mathcal{L}(\Sigma)$. If $u = x_{[i,i+|u|)}$, then $ux_{i+|u|} \sqsubseteq x$, so $ux_{i+|u|} \in \mathcal{L}(\Sigma)$. Thus we have proved that $\mathcal{L}(\Sigma)$ is an extendable language. We show $\mathcal{S}(\mathcal{L}(\Sigma)) = \Sigma$. If $x \in \Sigma$, then for any n, $x_{[0,n)} \in \mathcal{L}(\Sigma)$, so $x \in \mathcal{S}(\mathcal{L}(\Sigma))$. Thus $\Sigma \subseteq \mathcal{S}(\mathcal{L}(\Sigma))$. Suppose that $x \in \mathcal{S}(\mathcal{L}(\Sigma))$ and $x \notin \Sigma$. Since $A^\omega \setminus \Sigma$ is open, there exists n such that $[x_{[0,n)}] \subseteq A^\omega \setminus \Sigma$. Since $x \in \mathcal{S}(\mathcal{L}(\Sigma))$, $x_{[0,n)} \in \mathcal{L}(\Sigma)$ and there exists $y \in \Sigma$ such that $y_{[j,j+n)} = x_{[0,n)}$. Thus $\sigma^j(y) \in [x_{[0,n)}]$ and this is a contradiction. Thus $\mathcal{S}(\mathcal{L}(\Sigma)) \subseteq \Sigma$. □

If Σ is a subshift and $u \in \mathcal{L}(\Sigma)$ then we denote by

$$[u]_\Sigma = [u] \cap \Sigma = \{w \in \Sigma : \ w_{[0,|u|)} = u\}.$$

For a fixed subshift Σ we often drop the index and write $[u]$ instead of $[u]_\Sigma$. We often consider symbolic extensions $\Phi : \Sigma \to \overline{\mathbb{R}}$ and in this case we have a generalization of the redundancy test whose proof is the same as that of Theorem 2.18.

Theorem 2.28 *Let $\Sigma \subseteq A^\omega$ be a subshift and $\Phi : \Sigma \to Y$ a surjective continuous mapping such that for each $u \in \mathcal{L}(\Sigma)$,*

$$\{\mathrm{int}_{\Phi([u])}(\Phi([ua])) : \ ua \in \mathcal{L}(\Sigma)\}$$

is a cover of $\Phi([u])$. Then Φ is redundant.

2.5 Sofic Subshifts

When we work with a subshift, we want to know whether an infinite word belongs to the subshift or not. Since we can work only with finite prefixes of infinite words, we need a device which reads successively the letters of an infinite word and stops (or signals an error) when the word read does not belong to the language of the subshift. In the case of an SFT (and in a more general class of sofic subshifts) such a test can be performed by a **finite automaton**. A finite automaton is a device with a finite set B of inner states. When the automaton reads a letter $a \in A$, it changes its inner state according to a mapping $\delta_a : B \to B$. The change of state upon reading a word $u \in A^2$ is $\delta_u(p) = \delta_{u_1}(\delta_{u_0}(q))$, so $\delta_{u_0u_1} = \delta_{u_1} \circ \delta_{u_0}$. For $u \in A^n$ we get analogously $\delta_u = \delta_{u_{n-1}} \circ \cdots \circ \delta_{u_0}$. If we set $\delta_\lambda = \mathrm{Id}_B$, then $\delta_{uv} = \delta_v \circ \delta_u$. Thus $\delta_a : B \to B$ form an iterative systems, but in contrast to iterative systems of Sect. 2.3, the mappings are composed in the reverse order. We assume that the automaton has an initial state $\mathbf{i} \in B$ and a set of final (accepting) states $F \subseteq B$. A word $u \in A^*$ is accepted if $\delta_u(\mathbf{i}) \in F$. We say that $L \subseteq A^*$ is a **regular language**, if there exists a finite automaton $(B, \delta, \mathbf{i}, F)$ such that $u \in L$ iff $\delta_u(\mathbf{i}) \in F$.

If L is an extendable language and $\delta_u(\mathbf{i}) \in F$, then $\delta_v(\mathbf{i}) \in F$ for each prefix v of u: A word can be accepted only if all its prefixes have been accepted. This property leads to a simplification of the automaton since the rejecting states in $B \setminus F$ are not needed. We can remove them and leave $\delta_a(p)$ undefined whenever $\delta_a(p) \in B \setminus F$. Thus we get partial mappings $\delta_a : B \to B$ and we write $\exists \delta_a(p)$ when δ_a is defined at p. The compositions $\delta_u : B \to B$ are also partial mappings which are defined on $p \in B$ provided all δ_{u_i} are defined on $\delta_{u_{[0,i)}}(p)$.

Definition 2.29 An accepting automaton over an alphabet A is a triple $\mathcal{A} = (B, \delta, \mathbf{i})$, where B is a finite set of states, $\delta = \{\delta_a : B \to B : \ a \in A\}$ are partial mappings and $\mathbf{i} \in B$ is an initial state. The language and subshift accepted by $\mathcal{A}$ are

$$\mathcal{L}_{\mathcal{A}} = \{u \in A^* : \ \exists \delta_u(\mathbf{i})\},$$
$$\Sigma_{\mathcal{A}} = \{u \in A^\omega : \ \forall n, \exists \delta_{u_{[0,n)}}(\mathbf{i})\}.$$

A subshift $\Sigma \subseteq A^\omega$ is **sofic** iff $\mathcal{L}(\Sigma)$ is a regular language iff there exists an accepting automaton $\mathcal{A}$ such that $\Sigma = \Sigma_{\mathcal{A}}$ and $\mathcal{L}(\Sigma) = \mathcal{L}_{\mathcal{A}}$.

We represent accepting automata by oriented labelled graphs whose vertices are states of B and whose edges $p \xrightarrow{a} \delta_a(p)$ are labelled by letters of A. The initial state is enclosed in a circle. The SFT

$$\Sigma_{\{10\}} = \{0^n 1^\omega : \ n \geq 0\} \cup \{0^\omega\}$$

has an accepting automaton with $B = \{\lambda, 1\}$, $\mathbf{i} = \lambda\, \delta_0(\lambda) = \lambda$, $\delta_1(\lambda) = 1$, $\delta_1(1) = 1$, while $\delta_0(1)$ is undefined (Fig. 2.3 left). The SFT

$$\Sigma_{\{00,11\}} = \{(01)^\omega, (10)^\omega\}$$

has an accepting automaton with $B = \{\lambda, 0, 1\}$, initial state $\mathbf{i} = \lambda$, and transition function $\delta_a(\lambda) = a$, $\delta_a(a) = 1 - a$ for $a \in \{0, 1\}$ (Fig. 2.3 right).

We give examples of sofic subshifts which are not SFT. The **soliton subshift** in the binary alphabet $A = \{0, 1\}$ consists of words which contain at most one occurrence of the letter 1, so its forbidden set is

$$D = \{10^n 1 : \ n \geq 0\}.$$

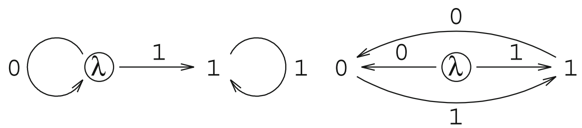

Fig. 2.3 Accepting automata for SFT $\Sigma_{\{10\}}$ (*left*) and $\Sigma_{\{00,11\}}$ (*right*)

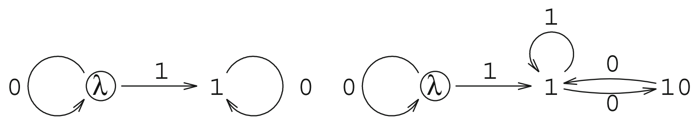

Fig. 2.4 Accepting automata for sofic subshifts: the soliton subshift (*left*) and the even subshift (*right*)

Its language is accepted by the automaton with states $B = \{\lambda, 1\}$, initial state λ and transition function $\delta_0(\lambda) = \lambda$, $\delta_1(\lambda) = 1$, $\delta_0(1) = 1$ (see Fig. 2.4 left). The **even subshift** in the binary alphabet $A = \{0, 1\}$ consists of words which do not contain an odd number of zeros between two ones, so its forbidden set is

$$D = \{10^{2n+1}1 : \; n \geq 0\}.$$

Its language is accepted by the automaton with states $B = \{\lambda, 1, 10\}$, initial state λ and transition function $\delta_0(\lambda) = \lambda$, $\delta_1(\lambda) = 1$, $\delta_1(1) = 1$, $\delta_0(1) = 10$, $\delta_0(10) = 1$ (see Fig. 2.4 right).

Definition 2.30 Given a subshift $\Sigma \subseteq A^\omega$, the **follower set** of $u \in A^*$ is

$$\mathcal{F}_u = \{v \in A^\omega : \; uv \in \Sigma\}.$$

Given an accepting automaton $\mathcal{A} = (B, \delta, \mathbf{i})$, the **follower set** of $p \in B$ is

$$\mathcal{F}_p = \{v \in A^\omega : \; \forall n, \exists \delta_{v_{[0,n)}}(p)\}.$$

Clearly $\mathcal{F}_u, \mathcal{F}_p \subseteq A^\omega$ are closed sets and $\mathcal{F}_u \neq \emptyset$ iff $u \in \mathcal{L}(\Sigma)$. For the empty word we have $\mathcal{F}_\lambda = \Sigma$. For the initial state $\mathbf{i} \in V$ we have $\mathcal{F}_{\mathbf{i}} = \Sigma_{\mathcal{A}}$. For the subshift $\Sigma_{\{11\}}$ there are just two follower sets: for each word $u \in \{0, 1\}^*$ we get $\mathcal{F}_{u0} = \Sigma$, $\mathcal{F}_{u1} = \{0u : \; u \in \Sigma\}$. For the soliton subshift we have also two follower sets: $\mathcal{F}_u = \Sigma$ provided $1 \not\sqsubseteq u$ and $\mathcal{F}_u = \{0^\omega\}$ otherwise.

Proposition 2.31 *If $u, v \in A^*$ and $\mathcal{F}_u = \mathcal{F}_v$, then $\mathcal{F}_{ua} = \mathcal{F}_{va}$ for each $a \in A$.*

Proof If $w \in \mathcal{F}_{ua}$ then $uaw \in \mathcal{L}(\Sigma)$, so $aw \in \mathcal{F}_u$, $aw \in \mathcal{F}_v$, and $w \in \mathcal{F}_{va}$. □

Theorem 2.32 *Σ is a sofic subshift iff the set $\{\mathcal{F}_u : \; u \in A^*\}$ of its follower sets is finite. In particular, every SFT is sofic.*

Proof If $\Sigma = \Sigma_{\mathcal{A}}$ with $\mathcal{A} = (B, \delta, \mathbf{i})$ and $u \in \mathcal{L}(\Sigma)$, then $\mathcal{F}_u = \mathcal{F}_p$ where $p = \delta_u(\mathbf{i}) \in B$. Since B is a finite set, $\{\mathcal{F}_u : \; u \in \mathcal{L}(\Sigma)\}$ is finite too. Conversely assume that $B = \{\mathcal{F}_u : \; u \in \mathcal{L}(\Sigma)\}$ is a finite set. We construct an accepting automaton $\mathcal{A} = (B, \delta, \mathcal{F}_\lambda)$ with initial state $\mathbf{i} = \mathcal{F}_\lambda = \Sigma$. Define the transition function by $\delta_a(\mathcal{F}_u) = \mathcal{F}_{ua}$ provided $ua \in \mathcal{L}(\Sigma)$, otherwise $\delta_a(\mathcal{F}_u)$ is undefined. By Proposition 2.31, this definition is correct. If $u \in \Sigma$ then $\delta_{u_{[0,n)}}(\mathcal{F}_\lambda) = \mathcal{F}_{u_{[0,n)}}$, so

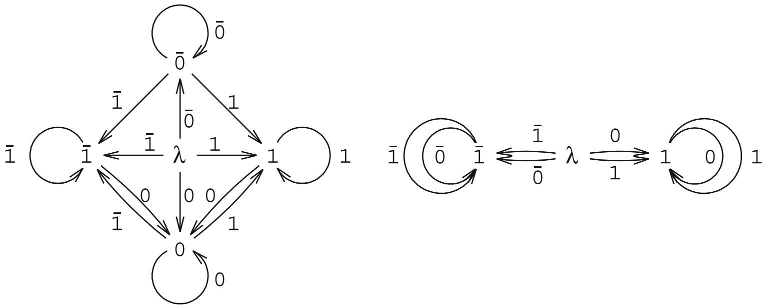

Fig. 2.5 Accepting automata for the binary signed subshift with forbidden set $D = \{\overline{1}0, \overline{1}1, 0\overline{0}, 1\overline{0}, 1\overline{1}, \overline{0}0\}$ (*left*) and symmetric continued fraction subshift $\Sigma = \{\overline{1}, \overline{0}\}^\omega \cup \{0, 1\}^\omega$ (*right*)

$u \in \Sigma_{\mathcal{A}}$. Conversely, if $\exists \delta_{u_{[0,n)}}(\mathcal{F}_\lambda)$ for each n, then $u_{[0,n)} \in \mathcal{L}(\Sigma)$, so $u \in \Sigma$. If Σ is a SFT of order $p \geq 2$, then $\mathcal{F}_{uv} = \mathcal{F}_v$ for every $u \in A^*$ and $v \in A^{p-1}$. Thus $\{\mathcal{F}_v : v \in A^{p-1}\}$ is the set of all follower sets. □

The construction of an accepting automaton is particularly simple for subshifts of finite type. If Σ is a SFT of order p we take

$$B = \{u \in \mathcal{L}(\Sigma) : |u| < p\}.$$

For $u \in B$, $a \in A$, let $\delta(u, a)$ be the longest suffix of ua which belongs to B. The initial state is $\mathbf{i} = \lambda$. Then $\mathcal{A} = (B, \delta, \mathbf{i})$ is an accepting automaton for Σ. However, this automaton need not be the minimal one since the follower sets of words of B need not be all different. This can be tested by a simple criterion: $\mathcal{F}_u = \mathcal{F}_v$ iff for all $w \in A^*$ with $|w| < p$, $uw \in \mathcal{L}(\Sigma)$ iff $vw \in \mathcal{L}(\Sigma)$. For example for the subshift $\{\overline{1}, \overline{0}\}^\omega \cup \{0, 1\}^\omega$ of the system of symmetric continued fractions from Definition 1.17 we get $B = \{\lambda, \overline{1}, 0, 1, \overline{0}\}$. However, $\mathcal{F}_{\overline{1}} = \mathcal{F}_{\overline{0}} = \{\overline{1}, \overline{0}\}^\omega$, $\mathcal{F}_0 = \mathcal{F}_1 = \{0, 1\}^\omega$, so there exists an accepting automaton with states $B = \{\lambda, \overline{1}, 1\}$ (Fig. 2.5 right).

2.6 Labelled Graphs

Let $\mathcal{A} = (B, \delta, \mathbf{i})$ be an accepting automaton. We say that a state $p \in B$ is reachable, if $\delta_u(\mathbf{i}) = p$ for some u. In an accepting computation, only the reachable states appear, so we can remove all nonreachable states without changing the accepted language. An accepting computation may start at any reachable state.

Proposition 2.33 *Let $\mathcal{A} = (B, \delta, \mathbf{i})$ be an accepting automaton whose every state is reachable. Then for each $u \in A^*$ we have $\exists \delta_u(\mathbf{i})$ iff $\exists p \in B, \exists \delta_u(p)$.*

Proof If $\delta_u(p) = q$, and $\delta_v(\mathbf{i}) = p$, then $\delta_{vu}(\mathbf{i}) = q$ so $vu \in \mathcal{L}(\Sigma)$ and $u \in \mathcal{L}(\Sigma)$. □

In an accepting automaton whose all states are reachable, the initial state need not be distinguished, since an accepting process can start at any state of B. The automaton is thus reduced to a partial iterative system $\delta_a : B \to B$. The accepted language of δ is $\mathcal{L}_\delta = \{u \in A^* : \exists p, \exists \delta_u(p)\}$. Since the computation may start at any state, we say that such an automaton is **nondeterministic**. A nondeterministic automaton may have fewer states that the deterministic one. For example if we remove from the accepting automaton of the even shift the initial state λ, we get a nondeterminsitic automaton which accepts the same language (see Fig. 2.4 right). Its states are $B = \{1, 10\}$, and the transition function is given by $\delta_1(1) = 1$, $\delta_0(1) = 10$, $\delta_0(10) = 1$. Conversely, a language accepted by a nondeterministic finite automaton is accepted also by a deterministic automaton (Theorem 2.36). However, the number of states of a deterministic automaton may be much (exponentially) larger. A nondeterministic finite automaton can be equivalently described by a finite labelled graph.

Definition 2.34 1. A **labelled graph** over an alphabet A is a pair $G = (B, E)$, where B is a finite set of vertices and $E \subseteq B \times A \times B$ is a set of labelled edges.
2. The source and target maps $s, t : E \to B$ are the projections $s(p, a, q) = p$, $t(p, a, q) = q$. We assume that $\forall p \in B, \exists e \in E, s(e) = p$. The labelling map $\ell : E \to A$ is the projection $\ell(p, a, q) = a$.
3. The **edge subshift** of G is

$$\Sigma_{|G|} = \{u \in E^\omega : \forall i \geq 0, t(u_i) = s(u_{i+1})\} \subseteq E^\omega.$$

3. The **subshift** of G is $\Sigma_G = \{\ell(u) : u \in \Sigma_{|G|}\} \subseteq A^\omega$.
4. The **language** of G is $\mathcal{L}_G = \mathcal{L}(\Sigma_G)$.

Note that $\Sigma_{|G|}$ is a SFT of order 2. A path is a finite or infinite word $u \in E^* \cup E^\omega$ such that $t(u_i) = s(u_{i+1})$. A finite path is equivalently described by a pair $(p, u) \in B^* \times A^*$ such that $|p| = |u| + 1$ and $(p_i, u_i, p_{i+1}) \in E$ for all $i < |u|$. An infinite path is a pair $(p, u) \in B^\omega \times A^\omega \approx (B \times A)^\omega$ such that $(p_i, u_i, p_{i+1}) \in E$ for all i. Thus the edge subshift may be equivalently defined as a subset of $(B \times A)^\omega$. The labelling map ℓ can be extended to the continuous mapping $\ell : E^\omega \to A^\omega$ defined by $\ell(u)_i = \ell(u_i)$. It follows that $\Sigma_G = \ell(\Sigma_{|G|})$ is compact and therefore it is a closed subset of A^ω. Since Σ_G is also shift-invariant, it is a subshift. Thus we have

Proposition 2.35 *If $\Sigma \subseteq A^\omega$ is a sofic subshift, then there exists a labelled graph G such that $\Sigma = \Sigma_G$.*

Proof Given an accepting automaton $\mathcal{A} = (B, \delta, \mathbf{i})$, we construct the labelled graph $G = (B_0, E)$, where $B_0 = \{\delta_u(\mathbf{i}) : u \in A^*\}$ is the set of reachable states and $E = \{(p, a, q) \in B_0 \times A \times B_0 : \delta_a(p) = q\}$. $\square$

Proposition 2.36 *Any subshift of any labelled graph is sofic.*

Proof Let $G = (B, E)$ be a labelled graph, let $Q = \mathcal{P}(B)\setminus\{\emptyset\}$ be the set of nonempty subsets of B. Define transition functions $\delta_a : Q \to Q$ by

$$\delta_a(M) = \{q \in Q : \exists e \in E, s(e) \in M, t(e) = q, \ell(e) = a\}$$

provided $\delta_a(M)$ is not empty, otherwise $\delta_a(M)$ is undefined. The initial state is $B \in Q$. We show that (Q, δ, B) accepts $\mathcal{L}(\Sigma_G)$. If

$$q_0 \xrightarrow{u_0} q_1 \cdots \xrightarrow{u_{n-2}} q_{n-1} \xrightarrow{u_{n-1}} q_n$$

is a path in G, then $q_n \in \delta_u(V)$, so $\delta_u(V) \neq \emptyset$ and u is accepted. Conversely, if $\delta_u(V) \neq \emptyset$, then pick some $q_n \in \delta_u(V)$. There exists $q_{n-1} \in \delta_{u_{[0,n-2]}}(V)$ such that $q_{n-1} \xrightarrow{u_{n-1}} q_n$ is a labelled edge in G. Continuing backwards, we obtain a path in G with label u. □

Definition 2.37 A **morphism** from a subshift $\Sigma \subseteq A^\omega$ to a subshift $\Theta \subseteq B^\omega$ is a continuous mapping $F : \Sigma \to \Theta$ such that for every $u \in \Sigma$, $\sigma(F(u)) = F(\sigma(u))$. If F is surjective, we say that Θ is a **factor** of Σ.

$$\begin{tikzcd}
\Sigma \arrow[r, "\sigma"] \arrow[d, "F"'] & \Sigma \arrow[d, "F"] \\
\Theta \arrow[r, "\sigma"] & \Theta
\end{tikzcd}$$

Proposition 2.38 *Any morphism $F : \Sigma \to \Theta \subseteq B^\omega$ is a* **sliding block code**. *This means that there exists $r \geq 0$ and a* **local rule** *$f : \mathcal{L}^r(\Sigma) \to B$ such that $F(x)_i = f(x_{[i,i+r)})$ for every $x \in \Sigma$.*

Proof Since F is uniformly continuous, for $\varepsilon = 1$ there exists $\delta > 0$ such that if $d(x, y) < \delta$, then $d(F(x), F(y)) < 1$. Take $r > 0$ with $2^{-r} < \delta$. Then

$$\begin{aligned} x_{[0,r)} = y_{[0,r)} &\Rightarrow d(x, y) \leq 2^{-r} < \delta \Rightarrow d(F(x), F(y)) < 1 \\ &\Rightarrow F(x)_0 = F(y)_0. \end{aligned}$$

Thus $F(x)_0$ depends only on the first r letters of x, and there exists a local rule $f : \mathcal{L}^r(\Sigma) \to B$ such that $f(x_{[0,r)}) = F(x)_0$. Since F is a morphism, we get $F(x)_n = \sigma^n(F(x))_0 = F(\sigma^n(x))_0 = f(\sigma^n(x)_{[0,r)}) = f(x_{[n,n+r)})$. □

Theorem 2.39 (Weiss [4]) *A subshift is sofic iff it is a factor of an SFT.*

Proof If Σ is sofic, then $\Sigma = \Sigma_G$ for some labelled graph G and $\ell : (\Sigma_{|G|}, \sigma) \to (\Sigma_G, \sigma)$ is a factor map with SFT $\Sigma_{|G|}$.
Conversely, let $F : (\Sigma, \sigma) \to (\Theta, \sigma)$ be a factor map, $\Sigma \subseteq A^\omega$ an SFT and $\Theta =$

$F(\Theta) \subseteq B^{\mathbb{N}}$. Let p be the order of Σ, so $u \in \Sigma$ iff $u_{[i,i+p)} \in \mathcal{L}(\Sigma)$ for all i. By Proposition 2.38, there exists a local rule $f : \mathcal{L}^r(\Sigma) \to B$ such that $F(x)_i = f(x_{[i,i+r)})$. We can assume $r \geq p$. Define a labelled graph $G = (V, E)$, where $V = \mathcal{L}^{r-1}(\Sigma)$,

$$E = \{(au, f(aub), ub) \in V \times B \times V : a, b \in A, aub \in \mathcal{L}^r(\Sigma)\}$$

We show that $\Sigma_G = \Theta$. If $v \in \Theta$ then there exists $u \in \Sigma$ such that $v = F(u)$ and we have a path

$$u_{[0,r-1)} \overset{f(u_{[0,r)})}{\longrightarrow} u_{[1,r)} \overset{f(u_{[1,r+1)})}{\longrightarrow} u_{[2,r+1)} \cdots$$

with label v, so $v \in \Sigma_G$. Conversely, if $v \in \Sigma_G$ then we have a path in (V, E) with label v. Then $u_{[i,i+r-1)} \in \mathcal{L}(\Sigma)$, so $u \in \Sigma$ and $v = F(u) \in \Theta$. Thus $\Sigma_G = \Theta$. ◻

Definition 2.40 1. We say that a graph $G = (B, E)$ is **right-resolving** if (p, a, q), $(p, b, r) \in E$ and $a = b$ implies $q = r$, i.e., if the edges with the same source carry different labels.
2. We say that $G = (B, E, \mathbf{i})$ is an **initialized graph**, if (B, E) is a graph, $\mathbf{i} \in B$, $\mathcal{F}_{\mathbf{i}} = \Sigma_G$, there is no edge with target $\mathbf{i}$ and for each $p \in B \setminus \{\mathbf{i}\}$ there exists a path $\mathbf{i} \overset{u}{\longrightarrow} p$.
3. We say that $G = (B, E, \mathbf{i})$ is a **deterministic graph**, if it is initialized and right-resolving.

For any graph G there exists an initialized graph with the same language. We just add to G a new vertex $\mathbf{i}$ and for any edge $p \overset{a}{\longrightarrow} q$ we add a new edge $\mathbf{i} \overset{a}{\longrightarrow} q$. Alternatively, if we allow edges with label λ, we may add edges $\mathbf{i} \overset{\lambda}{\longrightarrow} p$ for each vertex p of G. The deterministic graphs are exactly graphs of accepting automata, so each sofic subshift is a subshift of a deterministic graph. If G is a deterministic graph then there exists a continuous mapping $\nu : \Sigma_G \to \Sigma_{|G|}$ such that $\ell(\nu(u)) = u$ for each $u \in \Sigma_G$. For $u \in \Sigma_G$, $\nu(u)$ is the unique path with source $\mathbf{i}$ and label u. Note that ν is continuous but does not commute with the shift map, so it is not a morphism.

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