Preface

Current computers represent real numbers in the floating point format and the numbers are rounded up in each arithmetical operation. This usually works quite well but there are cases in which successive roundings yield wrong results. Exact arithmetical algorithms, on the other hand, work with real numbers specified to an arbitrary precision. The precision of the result depends on the precision of the operands. The theory of exact real computation is based on the concept of on-line algorithms whose inputs and outputs are infinite expansions of real numbers. The algorithms work in a loop in infinite time but each finite prefix of the output is computed in finite time from finite prefixes of the inputs.

Exact real algorithms work in general redundant Möbius number systems whose digits represent Möbius transformations of the form $M(x) = \frac{ax+b}{cx+d}$. For example, in a positional number system with base $\beta > 1$, a digit a represents the transformation $F_a(x) = \frac{x+a}{\beta}$. A Möbius number system specifies a subshift $\Sigma \subseteq A^\omega$ of admissible infinite sequences of digits and a value mapping $\Phi : \Sigma \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is the extended real line. The value mapping Φ should be surjective and continuous.

The computation of an exact real algorithm is a dynamic process. During the computation the algorithm updates its inner state which consists of matrices with integer entries. The algorithm repeatedly performs two kinds of actions: the absorption of a digit from an input word and the emission of a digit to the output word. In both kinds of actions, the inner state is updated by the transformations associated to the input or output digits.

The theory of on-line algorithms has been developed by Weihrauch [1]. The idea of on-line arithmetical algorithms with arbitrary precision has been suggested in an unpublished manuscript of Gosper [2] and developed by Kornerup and Matula [3] and Vuillemin [4]. On-line arithmetical algorithms are treated in the Ph.D. thesis of Potts [5] and in the last chapter of the monograph of Kornerup and Matula [6]. Exact real algorithms work in redundant systems, for example in positional number systems whose number of digits is larger than the base.

Chapter 1 of the book is introductory and treats classical positional number systems and number systems based on continued fractions. The correspondence between digits and transformations is explained.

Chapter 2 treats redundancy as a topological concept and shows that there exist redundant continuous surjective mappings $\Phi : \Sigma \to \overline{\mathbb{R}}$, where $\Sigma \subseteq A^{\omega}$ is a symbolic space of sequences of digits. In arithmetical algorithms, the symbolic space Σ is supposed to be a sofic subshift recognizable by a finite automaton, so the rest of the chapter deals with sofic subshifts.

Chapter 3 explains basic ideas of projective geometry which gives insight into the spaces connected with the number systems. The extended real line $\overline{\mathbb{R}}$ is identified with the one-dimensional projective space and the space of Möbius transformations is identified with the three-dimensional space of projective matrices. Geometrical properties of Möbius transformations are described in the context of hyperbolic geometry. Then we explain the concept of representation of a real number by a sequence of transformations.

Chapter 4 exposes the theory of Möbius number systems and shows several methods how to construct suitable subshifts $\Sigma \subseteq A^{\omega}$ and suitable value mappings $\Phi : \Sigma \to \overline{\mathbb{R}}$. A special treatment is given to sofic Möbius number systems for which arithmetical algorithms work.

Chapter 5 develops the calculus of bilinear tensors which represent binary arithmetical operations. Intervals are represented by projective matrices and operations with tensors and intervals are based on matrix calculus. The unary algorithm computes a Möbius transformation and the binary algorithm computes binary arithmetical operations like addition, multiplication or division.

Chapter 6 treats number systems whose matrices have integer entries. In particular, modular systems have transformations with unit determinant. In modular number systems, the unary algorithm can be computed by a finite state transducer.

Chapter 7 treats number systems with matrices whose entries are algebraic numbers. We review the theory of algebraic extension fields, algebraic integers and integral bases. Then we give classical results of Parry and Schmidt on positional number systems with algebraic bases $\beta > 1$ (so called β-systems introduced by Rényi [7]). Then we treat the algorithms of parallel addition in positional number systems.

Chapter 8 describes algorithms which compute transcendent functions like $e^x, \ln x, \tan x$ or $\arctan x$. We review the theory of Padé approximants and the representation of transcendent functions by general continued fractions. Computation of transcendent functions is based on algebraic tensors $T(x, y)$ which are rational functions in x for each fixed y and Möbius transformations in y for each fixed x.

The book is intended to be a source of valuable and useful information on the topics of dynamics of number systems and scientific computation with arbitrary precision. It is addressed to scholars, scientists and engineers, graduate students and all readers interested in this important area of science which is relevant both for theory and applications. The treatment is elementary and self-contained. The basic prerequisite is linear algebra and matrix calculus.

The book outgrew from a course I gave in 2013 to Ph.D. students and in 2015 to undergraduate students at the Faculty of Nuclear Sciences and Physical Engineering of the Czech Technical University in Prague. I thank all students who read parts of the book in various phases of its preparation, in particular Tomáš Hejda and Tomáš Vávra.

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