Chapter 2
Main Results and Strategy of Proof

Abstract In the first part of this chapter we gather the main results which follow from the analysis developed in this book. To start with, in Section 2.1, we discuss an example, in Theorem 2.1.1, of the leading large-N asymptotic expansion of $\ln \mathfrak{z}_N[W]$ where $\mathfrak{z}_N[W]$ is the unscaled partition function defined by (1.5.27). We shall also argue that the large-N asymptotic behaviour of (1.5.27)—whose integrand does not depend explicitly on N—can be deduced from the one of the rescaled model (2.5.1)—whose integrand depends explicitly on N—that we propose to study. Then, after presenting the *per se* model of interest and listing the assumptions on which our analysis builds in Section 2.2, we shall discuss the form of the large-N asymptotic expansion of the logarithm of the rescaled partition function $\ln Z_N[V]$ in Section 2.3. Then, in Section 2.4, we shall discuss the characterisation of the N-dependent equilibrium measure that is pertinent for our study as well as the form of the inverse $\mathcal{W}_N$ of a fundamental singular integral operator $\mathcal{S}_N$ that arises naturally in the study. Finally, Section 2.5, we outline the main steps of the proof.

2.1 A Baby Integral as a Motivation

Let $\mathcal{E}_{(\mathrm{ply})}$ be the functional, defined in $\mathbb{R} \cup \{+\infty\}$ for any probability measure $\mu \in \mathcal{M}^1(\mathbb{R})$ by:

$$\mathcal{E}_{(\mathrm{ply})}[\mu] \;=\; \int \Big\{ \frac{c_q}{2} \Big(|\xi|^q + |\eta|^q \Big) \;-\; \frac{\beta\pi(\omega_1+\omega_2)}{2} |\xi-\eta| \Big\} \, \mathrm{d}\mu(\xi) \mathrm{d}\mu(\eta) \; . \tag{2.1.1}$$

Theorem 2.1.1 *$\mathcal{E}_{(\mathrm{ply})}$ is a lower semi-continuous good rate function. Furthermore, given a potential W such that*

$$\lim_{|\xi| \to +\infty} |\xi|^{-q} \, W(\xi) \;=\; c_q > 0 \quad \textit{for some} \quad q > 1 \; , \tag{2.1.2}$$

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it holds

$$\lim_{N\to+\infty} \frac{\ln \mathfrak{z}_N[W]}{N^{2+\frac{1}{q-1}}} \;=\; -\inf_{\mu\in\mathcal{M}^1(\mathbb{R})} \mathcal{E}_{(\mathrm{ply})}\big[\mu\big]\,. \tag{2.1.3}$$

This infimum is attained at a unique probability measure $\mu_{\mathrm{eq}}^{(\mathrm{ply})}$. *This measure is continuous with respect to the Lebesgue measure and has density*

$$\rho_{\mathrm{eq}}^{(\mathrm{ply})}(\xi) \;=\; \frac{q(q-1)|\xi|^{q-2}}{2\pi\beta(\omega_1+\omega_2)} \cdot \mathbf{1}_{[a\,;b]}(\xi)\,. \tag{2.1.4}$$

$\mu_{\mathrm{eq}}^{(\mathrm{ply})}$ *is supported on the interval* $[a\,;b]$, *with* (a,b) *being the unique solution to the set of equations*

$$|b|^{q-1} \;=\; |a|^{q-1} \;=\; \frac{\pi\beta(\omega_1+\omega_2)}{q}\,. \tag{2.1.5}$$

We have, explicitly:

$$\lim_{N\to+\infty} \frac{\ln \mathfrak{z}_N[W]}{N^{2+\frac{1}{q-1}}} \;=\; \big(c_q\big)^{\frac{1}{q}} \cdot \left(\frac{\pi\beta}{q}(\omega_1+\omega_2)\right)^{\frac{q+1}{q}} \cdot \frac{2q^2-9q+6}{2(2q-1)}\,. \tag{2.1.6}$$

The proof of this proposition is postponed to Appendix B, and follows similar steps to, *e.g.* [1]. We now provide heuristic arguments to justify the occurrence of scaling in N in this problem. Just as discussed in the introduction, the repulsive effect of the sinh-2 body interactions will dominate over the confining effect of the potential as long as the integration variables will be located in some bounded set. Furthermore, in the same situation, the Lebesgue measure should contribute to the integral at most as an exponential in N. We thus look for a rescaling of the variables $y_a = T_N\lambda_a$ where the effects of the confining potential and the sinh-2 body interactions will be of the same order of magnitude in N. This recasts the partition function as

$$\begin{aligned}\mathfrak{z}_N[W] \;=\; \Big(T_N\Big)^N \int\limits_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh\big[\pi\omega_1 T_N(\lambda_a-\lambda_b)\big] \sinh\big[\pi\omega_2 T_N(\lambda_a-\lambda_b)\big]\Big\}^{\beta} \\ \times \prod_{a=1}^{N} \Big\{ \mathrm{e}^{-W(T_N\lambda_a)}\Big\}\, \mathrm{d}^N\boldsymbol{\lambda}\,,\end{aligned} \tag{2.1.7}$$

Taking into account the large-variable asymptotics of the potential, we have:

$$\sum_{a=1}^{N} W\big(T_N\lambda_a\big) \;\sim\; T_N^q\, N\,, \tag{2.1.8}$$

where the symbol $\sim$ means that for a "typical" distribution of the variables $\{\lambda_a\}_1^N$, the leading in N asymptotic behaviour of the sum in the right-hand side should be

of the order of the left-hand side. Similarly, assuming a typical distribution of the variables $\{\lambda_a\}_1^N$ such that most of the pairs $\{\lambda_a, \lambda_b\}$ satisfy $T_N|\lambda_a - \lambda_b| \gg 1$, one has

$$\sum_{a<b}^{N} \beta \ln \Big\{ \sinh\big[\pi\omega_1 T_N(\lambda_a - \lambda_b)\big] \sinh\big[\pi\omega_2 T_N(\lambda_a - \lambda_b)\big] \Big\} \ \sim \ C\, N^2\, T_N \ . \tag{2.1.9}$$

Thus, the confining potential and the two-body interaction will generate a comparable order of magnitude in N as soon as $N^2 \cdot T_N \ = \ T_N^q \cdot N$, *i.e.*

$$T_N \ = \ N^{\frac{1}{q-1}} \ . \tag{2.1.10}$$

Theorem 2.1.1 indeed justifies that the empirical distribution $L_N^{(\boldsymbol{\lambda})}$ of $\lambda_a \ = \ N^{\frac{-1}{q-1}} y_a$ concentrates around the equilibrium measure, with a large deviation principle governed by the rate function (2.1.1).

This observation implies that, in fact, $Z_N[V_N]$ with $V_N(\lambda) = N^{-\frac{q}{q-1}} \cdot W(N^{\frac{1}{q-1}}\lambda)$ is the good object to study in that it involves interactions that are already tuned to the proper scale in N. Due to the relation $\mathfrak{Z}_N[W] = N^{\frac{N}{q-1}} \cdot Z_N[V_N]$, one readily has access to the large-N asymptotic expansion of $\mathfrak{Z}_N[W]$.

2.2 The Model of Interest and the Assumptions

It follows from the arguments given in the previous section that, effectively, the analysis of the unrescaled model boils down to the one subordinated to the partition function

$$Z_N[V] \ = \ \int\limits_{\mathbb{R}^N} \prod_{a<b}^{N} \Big\{ \sinh\big[\pi\omega_1 N^{\alpha}(\lambda_a - \lambda_b)\big] \sinh\big[\pi\omega_2 N^{\alpha}(\lambda_a - \lambda_b)\big] \Big\}^{\beta} \prod_{a=1}^{N} \mathrm{e}^{-N^{1+\alpha} V(\lambda_a)} \cdot \mathrm{d}^N\boldsymbol{\lambda} \ , \tag{2.2.1}$$

with α some parameter—equal to $1/(q-1)$ in the previous paragraph—and V a potential that possibly depends on N. Due to such an effective reduction, in this book, we shall develop the general formalism to extract the large-N asymptotic behaviour. Therefore, we shall keep the complexity at minimum. In particular, we shall *not* consider the case of N-dependent potentials which would put the analysis of $Z_N[V]$ in complete correspondence with the one of $\mathfrak{Z}_N[W]$. Indeed, this would lead to numerous technical complication in our arguments, without bringing more light on the underlying phenomena. By focusing on (2.2.1), we believe that the new features and ideas of our methods are better isolated and illustrated.

In the present paper we obtain the large-N asymptotic expansion of $\ln Z_N[V]$ up to $o(1)$ under four hypothesis

- the potential V is confining, *viz.* there exists $\epsilon > 0$ such that

$$\liminf_{|\xi|\to+\infty} |\xi|^{-(1+\epsilon)}\, V(\xi) \;=\; +\infty \;; \tag{2.2.2}$$

- the potential V is smooth and strictly convex on $\mathbb{R}$;
- the potential is sub-exponential, namely there exists $\epsilon > 0$ and $C_V > 0$ such that

$$\forall \xi \in \mathbb{R}, \qquad \sup_{\eta\in[0\,;\epsilon]} \big|V'(\xi+\eta)\big| \;\le\; C_V\big(|V(\xi)|+1\big)\;, \tag{2.2.3}$$

  and given any $\kappa > 0$ and $p \in \mathbb{N}$, there exists $C_{\kappa,p}$ such that

$$\forall \xi \in \mathbb{R}, \qquad \big|V^{(p)}(\xi)\big|\, \mathrm{e}^{-\kappa V(\xi)} \;\le\; C_{\kappa,p}\;. \tag{2.2.4}$$

- the exponent α in N^α is neither too large nor too small:

$$0 < \alpha < 1/6\;. \tag{2.2.5}$$

The first hypothesis guarantees that the integral (2.2.1) is well-defined, and that the λ's will typically remain in a compact region of $\mathbb{R}$ independent of N. It could be weakened to study weakly confining potentials, for the price of introducing more technicalities, similar to those already encountered for β ensembles—see *e.g.* [2].

In the second assumption, V could be assumed $\mathcal{C}^k$ for k large enough. The convexity assumption guarantees that the support of the equilibrium measure is a single segment.[1] In principle, the multi-cut regime that may arise when the potential is not strictly convex can be addressed by importing the ideas of [3] to the present framework. We expect that the analysis of the Riemann–Hilbert problem in the multi-cut regime is very similar to the present case, but with a larger range of degrees for the polynomial freedom appearing in the solution (4.3.14). Though it would certainly represent some amount of work, the ideas we develop here should also be applicable to derive the fine large N analysis of the solution of the Riemann–Hilbert problem in the bulk and in the vicinity of all the edges of the support of the equilibrium measure.

The third assumption is not essential, but allows some simplification of the intermediate proofs concerning the equilibrium measure and the large deviation estimates, *e.g.* Theorem 3.1.6 and Corollary 3.1.10. It is anyway satisfied in physically relevant problems.

[1] See *e.g.* the expression of the $N = \infty$ equilibrium measure (2.4.3). Its proof is given in Appendix C.

In the fourth assumption, $\alpha = 0$ can already be addressed with existing methods [4]. The analysis of the transition regime $\alpha \to 0$ appears to be difficult. The upper limit $\alpha < \alpha^* = 1/6$ has a purely technical origin. The value of α^* could be increased by entering deeper into the fine structure of the analysis of the Schwinger–Dyson equation, and by finding more precise local and global bounds for the large N behaviour of the inverse of the master operator $\mathcal{U}_N^{-1}$, in more cunning norms. Intuitively, the genuine upper limit should be $\alpha^* = 1$, since in the $\alpha > 1$ case, we reach a regime where the particles do not feel the local repulsion any more. However, obtaining microscopic estimates is usually a difficult question—for β ensembles, it has been addressed *e.g.* in [5, 6]. So, one can expect important technical difficulties to extend our result to values of α increasing up to 1.

This set of hypothesis offers a convenient framework for our purposes, enabling us to focus on the technical aspects (i)–(vi) listed in Section 1.5.3 without adding extra complications.

2.3 Asymptotic Expansion of $Z_N[V]$ at $\beta = 1$

We now state one of the main results of the paper, namely the large-N asymptotic expansion of the partition function $Z_N[V]$ which holds for any potential V satisfying the hypothesis stated above

Theorem 2.3.1 *The below asymptotic expansion holds*

$$\begin{aligned}\ln\left(\frac{Z_N[V]}{Z_N[W_{G;N}]}\right)_{|\beta=1} &= -N^{2+\alpha}\sum_{p=0}^{\lfloor 2/\alpha\rfloor+1}\frac{\mathcal{O}_p[V]}{N^{\alpha p}} + N^{\alpha}\cdot\mathfrak{I}_0\cdot\Big(\wp[V,W_{G;N}](b_N)-\wp[V,W_{G;N}](a_N)\Big)\\ &+ \aleph_0\cdot\Big(\wp[V,W_{G;N}]'(b_N)+\wp[V,W_{G;N}]'(a_N)\Big) + \mathrm{o}(1).\end{aligned} \tag{2.3.1}$$

The whole V-dependence of this expansion is encoded in the coefficients $\mathcal{O}_p[V]$ and in the function $\wp[V, W_{G;N}](\xi)$. $\mathfrak{I}_0$ and $\aleph_0$ are numerical coefficients given, respectively, in terms of a single and four-fold integral. Also, the answer involves the Gaussian potential

$$W_{G;N}(\xi) = \frac{\pi\beta(\omega_1+\omega_2)\cdot\left[\xi^2-\big(a_N+b_N\big)\xi\right]}{b_N-a_N+\frac{1}{N^{\alpha}}\sum_{p=1}^{2}\frac{1}{\pi\omega_p}\ln\left(\frac{\omega_1\omega_2}{\omega_p(\omega_1+\omega_2)}\right)} \tag{2.3.2}$$

and sequences a_N and b_N that are given in Theorem 2.4.3. If we denote $V_N^{\pm} = V \pm W_{G;N}$, the coefficients $\mathcal{O}_p[V]$ take the form

$$\mathcal{O}_0[V] \;=\; \frac{-1}{4\pi(\omega_1+\omega_2)} \int_{a_N}^{b_N} V_N^-(\xi)\cdot (V_N^-)''(\xi)\,\mathrm{d}\xi \tag{2.3.3}$$

when $p = 0$ and, for any $p \geq 1$:

$$\begin{aligned}\mathcal{O}_p[V] = u_{p+1} &\int_{a_N}^{b_N} V_N^-(\xi)\cdot V^{(p+2)}(\xi)\,\mathrm{d}\xi \\ &+ \sum_{\substack{s+\ell=p-1\\ s,\ell\geq 0}} \frac{\beth_{s,\ell}}{s!}\Big\{(-1)^{\ell}\,(V_N^-)^{(\ell+1)}(a_N)\cdot (V_N^+)^{(s+1)}(a_N) \\ &+ (-1)^{s}\,(V_N^-)^{(\ell+1)}(b_N)\cdot (V_N^+)^{(s+1)}(b_N)\Big\}\,. \end{aligned}\tag{2.3.4}$$

The coefficients $\beth_{s,\ell}$ are defined by:

$$\beth_{s,\ell} \;=\; \frac{\mathrm{i}^{s+\ell+1}}{2\pi}\sum_{r=1}^{\ell+1}\frac{s!}{r!(s+\ell+1-r)!}\cdot\frac{\partial^r}{\partial\mu^r}\Big(\frac{\mu}{R_{\downarrow}(-\mu)}\Big)_{|\mu=0}\cdot\frac{\partial^{s+1+\ell-r}}{\partial\mu^{s+1+\ell-r}}\Big(\frac{1}{R_{\downarrow}(\mu)}\Big)_{|\mu=0}\,, \tag{2.3.5}$$

$R_{\downarrow}$ is the $\mathbb{H}^-$ Wiener–Hopf factor of $1/\mathcal{F}[S](\lambda)$, with S defined in (2.4.15), that reads

$$R_{\downarrow}(\lambda) \;=\; \frac{\lambda}{2\pi\sqrt{\omega_1+\omega_2}}\cdot\Big(\frac{\omega_2}{\omega_1+\omega_2}\Big)^{-\frac{\mathrm{i}\lambda}{2\pi\omega_1}}\cdot\Big(\frac{\omega_1}{\omega_1+\omega_2}\Big)^{-\frac{\mathrm{i}\lambda}{2\pi\omega_2}}\cdot\frac{\Gamma\Big(\frac{\mathrm{i}\lambda}{2\pi\omega_1}\Big)\cdot\Gamma\Big(\frac{\mathrm{i}\lambda}{2\pi\omega_2}\Big)}{\Gamma\Big(\frac{\mathrm{i}\lambda(\omega_1+\omega_2)}{2\pi\omega_1\omega_2}\Big)}\,. \tag{2.3.6}$$

The function $\wr$ describing the constant term is defined as

$$\wr[V, W_{G;N}](\xi) \;=\; \frac{V'(\xi)-W'_{G;N}(\xi)}{V''(\xi)-W''_{G;N}(\xi)}\,\ln\Big(\frac{V''(\xi)}{W''_{G;N}(\xi)}\Big). \tag{2.3.7}$$

The V-independent coefficient $\beth_0$ in front of the term N^{α} reads

$$\beth_0 = \int_0^{+\infty}\frac{\mathrm{d}u\;J(u)}{2\pi}\big(uS'(u)\;+\;S(u)\big)$$

with

$$J(u) = \int_{\mathscr{C}_{\mathrm{reg}}^{(+)}}\frac{2\sinh\big[\lambda/(2\omega_1)\big]\sinh\big[\lambda/(2\omega_2)\big]}{\sinh\big[\lambda(\omega_1+\omega_2)/(2\omega_1\omega_2)\big]}\cdot\frac{\mathrm{e}^{\mathrm{i}\lambda u}\,\mathrm{d}\lambda}{2\mathrm{i}\pi}\,. \tag{2.3.8}$$

Finally, the numerical prefactor $\aleph_0$ is expressed in terms of the four-fold integral

$$\begin{aligned}\aleph_0 = &- \frac{\omega_1+\omega_2}{2}\int_{\mathbb{R}}\frac{\mathrm{d}u\, J(u)}{2\pi}\int_{|u|}^{+\infty}\mathrm{d}v\,\partial_u\Big\{S(u)\cdot\Big(\mathfrak{r}\big[\frac{v-u}{2}\big]-\mathfrak{r}\big[\frac{v+u}{2}\big]\Big)\Big\}\\ &+\int_{\mathscr{C}_{\mathrm{reg}}^{(+)}}\frac{\mathrm{d}\lambda\,\mathrm{d}\mu}{(2\mathrm{i}\pi)^2}\,\frac{\mu(\omega_1+\omega_2)}{(\lambda+\mu)R_\downarrow(\lambda)R_\downarrow(\mu)}\\ &\times\int_0^{+\infty}\mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{\mathrm{i}\lambda x+\mathrm{i}\mu y}\,\partial_x\Big\{S(x-y)\Big(\mathfrak{r}(x)-\mathfrak{r}(y)-\frac{x-y}{2\pi(\omega_1+\omega_2)}\Big)\Big\}\,.\end{aligned} \tag{2.3.9}$$

The integrand of $\aleph_0$ involves the function $\mathfrak{r}$ which is given by

$$\mathfrak{r}(x) \;=\; \frac{\mathfrak{c}_1(x)+\mathfrak{c}_0(x)\Big[\sum_{p=1}^{2}\frac{1}{2\pi\omega_p}\ln\Big(\frac{\omega_1\omega_2}{\omega_p(\omega_1+\omega_2)}\Big)\Big]}{1+2\pi\beta(\omega_1+\omega_2)\mathfrak{c}_0(x)} \tag{2.3.10}$$

with

$$\mathfrak{c}_p(x) \;=\; \frac{\mathrm{i}^p}{2\mathrm{i}\pi\sqrt{\omega_1+\omega_2}}\int_{\mathscr{C}_{\mathrm{reg}}^{(+)}}\frac{\mathrm{e}^{\mathrm{i}\lambda x}}{\lambda}\,\frac{\partial^p}{\partial\lambda^p}\Big(\frac{1}{R_\downarrow(\lambda)}\Big)\cdot\frac{\mathrm{d}\lambda}{2\mathrm{i}\pi}\,. \tag{2.3.11}$$

The result for $\beta \neq 1$ contains two more terms, and is given in the body of the book, by Proposition 3.4.1, in terms of N-dependent simple and double integrals $\mathfrak{I}^{(2)}_{\mathrm{s};\beta}$. The final form for the asymptotics up to $o(1)$ of these extra terms can be worked out following the steps of Section 6.3, although we decided to leave it out of the scope of this book, since $\beta \neq 1$ does not seem to appear in quantum integrable systems.

2.4 The N-dependent Equilibrium Measure and the Master Operator

It is not hard to generalise the proof of Theorem 2.1.1 to the present setting so as to obtain the below characterisation of the leading in N asymptotic behaviour for $Z_N[V]$.

Theorem 2.4.1 *Let $\mathcal{E}_\infty$ be the lower semi-continuous good rate function*

$$\mathcal{E}_\infty[\mu] \;=\; \frac{1}{2}\int\big(V(\eta)+V(\xi)\;-\;\pi\beta(\omega_1+\omega_2)|\xi-\eta|\big)\,\mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta)\,. \tag{2.4.1}$$

Then, one has that

$$\lim_{N\to+\infty} \frac{\ln Z_N[V]}{N^{2+\alpha}} \;=\; -\inf_{\mu\in\mathcal{M}^1(\mathbb{R})} \mathcal{E}_\infty\big[\mu\big]\,. \tag{2.4.2}$$

The infimum is attained at a unique probability measure μ_{eq}. This measure is continuous with respect to the Lebesgue measure, and has density

$$\rho_{\mathrm{eq}}(\xi) \;=\; \frac{V''(\xi)}{2\pi\beta(\omega_1+\omega_2)}\cdot \mathbf{1}_{[a\,;b]}(\xi) \tag{2.4.3}$$

supported on the interval $[a\,;b]$, with (a,b) being the unique solution to the set of equations

$$V'(b) \;=\; -V'(a) \;=\; \pi\beta(\omega_1+\omega_2)\,. \tag{2.4.4}$$

One has, explicitly,

$$\lim_{N\to+\infty} \frac{\ln Z_N[V]}{N^{2+\alpha}} \;=\; -\frac{V(a)+V(b)}{2} + \frac{\big(V'(b)\big)^2(b-a) \;+\; \int_a^b \big(V'(\xi)\big)^2\,\mathrm{d}\xi}{4\pi\beta(\omega_1+\omega_2)}\,. \tag{2.4.5}$$

The strict convexity of V guarantees that the density (2.4.3) is positive and that it reaches a non-zero limit at the endpoints of the support. This behaviour differs from the situation usually studied in β ensembles with analytic potentials which leads to a generic square root (or inverse square root) vanishing (or divergence) of the equilibrium density at the edges.

Note that the function $\mathcal{E}_\infty$ defined in (2.4.1) arises as a good rate function in the large deviation estimates for the empirical measure $L_N^{(\lambda)}$, *c.f.* (1.2.2). In fact, a refinement of Theorem 2.4.1 would lead to the more precise estimates

$$\ln Z_N[V] \;=\; -N^{2+\alpha}\mathcal{E}_\infty\big[\mu_{\mathrm{eq}}\big] \;+\; \mathrm{O}(N^2)\,. \tag{2.4.6}$$

Thus, with respect to the usual varying weight β-ensemble case, there is a loss of precision by a $N^{1-\alpha}$ factor. This, in fact, takes its origin in that the purely asymptotic rate function $\mathcal{E}_\infty\big[\mu_{\mathrm{eq}}\big]$ does not absorb enough of the fine structure of the saddle-point. As a consequence, the remainder $\mathrm{O}(N^2)$ mixes both types of contributions: the deviation of the saddle-point with respect to its asymptotic position and the fluctuation of the integration variables around the saddle-point.

The fine, N-dependent, structure of the saddle-point is much better captured by the N-dependent deformation[2] of the rate functions $\mathcal{E}_\infty$:

[2]The property of lower semi-continuity along with the fact that $\mathcal{E}_N$ has compact level sets is verified exactly as in the case of β-ensembles, so we do not repeat the proof here.

$$\mathcal{E}_N[\mu] = \frac{1}{2}\int\left(V(\xi) + V(\eta) - \frac{\beta}{N^\alpha}\ln\Big\{\prod_{p=1}^{2}\sinh\big[\pi N^\alpha\omega_p(\xi-\eta)\big]\Big\}\right)\mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta)\,. \tag{2.4.7}$$

This N-dependent rate functions appear extremely effective for the purpose of our analysis. Namely, it allows us re-summing a whole tower of contributions into a single term. The use of $\mathcal{E}_N$ should not be considered as a mere technical simplification of the intermediate steps; it is, in fact, of prime importance. The use of the more classical object $\mathcal{E}_\infty$ would render the analysis of the Schwinger–Dyson equations impossible. This fact will become apparent in the core of the text.

As usual, this minimiser admits a characterisation in terms of a variational problem:

Proposition 2.4.2 *For any strictly convex potential V, the N-dependent rate function $\mathcal{E}_N$ admits its minimum on $\mathcal{M}^1(\mathbb{R})$ at a unique probability measure $\mu_{\mathrm{eq}}^{(N)}$. This equilibrium measure is supported on a segment $[a_N\,;b_N]$ and corresponds to the unique solution to the integral equations*

$$V(\xi) - \frac{\beta}{N^\alpha}\int\ln\left\{\prod_{p=1}^{2}\sinh\big[\pi N^\alpha\omega_p(\xi-\eta)\big]\right\}\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\eta) = C_{\mathrm{eq}}^{(N)} \quad \text{on } [a_N\,;b_N] \tag{2.4.8}$$

$$V(\xi) - \frac{\beta}{N^\alpha}\int\ln\left\{\prod_{p=1}^{2}\sinh\big[\pi N^\alpha\omega_p(\xi-\eta)\big]\right\}\mathrm{d}\mu_{\mathrm{eq}}^{(N)}(\eta) > C_{\mathrm{eq}}^{(N)} \quad \text{on } \mathbb{R}\setminus[a_N\,;b_N]\,, \tag{2.4.9}$$

with $C_{\mathrm{eq}}^{(N)}$ a constant whose determination is part of the problem (2.4.8)–(2.4.9). The equilibrium measure admits a density $\rho_{\mathrm{eq}}^{(N)}$, which is $\mathcal{C}^{k-2}$ in the interior $]a_N;b_N[$ if V is $\mathcal{C}^k$. Finally, one has the behaviour at the edges:

$$\rho_{\mathrm{eq}}^{(N)}(\xi) \underset{\xi\to a_N^+}{=} \mathrm{O}\big(\sqrt{\xi-a_N}\big)\,, \qquad \rho_{\mathrm{eq}}^{(N)}(\xi) \underset{\xi\to b_N^-}{=} \mathrm{O}\big(\sqrt{b_N-\xi}\big)\,. \tag{2.4.10}$$

The proof of this proposition is rather classical. It follows, for instance, from [4, Section 2.3] in what concerns the regularity, and from a convexity argument (see [7, Theorem 2.2]) in what concerns connectedness of the support and the strict inequality in (2.4.9). Elements of proof are nevertheless gathered in Appendix C. In fact, regarding to the equilibrium measure, we can be much more precise when N is large enough:

Theorem 2.4.3 *In the $N\to\infty$ regime, the equilibrium measure $\mu_{\mathrm{eq}}^{(N)}$:*

- *is supported on the single interval $[a_N\,;b_N]$ whose endpoints admit the asymptotic expansion*

$$a_N = a + \sum_{\ell=1}^{k} \frac{a_{N;\ell}}{N^{\ell\alpha}} + \mathrm{O}\Big(\frac{1}{N^{(k+1)\alpha}}\Big) \quad \text{and} \quad b_N = b + \sum_{\ell=1}^{k} \frac{b_{N;\ell}}{N^{\ell\alpha}} + \mathrm{O}\Big(\frac{1}{N^{(k+1)\alpha}}\Big), \tag{2.4.11}$$

where $k \in \mathbb{N}^*$ is arbitrary, (a, b) are as defined in (2.4.4) while

$$\begin{pmatrix} b_{N;1} \\ a_{N;1} \end{pmatrix} = \Big\{ \sum_{p=1}^{2} \frac{1}{2\pi\omega_p} \ln\Big(\frac{\omega_1\omega_2}{\omega_p(\omega_1+\omega_2)}\Big) \Big\} \cdot \begin{pmatrix} V''(a) \cdot \{V''(b)\}^{-1} \\ -V''(b) \cdot \{V''(a)\}^{-1} \end{pmatrix}; \tag{2.4.12}$$

- *is continuous with respect to Lebesgue. Its density is $\rho_{\mathrm{eq}}^{(N)}$ vanishes like a squareroot at the edges:*

$$\begin{aligned} \rho_{\mathrm{eq}}^{(N)}(\xi) &\underset{\xi \to a_N^+}{\sim} \Big(\frac{V''(a_N) + \mathrm{O}(N^{-\alpha})}{\pi\beta\sqrt{\pi(\omega_1+\omega_2)}} \Big) \sqrt{\xi - a_N}\,, \\ \rho_{\mathrm{eq}}^{(N)}(\xi) &\underset{\xi \to b_N^-}{\sim} \Big(\frac{V''(b_N) + \mathrm{O}(N^{-\alpha})}{\pi\beta\sqrt{\pi(\omega_1+\omega_2)}} \Big) \sqrt{b_N - \xi}\,, \end{aligned} \tag{2.4.13}$$

and there exists a constant $C > 0$ independent of N such that:

$$||\rho_{\mathrm{eq}}^{(N)}||_{L^\infty([a_N\,;b_N])} \le C\, ||V''||_{L^\infty([a_N\,;b_N])}\,. \tag{2.4.14}$$

This density takes the form $\rho_{\mathrm{eq}}^{(N)} = \mathcal{W}_N[V']$, with $\mathcal{W}_N$ as defined in (2.4.17).

If the potential V defining the equilibrium measures satisfies $V \in \mathcal{C}^k([a_N\,;b_N])$, then the density is of class $\mathcal{C}^{k-2}$ on $]a_N\,;b_N[$.

Note that the characterisation of $\rho_{\mathrm{eq}}^{(N)}$ in the theorem above comes from the fact that it is solution to the singular integral equation $\mathcal{S}_N\big[\rho_{\mathrm{eq}}^{(N)}\big](\xi) = V'(\xi)$ on $[a_N\,;b_N]$, where

$$\mathcal{S}_N\big[\phi\big](\xi) = \fint_{a_N}^{b_N} S\big[N^\alpha(\xi-\eta)\big]\phi(\eta)\,\mathrm{d}\eta \qquad \text{and} \qquad S(\xi) = \sum_{p=1}^{2} \beta\pi\omega_p \operatorname{cotanh}\big[\pi\omega_p\xi\big]. \tag{2.4.15}$$

The unknowns in this equation $(\rho_{\mathrm{eq}}^{(N)}, a_N, b_N)$ should be picked in such a way that $\rho_{\mathrm{eq}}^{(N)}$ has mass 1 on $[a_N\,;b_N]$ and is regular at the endpoints a_N, b_N. Thus, determining the equilibrium measure boils down to an inversion of the singular integral operator $\mathcal{S}_N$. In fact, the singular integral operator $\mathcal{S}_N$ also intervenes in the Schwinger–Dyson equations. The precise control on its inverse $\mathcal{W}_N$—defined between appropriate functional spaces—plays a crucial role in the whole asymptotic analysis.

These pieces of information can be obtained by exploiting the fact that the operator $\mathcal{S}_N$ is of truncated Wiener–Hopf type. As such, its inversion is equivalent to solving a 2×2 matrix valued Riemann–Hilbert problem. This Riemann–Hilbert problem

admits a solution for N large enough that can be constructed by means of a variant of the non-linear steepest descent method. By doing so, we are able to describe, quite explicitly, the inverse $\mathcal{W}_N$ by means of the unique solution χ to the 2×2 matrix valued Riemann–Hilbert problem given in Section 4.2.1. We will not discuss the structure of this solution here and, rather, refer the reader to the relevant section. We will, however, provide the main consequence of this analysis, *viz.* an explicit representation for the operator $\mathcal{W}_N$. For this purpose, we need to announce that χ_{11}, the $(1, 1)$ matrix entry of χ, is such that $\mu \mapsto \mu^{1/2} \cdot \chi_{11}(\mu) \in L^\infty(\mathbb{R})$.

Theorem 2.4.4 *Let $0 < s < 1/2$. The operator $\mathcal{S}_N : H_s\big([a_N\,;b_N]\big) \to \mathfrak{X}_s\big(\mathbb{R}\big)$ is continuous and invertible where, for any closed $A \subseteq \mathbb{R}$,*

$$\mathfrak{X}_s\big(A\big) \;=\; \Big\{ H \in H_s\big(A\big) \;:\; \int\limits_{\mathbb{R}+\mathrm{i}\epsilon} \chi_{11}(\mu)\mathcal{F}[H](N^\alpha\mu)\mathrm{e}^{-\mathrm{i}N^\alpha\mu b_N}\,\frac{\mathrm{d}\mu}{2\mathrm{i}\pi} \;=\; 0 \Big\} \tag{2.4.16}$$

is a closed subspace of $H_s\big(A\big)$ such that $\mathcal{S}_N\big(H_s\big([a_N\,;b_N]\big)\big) = \mathfrak{X}_s(\mathbb{R})$. The inverse is given by the integral transform $\mathcal{W}_N$ which takes, for $H \in \mathcal{C}^1([a_N\,;b_N]) \cap \mathfrak{X}_s(\mathbb{R})$, the form

$$\begin{aligned}\mathcal{W}_N[H](\xi) \;=\; & \frac{N^{2\alpha}}{2\pi\beta} \int\limits_{\mathbb{R}+2\mathrm{i}\epsilon} \frac{\mathrm{d}\lambda}{2\mathrm{i}\pi} \int\limits_{\mathbb{R}+\mathrm{i}\epsilon} \frac{\mathrm{d}\mu}{2\mathrm{i}\pi}\, \frac{\mathrm{e}^{-\mathrm{i}N^\alpha(\xi-a_N)\lambda}}{\mu-\lambda} \Big\{ \chi_{11}(\lambda)\chi_{12}(\mu) - \frac{\mu}{\lambda}\cdot\chi_{11}(\mu)\chi_{12}(\lambda) \Big\} \\ & \times \int\limits_{a_N}^{b_N} \mathrm{d}\eta \mathrm{e}^{\mathrm{i}N^\alpha\mu(\eta-b_N)} H(\eta)\,. \end{aligned} \tag{2.4.17}$$

In the above integral representations the parameter $\epsilon > 0$ is small enough but arbitrary. Furthermore, for any $H \in \mathcal{C}^1\big([a_N\,;b_N]\big)$, the transform $\mathcal{W}_N$ exhibits the local behaviour

$$\mathcal{W}_N[H](\xi) \underset{\xi\to a_N^+}{\sim} C_L H'(a_N)\sqrt{\xi - a_N} \quad \text{and} \quad \mathcal{W}_N[H](\xi) \underset{\xi\to b_N^-}{\sim} C_R H'(b_N)\sqrt{b_N - \xi}\,. \tag{2.4.18}$$

where $C_{L/R}$ are some H-independent constants.

Note that, within such a framework, the density of the equilibrium measure $\mu_{\mathrm{eq}}^{(N)}$ is expressed in terms of the inverse as $\rho_{\mathrm{eq}}^{(N)} = \mathcal{W}_N\big[V'\big]$. In this case, the pair of endpoints (a_N, b_N) of the support of $\mu_{\mathrm{eq}}^{(N)}$ corresponds to the unique solution to the system of equations

$$\int\limits_{a_N}^{b_N} \mathcal{W}_N\big[V'\big](\xi)\,\mathrm{d}\xi \;=\; 1 \qquad \text{and} \qquad \int\limits_{\mathbb{R}+\mathrm{i}\epsilon} \frac{\mathrm{d}\mu\;\chi_{11}(\mu)}{2\mathrm{i}\pi} \int\limits_{a_N}^{b_N} \mathrm{e}^{\mathrm{i}\mu N^\alpha(\eta-b_N)} V'(\eta)\,\mathrm{d}\eta \;=\; 0\,. \tag{2.4.19}$$

The first condition guarantees that $\mu_{\mathrm{eq}}^{(N)}$ has indeed mass 1, while the second one ensures that its density vanishes as a square root at the edges a_N, b_N. Using fine properties of the inverse, these conditions can be estimated more precisely in the large-N limit, hence enabling one to fix the large-N asymptotic expansion of the endpoints a_N, b_N as announced in (2.4.11) and (2.4.12).

2.5 The Overall Strategy of the Proof

In the following, we shall denote by $p_N(\boldsymbol{\lambda})$ the probability density on $\mathbb{R}^N$ associated with the partition function $Z_N[V]$ defined in (2.2.1):

$$p_N(\boldsymbol{\lambda}) \;=\; \frac{1}{Z_N[V]} \prod_{a<b}^{N} \Big\{ \sinh\big[\pi\omega_1 N^{\alpha}(\lambda_a-\lambda_b)\big] \sinh\big[\pi\omega_2 N^{\alpha}(\lambda_a-\lambda_b)\big] \Big\}^{\beta} \prod_{a=1}^{N} \mathrm{e}^{-N^{1+\alpha}V(\lambda_a)} \,. \tag{2.5.1}$$

$p_N(\boldsymbol{\lambda})$ gives rise to a probability measure $\mathbb{P}_N$ on $\mathbb{R}^N$. We also agree that, throughout the book, $L_N^{(\boldsymbol{\lambda})}$ refers to the empirical measure

$$L_N^{(\boldsymbol{\lambda})} \;=\; \frac{1}{N} \sum_{a=1}^{N} \delta_{\lambda_a} \tag{2.5.2}$$

associated with the stochastic vector $\boldsymbol{\lambda}$.

Definition 2.5.1 Let $\nu_1, \ldots, \nu_\ell$ be any (possibly depending on the stochastic vector $\boldsymbol{\lambda}$) measures and ψ a function in ℓ variables. Then we agree upon

$$\Big\langle \psi \Big\rangle_{\nu_1\otimes\cdots\otimes\nu_\ell} \equiv \Big\langle \psi(\xi_1,\ldots,\xi_\ell) \Big\rangle_{\nu_1\otimes\cdots\otimes\nu_\ell} \equiv \mathbb{P}_N\Big[\int_{\mathbb{R}^\ell} \psi(\xi_1,\ldots,\xi_\ell)\, \mathrm{d}\nu_1 \otimes\cdots\otimes \mathrm{d}\nu_\ell \Big] \tag{2.5.3}$$

whenever it makes sense. We shall add the superscript V whenever the functional dependence of the probability measure on the potential V needs to be made clear.

Note that if none of the measures $\nu_1, \ldots, \nu_\ell$ is stochastic, then the expectation versus $\mathbb{P}_N$ in (2.5.3) can be omitted.

As explained in Section 1.2.3, the Schwinger–Dyson equations constitute a tower of equations which relate expectation values of functions in many, non necessarily fixed, variables that are integrated versus the empirical measure (2.5.2). More precisely, the Schwinger–Dyson equations at rank k ($k \geq 1$) yield exact relations between various expectation values of a function in k variables and its transforms, this versus the empirical measure. The knowledge of these expectation values, yields an access to the derivatives of the partition function with respect of external parame-

ters. For instance, if $\{V_t\}_t$ is a smooth one parameter family of potentials, then

$$\partial_t \ln Z_N[V_t] \;=\; -N^{2+\alpha} \langle \partial_t V_t \rangle^{V_t}_{L_N^{(\lambda)}} \,. \tag{2.5.4}$$

The exponent V_t appearing in the right-hand side is there so as to emphasise that the expectation value is computed with respect to the probability measure subordinate to the t-dependent potential V_t.

Thus the problem boils down to obtaining a sufficiently precise control on the behaviour in N of the one-point expectation values. This can be achieved on the basis of a careful analysis of the system of Schwinger–Dyson equations associated with the present model. Since this machinery does not simplify much in the $\beta = 1$ case, we do this for general β. The result for some sufficiently regular function H and potentials V satisfying to the general hypothesis, is our Proposition 3.3.6.

In the $\beta = 1$ case, Proposition 3.3.6 reads:

$$-N^{2+\alpha} \langle H \rangle^{V}_{L_N^{(\lambda)}} \;=\; -N^{2+\alpha} \int_{a_N}^{b_N} H(\xi) \cdot \mathcal{W}_N\big[V'\big](\xi)\, \mathrm{d}\xi \;+\; \frac{1}{2} \mathfrak{I}_{\mathrm{d}}\big[H, V\big] \;+\; \mathrm{o}(1) \,. \tag{2.5.5}$$

and the proof shows that the remainder $\mathrm{o}(1)$ is uniform in H and V provided that H is regular enough and that V satisfies to the hypothesis given in (2.2.2)–(2.2.4).

Furthermore, the expansion (2.5.5) involves

$$\mathfrak{I}_{\mathrm{d}}[H, V] \;=\; \int_{a_N}^{b_N} \mathcal{W}_N\Big[\partial_\xi \big\{ S\big(N^\alpha(\xi - *)\big) \cdot \mathcal{G}_N\big[H, V\big](\xi, *) \big\}\Big](\xi)\, \mathrm{d}\xi \,, \tag{2.5.6}$$

with

$$\mathcal{G}_N\big[H, V\big](\xi, \eta) \;=\; \frac{\mathcal{W}_N[H](\xi)}{\mathcal{W}_N[V'](\xi)} \;-\; \frac{\mathcal{W}_N[H](\eta)}{\mathcal{W}_N[V'](\eta)} \,. \tag{2.5.7}$$

Note that, in (2.5.6), the $*$ indicates the variable of the function on which the operator $\mathcal{W}_N$ acts. Given sufficiently regular functions H, V, we obtain in Section 6.3 and more precisely in Proposition 6.3.10 the large-N asymptotic behaviour of $\mathfrak{I}_{\mathrm{d}}[H, V]$. We then have all the elements to calculate the large-N asymptotic behaviour of the partition function $Z_N[V]$. For this purpose, we observe that, when $\beta = 1$, the partition function associated to a quadratic potential can be explicitly evaluated as shown in Proposition D.0.19. One can also show (*cf.* Lemma D.0.18) that there exists a unique, up to a constant, quadratic potential $V_{G;N}$ such that its associated equilibrium measure has the same support $[a_N, b_N]$ as the one associated with V. Then $V_t = (1-t)V_{G;N} + tV$ is a one parameter t smooth family of strictly convex potentials, and $\mu^{(N)}_{\mathrm{eq};V_t} = (1-t)\mu^{(N)}_{\mathrm{eq};V_{G;N}} + t\mu^{(N)}_{\mathrm{eq};V}$. Furthermore, if follows from the details of the analysis that led to (2.5.5) that the remainder $\mathrm{o}(1)$ will be uniform in

$t \in [0\,;1]$. As a consequence, by combining all of the above results and integrating equation (2.5.4) over t, we get that, in the asymptotic regime,

$$\begin{aligned}\ln\left(\frac{Z_N[V]}{Z_N[V_{G;N}]}\right) = -N^{2+\alpha}\int_0^1 \mathrm{d}t \int \partial_t V_t(\xi)\,\mathrm{d}\mu^{(N)}_{\mathrm{eq};V_t}(\xi) \\ + N^{\alpha}\cdot \beth_0 \cdot \Big(\beth[V, V_{G;N}](b_N) - \beth[V, V_{G;N}](a_N)\Big) \\ + \aleph_0 \cdot \Big(\beth'[V, V_{G;N}](b_N) + \beth'[V, V_{G;N}](a_N)\Big) \;+\; \mathrm{o}(1).\end{aligned} \tag{2.5.8}$$

The constants $\beth_0$ and $\aleph_0$ were defined respectively in (2.3.8) and (2.3.9), while $\beth$ is as given by (2.3.7).

Note also that the first integral can be readily evaluated (integration of rational functions in t) on the asymptotic level by means of Proposition 6.1.6. It produces an expansion into inverse powers of N^{α} and, as such, does not contribute to the constant term unless α is of the form $2/n$ for some integer n. Note that it is this integral that gives rise to the functional $\mathcal{O}_p[V]$ in (2.3.1). Finally, the answer for the large-N asymptotic behaviour of the partition function $Z_N[V_{G;N}]_{|\beta=1}$ can be found in Proposition D.0.19. As follows from Lemma D.0.18 the quadratic potential $V_{G;N}$ is such that $V_{G;N} - W_{G;N} = \mathrm{O}\big(N^{-\infty}\big)$ with $W_{G;N}$ as defined in 2.3.2 and where the remainder is uniform on some N-independent relatively compact open neighbourhood of $[a_N\,; b_N]$. This allows one to replace $V_{G;N}$ by $W_{G;N}$ in the right hand side of (2.5.8). Also, it is clear form the large-N expansion of the partition function associated with quadratic potentials given in Proposition D.0.19 that $\ln Z_N[V_{G;N}]_{|\beta=1} - \ln Z_N[W_{G;N}]_{|\beta=1} = \mathrm{o}(1)$.

For $\beta \neq 1$, (2.5.5) is modified by the addition of two more terms $\mathfrak{I}^{(2)}_{\mathrm{s};\beta}$ and $\mathfrak{I}_{\mathrm{d};\beta}$. Their large N behaviour can be determined without difficulty—but with some algebra—along the lines of Section 6.1.6 and Section 6.3. Then, to arrive to a final answer for $Z_N[V]_{\beta\neq 1}$ similar to (2.5.8), we would need to compute exactly the partition function for the Gaussian potential $Z_N[V_{G;N}]_{\beta\neq 1}$. We do not know at present how to perform such a calculation. Thus, we would be able to derive the asymptotic behaviour of the partition function at $\beta \neq 1$ up to a universal, *i.e.* not depending on the potential V, function of β. However, since the values $\beta \neq 1$ do not seem to appear in quantum integrable systems, we shall limit ourselves in this book to the result of Proposition 3.4.1 for the case $\beta \neq 1$.

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