

Chapter 2

Fundamental Concepts of Mechanics

Before passing to the core lecture on continuum mechanics in three-dimensional (3D) space, let us first consider a simple one-dimensional model example of a deformable solid. This will be useful to define a number of intuitive notions to be appropriately generalized in subsequent chapters. This example will also serve to highlight problems that appear when one attempts to reach beyond one dimension in an analysis of deformable continuum—how far such an attempt complicates the mathematics needed to describe the phenomena typical of a more general case.

2.1 Statics of a Bar

Consider the system presented in Fig. 2.1. This is a bar of length l and a constant cross-section area A , stretched¹ along its axis by the force P . Let us assume that the bar is elastic, i.e. its length extension Δl is proportional to the force P ,

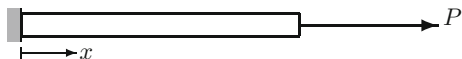
$$P = k \Delta l, \quad (2.1)$$

k being the proportionality factor (bar stiffness coefficient).

Let us discuss this system in a one-dimensional (1D) approach, treating each cross-section of the bar as its material point characterized by a coordinate x and the geometric property assumed constant along the bar's length, $A(x) = \text{const}$.

Due to the extension, a point with the initial location x moves to another location $\bar{x}$. There is a transformation $\bar{x} = \bar{x}(x)$ that describes the location of all points after the deformation of the bar. One may also define the *displacement* field of the bar points $u(x) = \bar{x} - x$, $x \in [0, l]$, assuming values $u = 0$ at $x = 0$ and $u = \Delta l$ at $x = l$.

¹The following discussion and conclusions are valid for the case of compressed bar too—in that case $P < 0$.

Fig. 2.1 Extended bar

In a continuum approach we postulate: (i) continuity of the field (function) $\bar{x}(x)$ (and, consequently, $u(x)$) and (ii) positive definition of the derivative $\frac{d\bar{x}}{dx}$. The two postulates are implied by the physical observation that neighbouring material particles are never separated and cannot occupy the same location in space.

The form of the function $u(x)$ within the interval $(0, l)$ is unknown and its determination generally requires solving the appropriate equations. It is, however, intuitively clear that in our case it can be assumed linear, $u(x) = x \Delta l / l$. Let us define the notions of *deformation* λ and *strain* ε , as

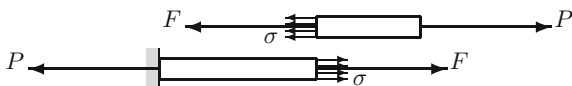
$$\lambda = \frac{d\bar{x}}{dx}, \quad \varepsilon = \frac{du}{dx}. \quad (2.2)$$

According to the assumed postulate, $\lambda > 0$. In the considered case the two quantities are constant with respect to x and equal $\lambda(x) = 1 + \Delta l / l$ and $\varepsilon(x) = \Delta l / l$, respectively.

Deformation and strain are fundamental geometrical notions in continuum kinematics. In subsequent chapters, their definitions will be generalized to the 3D case. In an undeformed medium, strain (i.e. relative extension) ε assumes the zero value while deformation λ equals one. In addition, positive values of strain indicate increase in the distance between material points of a medium (extension), while negative—its decrease (shortening).

The next introduced notion is *stress*. This is a resultant force of inter-particle reactions in material, related to the unit cross-section area. To determine stress in the considered bar, let us first mention the next fundamental law of mechanics, the rule of force equilibrium which implies that stresses in a material part must remain in equilibrium with other stresses and with external forces acting on it. In particular, let us imagine that we cut off a fragment of the bar (see Fig. 2.2) and replace internal forces acting in the material with equivalent external forces acting on the cross-section surface. To assure equilibrium, the load force P acting on its right end must be equilibrated by the cross-section forces, i.e. their resultant F must equal P and be directed outwards with respect to the cross-section surface. Assuming the forces are uniformly distributed over the cross-section, one can compute stress as

$$\sigma = \frac{F}{A}. \quad (2.3)$$

Fig. 2.2 Stress in the cross-section of the extended bar

An identical force F (although acting in the opposite direction) and corresponding stress appear on the adjacent cross-section surface of the remaining part of the bar. The force remains in equilibrium with the reaction force P acting at the support point of the bar. Let us assume the convention that stress is positive if it is directed outwards of the cross-section surface (tension) and negative in the opposite case (compression). This way, stresses in the two adjacent cross-sections are the same, both in terms of their value and sign. In subsequent chapters, the notion of stress will be generalized to the 3D case.

As can be seen in Fig. 2.2, the stress distribution along the bar's length is constant, $\sigma(x) = P/A$.

Let us finally introduce the notion of *constitutive equation*. This is the relationship between stress and strain, specific to a given material. For elastic materials, this relationship has a linear form known as the *Hooke law*,

$$\sigma = E\varepsilon, \quad (2.4)$$

where E is a material constant called the Young modulus or the longitudinal stiffness modulus. Looking at the formulae (2.4) and (2.1) and at the definitions of stress and strain, one can easily conclude that the stiffness coefficient of the bar in Eq. (2.1) can be expressed as

$$k = \frac{EA}{l}, \quad (2.5)$$

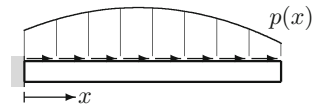
i.e., it depends on both the bar's material properties and its geometry.

Distributions of displacement and stress can be similarly determined in a bar loaded in a more complex way, e.g. with forces distributed along its length (Fig. 2.3). Stress in an arbitrary cross section initially located at x can be computed from the equilibrium equation as an integral of forces acting on the bar on one side of the cross-section (signs should be treated with careful attention), divided by the cross section area A , e.g.

$$\sigma(x) = \frac{1}{A} \int_x^l p(\xi) d\xi. \quad (2.6)$$

If the bar is additionally subjected to concentrated loads P_i acting at material points with initial locations x_i , respectively, these forces may be described as distributions of the Dirac delta function type, $\bar{p}(x) = \sum_i P_i \delta(x - x_i)$, which are added to the distribution $p(x)$ in the integrand in Eq. (2.6). Strains $\varepsilon(x)$ are determined by dividing σ by the stiffness modulus E (see Eq. (2.4)), while displacements determined by the

Fig. 2.3 Extended bar, general load case



integration of strains along x (see Eq. (2.2)₂) with the boundary conditions (here $u(0) = 0$) taken into account.

Note that Eq. (2.6) implies

$$\frac{d\sigma}{dx} = -\frac{p}{A}. \quad (2.7)$$

This is the differential *equilibrium equation* of the bar in the 1D case.

In summary, given the bar's geometry, material properties, loads and the way it is fixed, one can directly determine displacements, strains and stresses in particular material points (cross-sections) of the bar.

2.2 Trusses

In the previous section, probably the simplest of all solid mechanics problems has been described. Its solution scheme may now be a basis for solving a wide class of engineering structures named *trusses*. These are systems of several bars joined together with hinges located at the bars' ends and with loads and boundary conditions imposed only at the joints, called the *truss nodes*.

Consider a simple example of a planar truss consisting of four bars, fixed and loaded as presented in Fig. 2.4. The side length of the square composed by the bars is l . The cross-sections and mechanical properties of the bars are known. Let us try to determine displacements, strains and stresses in particular points of the system. Assuming linear displacement distributions along each bar's length which implies constant strain and stress in each bar, one may conclude that the task consists in determining the forces (stretching or compressing) acting in particular bars and the displacements of particular nodes. Additionally, it is necessary to determine unknown reaction forces in the fixed nodes. It will be shown now that the rules of force equilibrium and displacement continuity turn out to be sufficient to compute all these quantities.

Fig. 2.4 Planar four-bar truss

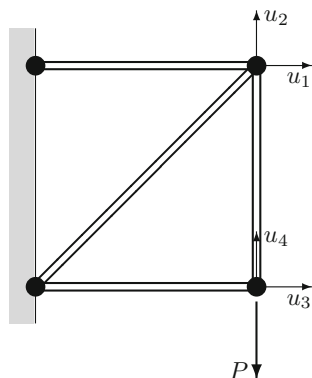
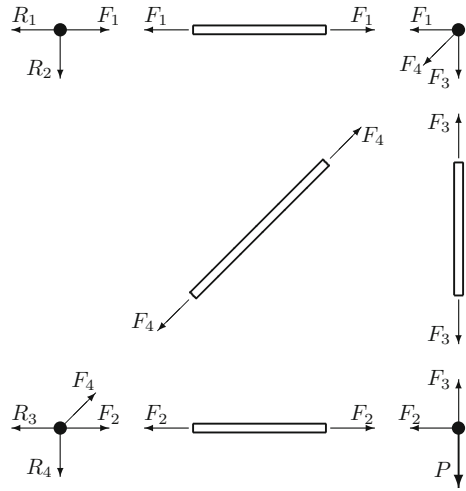


Fig. 2.5 Planar four-bar truss—decomposition into elements



Let us fictitiously decompose the truss into primary elements—bars and nodes—as shown in Fig. 2.5. It can be seen that there are 12 unknown quantities to be found:

$$u_1, u_2, u_3, u_4, F_1, F_2, F_3, F_4, R_1, R_2, R_3, R_4.$$

Equilibrium equations for each bar have been taken into account in the definition of unknowns, while the equilibrium equation of nodes written separately for the horizontal and vertical directions can be formulated as follows:

$$\begin{aligned} \text{left upper node:} & \quad F_1 - R_1 = 0, & -R_2 = 0, \\ \text{left lower node:} & \quad F_2 + \frac{1}{\sqrt{2}}F_4 - R_3 = 0, & \frac{1}{\sqrt{2}}F_4 - R_4 = 0, \\ \text{right upper node:} & \quad -F_1 - \frac{1}{\sqrt{2}}F_4 = 0, & -F_3 - \frac{1}{\sqrt{2}}F_4 = 0, \\ \text{right lower node:} & \quad -F_2 = 0, & F_3 - P = 0 \end{aligned} \quad (2.8)$$

(it is assumed that the nodal displacements are very small and thus the shape and geometric proportions of the system remain the same). The above 8 equations are sufficient to determine the unknown forces:

$$\begin{aligned} R_1 &= P, & F_1 &= P, \\ R_2 &= 0, & F_2 &= 0, \\ R_3 &= -P, & F_3 &= P, \\ R_4 &= -P, & F_4 &= -P\sqrt{2} \end{aligned} \quad (2.9)$$

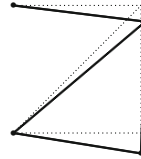
(negative value of F_4 obviously indicates compression of the particular bar while negative values of R_3 and R_4 mean that the forces act in the directions opposite to those shown in Fig. 2.5).

Nodal displacements can be determined from Eq. (2.1) in which the coefficients k for each bar can be determined from Eq. (2.5) while the length extension of each bar—from the geometry in Fig. 2.4. Four further equations take thus the form:

$$\begin{aligned} F_1 &= k_1 u_1, \\ F_2 &= k_2 u_3, \\ F_3 &= k_3 (u_2 - u_4), \\ F_4 &= k_4 \frac{1}{\sqrt{2}} (u_1 + u_2). \end{aligned} \quad (2.10)$$

Their solution (qualitatively illustrated in the sketch below) is:

$$\begin{aligned} u_1 &= \frac{P}{k_1}, \\ u_2 &= -P \left(\frac{1}{k_1} + \frac{2}{k_4} \right), \\ u_3 &= 0, \\ u_4 &= -P \left(\frac{1}{k_1} + \frac{2}{k_4} + \frac{1}{k_3} \right). \end{aligned} \quad (2.11)$$



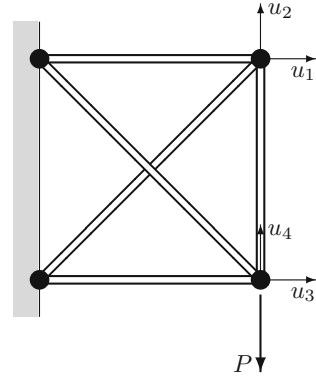
Now, the stresses in particular bars can be computed by dividing the forces F_i by appropriate cross-section areas A_i while strains by subsequently dividing stresses by E_i .

It is interesting to note that the lower horizontal bar does not participate in carrying the load—its force F_2 is null and its stiffness k_2 does not affect the solution. Note, however, that if the loading force P had a non-zero horizontal component, the bar would obviously become a significant load-carrying element of the system.

Remark. The system of equations for the 12 unknown quantities in the above discussed truss was quite easy to solve because equilibrium equations for bars (2.8) were actually independent of their elasticity equations (2.10). We took advantage of this feature by first determining the forces (8 unknowns) and only then displacements (4 unknowns). This is not always possible, however. Imagine a truss in which a fifth bar has been added as shown in Fig. 2.6. This makes another unknown force F_5 appear in the equation set (2.8)—terms containing this variable (the reader is encouraged to determine their form as an exercise) appear on the left-hand sides of equations for the upper left and lower right nodes. Determining the 9 unknown forces from the 8 equations is thus impossible. The problem of the truss statics has the solution, though, because the equations set (2.10) is extended by an additional equation, $F_5 = k_5 \frac{1}{\sqrt{2}} (u_3 - u_4)$. Hence, we have 13 equations with 13 unknowns altogether. It is easy to check in a computational exercise that in this case $F_2 \leq 0$ and $u_3 \leq 0$ and the values of the quantities depend on proportions between stiffness coefficients of particular bars.

Trusses for which the equations sets of equilibrium and elasticity cannot be decoupled are called *statically indeterminate*.

Fig. 2.6 Planar, statically indeterminate truss



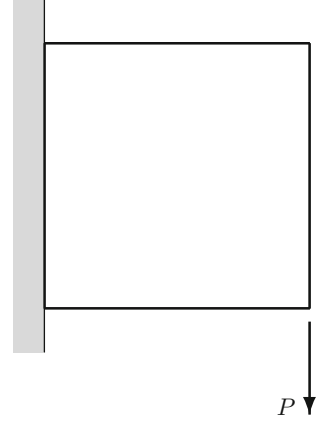
As it can be seen in the examples above, the solution of the problem of linear elastic truss statics consists in formulating the system equations of displacement continuity, force equilibrium and elasticity in the truss nodes and bars and solving the system. Extending this approach to 3D trusses is straightforward—in fact, the only change is a significant increase in the number of equations and unknowns. For actual truss structures the size of such a system ranges from about 10^4 to 10^6 equations and its coefficient matrix is usually very sparsely populated by non-zero terms. This makes the solution task quite easy and efficient for well-known numerical solution procedures dedicated to such systems.

2.3 Two-Dimensional Continuum Generalization

Consider a structure of the same shape as that shown in Fig. 2.4 or 2.6, made up not of bars but rather of a plane sheet with a specified thickness h and with a square shape (Fig. 2.7). Let us try to answer the question of the distribution of displacements and internal forces in the material points of the sheet, fulfilling all the equations postulated in the previous section.

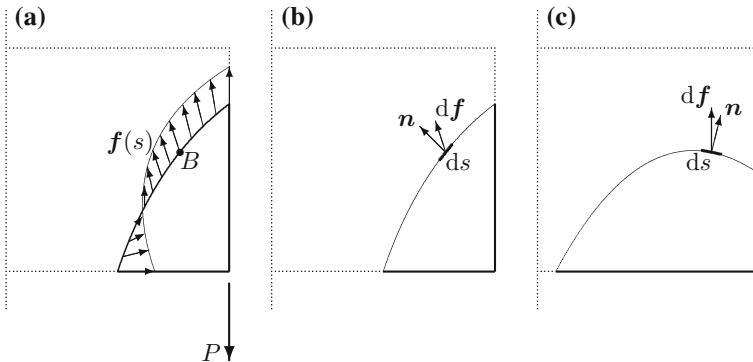
It can be easily seen that this task cannot be solved with the methods we have employed for bars and trusses. Firstly, this is because the system under consideration cannot be reasonably divided into simple primary elements in which the unknown quantities could be defined in a straightforward manner, as it was done in trusses. Secondly, the way we have defined certain fundamental notions in Sect. 2.1 is not applicable to the structure considered now. In fact, only the displacements u can be quite easily generalized to this case by defining

$$u = x^{\text{deformed}} - x, \quad (2.12)$$

Fig. 2.7 Plane sheet

where $\mathbf{x}$ and $\mathbf{x}^{\text{deformed}}$ stand for the initial and deformed location of a given material point, respectively (bold face indicates quantities defined by a set of components related to the assumed Cartesian coordinate system axes). The function $\mathbf{u}(\mathbf{x})$ is then a vector field in the 2D space. Defining stress, for example is a much more difficult task here, however. Following our experience from the example of a 1D bar, we could obviously imagine cutting the sheet into two parts and consider stresses as internal forces “revealed” by such a cut on the adjacent faces of the cross-section line, locally related to the section area unit. Such internal forces are continuously distributed along the cutting line, see Fig. 2.8a. A definition of stress that naturally comes to mind in this case would be

$$\sigma = \frac{1}{h} \frac{df}{ds} = \frac{df}{dA},$$

**Fig. 2.8** Stress in plane sheet

where s is a coordinate that parametrizes the section line and dA is an infinitesimal element of the cross-section area. This definition is ambiguous, though, which can be seen in Fig. 2.8b, c: depending on the direction of the section line at a given point B , the so-defined stress vector σ has different values. A thorough investigation of this issue leads to the conclusion that stress in a material point of a continuum is described by a second order tensor. The rigorous definition of this tensor will be given in Chap. 5. Analogous discussion allows us to conclude that strain and deformation are also second order tensors whose relations to the displacement field will be discussed in Chap. 4. Equilibrium equations written down in terms of so-defined quantities assume the form of partial differential equations with respect to scalar, vector and tensor fields.

To conclude this introductory chapter, let us finally mention that in the presented simple examples and fundamental definitions we have actually neglected the fact that the considered media (bars, sheets) are deformable in the sense of significantly changing their geometric shape. We have defined the displacements of material points and introduced the notion of strain but implicitly assumed that the deformation is small enough to neglect any substantial changes in geometry of the system. Hence, changes in geometry were not considered in formulating equilibrium equations, definition of stress, etc. However, the variable geometry of the system must be taken into account when dealing with a general case of a deformed continuum. These issues are going to be rigorously discussed in subsequent chapters of the book.

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