

Preface

This is a commented collection of some easily formulated open problems in the geometry and analysis on Banach spaces, focusing on basic linear geometry, convexity, approximation, optimization, differentiability, renormings, weak compact generating, Schauder bases and biorthogonal systems, fixed points, topology, and nonlinear geometry.

The collection consists of some commented questions that, to our best knowledge, are open. In some cases, we associate the problem with the person we first learned it from. We apologize if it may turn out that this person was not the original source. If we took the problem from a recent book, instead of referring to the author of the problem, we sometimes refer to that bibliographic source. We apologize that some problems might have already been solved. Some of the problems are long-standing open problems, some are recent, some are more important, and some are only “local” problems. Some would require new ideas, and some may go only with a subtle combination of known facts. All of them document the need for further research in this area. The list has of course been influenced by our limited knowledge of such a large field. The text bears no intentions to be systematic or exhaustive. In fact, big parts of important areas are missing: for example, many results in local theory of spaces (i.e., structures of finite-dimensional subspaces), more results in Haar measures and their relatives, etc. With each problem, we tried to provide some information where more on the particular problem can be found. We hope that the list may help in showing borders of the present knowledge in some parts of Banach space theory and thus be of some assistance in preparing MSc and PhD theses in this area. We are sure that the readers will have no difficulty to consider as well problems related to the ones presented here. We believe that this survey can especially help researchers that are outside the centers of Banach space theory. We have tried to choose such open problems that may attract readers’ attention to areas surrounding them.

Summing up, the main purpose of this work is to help in convincing young researchers in functional analysis that the theory of Banach spaces is a fertile field of research, full of interesting open problems. Inside the Banach space area, the text should help a young researcher to choose his/her favorite part to work in. This

way we hope that problems around the ones listed below may help in motivating research in these areas. For plenty of open problems, we refer also to, e.g., [AlKal06, BenLin00, BorVan10, CasGon97, DeGoZi93, Fa97, FHHMZ11, FMZ06, HaJo14, HMZ12, HVMZ08, HajZiz06, Kal08, LinTza77, MOTV09, Piet09, Woj91], and [Ziz03].

To assist the reader, we provided two indices and a comprehensive table referring to the listed problems by subject.

We follow the basic notation in the Banach space theory and assume that the reader is familiar with the very basic concepts and results in Banach spaces (see, e.g., [AlKal06, Di84, FHHMZ11, LinTza77, HVMZ08, Megg98]). For a very basic introduction to Banach space theory—“undergraduate style”—we refer to, e.g., [MZZ15, Chap. 11]. By a **Banach space** we usually mean an infinite-dimensional Banach space over the real field—otherwise we shall spell out that we deal with the finite-dimensional case. If no confusion may arise, the word **space** will refer to a Banach space. Unless stated otherwise, by a **subspace** we shall mean a closed subspace. The term **operator** refers to a bounded linear operator, and an operator with real values will be called a **functional**, understanding, except if it is explicitly mentioned, that it is continuous. A subspace Y of a Banach space X is said to be **complemented** if it is the range of a bounded linear projection on X . The **unit sphere** of the Banach space X , $\{x \in X : \|x\| = 1\}$, is denoted by S_X , and the **unit ball** $\{x \in X : \|x\| \leq 1\}$ is denoted by B_X . The words “smooth” and “differentiable” have the same meaning here. Unless stated otherwise, they are meant in the Fréchet (i.e., total differential) sense. If they are meant in the Gâteaux (i.e., directional) sense, we clearly mention it (for those concepts, see their definitions in, e.g., [FHHMZ11, p. 331]). We say that a norm is **smooth** when it is smooth at all nonzero points. Sometimes, we say that “a Banach space X admits a norm $\|\cdot\|$,” meaning that it admits an *equivalent* norm $\|\cdot\|$. By **ZFC** we mean, as usual, the Zermelo-Fraenkel-Choice standard axioms of set theory. Unless stated otherwise, we use this set of axioms. We say that some statement is **consistent** if its negation cannot be proved by the sole ZFC set of axioms. Cardinal numbers are usually denoted by \aleph , while ordinal numbers are denoted by α, β , etc. With the symbol \aleph_0 we denote the cardinal number of the set \mathbb{N} of natural numbers, and \aleph_1 is the first uncountable cardinal. Similarly, ω_0 (sometimes denoted ω) is the ordinal number of the set \mathbb{N} under its natural ordering, and ω_1 is the first uncountable ordinal. The continuum hypothesis then reads $2^{\aleph_0} = \aleph_1$. The cardinality 2^{\aleph_0} of the set of real numbers (the **continuum**) will be denoted by c . If no confusion may arise, we sometimes denote by ω_1 also its cardinal number \aleph_1 .

We prepared this little book as a working companion for [FHHMZ11] and [HVMZ08]. We often use this book to upgrade and update information provided in these two references.

Overall, we would be glad if this text helped in providing a picture of the present state of the art in this part of Banach space theory. We hope that the text may serve also as a kind of reference book for this area of research.

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The material comes from the interaction with many colleagues in meetings, in work, and in private conversations and, as the reader may appreciate in the comments to the problems, from many printed sources—papers, books, reviews, and even beamers from presentations—and, last, from our own research work. It is clear then that it will be impossible to explicitly thank so many influences. The authors prefer to carry on their own shoulders the responsibility for the selection of problems, eventual inaccuracies, wrong attributions, or lack of information about solutions. The names of authors appearing in problems, in comments, and in the reference list correspond to the panoply of mathematicians to whom thanks and acknowledgment usually appear in the introduction to a book.

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The authors wish the readers a pleasant time spent over this little book.

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