

Chapter 2

Basic Linear Geometry

2.1 Chebyshev Sets

A subset K of a Banach space X is said to be a **Chebyshev set** if every point in X has a unique nearest point in K . In such a case, the mapping that to $x \in X$ associates the point in K at minimum distance is called the **metric projection**.

V. Klee showed that *if the cardinality of Γ is, for example, the continuum c , then $\ell_1(\Gamma)$ can be covered by pairwise disjoint shifts of its closed unit ball* (we refer, e.g., to [FLP01]).

Note that the centers of these shifts in Klee's result clearly form a Chebyshev set that is not convex.

It is simple to prove that *every closed convex subset of a strictly convex reflexive Banach space is Chebyshev*. The norm $\|\cdot\|$ of a Banach space X is said to be **strictly convex** (or **rotund**) if its unit sphere does not contain nontrivial segments.

The existence of a nonconvex Chebyshev set in ℓ_2 is a longstanding open problem, which is equivalent with the statement that there is a nonsingleton set S in ℓ_2 such that each point of ℓ_2 has a unique **farthest point** in S (see, e.g., [FLP01]).

Thus we formulate

Problem 79. Is every Chebyshev set in ℓ_2 convex?

We refer to [FLP01]. It is known that *in a smooth finite-dimensional normed space X , every Chebyshev subset K is convex, and the metric projection is continuous on $X \setminus K$* . A result of Vlasov [Vla70] is that *in a Banach space X with a strictly convex dual norm, every Chebyshev subset with a continuous metric projection is convex*. Theorem 3.5.9 in [BorVan10] gives the result of V. Klee that *every weakly closed Chebyshev set in ℓ_2 is convex*.

We recommend the recent paper [FleMoo15]. In this paper, *a nonconvex Chebyshev set is constructed in a noncomplete inner product space.*



A subset K of a Banach space X is called **antiproximal**, if no point in $X \setminus K$ has a nearest point in K . The first such closed bounded convex set K with nonempty interior was constructed in c_0 by M. Edelstein and A. C. Thompson in [EdTh72]. It was also showed there that this means that *there is no nonzero functional on c_0 that would be supporting for K and B_X as well* (a nonzero functional f on a Banach space X is said to be a **support functional** of a bounded subset D of X if there exists a point $x_0 \in D$ such that $f(x_0) = \sup\{f(x) : x \in D\}$; the point x_0 is said to be a **support point** of D ; the set of all support functionals of D is denoted by $S(D)$). Thus such situation cannot happen in any space with the RNP, since there are supporting functionals forming residual sets in the dual space (cf., e.g., [FLP01, p. 634]).

Now, if $\|\cdot\|_1$ with closed unit ball B^1 and $\|\cdot\|_2$ with closed unit ball B^2 are two equivalent norms on a Banach space X , the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called **companion norms** if B^1 is an antiproximal set in $(X, \|\cdot\|_2)$, which obviously gives also that B^2 is an antiproximal set in $(X, \|\cdot\|_1)$ as well.

We refer to [BJM02], where the following open problem was posted:

Problem 80 (J. M. Borwein, M. Jiménez-Sevilla, and J. P. Moreno). Let X be a Banach space. Is the set of equivalent norms on X that admit a companion norm first Baire category in the space $N(X)$, where $N(X)$ denotes the metric space of all equivalent norms on X with the metric of uniform convergence on the unit ball of X ?

2.2 Tilings

A **tiling** of a Banach space X is a representation of X as a union of sets with pairwise disjoint interiors. We will restrict here to tilings with bounded convex closed sets having a nonempty interior. It follows directly that the space c_0 can be tiled with shifts of the $\frac{1}{2}$ -multiple of the closed unit ball to points with integer coordinates (see [FoLin98]). In this direction the following is an open problem:

Problem 81 (V. Fonf and J. Lindenstrauss). Does there exist a reflexive Banach space that can be tiled by shifts of a single closed convex bounded set with nonempty interior?

We refer to [FoLin98] and to [FLP01] for more on this problem.



We can also ask

Problem 82. For which separable reflexive Banach spaces does there exist a tiling such that there are positive numbers r and R such that every tile in this tiling has two properties: it contains a ball with radius r and is contained in a ball of radius R ?

We note that D. Preiss recently proved in [Pr10] that ℓ_2 is such a space.



2.3 Lifting Quotient Maps

Let T be a bounded linear operator from a Banach space X onto a Banach space Y , with kernel $Z \subset X$. Note that Y can be identified with X/Z by the Banach Open Mapping principle (and so T is identified to the quotient mapping $X \rightarrow X/Z$).

We search for a map f (in general nonlinear) from Y into X such that $T(f(y)) = y$ for every $y \in Y$. This mapping is called a **lifting** of the mapping T . The classical Bartle–Graves theorem (see, e.g., [FHHMZ11, Corollary 7.56]) says that *there always exists such a continuous lifting f* (we may assume $f(0) = 0$). Then the map h from X onto $Z \oplus Y$ defined by $h(x) = (x - f(Tx), Tx)$ is a nonlinear homeomorphism. Note that this map is a Lipschitz homeomorphism if f is a Lipschitz map.

In general there is no uniformly continuous lifting of such T (see, e.g., [BenLin00, p. 24]). Note that if T has a bounded linear lifting f then the mapping $P = f \circ T$ from X into X is a bounded linear projection from X onto $f(Y)$, and $f(Y)$ is linearly isomorphic to Y .

I. Aharoni and J. Lindenstrauss proved in [AhaLi78] (see, e.g., [FHHMZ11, p. 640]) that *there is a nonseparable Banach space X , a Banach space Y isomorphic to $c_0(\Gamma)$ for Γ uncountable, and a quotient map T from X onto Y , such that (1) the kernel of T is isomorphic to c_0 , (2) T has a Lipschitz lifting, and (3) X is not linearly isomorphic to $c_0(I)$ for any uncountable I* . Since $c_0(\Gamma) \oplus c_0$ is isomorphic to such $c_0(I)$, we get an example of two nonseparable spaces that are Lipschitz homeomorphic and not linearly isomorphic. We will come to these questions later on again.

Due to the power of the concept of differentiability, the situation in separable spaces is completely different. This is due to the following result of G. Godefroy and N. J. Kalton in [GoKal03] (see also [Kal08]): *If Y is separable and Q is a bounded linear operator from X onto Y that admits a Lipschitz lifting f , then Q admits a bounded linear lifting T with $\|T\| \leq \text{Lip}(f)$, where $\text{Lip}(f)$ is the Lipschitz constant of f , i.e., $\sup\{\|f(x_1) - f(x_2)\| / \|x_1 - x_2\| : x_1, x_2 \in X, x_1 \neq x_2\}$. This has the following corollary [GoKal03] (see also [Kal08]): *If a separable Banach space X is nonlinearly isometric to a subset of a Banach space Y then X is linearly isometric to a **contractively complemented subspace** of Y , i.e., by a projection of norm 1. Indeed, denote the nonlinear isometry of X into Y by f and assume without loss of generality that the closed linear hull of $f(X)$ is Y . Assume that $f(0) = 0$. Then by a result of Figiel (see, e.g., [BenLin00, p. 342]), there is a bounded linear operator T from Y onto X such that $T(f(x)) = x$ for all $x \in X$ and $\|T\| = 1$. So, T is a quotient map from Y onto X that has a nonlinear lifting that is in fact an isometry, and we can use the Godefroy–Kalton result mentioned above.**

The following is a slightly modified open question by G. Godefroy in [Go10]:

Problem 83 (G. Godefroy). Let X and Y be separable Banach spaces. Does there exist a sequence of contractively complemented subspaces of Y , each of them linearly isometric to X , that would guarantee the existence of a nonlinear isometry f from X into Y with the closed linear hull of $f(X)$ being Y ?

2.4 Isometries

The following is an old classical open problem:

Problem 84. Assume that X is an infinite-dimensional separable Banach space such that for any pair of points x and y in the unit sphere of X there is a linear isometry T from X onto X such that $Tx = y$. Is X linearly isometric to a Hilbert space?

It does hold for finite-dimensional spaces. However, it does not hold true for nonseparable spaces. We refer to [PeBe79, p. 255].



We do not know if the following is still open:

Problem 85. Assume that an infinite-dimensional Banach space X is such that both X and X^* are linearly isometric to subspaces of L_1 . Is X linearly isometric to a Hilbert space?

The isomorphic version works, see [BePe75, p. 257].



The following problem is related to the classical Mazur–Ulam theorem, stating that *if X and Y are Banach spaces, and $T : X \rightarrow Y$ is a surjective isometry, then T is affine* (cf., e.g., [FHHMZ11, p. 548]).

Problem 86 (D. Tingley [Tin87]). Let X and Y be real Banach spaces. Suppose that $T_0 : S_X \rightarrow S_Y$ is a surjective isometry. Does T_0 have a linear isometric extension $T : X \rightarrow Y$?

P. Mankiewicz [Man72] proved that *if $U \subset X$ and $V \subset Y$ are open and connected, then every isometry from U onto V can be extended to an affine isometry from X onto Y* . Tingley’s problem has an affirmative answer for finite-dimensional polyhedral Banach spaces [KaMar12]. For references and further information, see [Tana14].



Surprisingly, the next two-dimensional problem, a particular case of Problem 86, remains apparently open.

Problem 87. Does Problem 86 have a positive answer for $Y = X$ and $\dim(X) = 2$?

We mention [Tana14], where a history of these problems and a list of references are included.

We finish this section by mentioning in passing that K. Jarosz showed that *every real Banach space X can be equivalently renormed so that all linear isometries of X onto itself are $\pm Id_X$* [Ja88]. See, e.g., [HMOVZ08, p. 297].

2.5 Quasitransitive Norms

A norm of a Banach space X is called **quasitransitive** if for every $x, x' \in S_X$ and every $\varepsilon > 0$ there is a linear isometry S from X onto X such that $\|S(x) - x'\| < \varepsilon$. It follows that *the canonical norm on L_p is quasitransitive* (see, e.g., [DeGoZi93, p. 163]), while S. J. Dilworth and B. Randrianantoanina proved in [DilRan15] that ℓ_p for $1 < p < \infty$, $p \neq 2$, *does not have any equivalent quasitransitive norm*.

Problem 88. Assume that X is an infinite-dimensional separable Banach space such that any subspace of X admits an equivalent quasitransitive norm. Is X necessarily isomorphic to a Hilbert space?

We refer to [DilRan15].

2.6 Quasi-Banach Spaces

A map $x \rightarrow \|x\|$ from a vector space X into $[0, \infty)$ is called a **quasi-norm** if

1. $\|x\| > 0$ if $x \neq 0$,
2. $\|\alpha x\| = |\alpha| \|x\|$ for a scalar α and for $x \in X$,
3. There is a constant $C \geq 1$ such that $\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|)$ for all $x_1, x_2 \in X$.

The constant C is called the **modulus of concavity of the quasi-norm** $\|\cdot\|$.

For $0 < p \leq 1$, we call $\|\cdot\|$ a **p -norm** if, in addition of being a quasi-norm, it is **p -subadditive**, that is

4. $\|x_1 + x_2\|^p \leq \|x_1\|^p + \|x_2\|^p$ for all $x_1, x_2 \in X$.

It is known that *every quasi-norm is equivalent to a p -norm for some $0 < p \leq 1$* .

If $\|\cdot\|$ is a p -norm, then the function $\|x - y\|^p$ is a metric on X , and X equipped with $\|\cdot\|$ is called a **quasi-Banach space** if it is complete in this metric.

An example of a quasi-Banach space is the L_p space of all p -power integrable functions with the quasi-norm $\|f\| := (\int |f|^p)^{\frac{1}{p}}$ for $0 < p < 1$. Its modulus of concavity is $C := 2^{\frac{1}{p}-1}$, and $\|\cdot\|$ is a p -norm.

Quasi-Banach spaces suffer from the lack of the Hahn–Banach theorem. For example, *if $0 < p < 1$, then $L_p^* = \{0\}$* (M. M. Day). This leads for example to the following open problem:

Problem 89. Does every infinite-dimensional quasi-Banach space have a proper infinite-dimensional closed subspace?

By “proper” we mean here “different from the whole space.”

N. J. Kalton constructed in [Kal65] a quasi-Banach space X that contains a vector $x \neq 0$ such that every closed infinite-dimensional subspace of X contains x .



Problem 90 (Kwapień). Assume that X is a Banach space that is isomorphic to a subspace of L_p , $0 < p < 1$. Is X necessarily isomorphic to a subspace of L_1 ?

We refer to [BenLin00, p. 195]. *The answer is positive if X is reflexive, and there is a 3-dimensional Banach space that is not isometric to any subspace of L_1 (A. Koldobsky).* For both results see [BenLin00, p. 195]. In this direction note that *any 2-dimensional Banach space is isometric to a subspace of L_1 (J. Lindenstrauss, see, e.g., [FHHMZ11, p. 266]).*



2.7 Banach–Mazur Distance

If X and Y are two isomorphic Banach spaces, recall that $d(X, Y)$ denotes the **Banach–Mazur distance from X to Y** , i.e., the infimum over all isomorphisms T from X onto Y of the set $\{\|T\| \cdot \|T^{-1}\|\}$. If X is a Banach space, let

$$D(X) := \sup\{d(Y, Z) : Y, Z \text{ are isomorphic to } X\}.$$

It is known that *there is a constant $C > 0$ such that if the dimension of X is n , then $Cn \leq D(X) \leq n$ (see, e.g., [JoOd05]).*

Problem 91 (W. B. Johnson and E. Odell). Let X be a nonseparable Banach space. In the notation above, is $D(X)$ infinite?

We note that W. B. Johnson and E. Odell showed in [JoOd05] that *the answer to Problem 91 is positive for separable infinite-dimensional spaces*. It is known that $D(X)$ is also infinite for superreflexive spaces (V. I. Gurarii, see [JoOd05]). G. Godefroy showed in [Go10a] that $D(X)$ is infinite for all nonseparable subspaces of ℓ_∞ under Martin's MM axiom (see Chap. 3), and that in ZFC, it is infinite for all nonseparable so-called **representable** subspaces of ℓ_∞ .



Problem 92. Is Godefroy's result mentioned in comments to Problem 91 valid in ZFC for all nonseparable subspaces of ℓ_∞ ?

This question is raised in [Go10a].



Using projectional resolutions of the identity (cf., e.g., [DeGoZi93] and [HMOVZ08]) the following special case of Problem 91 may be perhaps easier to decide on:

Problem 93. If X is a nonseparable WCG space, is $D(X)$ infinite?



Problem 94. Let X be a Banach space such that X^{**}/X is infinite-dimensional and let $\lambda \geq 1$. Does there exist an equivalent norm $\|\cdot\|_\lambda$ on X such that the Banach–Mazur distance of $(X, \|\cdot\|_\lambda)$ to any isometric dual space is $\geq \lambda$?

We refer to [Sin78] and to [GoSa88].



A topological space is **hereditarily separable** if every subspace of it is separable. A compact space is called **scattered** if every closed subspace L of it has an isolated point in L .

Under a consistent axiom, there is a scattered nonmetrizable compact space K such that $C(K)^$ is hereditarily separable in its w^* -topology, and $C(K)$ is hereditarily Lindelöf in its weak topology.*

We will call such compact the **Kunen compact**, see, e.g., [HMOV08, p. 151].

Since $C(K)$ has a special renorming property (see [Go10a]), the following problem is of a special interest:

Problem 95. If K denotes the Kunen compact, is $D(C(K))$ infinite?



Problem 96 (E. Odell). Is it true that for every infinite-dimensional separable Banach space X there exist two spaces Y and Z isomorphic to X such that the Banach–Mazur distance $d(Y, Z) = 1$ and Y is not isometric to Z ?

For the space $X = c_0$ the solution in the positive (due to C. Bessaga and A. Pełczyński) is in [FHHMZ11, p. 276].



Problem 97 (A. Pełczyński). Determine the asymptotic behavior of the Banach–Mazur distance function from the cube, i.e., of the function $R_\infty^n := \max\{\text{dist}(X, \ell_\infty^n) : \dim(X) = n\}$ as $n \rightarrow \infty$.

It is known that for some absolute constants c and C , $c\sqrt{n} \ln n \leq R_\infty^n \leq Cn^{\frac{5}{6}}$. We refer to [To97].



Following [JoOd05], call a Banach space X **K -elastic** if there is a constant $K > 0$ such that for every equivalent norm $\|\cdot\|$ on X , $(X, \|\cdot\|)$ is K -linearly isomorphic to a subspace of X . For example, the space $C[0, 1]$ is 1-elastic (see, e.g., [FHHMZ11, p. 240]).

Problem 98 (W. B. Johnson and E. Odell). Assume that an infinite-dimensional Banach space X is K -elastic for some $K > 0$. Is it true that every Banach space of density equal to that of X isomorphically embeds into X ?

W. B. Johnson and E. Odell showed in [JoOd05] that c_0 *isomorphically embeds into X if X is separable and K -elastic for some $K > 0$.*

D. Alspach and B. Sari recently showed that *Problem 98 has a positive solution for $\text{dens } X = \aleph_0$* [AlsSa16].



Problem 99. Does every separable Banach space X contain a compact set K such that any Banach space Z that isometrically contains a copy of K necessarily contains a linear isometric copy of X ?

We refer to [DuLan08].

2.8 Rotund Renormings of Banach Spaces

The norm $\|\cdot\|$ of a Banach space X is said to be **locally uniformly rotund (LUR, in short)** if the following holds: Given an arbitrary $x \in S_X$ and a sequence $\{x_n\}_{n=1}^\infty$ in S_X such that $\|x + x_n\| \rightarrow 2$, then $\|x - x_n\| \rightarrow 0$. Clearly, an LUR norm is strictly convex. Note that if the dual norm of a Banach space $(X, \|\cdot\|)$ is LUR, then $\|\cdot\|$ is Fréchet differentiable (see, e.g., [FHHMZ11, Corollary 7.25]).

The notion of LUR was introduced in the thesis of A. Lovaglia [Lov55] under the supervision of R. C. James. Soon it became extremely useful in many areas of Banach spaces. M. I. Kadets *renormed every separable space by such a norm* (see, e.g., [FHHMZ11, p. 383]) and used it to solve a 40-year-old M. Fréchet open problem in the positive, namely whether *all infinite-dimensional separable Banach spaces are mutually homeomorphic* (see, e.g., [FHHMZ11, p. 543]). J. Lindenstrauss used it in a substantial strengthening of the Krein–Milman theorem [Lin63]. J. Lindenstrauss [Lin63] and independently E. Asplund [As68] used it to show *the Fréchet differentiability at dense sets of points of continuous convex functions on separable reflexive spaces*. S. L. Troyanski made a crucial step forward when he proved that *every WCG space can be renormed by an LUR norm* (see, e.g., [FHHMZ11, p. 587]). S. L. Troyanski and, independently, J. Lindenstrauss, showed that ℓ_∞ *does not admit any LUR norm* (see, e.g., [DeGoZi93b], [FHHMZ11, p. 409], and [HMOVZ08]). M. I. Kadets and S. L. Troyanski asked if

conversely, every space that admits no LUR norm must contain a copy of ℓ_∞ . This question was first negatively answered in [HayZi89]; this paper originated from a J. Lindenstrauss' suggestion. Now is known that there is a space that does not contain a copy of ℓ_∞ and does not admit any strictly convex renorming [Hay99].

In [HMZ12, Sect. 5] several similar notions regarding some type of uniformity in convexity are collected, and known relationships among them are also presented, together with a list of related problems. In this section we repeat one of the problems there and add to that list a few more.

Problem 100 (Problem 15, [HMZ12]). Is it true that a separable Banach space X does not contain a copy of ℓ_1 if, and only if, X can be renormed by a norm that satisfies the following: whenever a sequence $\{x_n\}_{n=1}^\infty$ in S_X satisfies $\lim_{n,m} \|x_n + x_m\| = 2$, then $\{x_n\}_{n=1}^\infty$ is weakly Cauchy.

The property of the norm in Problem 100 is called in [HMZ12] **weakly Cauchy rotundness (WCR, in short)**. It is not known whether the property WCR of a norm is equivalent to the property W^*CR (replace the weak by the w^* -topology in the definition) of its bidual norm (see [HMZ12, Problem 14]).

It is known that reflexivity implies WCR renormability (via a W2R renorming; the norm of a Banach space X is **weakly 2-rotund (W2R, in short)** if given a sequence $\{x_n\}_{n=1}^\infty$ in B_X such that $\|x_n + x_m\| \rightarrow 2$ as $n, m \rightarrow \infty$, there exists $x \in X$ such that $x_n \rightarrow x$ in the weak topology; *a Banach space is reflexive if, and only if, it is weakly 2-rotund renormable* [HaJo04]).



A Banach space X is called an **Asplund space** if Y^* is separable for every separable subspace Y of X (see also Sect. 4.1 for an equivalent definition, in fact the original one).

The norm of a Banach space X is **weakly uniformly rotund (WUR, in short)** whenever $x_n - y_n \rightarrow 0$ in the weak topology as soon as $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are sequences in B_X such that $\|x_n + y_n\| \rightarrow 2$. If X is separable, WUR renormability is equivalent to X being an Asplund space. See, e.g., [FHHMZ11, p. 387].

For further details, see, e.g., [HMZ12].

If a Banach space X is WUR, then so are its separable subspaces, and thus X is Asplund. The space JL_0 of Johnson and Lindenstrauss (see, e.g., [Ziz03]) is an example of an Asplund space that has no equivalent WUR norm.

Problem 101. Assume that X is a Banach space and Y is a subspace of X . Assume that both Y and X/Y admit equivalent WUR norms. Does X admit an equivalent WUR norm?

If the WUR property is replaced by the uniform convexity, the answer to Problem 101 is positive, see [EnLiPi75]. See also, e.g., [CasGon97].



The recent paper [SmiTr10] gives an account of the state of the art in LUR renormability for $C(K)$ -spaces up to this date. It is mentioned there the following result of R. Haydon [Hay08]: *a Banach space X is LUR renormable whenever X^* admits a dual LUR norm*. Problem 1 in [SmiTr10] is related to the following question. If X is a Banach space, the dual norm $\|\cdot\|$ on X^* is called w^* -LUR if $x_n^* - x^* \rightarrow 0$ in the w^* -topology of X^* whenever $x_n^*, x^*, n \in \mathbb{N}$, are on the unit sphere and $\|x_n^* + x^*\| \rightarrow 2$.

Problem 102. Assume that X admits a norm the dual of which is w^* -LUR. Does X admit an LUR norm?

We took this problem from [Hay08, p. 2026].

It is known that *a Banach space X admits a norm whose dual norm is w^* -LUR if and only if the dual unit ball endowed with the w^* -topology is a descriptive compactum* [SmiTr10]. Therefore, Problem 102 could be stated as follows: does (B_{X^*}, w^*) descriptive compactum implies that X is LUR renormable?

There exists a conjecture that constitutes a particular case of problem 102 saying that $C(K)$ admits an LUR norm if K is descriptive. In [Haj98] it is shown that $C(K)$ has an equivalent LUR norm if K is a Namioka–Phelps compactum (these form a subclass of the class of descriptive compacta). An alternative approach to prove this can be found in [MMOT10]. For definitions and more on this matter, see also [SmiTr10].



The following problem is posed in [MOTV09].

Problem 103. Assume that the norm of a Banach space is strictly convex and that the weak topology on the unit sphere of X is metrizable. Does X admit an equivalent LUR norm?



The norm $\|\cdot\|$ of a Banach space is said to be **midpoint locally uniformly rotund** (**MLUR**, in short), if for every $x \in S_X$ and every sequence $\{x_n\}_{n=1}^\infty$ in B_X such that

$\|x + x_n\| \rightarrow 1$ and $\|x - x_n\| \rightarrow 1$, then $\|x_n\| \rightarrow 0$. Every LUR norm is MLUR, and certainly MLUR implies strict convexity. However, there are MLUR norms that are not LUR (see, e.g., [Sm81]).

Problem 104 (R. Haydon). Assume that X is an Asplund space that admits an equivalent strictly convex norm. Does X admit an equivalent MLUR norm?

We refer to [Hay99].



A norm $\|\cdot\|$ on a Banach space X is called **average locally uniformly rotund** (**ALUR**, in short) if each point x of S_X is an extreme point of the ball and the identity map from (B_X, w) onto $(B_X, \|\cdot\|)$ is continuous at x .

Problem 105. Let X be a separable Banach space. Is it true that X does not contain a copy of ℓ_1 if and only if X has an equivalent norm the dual of which is ALUR?

We refer to [La04], where a characterization of Banach spaces having a renorming with dual ALUR norm was provided. A later paper by the same author [La11] characterizes the same property in the case that the space does not contain a copy of ℓ_1 .



There is a need for a comprehensive list of practical examples of norms with several different rotundity properties, in our opinion a good MSc project in this area.

2.9 More on the Structure of Banach Spaces

If X is a closed subspace of $C[0, 1]$ and $t_0 \in [0, 1]$, the point t_0 is called an **oscillating point** for X if there is $a > 0$ such that for every nondegenerated interval I around t_0 , there is $x \in X$ such that the oscillation of x on I is greater than or equal to a . The set of all such points is called the **oscillation spectrum** of X and is denoted $\Omega(X)$.

Problem 106. In the notation above, assume that $\Omega(X)$ is countable. Is then X isomorphic to a subspace of c_0 ?

We took this problem from [EnGuSe14]. In this paper it is proved that *the answer to this problem is positive if we assume that $\Omega(X)$ is finite.*



Problem 107 (R. Aron, V. Gurarii). Is the subset of ℓ_∞ formed by all the elements that have only finite number of zero coordinates, spaceable in ℓ_∞ ?

Spaceability was defined in Problem 47.

We took this problem from [EnGuSe14]. In this paper, the following known results are discussed:

1. *The set M of all continuous functions on $[0, 1]$ that are nowhere differentiable is spaceable in $C[0, 1]$. In fact, $M \cup \{0\}$ contains a linear isometric copy of any separable Banach space.*
2. *The set of all differentiable functions on $[0, 1]$ is not spaceable in $C[0, 1]$.*
3. *The set of all continuous functions on $[0, 1]$ that are differentiable on $(0, 1)$ is spaceable in $C[0, 1]$.*

For references, see [EnGuSe14].

In this paper it is proved among other things that *for every infinite-dimensional subspace X of $C[0, 1]$, the set of functions in X that have infinite number of zeros is spaceable in X .*



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