Chapter 2
Fixed Point Results and Their Applications in Right Multivariate Fractional Calculus

We introduce a fixed point iterative scheme and use it to approximate a solution of a nonlinear operator equation. Applications are suggested involving in particular right multivariate fractional calculus. It follows [8].

2.1 Introduction

Let B_1, B_2 denote Banach spaces and Ω be a subset of B_1. Let also $\mathcal{L}(B_1, B_2)$ stand for the space of bounded linear operators from B_1 into B_2.

Problems in applied sciences, engineering and other disciplines can be written like

$$S(x) = 0, \tag{2.1.1}$$

where $S : \Omega \to B_2$ is a continuous operator in many cases using Mathematical Modelling [1, 7, 12, 13, 17, 19].

Solving such equations is a challenge. Closed form solutions x^* can be obtained only in some special cases. Therefore, researchers resort mostly to the utilization of iterative methods [1, 7, 12].

In the present chapter we shall approximate x^* with a very general iterative process allowing applications in diverse areas including right multivariate fractional calculus as follows: Let $A(\cdot) : \Omega \to \mathcal{L}(B_1, B_2)$ be a continuous operator and set

$$F = LS, \tag{2.1.2}$$

for some $L \in \mathcal{L}(B_2, B_1)$. The solution x^* is approximated as a limit of the sequence $\{x_n\}$ given for $x_0 \in \Omega$ by the fixed point scheme:

$$\begin{aligned} & x_{n+1} := x_n + w_n, \ A(x_n) w_n + F(x_n) = 0 \\ & \Leftrightarrow w_n = Q(w_n) := (I - A(x_n)) w_n - F(x_n). \end{aligned} \tag{2.1.3}$$

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Clearly, the sequence $\{x_n\}$ given by

$$x_{n+1} = Q(x_n) = Q^{(n+1)}(x_0) \tag{2.1.4}$$

is well defined. Suppose that sequence $\{x_n\}$ converges. Then, we can write:

$$Q^{\infty}(x_0) := \lim_{n\to\infty} \left(Q^n(x_0)\right) = \lim_{n\to\infty} x_n. \tag{2.1.5}$$

Many methods in the literature can be considered special cases of method (2.1.3). We can choose A to be: $A(x) = F'(x)$ (Newton's method), $A(x) = F'(x_0)$ (Modified Newton's method), $A(x) = [x, g(x); F]$, $g : \Omega \to B_1$ (Steffensen's method). Many other choices for A can be found in [1–21] and the references there in. Therefore, it is important to study the convergence of method (2.1.3) under generalized conditions. In particular, we present the semi-local convergence of method (2.1.3) using only continuity assumptions on operator F and for a so general operator A as to allow applications to right multivariate fractional calculus and other areas.

The rest of the chapter is organized as follows: Sect. 2.2 contains the semi-local convergence of method (2.1.3). In the concluding Sect. 2.3, we suggest some applications to right multivariate fractional calculus.

2.2 Convergence

Let $B(x, \xi)$, $\overline{B}(x, \xi)$ stand, respectively for the open and closed balls in B_1 with center $x \in B_1$ and of radius $\xi > 0$.

We present the semi-local convergence of method (2.1.3) in this section.

Theorem 2.1 *Let $F : \Omega \subset B_1 \to B_2$, $A(\cdot) : \Omega \to \mathcal{L}(B_1, B_1)$ and $x_0 \in \Omega$ be as defined in the Introduction. Suppose: there exist $\delta_0 \in (0, 1)$, $\delta_1 \in (0, 1)$, $\eta \geq 0$ such that for each $x, y \in \Omega$*

$$\delta := \delta_0 + \delta_1 < 1, \tag{2.2.1}$$

$$\|F(x_0)\| \leq \eta, \tag{2.2.2}$$

$$\|I - A(x)\| \leq \delta_0, \tag{2.2.3}$$

$$\|F(y) - F(x) - A(x)(y - x)\| \leq \delta_1 \|y - x\| \tag{2.2.4}$$

and

$$\overline{B}(x_0, \delta) \subseteq \Omega, \tag{2.2.5}$$

where

$$\rho = \frac{\eta}{1 - \delta}. \tag{2.2.6}$$

Then, sequence $\{x_n\}$ *generated for* $x_0 \in \Omega$ *by*

$$x_{n+1} = x_n + Q_n^{\infty}(0), \;\; Q_n(w) := (I - A(x_n))\, w - F(x_n) \tag{2.2.7}$$

is well defined in $\overline{B}(x_0, \rho)$*, remains in* $\overline{B}(x_0, \rho)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to* x^* *which is the only solution of equation* $F(x) = 0$ *in* $\overline{B}(x_0, \rho)$*. Moreover, an apriori error estimate is given by the sequence* $\{\rho_n\}$ *defined by*

$$\rho_0 := \rho, \quad \rho_n = T_n^{\infty}(0), \quad T_n(t) = \delta_0 + \delta_1 \rho_{n-1} \tag{2.2.8}$$

for each $n = 1, 2, \ldots$ *and satisfying*

$$\lim_{n\to\infty} \rho_n = 0. \tag{2.2.9}$$

Furthermore, an aposteriori error estimate is given by the sequence $\{\sigma_n\}$ *defined by*

$$\sigma_n := H_n^{\infty}(0), \quad H_n(t) = \delta t + \delta_1 p_{n-1}, \tag{2.2.10}$$

$$q_n := \|x_n - x_0\| \le \rho - \rho_n \le \rho, \tag{2.2.11}$$

where

$$p_{n-1} := \|x_n - x_{n-1}\| \text{ for each } n = 1, 2, \ldots \tag{2.2.12}$$

Proof We shall show using mathematical induction the following assertion is true:

(A_n) $x_n \in X$ and $\rho_n \ge 0$ are well defined and such that

$$\rho_n + p_{n-1} \le \rho_{n-1}. \tag{2.2.13}$$

By the definition of ρ, (2.2.3)–(2.2.6) we have that there exists $r \le \rho$ (Lemma 1.4 [7, pp.3]) such that

$$\delta_0 \tau + \|F(x_0)\| = r$$

and

$$\delta_0^k r \le \delta_0^k \rho \to 0 \text{ as } k \to \infty.$$

That is (Lemma 1.5 [7, pp.4]) x_1 is well defined and $p_0 \le r$.

We need the estimate:

$$T_1(\rho - r) = \delta_0(\rho - r) + \delta_1 \rho_0 =$$

$$\delta_0 \rho - \delta_0 r + \delta_1 \rho = G_0(\rho) - r = \rho - r.$$

That is (Lemma 1.4 [7, pp.3]) ρ_1 exists and satisfies

$$\rho_1 + p_0 \leq \rho - r + r = \rho = \rho_0.$$

Hence (I_0) is true. Suppose that for each $k = 1, 2, \ldots, n,$ assertion (I_k) is true. We must show: x_{k+1} exists and find a bound r for p_k. Indeed, we have in turn that

$$\begin{aligned}\delta_0 \rho_k + \delta_1 (\rho_{k-1} - \rho_k) &= \delta_0 \rho_k + \delta_1 \rho_{k-1} - \delta_1 \rho_k \\ &= T_k (\rho_k) - \delta_1 \rho_k \leq \rho_k.\end{aligned}$$

That is there exists $r \leq \rho_k$ such that

$$r = \delta_0 r + \delta_1 (\rho_{k-1} - \rho_k) \text{ and } (\delta_0 + \delta_1)^i r \to 0 \tag{2.2.14}$$

as $i \to \infty$.

The induction hypothesis gives that

$$q_k \leq \sum_{m=0}^{k-1} p_m \leq \sum_{m=0}^{k-1} (\rho_m - \rho_{m+1}) = \rho - \rho_k \leq \rho,$$

so $x_k \in \overline{B} (x_0, \rho) \subseteq \Omega$ and x_1 satisfies $\|I - A (x_1)\| \leq \delta_0$ (by (2.2.3)).

Using the induction hypothesis, (2.1.3) and (2.2.4), we get

$$\|F (x_k)\| = \|F (x_k) - F (x_{k-1}) - A (x_{k-1}) (x_k - x_{k-1})\| \tag{2.2.15}$$

$$\leq \delta_1 p_{k-1} \leq \delta_1 (\rho_{k-1} - \rho_k)$$

leading together with (2.2.14) to:

$$\delta_0 r + \|F (x_k)\| \leq r,$$

which implies x_{k+1} exists and $p_k \leq r \leq \rho_k$. It follows from the definition of ρ_{k+1} that

$$T_{k+1} (\rho_k - r) = T_k (\rho_k) - r = \rho_k - r,$$

so ρ_{k+1} exists and satisfies

$$\rho_{k+1} + p_k \leq \rho_k - r + r = \rho_k$$

so the induction for (I_n) is completed.

Let $j \geq k$. Then, we obtain in turn that

$$\left\|x_{j+k} - x_k\right\| \leq \sum_{i=k}^{j} p_i \leq \sum_{i=k}^{j} \left(\rho_j - \rho_{j+1}\right) = \rho_k - \rho_{j+k} \leq \rho_k. \tag{2.2.16}$$

We also get using induction that

$$\rho_{k+1} = T_{k+1}\left(\rho_{k+1}\right) \leq T_{k+1}\left(\rho_k\right) \leq \delta\rho_k \leq \ldots \leq \delta^{k+1}\rho. \tag{2.2.17}$$

Hence, by (2.2.1) and (2.2.17) $\lim_{k\to\infty} \rho_k = 0$, so $\{x_k\}$ is a complete sequence in a Banach space X and as such it converges to some x^*. By letting $j \to \infty$ in (2.2.16), we conclude that $x^* \in \overline{B}\left(x_k, \rho_k\right)$. Moreover, by letting $k \to \infty$ in (2.2.15) and using the continuity of F we get that $F\left(x^*\right) = 0$. Notice that

$$H_k\left(\rho_k\right) \leq T_k\left(\rho_k\right) \leq \rho_k,$$

so the apriori bound exists. That is σ_k is smaller in general than ρ_k. Clearly, the conditions of the theorem are satisfied for x_k replacing x_0 (by (2.2.16)). Hence, by (2.2.8) $x^* \in \overline{B}\left(x_n, \sigma_n\right)$, which completes the proof for the aposteriori bound. ■

Remark 2.2 (a) It follows from the proof of Theorem 2.1 that the conclusions hold, if $A(\cdot)$ is replaced by a more general continuous operator $A : \Omega \to B_1$.

(b) In the next section some applications are suggested for special choices of the "A" operators with $\gamma_0 := \delta_0$ and $\gamma_1 := \delta_1$.

2.3 Applications to Right Multivariate Fractional Calculus

Our presented earlier semi-local convergence results, see Theorem 2.1, apply in the next two multivariate fractional settings given that the following inequalities are fulfilled:

$$\|1 - A(x)\|_{\infty} \leq \gamma_0 \in (0, 1), \tag{2.3.1}$$

and

$$\left\|\left(F(y) - F(x)\right)\overrightarrow{i} - A(x)(y - x)\right\| \leq \gamma_1 \|y - x\|, \tag{2.3.2}$$

where $\gamma_0, \gamma_1 \in (0, 1)$, furthermore

$$\gamma = \gamma_0 + \gamma_1 \in (0, 1), \tag{2.3.3}$$

for all $x, y \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$, where $a_i < a_i^* < b_i^* < b_i$, $i = 1, \ldots, k$.

Above $\overrightarrow{i}$ is the unit vector in $\mathbb{R}^k$, $k \in \mathbb{N}$, $\left\|\overrightarrow{i}\right\| = 1$, and $\|\cdot\|$ is a norm in $\mathbb{R}^k$.

The specific functions $A(x)$, $F(x)$ will be described next.

(I) Consider the right multidimensional Riemann–Liouville fractional integral of order $\alpha = (\alpha_1, \ldots, \alpha_k)$ ($\alpha_i > 0$, $i = 1, \ldots, k$):

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \int_{x_1}^{b_1} \ldots \int_{x_k}^{b_k} \prod_{i=1}^{k} (t_i - x_i)^{\alpha_i - 1} f(t_1, \ldots, t_k)\, dt_1 \ldots dt_k, \tag{2.3.4}$$

where Γ is the gamma function, $f \in L_\infty\left(\prod_{i=1}^{k}[a_i, b_i]\right)$, $b = (b_1, \ldots, b_k)$, and $x = (x_1, \ldots, x_k) \in \prod_{i=1}^{k}[a_i, b_i]$.

By [6], we get that $\left(I_{b-}^{\alpha} f\right)$ is a continuous function on $\prod_{i=1}^{k}[a_i, b_i]$. Furthermore by [6] we get that I_{b-}^{α} is a bounded linear operator, which is a positive operator, plus that $\left(I_{b-}^{\alpha} f\right)(b) = 0$.

In particular, $\left(I_{b-}^{\alpha} f\right)$ is continuous on $\prod_{i=1}^{k}\left[a_i^*, b_i^*\right]$.

Thus there exist $x_1, x_2 \in \prod_{i=1}^{k}\left[a_i^*, b_i^*\right]$ such that

$$\begin{aligned} \left(I_{b-}^{\alpha} f\right)(x_1) &= \min \left(I_{b-}^{\alpha} f\right)(x), \\ \left(I_{b-}^{\alpha} f\right)(x_2) &= \max \left(I_{b-}^{\alpha} f\right)(x), \end{aligned} \tag{2.3.5}$$

over all $x \in \prod_{i=1}^{k}\left[a_i^*, b_i^*\right]$.

We assume that

$$\left(I_{b-}^{\alpha} f\right)(x_1) > 0. \tag{2.3.6}$$

Hence

$$\left\| I_{b-}^{\alpha} f \right\|_{\infty, \prod_{i=1}^{k}\left[a_i^*, b_i^*\right]} = \left(I_{b-}^{\alpha} f\right)(x_2) > 0. \tag{2.3.7}$$

Here we define

$$Jf(x) = mf(x), \quad 0 < m < \frac{1}{2}, \tag{2.3.8}$$

for any $x \in \prod_{i=1}^{k}\left[a_i^*, b_i^*\right]$.

Therefore the equation

$$Jf(x) = 0, \quad x \in \prod_{i=1}^{k}\left[a_i^*, b_i^*\right], \tag{2.3.9}$$

has the same solutions as the equation

$$F(x) := \frac{Jf(x)}{2\left(I_{b-}^{\alpha} f\right)(x_2)} = 0, \quad x \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]. \tag{2.3.10}$$

Notice that

$$I_{b-}^{\alpha}\left(\frac{f}{2\left(I_{b-}^{\alpha} f\right)(x_2)}\right)(x) = \frac{\left(I_{b-}^{\alpha} f\right)(x)}{2\left(I_{b-}^{\alpha} f\right)(x_2)} \leq \frac{1}{2} < 1, \ x \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]. \tag{2.3.11}$$

Call

$$A(x) := \frac{\left(I_{b-}^{\alpha} f\right)(x)}{2\left(I_{b-}^{\alpha} f\right)(x_2)}, \quad \forall x \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]. \tag{2.3.12}$$

We notice that

$$0 < \frac{\left(I_{b-}^{\alpha} f\right)(x_1)}{2\left(I_{b-}^{\alpha} f\right)(x_2)} \leq A(x) \leq \frac{1}{2}, \quad \forall x \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]. \tag{2.3.13}$$

Hence the first condition (2.3.1) is fulfilled by

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{\left(I_{b-}^{\alpha} f\right)(x_1)}{2\left(I_{b-}^{\alpha} f\right)(x_2)} =: \gamma_0, \quad \forall x \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]. \tag{2.3.14}$$

Hence $\|1 - A(x)\|_{\infty} \leq \gamma_0$, where $\|\cdot\|_{\infty}$ is over $\prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$. Clearly $\gamma_0 \in (0, 1)$.

Next we assume that $\frac{f(x)}{2\left(I_{b-}^{\alpha} f\right)(x_2)}$ is a contraction, that is

$$\left| \frac{f(x)}{2\left(I_{b-}^{\alpha} f\right)(x_2)} - \frac{f(y)}{2\left(I_{b-}^{\alpha} f\right)(x_2)} \right| \leq \theta \|x - y\|, \quad \text{all } x, y \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right], \tag{2.3.15}$$

$0 < \theta < 1$.

Hence

$$\left| \frac{mf(x)}{2\left(I_{b-}^{\alpha} f\right)(x_2)} - \frac{mf(y)}{2\left(I_{b-}^{\alpha} f\right)(x_2)} \right| \leq m\theta \|x - y\| \leq \frac{\theta}{2} \|x - y\|, \tag{2.3.16}$$

all $x, y \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$.

Set $\lambda = \frac{\theta}{2}$, it is $0 < \lambda < \frac{1}{2}$. We have that

$$|F(x) - F(y)| \leq \lambda \|x - y\|, \tag{2.3.17}$$

all $x, y \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$.

Equivalently we have

$$|Jf(x) - Jf(y)| \leq 2\lambda \left(I_{b-}^{\alpha} f\right)(x_2) \|x - y\|, \text{ all } x, y \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]. \tag{2.3.18}$$

We observe that

$$\left\| (F(y) - F(x)) \overrightarrow{i} - A(x)(y - x) \right\| \leq$$

$$|F(y) - F(x)| + |A(x)| \|y - x\| \leq \tag{2.3.19}$$

$$\lambda \|y - x\| + |A(x)| \|y - x\| = (\lambda + |A(x)|) \|y - x\| =: (\psi_1),$$

$\forall\, x, y \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$.

By [6], we have that

$$\left|\left(I_{b-}^{\alpha} f\right)(x)\right| \leq \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right) \|f\|_{\infty}, \tag{2.3.20}$$

$\forall\, x \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$, where $\|\cdot\|_{\infty}$ now is over $\prod_{i=1}^{k} [a_i, b_i]$.

Hence

$$|A(x)| = \frac{\left|\left(I_{b-}^{\alpha} f\right)(x)\right|}{2\left(I_{b-}^{\alpha} f\right)(x_2)} \leq \frac{1}{2\left(I_{b-}^{\alpha} f\right)(x_2)} \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right) \|f\|_{\infty} < \infty, \tag{2.3.21}$$

$\forall\, x \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$.

Therefore we get

$$(\psi_1) \leq \left(\lambda + \frac{1}{2\left(I_{b-}^{\alpha} f\right)(x_2)} \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right) \|f\|_{\infty}\right) \|y - x\|, \tag{2.3.22}$$

$\forall\, x, y \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$.

Call

$$0 < \gamma_1 := \lambda + \frac{1}{2\left(I_{b-}^{\alpha} f\right)(x_2)} \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \| f \|_\infty , \tag{2.3.23}$$

and by choosing $(b_i - a_i)$ small enough, $i = 1, \ldots, k$, we can make $\gamma_1 \in (0, 1)$, fulfilling (2.3.2).

Next we call and we need that

$$0 < \gamma := \gamma_0 + \gamma_1 = \left(1 - \frac{\left(I_{b-}^{\alpha} f\right)(x_1)}{2\left(I_{b-}^{\alpha} f\right)(x_2)} \right) +$$

$$\left(\lambda + \frac{1}{2\left(I_{b-}^{\alpha} f\right)(x_2)} \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \| f \|_\infty \right) < 1, \tag{2.3.24}$$

equivalently,

$$\lambda + \frac{1}{2\left(I_{b-}^{\alpha} f\right)(x_2)} \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \| f \|_\infty < \frac{\left(I_{b-}^{\alpha} f\right)(x_1)}{2\left(I_{b-}^{\alpha} f\right)(x_2)}, \tag{2.3.25}$$

equivalently,

$$2\lambda \left(I_{b-}^{\alpha} f\right)(x_2) + \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right) \| f \|_\infty < \left(I_{b-}^{\alpha} f\right)(x_1) , \tag{2.3.26}$$

which is possible for small λ and small $(b_i - a_i)$, all $i = 1, \ldots, k$. That is $\gamma \in (0, 1)$, fulfilling (2.3.3). So our numerical method converges and solves (2.3.9).

(II) Let $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\alpha_i > 0$, $m_i = \lceil \alpha_i \rceil$ ($\lceil \cdot \rceil$ ceiling function), $\alpha_i \notin \mathbb{N}$, $i = 1, \ldots, k \in \mathbb{N}$, and $G \in C^{\sum_{i=1}^{k} m_i - 1} \left(\prod_{i=1}^{k} [a_i, b_i] \right)$, such that

$$0 \neq \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} \ldots \partial x_k^{m_k}} \in L_\infty \left(\prod_{i=1}^{k} [a_i, b_i] \right).$$

Here we consider the multivariate right Caputo type fractional mixed partial derivative of order α:

$$D_{b-}^{\alpha} G(x) = \frac{(-1)^{\sum_{i=1}^{k} m_i}}{\prod_{i=1}^{k} \Gamma(m_i - \alpha_i)} \int_{x_1}^{b_1} \ldots \int_{x_k}^{b_k} \prod_{i=1}^{k} (t_i - x_i)^{m_i - \alpha_i - 1} \cdot \tag{2.3.27}$$

$$\frac{\partial^{\sum_{i=1}^{k} m_i} G(t_1, \ldots, t_k)}{\partial t_1^{m_1} \ldots \partial t_k^{m_k}} dt_1 \ldots dt_k,$$

where again Γ is the gamma function, $b = (b_1, \ldots, b_k)$, $\forall\ x = (x_1, \ldots, x_k) \in \prod_{i=1}^{k} [a_i, b_i]$. Notice here that $m_i - \alpha_i > 0$, $i = 1, \ldots, k$.

By [6], we get that $D_{b-}^{\alpha} G$ is a continuous function on $\prod_{i=1}^{k} [a_i, b_i]$, and it holds that $D_{b-}^{\alpha} G(b) = 0$.

In particular $D_{b-}^{\alpha} G$ is continuous on $\prod_{i=1}^{k} [a_i^*, b_i^*]$, where $a_i < a_i^* < b_i^* < b_i$, $i = 1, \ldots, k$.

Therefore there exist $x_1, x_2 \in \prod_{i=1}^{k} [a_i^*, b_i^*]$ such that

$$\begin{aligned} \left(D_{b-}^{\alpha} G\right)(x_1) &= \min \left(D_{b-}^{\alpha} G\right)(x), \\ \left(D_{b-}^{\alpha} G\right)(x_2) &= \max \left(D_{b-}^{\alpha} G\right)(x), \end{aligned} \tag{2.3.28}$$

over all $x \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

We assume that

$$\left(D_{b-}^{\alpha} G\right)(x_1) > 0. \tag{2.3.29}$$

Hence

$$\left\| D_{b-}^{\alpha} G \right\|_{\infty, \prod_{i=1}^{k} [a_i^*, b_i^*]} = \left(D_{b-}^{\alpha} G\right)(x_2) > 0. \tag{2.3.30}$$

Here we define

$$JG(x) = mG(x), \quad 0 < m < \frac{1}{2}, \tag{2.3.31}$$

for any $x \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

Therefore the equation

$$JG(x) = 0, \quad x \in \prod_{i=1}^{k} [a_i^*, b_i^*], \tag{2.3.32}$$

has the same solutions as the equation

$$F(x) := \frac{JG(x)}{2D_{b-}^{\alpha} G(x_2)} = 0, \quad x \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{2.3.33}$$

Notice that

$$D_{b-}^{\alpha}\left(\frac{G(x)}{2D_{b-}^{\alpha}G(x_2)}\right)=\frac{D_{b-}^{\alpha}G(x)}{2D_{b-}^{\alpha}G(x_2)}\leq\frac{1}{2}<1,\quad x\in\prod_{i=1}^{k}\left[a_i^*,b_i^*\right]. \tag{2.3.34}$$

We call

$$A(x):=\frac{D_{b-}^{\alpha}G(x)}{2D_{b-}^{\alpha}G(x_2)},\quad\forall\, x\in\prod_{i=1}^{k}\left[a_i^*,b_i^*\right]. \tag{2.3.35}$$

We notice that

$$0<\frac{D_{b-}^{\alpha}G(x_1)}{2D_{b-}^{\alpha}G(x_2)}\leq A(x)\leq\frac{1}{2}. \tag{2.3.36}$$

Hence the first condition (2.3.1) is fulfilled by

$$|1-A(x)|=1-A(x)\leq 1-\frac{D_{b-}^{\alpha}G(x_1)}{2D_{b-}^{\alpha}G(x_2)}=:\gamma_0,\quad\forall\, x\in\prod_{i=1}^{k}\left[a_i^*,b_i^*\right]. \tag{2.3.37}$$

Hence

$$\|1-A(x)\|_{\infty}\leq\gamma_0, \tag{2.3.38}$$

where $\|\cdot\|_{\infty}$ is over $\prod_{i=1}^{k}\left[a_i^*,b_i^*\right]$.

Clearly $\gamma_0\in(0,1)$.

Next we assume that $\frac{G(x)}{2\left(D_{b-}^{\alpha}G\right)(x_2)}$ is a contraction, that is

$$\left|\frac{G(x)}{2\left(D_{b-}^{\alpha}G\right)(x_2)}-\frac{G(y)}{2D_{b-}^{\alpha}G(x_2)}\right|\leq\theta\|x-y\|,\ \text{all } x,y\in\prod_{i=1}^{k}\left[a_i^*,b_i^*\right], \tag{2.3.39}$$

with $0<\theta<1$.

Hence

$$\left|\frac{mG(x)}{2\left(D_{b-}^{\alpha}G\right)(x_2)}-\frac{mG(y)}{2\left(D_{b-}^{\alpha}G\right)(x_2)}\right|\leq m\theta\|x-y\|\leq\frac{\theta}{2}\|x-y\|, \tag{2.3.40}$$

all $x,y\in\prod_{i=1}^{k}\left[a_i^*,b_i^*\right]$.

Set $\lambda=\frac{\theta}{2}$, it is $0<\lambda<\frac{1}{2}$. We have that

$$|F(x)-F(y)|\leq\lambda\|x-y\|, \tag{2.3.41}$$

all $x, y \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$.

Equivalently we have

$$|JG(x) - JG(y)| \leq 2\lambda \left(D_{b-}^{\alpha} G\right)(x_2) \|x - y\|, \text{ all } x, y \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]. \quad (2.3.42)$$

We observe that

$$\left\| (F(y) - F(x)) \overrightarrow{i} - A(x)(y - x) \right\| \leq$$

$$|F(y) - F(x)| + |A(x)| \|y - x\| \leq$$

$$\lambda \|y - x\| + |A(x)| \|y - x\| = (\lambda + |A(x)|) \|y - x\| =: (\psi_2), \quad (2.3.43)$$

$\forall\, x, y \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$.

By (2.3.27), we notice that

$$\left|D_{b-}^{\alpha} G(x)\right| \leq \frac{1}{\prod_{i=1}^{k} \Gamma(m_i - \alpha_i)} \cdot$$

$$\left(\int_{x_1}^{b_1} \ldots \int_{x_k}^{b_k} \prod_{i=1}^{k} (t_i - x_i)^{m_i - \alpha_i - 1} dt_1 \ldots dt_k\right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} \ldots \partial x_k^{m_k}} \right\|_{\infty}$$

$$= \frac{1}{\prod_{i=1}^{k} \Gamma(m_i - \alpha_i)} \left(\prod_{i=1}^{k} \frac{(b_i - x_i)^{m_i - \alpha_i}}{m_i - \alpha_i}\right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} \ldots \partial x_k^{m_k}} \right\|_{\infty}$$

$$= \left(\prod_{i=1}^{k} \frac{(b_i - x_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)}\right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} \ldots \partial x_k^{m_k}} \right\|_{\infty}. \quad (2.3.44)$$

We have proved that

$$\left|D_{b-}^{\alpha} G(x)\right| \leq \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)}\right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} \ldots \partial x_k^{m_k}} \right\|_{\infty}, \quad (2.3.45)$$

$\forall\, x \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$, where $\|\cdot\|_\infty$ now is over $\prod_{i=1}^{k} [a_i, b_i]$.

Hence we get

$$|A(x)| \le \frac{1}{2D_{b-}^{\alpha} G(x_2)} \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} \dots \partial x_k^{m_k}} \right\|_\infty < \infty, \tag{2.3.46}$$

$\forall\, x \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$.

Therefore we obtain

$$(\psi_2) \le \left(\lambda + \frac{1}{2D_{b-}^{\alpha} G(x_2)} \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} \dots \partial x_k^{m_k}} \right\|_\infty \right) \|y - x\|, \tag{2.3.47}$$

$\forall\, x, y \in \prod_{i=1}^{k} \left[a_i^*, b_i^*\right]$.

Call

$$0 < \gamma_1 := \lambda + \frac{1}{2D_{b-}^{\alpha} G(x_2)} \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} \dots \partial x_k^{m_k}} \right\|_\infty, \tag{2.3.48}$$

and by choosing $(b_i - a_i)$ small enough, $i = 1, \dots, k$, we can make $\gamma_1 \in (0, 1)$, fulfilling (2.3.2).

Next we call and we need that

$$0 < \gamma := \gamma_0 + \gamma_1 = \left(1 - \frac{D_{b-}^{\alpha} G(x_1)}{2D_{b-}^{\alpha} G(x_2)}\right) +$$

$$\left\{ \lambda + \frac{1}{2D_{b-}^{\alpha} G(x_2)} \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} \dots \partial x_k^{m_k}} \right\|_\infty \right\} < 1, \tag{2.3.49}$$

equivalently,

$$\lambda + \frac{1}{2D_{b-}^{\alpha} G(x_2)} \left(\prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} \dots \partial x_k^{m_k}} \right\|_\infty < \frac{D_{b-}^{\alpha} G(x_1)}{2D_{b-}^{\alpha} G(x_2)}, \tag{2.3.50}$$

equivalently,

$$2\lambda D_{b-}^{\alpha}G(x_2)+\left(\prod_{i=1}^{k}\frac{(b_i-a_i)^{m_i-\alpha_i}}{\Gamma(m_i-\alpha_i+1)}\right)\left\|\frac{\partial^{\sum_{i=1}^{k}m_i}G}{\partial x_1^{m_1}\ldots\partial x_k^{m_k}}\right\|_{\infty}<D_{b-}^{\alpha}G(x_1), \tag{2.3.51}$$

which is possible for small λ and small (b_i-a_i), all $i=1,\ldots,k$. That is $\gamma\in(0,1)$, fulfilling (2.3.3). So our numerical method converges and solves (2.3.32).

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