

## Chapter 2

# Tools and Concepts

One of the most interesting features of the PAM is that, being a partial differential equation with random coefficients, it lies in the intersection of probability and functional analysis, which opens up exciting possibilities for combining tools from these two different parts of mathematics. Furthermore, there are classic and well-developed mathematical theories that enable explicit solution formulas and the application of further techniques to the study of the PAM. In this chapter, we give an account on these tools and pave the way for a deep understanding and a powerful analysis of the PAM. We bring the probabilistic side in Sect. 2.1 and the functional analytic side in Sect. 2.2. In Sect. 2.3, we discuss a number of aspects and conclusions that immediately follow from a combination of these tools; a panorama of precise conjectures arises.

### 2.1 Probabilistic Aspects

The PAM has a lot of relations to other questions and models, which explains the great interest that the PAM receives. We briefly survey the most important ones. Furthermore, we provide the most important probabilistic tools for the treatment of the PAM.

#### 2.1.1 *Branching Process with Random Branching Rates*

The solution  $u$  to (1.1) admits an interpretation that arises from branching particle dynamics, see [GärMol90] and [CarMol94]. The following model is one important representative of a class of models called *branching random walk in random environment (BRWRE)*.

Imagine that initially, at time  $t = 0$ , there is a single particle at the origin, and all other sites are vacant. This particle moves according to a continuous-time symmetric random walk with generator  $\Delta^d$ . When present at site  $z$ , the particle splits into two particles with rate  $\xi_+(z) \in (0, \infty)$  and is killed with rate  $\xi_-(z) \in (0, \infty]$ , where  $\xi_+ = (\xi_+(z))_{z \in \mathbb{Z}^d}$  and  $\xi_- = (\xi_-(z))_{z \in \mathbb{Z}^d}$  are independent random i.i.d. fields. Every particle continues from its birth site in the same way as the parent particle, and their movements are independent. Put  $\xi(z) = \xi_+(z) - \xi_-(z) \in [-\infty, \infty)$ . Then, given  $\xi_-$  and  $\xi_+$ , the expected number of particles present at the site  $z$  at time  $t$ , as a function of  $(t, z) \in [0, \infty) \times \mathbb{Z}^d$ , solves the equation (1.1) and is therefore, by uniqueness of the solution, equal to  $u(t, z)$  [GärMol90]. Here the expectation is taken over the particle motion and over the splitting resp. killing mechanism, but not over the random medium  $(\xi_-, \xi_+)$ . The fact that the expected particle number solves (1.1) is standard in the study of branching processes; see [Hol00] for an elementary derivation.

The successful work on the PAM since 1990 has fertilized also the study of the BRWRE, but to a surprisingly little extent yet. In Sect. 7.11 below, we survey some heuristics and results on the BRWRE that are influenced by the study of the PAM.

**Remark 2.1 (Applications)** The mathematical concept of spatial branching processes as described above has the following main applications.

- **Population dynamics.** It is a basic model for the spatial movement, branching and killing of indistinguishable particles in space. It may be seen as a drawback for realistic applications that the production of new particles at site does not depend on the current number of particles there, in particular the number of particles at a site is unbounded.
- **Mutation and selection.** If the space  $\mathbb{Z}^d$  is replaced by a space of phenotypes of an individual, then the underlying branching process is a popular model for population genetics. Indeed, each jump within this space is interpreted as a mutation, as some detail in its biologic properties is changed, and the particles in the branching process are not registered according to the spatial location of the individuals, but according to their biological properties. One example is the replacement of  $\mathbb{Z}^d$  by the  $N$ -dimensional hypercube  $\{-1, 1\}^N$ , which is a simple model space for the set of gene sequences that we mentioned in Remark 1.8.
- **Chemical reactions.** The particle model with migration, branching and killing also serves as a (very simple) model for chemical reactions. Indeed, imagine that particles are randomly distributed over  $\mathbb{Z}^d$  that have an action as *catalysts* for a certain type of chemical reaction; that is, their presence at a given site supports the reaction of a certain *reactant* and helps producing new substance of it. In mathematical terms, we assume that a reactant particle at  $z$  is, for any catalyst particle present at  $z$ , split into two at a given rate  $\gamma \in (0, \infty)$ , say. That is, the rate of the reaction is linear in the number of catalysts. Additionally, assume that each reactant particle dies with fixed rate  $\delta \in (0, \infty)$ . Let  $\xi_*(z)$  denote the

number of catalyst particles at  $z$ , whose presence we want to assume as random. Then  $u(t, z)$  is the expected number of reactant particles at time  $t$  in the site  $z$ , where the random potential is given as  $\xi = \gamma\xi_* - \delta$ .

◇

**Remark 2.2 (The (non-parabolic) Anderson equation)** As we announced in Remark 1.1, there is an interpretation of the solution of the non-parabolic version of the PAM, the original quantum mechanic Schrödinger equation (1.4), in terms of a branching processes, see [Wag14, Wag15a, Wag15]. The underlying branching process is indeed a *marked* branching process with migration in  $\mathbb{Z}^d$ , and the marks are taken from the set  $\{1, -1\} \times \{+, -\}$ . While the marks 1 and  $-1$  appear very natural to mark, in some way, the real part and the imaginary part, respectively, the introduction of the marks  $+$  and  $-$  are at the first sight surprising. They can be interpreted as ‘present’ and ‘vanishing’ or as ‘visible’ and ‘hidden’. See [Wag14] for the precise mechanism. Then the solution to (1.4) is given as

$$u(t, z) = \mathbb{E}[\eta_{1,+}(t, z) - \eta_{1,-}(t, z)] + i \mathbb{E}[\eta_{-1,+}(t, z) - \eta_{-1,-}(t, z)], \quad (2.1)$$

where  $\eta_m(t, z)$  denotes the number of particles with mark  $m$  at time  $t$  in the site  $z$ . In [Wag15], a kind of Feynman-Kac formula is formulated in terms of a simple random walk on  $\mathbb{Z}^d$ . However, it seems more appropriate to formulate one for a suitable random walk on  $\mathbb{Z}^d \times \{1, -1\} \times \{+, -\}$ , but this is currently open. ◇

### 2.1.2 Feynman-Kac Formula

Some of the most interesting applications of the PAM are best explained in terms of an explicit formula for the solution in terms of random walks. A very useful standard tool for the probabilistic investigation of (1.1) is the well-known *Feynman-Kac formula* for the solution  $u$ , which reads

**Feynman-Kac formula.** Under the moment condition (1.5),

$$u(t, z) = \mathbb{E}_0 \left[ \exp \left\{ \int_0^t \xi(X(s)) \, ds \right\} \delta_z(X(t)) \right], \quad (t, z) \in [0, \infty) \times \mathbb{Z}^d. \quad (2.2)$$

Here  $(X(s))_{s \in [0, \infty)}$  is a continuous-time random walk on  $\mathbb{Z}^d$  with generator  $\Delta^d$  starting at  $z \in \mathbb{Z}^d$  under  $\mathbb{E}_z$ . One can also write  $\delta_z(X(t)) = \delta_{X(t)}(z) = \mathbb{1}\{X(t) = z\}$  for the indicator variable on the event that the endpoint of the path is located at  $z$ .

In words, in (2.2) a random walk path runs from the origin to  $z$ , and in the exponential we evaluate the sum of all the potential values that the walker sees on this way, weighted with the time that it spends in the respective lattice site. Actually, we used a time-reversal here, e.g., the initial condition  $\delta_0$  and the evaluation at  $z$  at time  $t$  may be interchanged. For  $u$  a solution with initial condition  $u(0, \cdot) = u_0(\cdot)$ , one would have to start the random walk at  $z$  and replace  $\delta_z$  by  $u_0$ . By summing up over all  $z \in \mathbb{Z}^d$ , we see that the total mass  $U(t)$  admits the Feynman-Kac representation

$$U(t) = \mathbb{E}_0 \left[ \exp \left\{ \int_0^t \xi(X(s)) \, ds \right\} \right], \quad t \in [0, \infty). \quad (2.3)$$

We refer the reader to [GärMol90, Theorem 2.1] for a proof of (2.2) (or its finite-space version in (2.7)), which is intimately connected with the almost sure existence and uniqueness of a solution to (1.1). Actually, the restriction to a finite box is technically much easier to handle (see Sect. 2.1.3); for a proof of finiteness of the infinite-space version (2.2) one has to control the decay of the potential  $\xi$  at infinity, and some percolation arguments are necessary for proving the uniqueness part, see our remarks after Theorem 1.2.

**Remark 2.3 (Relation with semigroup theory)** For the sake of better understanding, we give an explanation why the Feynman-Kac representation given in (2.2) actually solves problem (1.1) by making a connection to the theory of semigroups of operators. Consider the family of operators  $(\mathcal{P}_t)_{t \geq 0}$  acting on the bounded functions on the lattice as

$$\mathcal{P}_t f(z) = \mathbb{E}_z \left[ \exp \left\{ \int_0^t \xi(X(s)) \, ds \right\} f(X(t)) \right], \quad (t, z) \in [0, \infty) \times \mathbb{Z}^d. \quad (2.4)$$

By time reversal, we see that (2.2) is tantamount to  $u(t, z) = \mathcal{P}_t \delta_0(z)$ . An application of the Markov property shows that the family  $(\mathcal{P}_t)_{t \geq 0}$  is a semi-group, i.e.,  $\mathcal{P}_0$  is the identical operator and  $\mathcal{P}_s \circ \mathcal{P}_t = \mathcal{P}_{s+t}$  for any  $s, t \in [0, \infty)$ . Elementary calculations using the theory of continuous-time Markov chains reveal that the corresponding generator is equal to  $\Delta^d + \xi$  showing up on the right hand side of (1.1), i.e.,  $\partial_t \mathcal{P}_t f|_{t=0} = \Delta^d f + \xi f$  for many functions  $f$ . (We do not enter here a discussion about the largest class of validity, i.e., a characterisation of the domain of the generator, but recall the criterion in (1.6).) Then we obtain the forward equation  $\frac{\partial}{\partial t} \mathcal{P}_t f = (\Delta^d + \xi) \mathcal{P}_t f$ , which means that  $u(t, z) = \mathcal{P}_t f(z)$  solves the parabolic Anderson problem (1.1) with initial condition  $u(0, \cdot) = \mathcal{P}_0 f = f$ .  $\diamond$

**Remark 2.4 (Large- $t$  asymptotics  $\approx$  maximisation)** By (2.2),  $U(t)$  is the  $t$ -th exponential moment of the random variable  $\frac{1}{t} \int_0^t \xi(X(s)) \, ds$ . Hence, its large- $t$  asymptotics is intimately connected with the maximisation of this random variable, according to the well-known fact that the  $t$ -th exponential moment of a random variable behaves exponentially in  $t$  with rate equal to the essential supremum of that random variable. We will elaborate on this thought in Sect. 2.3.1. For the moment,

we keep in mind that the large- $t$  asymptotics of  $U(t)$  are determined by paths  $X$  that make  $\int_0^t \xi(X(s)) ds$  large.  $\diamond$

**Remark 2.5 (Self-attractiveness)** We see from the Feynman-Kac formula that the interaction with a random potential  $\xi$  induces a self-attractive effect to the path, when one takes the expectation with respect to  $\xi$ . Indeed, according to Remark 2.4, the paths that really count in the limit  $t \rightarrow \infty$  are those who make the exponent  $\int_0^t \xi(X(s)) ds$  large. When taking the expectation w.r.t.  $\xi$ , then the path and the potential jointly make  $\int_0^t \xi(X(s)) ds$  large in a coordinated strategy, which consist of the following:

- $\xi$  is very large in an area  $B \subset \mathbb{Z}^d$ , and
- the path does not leave  $B$  by time  $t$ .

Achieving large values is probabilistically costly, hence the potential would like to be large only in a small set  $B$ . On the other hand, not leaving a small set is probabilistically costly as well, so the path would like to do this only with a large set  $B$ . Potential and path have to find a compromise, i.e., an optimal size of  $B$ . This depends mainly on the costs for the potential to achieve extremely high values, i.e., on the upper tails of  $\xi(0)$  (the asymptotics of  $\text{Prob}(\xi(0) > r)$  as  $r \uparrow \text{esssup } \xi(0)$ ). We will see in Chap. 3 how this compromise will be found, but it is clear from the central limit theorem that the diameter of an optimal set  $B$  will be much smaller than  $\sqrt{t}$ .  $\diamond$

It should be stressed that the random walk path in (2.2) is *not* to be interpreted as the trajectory along which the mass flows through the random potential, even though this association may be tempting. It does *not* reflect the time-evolution of the heat flow that is described by the solution  $u(t, \cdot)$ , but should be seen only as some mathematical object that enables an explicit description of  $u(t, \cdot)$ . Nevertheless, this path is often studied as an object on its own interest as some random path under the influence of the random environment  $\xi$ ; see Sect. 2.1.5.

Certainly, there is a Feynman-Kac formula in the spatially continuous case as well, see many standard texts on Brownian motion, e.g., [Szn98, Section 1.1]. If  $f: \mathbb{R}^d \rightarrow [-\infty, \infty)$  is a continuous function that is bounded from above, then the solution  $u: (0, \infty) \times \mathbb{R}^d \rightarrow [0, \infty)$  of the PAM (1.7) (with  $V$  replaced by  $f$ ) satisfies the formula

$$u(t, x) = \mathbb{E}_0 \left[ \exp \left\{ \int_0^t f(Z(s)) ds \right\}; Z(t) \in dx \right] / dx, \quad t \in [0, \infty), \quad (2.5)$$

where  $Z = (Z(s))_{s \in [0, \infty)}$  is a Brownian motion with generator  $\Delta$  in  $\mathbb{R}^d$  starting from  $x$  under  $\mathbb{E}_x$ . (Note that we dropped the factor  $\frac{1}{2}$  in front of the Laplace operator, in accordance with (1.7).) This time, we need to conceive the expectation as a density of the terminal site  $Z(t)$  in  $x$ , therefore we do not use the time-reversal property. In words, the right-hand side is the density of the random variable  $Z(t)$  under the measure with density given by the exponential. The formula in (2.5) holds also under much weaker assumptions than upper boundedness of  $f$  [Szn98, Section 1.2];

it suffices to assume that  $\limsup_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ \int_0^t |f(Z_s)| ds \right] = 0$ , i.e., that  $f$  lies in the *Kato class*.

**Remark 2.6 (Renormalized Poisson potential)** As we said in Example 1.15, one of the most natural and most-studied potentials is the Poisson trap potential  $V(x) = -\int_{\mathbb{R}^d} W(x-y) \omega(dy)$ , where  $\omega = \sum_i \delta_{x_i}$  is a standard Poisson point process in  $\mathbb{R}^d$  with intensity  $\nu \in (0, \infty)$ , and  $W: \mathbb{R}^d \rightarrow [0, \infty)$  is a continuous potential. However, for the choice  $W(x) = C|x|^{-q}$  with  $q \in (0, d]$ , this potential is equal to  $-\infty$  almost everywhere, almost surely [CheKul12, Proposition 2.1]. This type of potentials is worth being studied, since, in  $d \geq 3$ , for the choices  $q = d-1$  and  $q = d-2$  the potential has the interpretation of the gravitational force and potential, respectively. One way out of the problem is to consider the renormalised Poisson potential  $V(x) = -\int_{\mathbb{R}^d} W(x-y)(\omega(dy) - dy)$ . Indeed, if  $\int_{\mathbb{R}^d} (e^{-W(x)} - 1 + W(x)) dx < \infty$  (this is satisfied for  $W(x) = C|x|^{-q}$  precisely in the case  $q \in (d/2, d)$ ), then [CheKul12, Theorem 1.1] the renormalised potential can be properly defined, and [CheKul12, Proposition 1.2] the corresponding Feynman-Kac formula is a solution to (1.7). However, it is a solution possibly only in the weak sense, i.e., in the sense that

$$u(t, x) = u_0(x) + \int_0^t ds \int_{\mathbb{R}^d} dy p_{t-s}(x-y) u(s, y) \xi(y), \quad x \in \mathbb{R}^d, t > 0,$$

and the integral on the right-hand side converges absolutely, where  $p_s(x) = (4\pi s)^{-d/2} e^{-|x|^2/4s}$  is the Gaussian density. In particular, the expectation of the Feynman-Kac formula (taken over the Poisson process) is finite, i.e., the first moment of the solution is finite. Let us also remark that one of the main formulas in [CheKul12] is

$$\langle U(t) \rangle = \left\langle \mathbb{E}_0 \left[ \exp \left\{ \int_0^t V(Z_s) ds \right\} \right] \right\rangle = \mathbb{E}_0 \left[ \exp \left\{ \nu \int_{\mathbb{R}^d} F(w(t, x)) dx \right\} \right], \quad (2.6)$$

where  $F(x) = e^{W(x)} - 1 + W(x)$ ,  $w(t, x) = \int_0^t W(Z_s - x) ds$ , and  $(Z_s)_{s \in [0, \infty)}$  is a Brownian motion in  $\mathbb{R}^d$  with generator  $\Delta$  starting from  $x$  under  $\mathbb{E}_x$ . We will discuss the large- $t$  behaviour of the moments in Sect. 7.3.4 below.

The first moment of the solution turns out to be infinite [CheKul12, Theorem 1.4] if we change the sign in front of the potential, i.e., if we consider  $V(x) = C \int_{\mathbb{R}^d} |x-y|^{-q} (\omega(dy) - dy)$  with  $C > 0$  and  $q \in (d/2, d)$ . However, it was shown in [CheKul12, Theorem 1.5] that the Feynman-Kac formula representing the solution to the PAM with this potential is almost surely finite for  $q \in (d/2, \min\{2, d\})$ , but infinite if  $q > 2$ . In the critical case  $q = 2$  and  $d = 3$  [CheRos11], the finiteness depends on whether  $C < 1/16$  or not. The effect of another additional ingredient is studied in [CheXio15], where the Poisson process  $\omega$  is assumed time-dependent, more precisely, it is replaced by the process  $(\omega_s)_{s \in [0, \infty)}$  of independent Brownian motions, which is a Poisson point process at every time  $s$ . We refer to Sect. 7.3.4 for some asymptotic results for the moments for this model.  $\diamond$

### 2.1.3 Finite-Space Feynman-Kac Formulas

For many proofs, it will be important later to approximate the PAM with finite boxes. Luckily, the two most important types of boundary conditions turn out to serve for very useful lower and upper bounds, respectively.

If we equip the Anderson operator  $\Delta^d + \xi$  with zero boundary condition in some finite set  $B \subset \mathbb{Z}^d$ , then the corresponding solution  $u_B$  (see Remark 1.6) may be represented as

$$u_B(t, z) = \mathbb{E}_0 \left[ \exp \left\{ \int_0^t \xi(X(s)) ds \right\} \mathbb{1}_{\{X([0, t]) \subset B\}} \mathbb{1}_{\{X(t) = z\}} \right], \quad (2.7)$$

i.e., the zero boundary condition is translated into the condition that the random walk does not leave  $B$  by time  $t$ . The Laplace operator with zero boundary condition in  $B$  generates the simple random walk before it exits  $B$ , i.e., restricted to not leaving  $B$ . It is clear that  $u_B \leq u$  and that the total mass of  $u_B$ ,

$$U_B(t) = \mathbb{E}_0 \left[ \exp \left\{ \int_0^t \xi(X(s)) ds \right\} \mathbb{1}_{\{X([0, t]) \subset B\}} \right], \quad (2.8)$$

satisfies  $U_B \leq U$ .

Now let  $B = (-R, R]^d \cap \mathbb{Z}^d$  with  $R \in \mathbb{N}$  be a centred box and consider the Anderson operator  $\Delta^d + \xi$  with periodic boundary condition in  $B$ . We obtain a Feynman-Kac formula by noting that the Laplace operator with periodic boundary condition generates the periodised simple random walk,  $X^{(R)} = (X^{(R)}(s))_{s \in [0, \infty)}$ , which can be pathwise realised as  $X^{(R)}(s) = X(s) \bmod B$ . This walk never leaves  $B$ . In plain words, if the walker is at the boundary of  $B$  and decides to jump to the outside of  $B$ , then it re-appears at the opposite side of  $B$ . Hence, we obtain

$$u_B^{(\text{per})}(t, z) = \mathbb{E}_0 \left[ \exp \left\{ \int_0^t \xi(X^{(R)}(s)) ds \right\} \mathbb{1}_{\{X^{(R)}(t) = z\}} \right], \quad (2.9)$$

and for its total mass:

$$U_B^{(\text{per})}(t) = \sum_{z \in B} u_B^{(\text{per})}(t, z) = \mathbb{E}_0 \left[ \exp \left\{ \int_0^t \xi(X^{(R)}(s)) ds \right\} \right]. \quad (2.10)$$

We will see in Sect. 4.3 that, after taking expectation with respect to  $\xi$ ,  $U_B^{(\text{per})}(t)$  turns out to be an upper bound for  $U(t)$ , i.e.,  $\langle U(t) \rangle \leq \langle U_B^{(\text{per})}(t) \rangle$ .

**Remark 2.7 (One particle with random mass)** The Feynman-Kac formulas in (2.2) and (2.7) and all variants will serve not only as starting points for several proofs, but also as settings for our intuition for the interpretation of the solution to the PAM (like also the branching process setting of Sect. 2.1.1). Indeed, we can now imagine that we start with one particle at the origin, carrying a unit mass

at time zero. Then the particle starts its random walking along the trajectory in the Feynman-Kac formula and increases and decreases the mass that it carries according to the potential values that it sees on the way. At time  $t$ , its mass has the current value  $\exp\{\int_0^t \xi(X_s) ds\}$ . Then  $u(t, z)$  is its expectation, restricted to being at site  $z$  at that time. We will often refer to this picture and will talk about the ‘random walker’ or the ‘trajectory of the Feynman-Kac formula’. When we talk about the ‘optimal’ one, then we mean those realisations of the trajectory that gives the best contribution to the expectation, i.e., maximising the value of  $\exp\{\int_0^t \xi(X_s) ds\}$  in comparison to the probability of that trajectory.  $\diamond$

### 2.1.4 Local Times and Moments

The functional in the exponent in the above Feynman-Kac formulas,  $\int_0^t \xi(X(s)) ds$ , is indeed a functional of the *local times* of the walk,

$$\ell_t(z) = \int_0^t \delta_z(X_s) ds, \quad t > 0, z \in \mathbb{Z}^d. \quad (2.11)$$

The family  $(\ell_t(z))_{z \in \mathbb{Z}^d}$  is a random measure on  $\mathbb{Z}^d$  with total mass equal to  $t$ . It registers the amount of time that the random walk spends in  $z$  up to time  $t$ . The *occupation times formula* says that

$$\int_0^t \xi(X_s) ds = \sum_{z \in \mathbb{Z}^d} \xi(z) \ell_t(z). \quad (2.12)$$

Taking into account that the random potential  $\xi$  is i.i.d., we may easily calculate the expectation of the main term in the Feynman-Kac formula:

$$\begin{aligned} \langle e^{\int_0^t \xi(X_s) ds} \rangle &= \left\langle \prod_{z \in \mathbb{Z}^d} e^{\xi(z) \ell_t(z)} \right\rangle = \prod_{z \in \mathbb{Z}^d} \langle e^{\xi(0) \ell_t(z)} \rangle = \prod_{z \in \mathbb{Z}^d} e^{H(\ell_t(z))} \\ &= \exp \left\{ \sum_{z \in \mathbb{Z}^d} H(\ell_t(z)) \right\}, \end{aligned} \quad (2.13)$$

where we recall the logarithmic moment generating function  $H(t) = \log \langle e^{t\xi(0)} \rangle$  from (1.13). Certainly, for this calculation we have to assume that  $H(t)$  is finite for all positive  $t$ , i.e., that all positive exponential moments of  $\xi(0)$  are finite. Using Fubini’s theorem for interchanging the two expectations, we arrive at

$$\langle U(t) \rangle = \mathbb{E}_0 \left[ \exp \left\{ \sum_{z \in \mathbb{Z}^d} H(\ell_t(z)) \right\} \right], \quad (2.14)$$



and similar formulas for the expectations of  $u$ , also for zero and periodic boundary conditions in some box  $B$ .

**Remark 2.8 (Random motions in random media)** We want to give a little guidance to the classification of the PAM within the world of random motions in random media; the vocabulary used in the probability literature has achieved a certain stability in this respect.

By the virtue of the Feynman-Kac formula in (2.2), the PAM is often called a *random walk in random potential*, and  $\xi$  is often called a *potential*. This is one of a handful of fundamental examples of a random motion in the presence of a random medium. Another one is the process  $(\int_0^t \xi(X(s)) ds)_{t \in [0, \infty)}$  that appears in the exponent of the Feynman-Kac formula, which is sometimes called the *random walk in random scenery (RWRS)*. This is an interesting object to study on its own, also in discrete time and for Brownian motion instead of random walks. In recent years, several authors got interested in the description of its extreme behaviour, which, on a technical level, has much to do with the analysis of the PAM, see Sect. 7.4. A third example is the *random walk in random environment (RWRE)*, whose transition probabilities are given by a random field of probability measures in the sites of  $\mathbb{Z}^d$ . Conditional on the environment (i.e., in the quenched setting), is a Markov process, but not in the annealed setting, i.e., when the environment is averaged out. Important examples are the *random walk among random conductances (RWRC)*, which we consider in Sect. 7.9, and the *Bouchaud random walk*, see Sect. 7.9.2. To complete this small list (without considering them further), also *self-interacting random walks* are fundamental, which evolve in time according to random mechanisms depending on the past, often only on the local times produced so far. Important examples here are the *reinforced random walk* and the *myopic random walk* or *true self-repellent random walk*.  $\diamond$

### 2.1.5 Quenched and Annealed Transformed Path Measures

Starting from the Feynman-Kac formula (2.2), it is rather natural to consider the *quenched path measure*

$$Q_{\xi,t}(dX) = \frac{e^{\int_0^t \xi(X(s)) ds}}{U(t)} \mathbb{P}_0(dX), \quad t \geq 0, \quad (2.15)$$

on the set of trajectories  $[0, t] \rightarrow \mathbb{Z}^d$ ,  $[0, t] \ni s \mapsto X(s)$ . Note that  $Q_{\xi,t}$  is a probability measure by (2.2). It obviously depends on the realisation of the potential  $\xi$  and does in general not constitute a consistent family of measures in  $t$ , i.e., they do not come from a path measure for paths  $[0, \infty) \rightarrow \mathbb{Z}^d$  by projection on the time interval  $[0, t]$ .

Like in many models of statistical mechanics, the study of the large- $t$  asymptotics of  $U(t)$  is intimately connected with the question of the behaviour of  $(X(s))_{s \in [0, t]}$  under  $Q_{\xi,t}$ . In particular, we will often discuss intermittency (see Sect. 1.4) in this

view. Plainly, intermittency is reflected by the behaviour of the random walker  $X$  under  $Q_{\xi,t}$  to move quickly through the potential landscape to reach some region(s) of exceptionally high potential and then stay there up to time  $t$ . This would make the integral in the numerator on the right of (2.15) especially large, and it would give much weight to trajectories that end at time  $t$  in such regions. Certainly, these regions are the intermittent islands, and it may a priori be that different trajectories choose different islands. On the other hand, the probability (under  $\mathbb{P}_0$ ) to quickly reach such a distant potential peak may be rather small, since the optimal regions are typically far away. Hence, the main mass in  $Q_{\xi,t}$  comes from paths that find a good compromise between the high potential values and the far distance, and so does the main contribution to  $U(t)$ . This contribution is mainly given by the absolute height of the peak. The second-order contribution to  $U(t)$  is determined by some finer information, for example, by geometric properties of the potential in that peak.

In analogy, the *annealed path measures* are defined as

$$Q_t(dX) = \frac{\langle e^{\int_0^t \xi(X_s) ds} \rangle}{\langle U(t) \rangle} \mathbb{P}_0(dX) = \frac{e^{\sum_{z \in \mathbb{Z}^d} H(\ell_t(z))}}{\mathbb{E}_0[e^{\sum_{z \in \mathbb{Z}^d} H(\ell_t(z))}]} \mathbb{P}_0(dX), \quad t \geq 0, \quad (2.16)$$

where we recall the local times and the cumulant generating function from Sect. 2.1.4. It might be confusing that the density in (2.16) is *not* chosen as the expectation of the density in (2.15), which would also make perfect sense as an annealed path measure (note that the term ‘annealed’ is not a mathematical term, but depends on the taste and the view of the author). However, the main objective is the analysis of the partition function, i.e., the term  $e^{\int_0^t \xi(X_s) ds}$ , and therefore the impact of its expectation is of principal interest.

In (4.5) below we will see that the density in (2.16) has an attractive effect on the path, as the functional  $\mu \mapsto \exp\{\sum_z H(t\mu(z))\}$ , seen as a map on probability measures on  $\mathbb{Z}^d$ , is convex. Hence, one may already here expect that the walk will, under  $Q_t$ , spread out on a smaller area than the free random walk, i.e., we may expect that  $X_t \ll \sqrt{t}$  as  $t \rightarrow \infty$ , typically under  $Q_t$ . See Sect. 7.5 for results on upper tails for the functional in the exponent and Sect. 7.6 for results on such path measures.

**Example 2.9 (Simple random walk among Bernoulli traps)** The simple case where  $\xi$  is an i.i.d. field and each potential value is either  $-\infty$  or 0 (see Example 1.10) is called *simple random walk among Bernoulli traps*. The solution to the PAM may be seen as the survival probability of the walk. Indeed, let

$$\mathcal{O} = \{z \in \mathbb{Z}^d : \xi(z) = -\infty\}$$

be the set of obstacles or traps, then it is clear that the exponent  $\int_0^t \xi(X(s)) ds$  in the Feynman-Kac formula is equal to  $-\infty$  as soon as the path  $X([0, t])$  hits  $\mathcal{O}$ . This implies that

$$u(t, z) = \mathbb{P}_0(X([0, t]) \subset \mathcal{O}^c, X(t) = z)$$

is the probability that the path does not hit any trap by time  $t$  and ends up at the site  $z$ , and  $U(t)$  is the survival probability. Introducing the stopping time  $T_{\mathcal{O}} = \inf\{t > 0: X(t) \in \mathcal{O}\}$  of the first visit to the obstacles, we may also write  $u(t, z) = \mathbb{P}_0(T_{\mathcal{O}} > t, X(t) = z)$ , and  $U(t) = \mathbb{P}_0(T_{\mathcal{O}} > t)$  is the upper tail of  $T_{\mathcal{O}}$ . Hence, the measure  $Q_{\xi, t}$  defined in (2.15) has the density  $\mathbb{1}\{T_{\mathcal{O}} > t\}/\mathbb{P}_0(T_{\mathcal{O}} > t)$ .

The density of the annealed measure,  $Q_t$ , can easily be calculated from (2.13), since

$$\sum_{z \in \mathbb{Z}^d} H(\ell_t(z)) = (\log p) \sum_{z \in \mathbb{Z}^d} \mathbb{1}\{\ell_t(z) > 0\} = R_t \log p,$$

where  $R_t = |\{X(s): s \in [0, t]\}|$  is the *range* of the walk by time  $t$ , the number of visited sites. Hence, the density of  $Q_t$  with respect to the simple random walk measure is equal to  $e^{-\nu R_t}/\mathbb{E}_0[e^{-\nu R_t}]$ , where  $\nu = -\log p \in (0, \infty)$ . That is, the expected total mass of the solution of the PAM,  $\langle U(t) \rangle = \mathbb{E}_0[e^{-\nu R_t}]$ , is equal to a negative exponential moment of the range. The large- $t$  study of the latter has been called the *range problem*.

The intermittent islands are the ones where  $u(t, \cdot)$  achieves its maximum zero, the trap-free zones. It will turn out that these islands depend on  $t$  and are rather large; in fact, in the annealed setting their radius is of order  $t^{1/(d+2)}$ , and in the quenched setting they are of order  $(\log t)^{1/(d+2)}$ .

Let us mention that a discussion of general trapping problems from a physicist's and a chemist's point of view, including a survey on related mathematical models and a collection of open problems, is provided in [HolWei94].  $\diamond$

**Example 2.10 (Brownian motion among Poisson traps)** Recall the trapped Brownian motion of Example 1.15 and look at the special case  $V(x) = -\infty \times \sum_i \mathbb{1}\{x \in K_a(x_i)\}$ , where we use  $K_a(x)$  to denote the ball with radius  $a$  around  $x$ , and  $(x_i)_i$  is a Poisson point process in  $\mathbb{R}^d$  with intensity  $\nu \in (0, \infty)$ . Hence,  $V(x)$  is equal to  $-\infty$  in the  $a$ -neighbourhood of the union of the Poisson points. The solution  $u$  to the PAM is also known under the name *Brownian motion among Poisson traps* or *Brownian motion among Poisson obstacles*, as it is equal to a survival probability. Indeed, let  $\mathcal{O} = \bigcup_i K_a(x_i)$  be the union of the  $a$ -balls around the Poisson points, then  $V(x) = -\infty \mathbb{1}\{x \in \mathcal{O}\}$ . Consider the stopping time  $T_{\mathcal{O}} = \inf\{t > 0: Z_t \in \mathcal{O}\}$ , the first entry time of the Brownian motion  $(Z(s))_{s \in [0, \infty)}$  into the obstacle set  $\mathcal{O}$ , then we have  $\int_0^t V(Z(s)) ds = -\infty \mathbb{1}\{T_{\mathcal{O}} \leq t\}$ . The Feynman-Kac representation reads

$$u(t, x) = \mathbb{P}_0(T_{\mathcal{O}} > t, Z(t) \in dx) / dx,$$

i.e.,  $u(t, x)$  is equal to the sub-probability density of  $Z(t)$  on survival in the Poisson field of traps by time  $t$ . The total mass  $U(t) = \mathbb{P}_0(T_{\mathcal{O}} > t)$  is the survival probability by time  $t$ . The analogue of the path measure  $Q_{\xi, t}$  is the conditional distribution given the event  $\{T_{\mathcal{O}} > t\}$ , i.e., it transforms with the Radon-Nikodym density  $\mathbb{1}\{T_{\mathcal{O}} > t\}/U(t)$ .

It is easily seen that the first moment of  $U(t)$  coincides with a negative exponential moment of the volume of the *Wiener sausage*  $S_a(t) = \bigcup_{s \in [0, t]} K_a(Z(s))$ , i.e.,

$$\begin{aligned} \langle U(t) \rangle &= \mathbb{E}_0[\{\mathbb{1}\{Z([0, t]) \cap \mathcal{O} = \emptyset\}\}] = \mathbb{E}_0[\{\mathbb{1}\{\#\{i: x_i \in S_a(t)\} = 0\}\}] \\ &= \mathbb{E}_0[e^{-\nu|S_a(t)|}], \end{aligned} \tag{2.17}$$

where  $\nu \in (0, \infty)$  is the intensity of the Poisson process, and  $|\cdot|$  denotes the Lebesgue measure. For this reason, the analysis of the annealed transformed path measure  $Q_t$  is sometimes called the *Wiener sausage problem*; it was historically the first special case of a PAM for which substantial asymptotic results were derived [DonVar75]; see Sect. 3.5.1.  $\diamond$

## 2.2 Functional Analytic Aspects

It belongs to the standard knowledge of functional analysis that the solution to the heat equation with potential  $\xi$  in a finite box can be represented in terms of an *eigenvalue expansion* (also called *Fourier expansion*), i.e., an expansion with respect to the spectrum of the operator on the right-hand side of (1.1), the *Anderson Hamiltonian*  $\Delta^d + \xi$ . This is one of the most important and fruitful connections of the heat equation with analytic theory. In this section we introduce the relevant notions and recall the most important facts. Recall that we do not put a minus sign in front of the Laplace operator, unlike the mathematical physics community. In particular, we do not speak of the ‘bottom of the spectrum’ but of the ‘top’, and ‘deep valleys’ of the potential are here ‘high exceedances’.

### 2.2.1 Eigenvalue Expansion

We introduce Dirichlet (i.e., zero) boundary condition in a finite set  $B \subset \mathbb{Z}^d$  and denote the Hamilton operator  $\Delta^d + \xi$  by  $\mathcal{H}_B$ . This operator is symmetric and non-negative definite on the Hilbert space  $\ell^2(B)$  of square-integrable functions (vectors) on  $B$ . Furthermore, it has precisely  $|B|$  eigenvalues  $\lambda_1(B) > \lambda_2(B) \geq \lambda_3(B) \geq \dots \geq \lambda_{|B|}(B)$ , and there is an orthonormal basis of  $\ell^2(B)$  consisting of corresponding eigenfunctions (eigenvectors)  $v_1, v_2, v_3, \dots, v_{|B|}$ , which also depend certainly on  $B$ . We always take the *principal eigenfunction*  $v_1$  positive everywhere in  $B$ , while all the other eigenvectors may have positive and negative values in  $B$ . We think of  $v_k$  as being defined on the entire space  $\mathbb{Z}^d$  with  $v_k(z) = 0$  in  $B^c$ . Certainly, all these eigenvalues and eigenvectors are random, as they depend on  $\xi$ , but for a while this will not be important.

Now we consider the solution  $u_B$  of (1.1) in  $B$  with localised initial condition  $u_B(0, \cdot) = \delta_0(\cdot)$ ; see Remark 1.6. It admits the spectral representation (sometimes also called *Fourier expansion* or *spectral decomposition*)

$$u_B(t, \cdot) = \sum_{k=1}^{|B|} e^{t\lambda_k(B)} v_k(0) v_k(\cdot), \quad t \in (0, \infty), \quad (2.18)$$

with respect to the eigenvalues and eigenfunctions. This can also be written as

$$u_B(t, z) = \langle \delta_z, e^{t\mathcal{H}_B} \delta_0 \rangle = (e^{t\mathcal{H}_B} \delta_0)(z), \quad (2.19)$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product in  $\ell^2(\mathbb{Z}^d)$ . The  $\delta_0$  is the initial condition, the  $\delta_z$  refers to the evaluation at  $z$ , which can also be seen as a terminal condition.

One of the most obvious nice things about (2.18) is that the time-dependence sits exclusively in the exponent as a prefactor of the eigenvalues. In particular, one immediately sees that the main part of the large- $t$  behaviour of  $u_B(t, \cdot)$  should come from the principal eigenvalue,  $\lambda_1(B)$ . This fact is not drastically altered if the box  $B$  depends on  $t$  and grows to  $\mathbb{Z}^d$  for large  $t$ , but will need some more care; actually there are cases in which the main contribution does not come from the first eigenvalue, but from another eigenvalue  $\lambda_k$  that has a better value of  $v_k(0)$ .

Some drawback about (2.18) is that there is a priori no version for  $B = \mathbb{Z}^d$ , at least not for random i.i.d. potentials  $\xi$ , unlike the Feynman-Kac formula. Versions of (2.18) on the entire space  $\mathbb{Z}^d$  require that the potential decreases to  $-\infty$  far out (i.e.,  $\lim_{|z| \rightarrow \infty} \xi(z) = -\infty$ ), as this implies that the Hamiltonian  $\Delta^d + \xi$  has a compact resolvent on  $\ell^2(\mathbb{Z}^d)$ , but this is not satisfied for an i.i.d. potential  $\xi$ , unlike in trivial cases. I am not aware of versions of (2.18) on  $\mathbb{Z}^d$  that work under conditions like (1.5), but possibly they would not be too helpful.

### 2.2.2 Relation Between Eigenvalue Expansion and the PAM

The eigenvalue expansion in (2.18) yields an instructive explanation of the large- $t$  asymptotics from a spectral point of view and serves as a starting point for powerful proofs, see also Sect. 2.2.3. Let us illustrate some of the benefits for the study of the PAM that (2.18) offers.

**Rayleigh-Ritz formula.** One certainly guesses that the large- $t$  asymptotics of the function  $u_B(t, \cdot)$  should be mainly governed by the principal eigenvalue,  $\lambda_1 = \lambda_1(B)$ , and this is true for many levels of precision. Therefore, the *Rayleigh-Ritz formula*

or *Rayleigh-Ritz principle* is of high interest:

$$\begin{aligned}
 \lambda_1(B) &= \sup_{v \in \ell^2(\mathbb{Z}^d): \text{supp}(v) \subset B, \|v\|_2=1} \langle \mathcal{H}_B v, v \rangle \\
 &= - \inf_{v \in \ell^2(\mathbb{Z}^d): \text{supp}(v) \subset B, \|v\|_2=1} \left( \frac{1}{2} \sum_{x,y \in \mathbb{Z}^d: x \sim y} (v_x - v_y)^2 - \sum_{z \in B} \xi(z) v_z^2 \right),
 \end{aligned} \tag{2.20}$$

where we wrote  $x \sim y$  to denote that  $x$  and  $y$  are nearest neighbours, i.e., they differ in precisely one component and precisely by one. (In the sum on  $x, y$  we mean the sum on the ordered pairs  $(x, y)$ , i.e.,  $(x, y) \neq (y, x)$  for  $x \neq y$ , which gives rise to the prefactor of  $\frac{1}{2}$ .) We remark that the first sum in the second line can be restricted to the sum on  $x$  and  $y$  in  $B$  and its outer boundary. The variational formula on the right-hand side has precisely one solution up to a multiplicative constant, and this is  $v_1$ , which is a positive vector.

**Upper estimates for  $u_B$ .** There is a standard way to estimate the total mass of  $u_B$  in terms of the principal eigenvalue with the help of the Cauchy-Schwarz inequality ( $\langle f, g \rangle \leq \|f\|_2 \|g\|_2$  for any  $f, g \in \ell^2(B)$ ) and Parseval's identity ( $\sum_{k=1}^{|B|} \langle v_k, f \rangle^2 = \|f\|_2^2$  for any  $f \in \ell^2(B)$ ) as follows:

$$\begin{aligned}
 U_B(t) &= \sum_{z \in B} u_B(t, z) = \sum_{k=1}^{|B|} e^{t\lambda_k} \langle v_k, \delta_0 \rangle \langle v_k, \mathbb{1} \rangle \\
 &\leq \left( \sum_{k=1}^{|B|} e^{t\lambda_k} \langle v_k, \delta_0 \rangle^2 \right)^{1/2} \left( \sum_{k=1}^{|B|} e^{t\lambda_k} \langle v_k, \mathbb{1} \rangle^2 \right)^{1/2} \\
 &\leq e^{t\lambda_1} \left( \sum_{k=1}^{|B|} \langle v_k, \delta_0 \rangle^2 \right)^{1/2} \left( \sum_{k=1}^{|B|} \langle v_k, \mathbb{1} \rangle^2 \right)^{1/2} \\
 &= e^{t\lambda_1} \|\delta_0\|_2 \|\mathbb{1}\|_2 = e^{t\lambda_1} \sqrt{|B|}.
 \end{aligned} \tag{2.21}$$

See Remark 3.1 for the approximation of  $U$  with  $U_B$  for large boxes  $B$ .

**Lower estimates for  $u_B$ .** In the investigation of the PAM, it turned out useful to reverse the estimate in (2.21), i.e., to estimate the eigenvalue  $\lambda_1$  in terms of the solution  $u_B$ , with the help of the expansion in (2.18). This seems difficult on the first sight, since all eigenfunctions  $v_k$ , with the exception of  $v_1$ , assume positive and negative values. However, if one plays with the initial condition, this problem is removed. So let us denote by  $u_B^{(y)}$  the solution to (1.1) with initial condition

$u_B^{(y)}(0, \cdot) = \delta_y(\cdot)$  instead of  $\delta_0(\cdot)$ , then we can estimate, using that every  $v_k$  is  $\ell^2$ -normalised,

$$e^{t\lambda_1} \leq \sum_{k=1}^{|B|} e^{t\lambda_k} = \sum_{k=1}^{|B|} e^{t\lambda_k} \sum_{x \in B} v_k(x)^2 = \sum_{x \in B} \sum_{k=1}^{|B|} e^{t\lambda_k} \langle v_k, \delta_x \rangle^2 = \sum_{x \in B} u_B^{(x)}(t, x). \quad (2.22)$$

Applying the Feynman-Kac formula in (2.7), adapted to the initial condition  $\delta_y$ , we arrive at expressions that can be handled further with the same means as  $U_B(t) = \sum_{x \in B} u_B(t, x)$ , and they give the same leading asymptotics.

### 2.2.3 Anderson Localisation

One of the great sources of interest in the random Schrödinger operator  $\Delta^d + \xi$  is the fact that its spectral properties help describing electrical conductance properties of alloys of metals or optical properties of glasses with random impurities. Therefore, one is naturally interested in *bounded* potentials, as the potential generically models the concentration ratio of the two metals in the conductance application.

Another source of interest in the spectrum of  $\Delta^d + \xi$  is the study of the Anderson Schrödinger equation in (1.4), which differs from the PAM by the additional prefactor of the imaginary unit  $i$  on the left-hand side of (1.1) and models dynamical quantum mechanical processes in a random environment. Note that in the large- $t$  analysis of this equation, the entire spectrum is involved, in contrast to the large- $t$  asymptotics of the PAM, where only the top of the spectrum is involved.

The driving force for studying the spectral properties of  $\Delta^d + \xi$  comes from the exciting prediction of P.W. Anderson [And58] that the spectrum should have a peculiar behaviour, which in a way interpolates between the smoothing effect of the Laplace operator, whose ‘eigenfunctions’ are infinitely spread out in  $\mathbb{Z}^d$ , and the localising effect of the multiplication operator  $\xi$ , which has only delta functions as eigenfunctions. He predicted that, at least in its spectrum close to the spectral ends (we are thinking of a bounded random potential  $\xi$ , for which also the spectrum of  $\Delta^d + \xi$  is bounded), all eigenfunctions of  $\Delta^d + \xi$  should be exponentially localised. More precisely, for all eigenvalues close to any of the two boundaries of the spectrum, the corresponding eigenfunction should decay exponentially fast away from its individual (random) localisation centre. This predicted phenomenon is nowadays called *Anderson localisation*. It was the motivation of an intense research activity in the last decades, and its validity has meanwhile been confirmed in a great number of cases, after the invention of deep mathematical tools. See [Kir10] for an extensive, and pedagogically written, survey on Anderson localisation and further reading.

### 2.2.4 Intermittency and Anderson Localisation

Let us explain how Anderson localisation is related with intermittency in the PAM. The starting point is the spectral representation in (2.18) with a large box  $B$  (depending on  $t$ ) such that  $u_B$  is a good approximation for  $u$  (see Remark 3.1). In the limit  $t \rightarrow \infty$ , we can neglect all the summands in (2.18) with large  $k$ , because the exponential term  $e^{i\lambda_k}$  makes them negligible in comparison to the leading terms  $e^{i\lambda_1}, e^{i\lambda_2}, e^{i\lambda_3}, \dots$ . According to the Anderson localisation prediction, at least for small  $k$ , the eigenfunctions  $v_k$  should be exponentially localised in centres  $x_k$ . Here we anticipate that the localisation property, which is predicted by Anderson localisation theory only in the entire space  $\mathbb{Z}^d$ , persists to large boxes. Moreover, as extreme-value statistics predicts (see Sect. 6.4 below), these centres are far away from each other, since they form a Poisson point process, after rescaling (see Sect. 6.3 below). Hence,  $v_k$  should be small outside a finite neighbourhood of  $x_k$  and even extremely small in neighbourhoods of the other  $x_i$ s and in the origin. Hence,  $u_B(t, x_k + \cdot)$  is well-approximated in a neighbourhood of zero by just the  $k^{\text{th}}$  term,  $e^{i\lambda_k} v_k(0) v_k(x_k + \cdot)$ . As a consequence, the field  $u_B(t, \cdot)$  has high peaks in small islands (the neighbourhoods of the localisation centres of the leading eigenvalues), which are far away from each other, and is much smaller outside these islands. This is a clear picture of intermittency. Additionally, we also see that the solution  $u(t, \cdot)$  should be shaped like the eigenfunctions in these islands.

### 2.2.5 Integrated Density of States

We saw in Sect. 2.2.4 that large- $t$  asymptotics of the PAM have much to do with the top of the spectrum (eigenvalues and the corresponding eigenfunctions) of the Anderson Hamiltonian  $\Delta^d + \xi$  with zero boundary condition in large boxes. Another explicit manifestation of this relation is in terms of *Lifshitz tails*, which describe the upper tails of the *integrated density of states (IDS)*.

One definition of the integrated density of states is as follows, see [CarLac90, Kir10]. In order to be consistent with the literature, we consider the operator  $-\Delta^d - \xi$ . By  $(-\Delta^d - \xi)_{B_R}$  we denote its restriction to the box  $B_R = [-R, R] \cap \mathbb{Z}^d$  with zero boundary condition. Denote by  $E_1 < E_2 \leq E_2 \leq \dots \leq E_{|B_R|}$  its eigenvalues, counted with multiplicity (and of course depending on  $R$ ). Let  $N_R = \sum_k \delta_{E_k}$  denote its spectral measure. For an energy  $E \in \mathbb{R}$ , let

$$\mu_R(E) = N_R((-\infty, E]) \quad (2.23)$$

denote the number of eigenvalues  $\leq E$  of  $(-\Delta^d - \xi)_{B_R}$ . Then, by the subadditive ergodic theorem, the limit

$$\mu(E) = \lim_{R \rightarrow \infty} \frac{1}{|B_R|} \mu_R(E) \quad (2.24)$$



exists and is almost surely constant. The function  $\mu$  is called the *integrated density of states (IDS)*. The interpretation of  $\mu(E)$  is the number of energy levels of  $-\Delta^d - \xi$  below  $E$  per unit volume. Note that  $\mu(E) \in [0, 1]$ , since the  $B_R \times B_R$ -matrix  $(-\Delta^d - \xi)_{B_R}$  cannot have more eigenvalues than the cardinality of  $B_R$ . After shifting and rescaling,  $\mu$  is a distribution function, i.e., it is increasing and right-continuous with left limits and boundary values 1 as  $E \uparrow \sup \sigma(-\Delta^d - \xi)$  and 0 as  $E \downarrow \inf \sigma(-\Delta^d - \xi)$ , where  $\sigma(\mathcal{H})$  denotes the spectrum of an operator  $\mathcal{H}$ .

The IDS is related to the PAM as follows. Let

$$\mathcal{L}(N_R, t) = \int_{\mathbb{R}} e^{-\lambda t} N_R(d\lambda) = \sum_k e^{-tE_k} \quad (2.25)$$

be the Laplace transform of  $N_R$  evaluated at  $t > 0$ . Using the eigenvalue expansion in (2.18) and the finite-box Feynman-Kac formula in (2.7), we have the representation

$$\mathcal{L}(N_R, t) = \sum_{z \in B_R} \mathbb{E}_z \left[ e^{\int_0^t \xi(X_s) ds} \mathbb{1}\{X_{[0,t]} \subset B_R\} \mathbb{1}\{X_t = z\} \right], \quad (2.26)$$

i.e., the sum over  $z \in B_R$  of solutions to the PAM with initial condition  $\delta_z$ , evaluated at  $z$ ; see (2.22). The existence of the limit in (2.24) is proved by showing that  $\frac{1}{|B_R|} N_R$  has an almost sure limit  $N$ , and this in turn is proved by showing that  $\frac{1}{|B_R|} \mathcal{L}(N_R, t)$  has a non-trivial limit for any  $t$ . Using the ergodic theorem in (2.26), it is not difficult to prove that, almost surely,

$$\lim_{R \rightarrow \infty} \frac{1}{|B_R|} \mathcal{L}(N_R, t) = \left\langle \mathbb{E}_0 \left[ e^{\int_0^t \xi(X_s) ds} \mathbb{1}\{X_t = 0\} \right] \right\rangle = \langle u(t, 0) \rangle. \quad (2.27)$$

Hence,  $\frac{1}{|B_R|} N_R$  has a limit  $N$  as  $R \rightarrow \infty$ , whose Laplace transform  $\mathcal{L}(N, t)$  is given by the right-hand side of (2.27), and this is equal to the expectation of the solution to the PAM as in (1.1) evaluated at zero. Certainly,  $\mu(E) = N((-\infty, E])$  is the distribution function of  $N$ .

There is also a useful connection between the IDS and the principal eigenvalue in a fixed box [CarLac90, VI.15, p. 311]. Indeed, for any  $R \in \mathbb{N}$ ,

$$\mu(E) \geq \frac{1}{|B_R|} \langle \mu_R(E) \rangle \geq \frac{1}{|B_R|} \sum_k \text{Prob}(E_k \leq E) \geq \frac{1}{|B_R|} \text{Prob}(E_1 \leq E). \quad (2.28)$$

This connection was utilized in [Fuk09b] for deriving relations between asymptotics of  $\mu(E)$  for  $E \downarrow \inf \sigma(-\Delta^d - \xi)$  (see Sect. 2.2.6) and the almost sure asymptotics of the principal eigenvalue in large boxes.

### 2.2.6 Lifshitz Tails

Roughly speaking, the logarithmic asymptotics of the IDS  $\mu(E)$  (as defined in (2.24) above) for  $E \downarrow \inf \sigma(-\Delta^d - \xi)$  are called the *Lifshitz tails* of the operator  $-\Delta^d - \xi$ , see [CarLac90, Kir10]. They are of high interest for the description of the spectrum of  $-\Delta^d - \xi$  close to its bottom. But they are intimately connected with the large- $t$  asymptotics of the PAM by the fact that they stand in a one-to-one connection with its Laplace transform  $\mathcal{L}(N, t)$  of the IDS  $N$ . According to (2.27), we have  $\mathcal{L}(N, t) = \langle u(t, 0) \rangle$ , where  $u$  is the solution to the PAM with initial condition  $\delta_0$ .

In this book we are mostly concerned with moment asymptotics for the total mass (rather than  $u(t, 0)$ ), in particular in Chap. 3. However, it is not difficult (see below) to see that

$$\mathcal{L}(N, t) \approx \langle U(t) \rangle \quad \text{as } t \rightarrow \infty, \quad (2.29)$$

where we mean by ‘ $\approx$ ’ logarithmic equivalence, i.e., the quotient of the logarithms of the two sides converges to one. Based on (2.29), one easily derives Lifshitz tails from large- $t$  moment asymptotics of the total mass. We refer to Remark 3.21 for an explicit example of such assertion that has been proved in the literature.

A proof of (2.29) is not very difficult. Indeed, ‘ $\leq$ ’ is obvious from the Feynman-Kac formula in (2.2) (just drop the indicator on  $\{X(t) = 0\}$ ), and one obtains a lower bound for  $\mathcal{L}(N, t)$  by inserting, on the right-hand side of (2.27), the indicator on  $\{X([0, t]) \subset B\}$  for any set  $B$ . A good choice for  $B$  is a  $t$ -dependent large centred box; see Remark 3.1. Expanding this in an eigenvalue series, we obtain

$$\mathcal{L}(N, t) \geq \left\langle \sum_k e^{t\lambda_k(B)} v_k(0)^2 \right\rangle \geq \langle e^{t\lambda_1(B)} v_1(0)^2 \rangle.$$

(Note that it was the coincidence of the initial and terminal conditions that enabled us to drop all other summands.) Now some technical work is required to deduce that the term  $v_1(0)^2$  is negligible. The fact that  $\langle e^{t\lambda_1(B)} \rangle \approx \langle U(t) \rangle$  can be proved starting from the estimate in (2.21) and using Remark 3.1.

## 2.3 First Heuristic Observations

Based on the probabilistic and the functional analytic considerations in Sects. 2.1 and 2.2, let us give now some heuristics about what to expect in the description of the solution of the PAM in the long-time limit.

### 2.3.1 The Total Mass as an Exponential Moment

The first observation is that, via the Feynman-Kac formula in (2.3),  $U(t)$  is equal to the  $t$ -th positive exponential moment of the quantity

$$Y_t = \frac{1}{t} \int_0^t \xi(X_s) ds,$$

the average of the potential values along the random walk path. (The quantity  $tY_t$  is sometimes called *random walk in a random scenery*, see Remark 2.8.) It is a well-known fact from standard probability theory that, for any random variable  $Y$ , we have  $\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}[e^{tY}] = \text{esssup } Y \in (-\infty, \infty]$ . Hence, the limiting exponential growth rate of  $U(t)$  as  $t \rightarrow \infty$  will have much to do with the maximisation of  $Y_t$  over the probability space.

Actually, this maximisation has to be put into the right balance with the limiting behaviour of  $Y_t$  as  $t \rightarrow \infty$ , i.e., with the prefactor  $t$  in the exponent. If one considers the expectation of  $U(t)$  with respect to  $\xi$ , one has to find a balance between the two random objects, the path and the potential. Certainly, an optimisation of  $Y_t$  is achieved by confining the random walk path  $(X(s))_{s \in [0, t]}$  to an area in which the potential  $\xi$  is extremely large, and in which it does not cost the path too much to stay a long time. This area will be centred around the starting point of the motion, and it will be much smaller than of the size that one knows from the central limit theorem, i.e., its diameter will be much smaller than  $\sqrt{t}$ . On the other hand, picking just one site in which the potential is extremely huge will not necessarily be optimal.

Hence, the upper tails of  $\xi$  (i.e., the asymptotics of  $\text{Prob}(\xi(0) > r)$  for  $r \uparrow \text{esssup } \xi(0)$ ) will be one of the most decisive criteria, since they quantify the probabilistic cost of making the potential large, and they give information about the size of the highest peaks of the potential. The second relevant criterion is the probabilistic cost to confine the motion to the optimal area. The balance between the two strategies is subtle and will be described in detail in Sect. 3.2.

### 2.3.2 Moment Asymptotics Versus Almost Sure Asymptotics

Let us now explain the difference in the thinking about the annealed and the quenched setting. The asymptotics of the moments of  $U(t)$  and its almost sure asymptotics are based on quite different (but related) arguments. The phenomenological difference between the two is the following.

First we consider the moments, i.e., the annealed setting, see also Remark 2.5. From the Feynman-Kac formula in (2.2) we see that the moments of  $U(t)$  are the joint expectations over the path and over the potential. Hence, both random objects can ‘work together’ according to a joint strategy that is a compromise between the two; each of them gives a contribution that is exponentially costly: the potential assumes high values in a suitable area, and the path does not leave that area during the time interval  $[0, t]$ . For making the latter not too costly, the area should be a centred ball. Hence, the main contribution to the moments of  $U(t)$  should come from a self-attractive behaviour of the random walk and an extreme behaviour of the potential. Phrasing it in terms of intermittency, it will turn out that the moments of the total sum over  $z \in \mathbb{Z}^d$  of the solution  $u(t, z)$  is asymptotically already well approximated by just the sub- sum on a much smaller region, which is centred at the origin and has a radius that we will call  $R\alpha(t)$  in Chap. 3. This is an intermittent island, and we want to stress here that

In the annealed setting, just one intermittent island is sufficient, and this island is centred.

**Remark 2.11 (Estimating the probabilistic costs)** Here is a simple rule of thumb for estimating the probabilistic cost for the random walk to stay in a ball of ( $t$ -dependent) radius  $1 \ll r_t \ll \sqrt{t}$  until time  $t$ , the *non-exit probability*. Namely, it is of order  $e^{-O(t/r_t^2)}$ . More precisely, writing ‘ $\asymp$ ’ if the quotient is bounded and bounded away from zero, we have

$$-\log \mathbb{P}_0(X([0, t]) \subset [-r_t, r_t]^d) \asymp \frac{t}{r_t^2}, \quad t \rightarrow \infty, \quad (2.30)$$

where  $\asymp$  means that the quotient of the two sides is bounded and bounded away from zero. This can be seen, with the help of the central limit theorem, as follows. Chop the random walk path into  $t/r_t^2$  pieces of length  $r_t^2$  to see that staying  $t$  time units in a ball with radius  $r$  is equivalent to each of these  $t/r_t^2$  pieces staying in that ball. For each piece, the probability for doing this converges towards some fixed number in  $(0, 1)$ , according to the central limit theorem. According to the Markov property, the total probability for all the pieces to stay within that ball, should be roughly the product of the single probabilities, i.e., a product of  $O(t/r_t^2)$  terms of finite size in  $(0, 1)$ , i.e., of size  $\exp\{-O(t/r_t^2)\}$ .

Another way to see that (2.30) holds (even with explicit identification of the prefactor) is to write  $\mathbb{P}_0(X([0, t]) \subset [-r, r]^d)$  as the total mass  $U_{B_r}(t)$  of the solution to the PAM with potential  $\xi \equiv 0$  in  $B_r = [-r, r]^d \cap \mathbb{Z}^d$  and to use the estimate in (2.21) to obtain that the left-hand side of (2.30) is not smaller than  $-\log \lambda_1(B_r) - \log(2r + 1)^{d/2}$ . Now one needs a rescaling argument for the principal eigenvalue  $\lambda_1(B_r)$  of  $\Delta^d$  in  $B_r$ ; actually, one can show that  $\lambda_1(B_r) \sim r^{-2} \lambda_1^c([-1, 1]^d)$  as  $r \rightarrow \infty$ , where  $\lambda_1^c([-1, 1]^d)$  is the principal eigenvalue of the Laplace operator  $\Delta$  in  $[-1, 1]^d$  with Dirichlet boundary condition. This shows that the left-hand side of (2.30) is

even  $\sim \lambda_1^c([-1, 1]^d)^{\frac{t}{r_t^2}}$ . (Because of the appearance of the term  $-\log(2r+1)^{d/2}$  in the above estimate, the upper limitation  $r_t \ll \sqrt{t}$  has to be sharpened by adding a suitable logarithmic correction.)

Much more precise assertions are possible using the large-deviations principle in Lemma 4.3.  $\diamond$

In contrast, in the almost sure setting (see Chap. 5 for details), the quenched setting, the potential makes no particular effort of any kind, but behaves ‘as usual’. The random path has to cope with that and must ‘make the best’ out of it. Hence, the identification of the almost sure asymptotics depends on a closer analysis of the potential landscape, almost surely for every sufficiently large  $t$ . In fact, within some large ‘macrobox’, one derives the existence of ‘microboxes’ (local, much smaller regions) in which  $\Delta^d + \xi$  possesses particularly large principal Dirichlet eigenvalues. Then one either lower bounds the Feynman-Kac formula for  $U(t)$  by requesting the path to spend almost all its time there (neglecting the travel time to that place), or one lower bounds the principal eigenvalue of the macrobox against the local principal eigenvalue of one of these microboxes.

For the argument to work, one needs a good control on the upper tails of the eigenvalues of  $\Delta^d + \xi$  in local subregions inside a given large a priori box. In particular, one needs control on the probability that the local principal eigenvalues are extremely large. This is achieved by a control on their exponential moments with large prefactor, by use of the exponential Chebyshev inequality. This control in turn is a by-product of the proof of the asymptotics of the moments, since  $\langle U(t) \rangle \approx \langle e^{t\lambda_1} \rangle$ . In this way, the analysis of the moments gives the necessary control on the upper tails of the eigenvalue and serves as an important input in the proof of the almost sure asymptotics.

In this way, we will use just one island (microbox) for a lower bound for the total mass. This estimate turns out to be very satisfactory, as it matches with a corresponding upper bound. However, this a priori does not mean that this island alone approximates the total mass  $U(t)$  so well that the sum coming from the complement of this island is negligible with respect to the sum from that island. In order to achieve this, we a priori need to collect much more such islands, whose family is then called the *intermittent islands* in the sense of Sect. 1.4. Such assertions are handled with the help of spatial extreme-value analysis, which is used to identify the number, location, size and form of such islands. At this point, we would like to stress that

In the quenched setting, there are a priori many intermittent islands, and they are widely spread, randomly located and much smaller than the annealed intermittent island.

It is another, much deeper story to prove that finally indeed just one of these islands (carefully picked) is sufficient to asymptotically exhaust the total mass, see Sect. 6.4.

The above explains only lower bounds (but very good ones). Most of the proofs in the literature for the corresponding upper bound do not reflect any details about the potential landscape and are quite abstract. We present the most successful proof strategies in Chap. 4.

### 2.3.3 Mass Concentration

In Chap. 6, we will go much deeper into the description of the PAM and will reveal much more information about the intermittent islands. This is well explained in terms of the eigenvalue expansion in (2.18):

$$u(t, \cdot) \approx u_{B^{(t)}}(t, \cdot) = \sum_{k=1}^{|B^{(t)}|} e^{t\lambda_k} v_k(0) v_k(\cdot), \quad (2.31)$$

where  $B^{(t)}$  is a centred box that is so large that the first approximation is good enough; see Remark 3.1.

The intermittent islands are equal to the regions where one of the eigenfunctions  $v_k$  has its main mass. More precisely, as Anderson localisation theory predicts, all the leading eigenfunctions  $v_k$  are highly concentrated in a region of small size somewhere in  $B^{(t)}$  and are extremely close to zero everywhere outside. Each of these regions gives rise to a lower bound of the kind that we explained in Sect. 2.3.2. Furthermore, using extreme-value statistics, one can understand and prove that the shape of the potential  $\xi$  and the one of the solution  $u(t, \cdot)$  in these islands approaches a certain deterministic form.

For the lower bound for  $U(t)$  we just considered the first term in the above sum, the one with the largest eigenvalue. However, also the distance of the island to the starting point of the motion plays a rôle. Looking at the eigenvalue expansion in (2.31), this distance is roughly expressed by the term  $v_k(0)$ , by the fact that  $v_k$  is exponentially decreasing away from the centre of that island. Hence, the above heuristics gives the best lower bound by taking that  $k$  that maximises the term  $e^{t\lambda_k} v_k(0)$ . This maximal  $k$  may be different from one. Then the conjecture is tempting that it is just this single summand that gives the overwhelming contribution, which is a rather strong form of intermittency, it is indeed a concentration assertion. This is indeed known for a number of potential distributions, see Chap. 6.

### 2.3.4 Time-Evolution of the Mass Flow

All heuristics so far considered only the situation of the mass flow at a given fixed, large time, i.e., a snapshot. However, one of the main goals is to describe the evolution of the mass flow, i.e., the function  $t \mapsto u(t, \cdot)$ . Making qualitative

statements in this general view is rather difficult. However, in those cases in which the concentration property in just one island holds, we can make much more precise assertions. Indeed, we here can restrict the description of the main mass flow to a description of the time-evolution of the centre of that island. For this process there are explicit formulas available for a number of potential distributions; see Sect. 6.5.

An intuitive description of the main mass flow is as follows (see also [Mör11]). The box  $B^{(t)}$  in (2.31), if it is picked sufficiently large, is the space horizon of the main mass at time  $t$ , i.e., the space in which it is located with high probability at time  $t$ . Now imagine that we look at a movie and let  $t$  increase, assuming that the radius of  $B^{(t)}$  increases accordingly. Then from time to time it happens that this increasing horizon suddenly includes a new, much better local island than all islands that it contained before. Here ‘better’ refers to the relation between size of the local eigenvalue and the distance to the origin, as is expressed by the term  $e^{i\lambda_k} v_k(0)$ . During a small time interval, this island becomes relevant and replaces the island that was optimal before. As a result, the main mass ‘jumps’ to the new island, and the Feynman-Kac formula is from now mainly concentrated on paths that go in short time to this new island and spend there most of the time.

There are two interesting time scales in this picture: the time lag during which a certain island is the optimal one, and the time lag during which the main mass moves from one optimal island to the next one. In all cases that we present in Sect. 6.5, the latter is much shorter than the first one. Furthermore, we also see there that the time lag during which an island is optimal increases from one optimal island to the next one, i.e., the mass flow *ages*; it behaves differently at late times than at early times. The effect of ageing will be highlighted and deepened in Sect. 6.5.

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