Chapter 2
Introduction to Part I

In Part I of this work we study a model for mass transport where Fick's law is satisfied. Fick's law is the analogue for mass of Fourier's law for heat conduction. Fourier's law, see [1], specifies the amount of heat flux in a metal bar when we heat it from one side and cool it from the other. Its analogue for mass fluxes is Fick's law, formally described by the same equation. Since the transversal direction to the flow is not relevant we model our system as one dimensional. The ideal experiment of mass transport that we have in mind is the following: for $t \geq 0$ the system occupies a time varying space interval $[0, X_t]$, where X_t is a given positive, continuous and piecewise C^1 function; for instance we move the edge X_t with constant velocity for some time, then we change velocity and so on. We act on the system by injecting mass from its left boundary 0 at rate $j > 0$ while we remove mass from the right boundary X_t in such a way as to keep the mass density at X_t equal to 0 for all $t \geq 0$. The evolution of the mass density $\rho(r, t)$ in the interior of the spatial domain is ruled by combining the continuity equation and Fick's law, so that, supposing a constant conductivity (set equal to 1/2), we have

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial r}\,, \qquad J = -\frac{1}{2}\frac{\partial \rho}{\partial r} \tag{2.0.1}$$

where $J(r, t)$ is the local mass-flux and $\rho(r, t)$ the mass density. Thus $\rho(r, t)$ solves the heat equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2} \tag{2.0.2}$$

in the time varying domain $[0, X_t]$ with some initial condition $\rho(r, 0) = \rho_0(r)$ and boundary conditions

$$J(0, t) = j, \quad \rho(X_t, t) = 0\,. \tag{2.0.3}$$

Physically these boundary conditions mean that the system is in contact with a *current reservoir* which sends in mass at rate j and thus imposes a current j at the origin; instead at the other endpoint X_t there is a *density reservoir* which removes

G. Carinci et al., *Free Boundary Problems in PDEs and Particle Systems*,
SpringerBriefs in Mathematical Physics 12, DOI 10.1007/978-3-319-33370-0_2

mass as fast as needed to fix the mass density to be constantly equal to zero. As a consequence, in this setting, the total mass of the system is not a conserved quantity.

The main question we want to study here arises when we require mass conservation at all times. To achieve this, one needs to regard X_t as a control parameter and one is lead to study the following *control problem*:

> Is it possible to choose X_t in such a way that the total mass in the system is constant ?

We clearly succeed if we can solve the free boundary problem (FBP) given by (2.0.2) with initial datum $\rho(r, 0) = \rho_0(r)$, $r \in [0, X_0]$, and

$$-\frac{1}{2}\frac{\partial \rho}{\partial r}(0, t) = j, \qquad -\frac{1}{2}\frac{\partial \rho}{\partial r}(X_t, t) = j, \qquad \rho(X_t, t) = 0\ . \tag{2.0.4}$$

In fact the rate at which mass is taken out of the system from X_t is

$$J(X_t, t) = -\frac{1}{2}\frac{\partial \rho}{\partial r}(X_t, t)$$

which, by (2.0.4), is exactly equal to the rate at which we inject mass at 0 so that the total mass is constant.

As discussed in the next chapter (see Sect. 3.3) we can find in the existing literature on FBP an affirmative answer for special initial data and for finite times. In fact one can readily check (see Sect. 3.3 for details) that the current $J(r, t)$ solves the classical Stefan problem for which the theory (in particular in one dimension) is very rich with many detailed results available [2–6]. As a consequence local existence and uniqueness of classical solutions can be proved for the FBP defined by (2.0.2) and (2.0.4) for smooth initial data which satisfy the boundary conditions. In some cases the classical solution is global extending to all times, but this is not true in general as it is known that singularities may develop.

Thus our control problem when stated for an arbitrarily long time interval $[0, T]$ and for general initial data cannot always be solved via the above FBP. Take for instance $\rho_0 \in L^1(\mathbb{R}_+)$, bounded, continuous and everywhere strictly positive: in such a case the whole problem has to be redefined. As usual the idea is to study a *relaxed* version: we thus introduce an accuracy parameter $\epsilon > 0$ and replace ρ_0 by a nice function $\rho_0^{(\epsilon)}$, smooth, non-negative and with compact support, requiring however that $\int |\rho_0(r) - \rho_0^{(\epsilon)}(r)|\, dr \le \epsilon$. We may also ask that $\rho_0^{(\epsilon)}$ satisfies (2.0.4) so that, for what said above, we have a classical solution of FBP for some time $[0, S]$. However this could be shorter than the interval $[0, T]$ we have fixed initially, in which case the problem still remains. Moreover even if $S \ge T$ we have a poor control of the solution and it is hard to see how this behaves when we remove the relaxation taking $\epsilon \to 0$. The idea then is to further simplify the problem by relaxing also the boundary condition at the edge. We refer to the next chapter for a precise definition of suitably relaxed solutions. Here we just say that in Part I we will prove that any ϵ-relaxed solution converges to a unique limit when $\epsilon \to 0$. This will allow us to define a notion of relaxed solution of the problem which is global in time and applies to a large class of initial data.

In the last chapter of Part I we study a particles version of the above basic model. The system has N particles so that the mass distribution is no longer continuous but instead concentrated on points (the positions of the N particles). To simulate an initial condition $\rho_0(r)$ (we assume $\int \rho_0(r)dr = 1$ for simplicity), we distribute the N particles independently of each other and with law $\rho_0(r)dr$. We then define the "empirical mass density measure"

$$\pi_0^{(N)}(dr) = \frac{1}{N}\sum_{i=1}^{N} \delta_{B_i(0)}(r)dr \tag{2.0.5}$$

where $B_i(0)$ are the random positions of the N particles and $\delta_a(r)$ is the Dirac delta at a. The value 0 refers to time, so far we have been describing the situation at time 0. Thus $\pi_0^{(N)}(dr)$ is a probability measure on $\mathbb{R}$ which is random as the terms $B_i(0)$ are the random positions of the particles. If we denote by E the expectation with respect to the law of the $B_i(0)$ and by $f(r)$ a test function, we have

$$E\Big[\int \pi_0^{(N)}(dr) f(r)\Big] = \int \rho_0(r) f(r)dr. \tag{2.0.6}$$

By the law of large number if N is large we do not need to take the expectation because, with large probability, $\int \pi_0^{(N)}(dr)f(r)$ is close to $\int \rho_0(r)f(r)dr$.

Let us now make the particles move. We first consider the free case where the particles are independent Brownian motions $B_i(t)$ on $\mathbb{R}_+$ with reflection at 0. Call $\pi_t^{(N)}(dr)$ the random mass distribution at time t and denote now by E the joint law of the initial distribution of the particles and of their Brownian evolution. We then have

$$E\Big[\int \pi_t^{(N)}(dr) f(r)\Big] = \int \rho(r,t) f(r)dr \tag{2.0.7}$$

where $\rho(r,t)$ is the solution of (2.0.2) on $\mathbb{R}_+$ with Neumann boundary condition at 0 given by $\frac{\partial \rho}{\partial r}(0,t) = 0$. All that is the well known relation between heat equation and Brownian motions.

We next go to the injection-removal of mass mechanism. This is simply done as follows: at exponential times of intensity jN the rightmost particle moves to the origin (which is the same as saying that we add a new Brownian particle at 0 and simultaneously we take out the particle which at that time is the rightmost one). In between such actions the particles move as independent Brownian motions (with reflection at the origin). We denote again by E the expectation with respect to the law of this process (which includes the initial distribution of the particles, their motion and the injection-removal of particles). Thus the total mass (i.e. the total number of particles) is conserved but as in the continuum we are injecting mass at 0 and removing mass on the right. Such a simple action however creates strong correlations among the particles: the choice of the rightmost particle requires knowledge of the positions of all the others. We thus lose the independency property and the analysis of the left-hand side of (2.0.7) in this case becomes highly non-trivial. Existence of the process is easy but the relation with the continuum version is harder. The question becomes simpler if we study the asymptotic behavior of the system as $N \to \infty$, namely its "hydrodynamic limit". We would like that:

$$\lim_{N\to\infty} E\Big[\int \pi_t^{(N)}(dr)f(r)\Big] = \int \rho(r,t)f(r)dr \tag{2.0.8}$$

where $\rho(r, t)$ is the solution of the control problem described previously and in particular of the FBP when this has a classical solution. In Chap. 11 we prove (2.0.8).

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http://www.springer.com/978-3-319-33369-4

Free Boundary Problems in PDEs and Particle Systems
Carinci, G.; De Masi, A.; Giardina, C.; Presutti, E.
2016, VII, 110 p. 2 illus., 1 illus. in color., Softcover
ISBN: 978-3-319-33369-4