Chapter 2
Hydrostatics

Abstract Fluid pressure is introduced based on the formulation of the equilibrium equations (force and moment balances). This gives rise to the introduction of body and surface specific forces and the definition of normal shear tractions on surfaces. Liquids in equilibrium are based on the assumption that shear tractions vanish, which, through the equilibrium conditions, yield a unique definition of the concept of 'hydrostatic pressure'. This leads, naturally, to the fundamental equation of hydrostatics, which subsequently is applied to various examples of density preserving liquids: among these are communicating vessels, PASCAL's paradoxon, manometers, hydraulic heavers, buoyancy and stability of floating bodies. Two sections extend this to hydrostatics in accelerated reference systems and pressure distribution in a still atmosphere.

Keywords Equilibrium equations for continuous bodies · Pressure · Hydrostatic equation · PASCAL's paradoxon · Buoyancy of floating bodies · ARCHIMEDES' principle · Hydrostatics in accelerated frames · Pressure distribution in still atmosphere

List of Symbols

Roman Symbols

A, A_x, A_y, A_z	Surfaces with arbitrary spatial orientation, perpendicular to the x-axis, …, ...
$\mathrm{d}A, \mathrm{d}A_x, \ldots$	Surface elements on general plane,—on plane perpendicular to the x-direction
A, a	Area
$\boldsymbol{A}$	Buoyancy force
$\boldsymbol{a}$	Vector in $\mathcal{R}^3$ (mostly an acceleration)
$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}$	Time rate of change of $\boldsymbol{a}$ as referred to an inertial frame
$\frac{\delta \boldsymbol{a}}{\delta t}$	Time rate of change of $\boldsymbol{a}$ relative to a non-inertial frame
$\boldsymbol{b} = \frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}$	Acceleration vector, absolute acceleration
$\boldsymbol{b}_r = \frac{\delta^2\boldsymbol{r}}{\delta t^2}$	Relative acceleration vector
$\boldsymbol{b}_c$	CORIOLIS acceleration: $\boldsymbol{b}_c = 2\boldsymbol{\omega} \times \frac{\delta \boldsymbol{r}}{\delta t}$

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$\boldsymbol{b}_f$	'Guiding' acceleration: $\boldsymbol{b}_f = \frac{\mathrm{d}^2\boldsymbol{r}_0}{\mathrm{d}t^2} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \frac{\delta\boldsymbol{\omega}}{\delta t} \times \boldsymbol{r}$
C_{xy}	Deviator moment of inertia
$D = \rho_K/\rho_f$	Density ratio of a body suspended in a fluid with density ρ_f
$\mathcal{D}$	Symbol for a domain
$\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$	Cartesian basis; unit vectors in the x-, y-, z-directions
F	Force
$\boldsymbol{G}, \boldsymbol{g}$	Gravity force, specific gravity
h, h_1, h_2	Height, depth variables
$h_M = I_S/V_{\mathbb{E}}$	Metacentric height
δh	Difference of heights
I_S	Moment of inertia of the area enclosed by the water line of a ship
I_x, I_y	Moments of inertia
K_O	Total surface force on the body surface ∂V
K_V	Total volume force for the volume V
$\boldsymbol{k}$	Body force per unit mass: $\boldsymbol{k} = (k_x, k_y, k_z)$
M	Mass of a body
ΔM	Mass increment
M_V	Total moment of $\rho\boldsymbol{k}$ for the volume V
$\mathcal{M}$	Metacenter
N^2	Square of the buoyancy frequency: $N^2 = -\frac{\frac{\mathrm{d}\rho}{\mathrm{d}\zeta}(0)g}{\rho(0)}$
n	Polytrope exponent
$\boldsymbol{n}$	Unit normal vector
p	Pressure
$p(\rho)$	Barotropic pressure
R	Gas constant
r	Radius
S	Center of mass of a body
$S_{\mathbb{E}}$	Center of mass of the displaced fluid center
T	Temperature
$\boldsymbol{t}$	CAUCHY Stress tensor
$\boldsymbol{tn}$	Traction on a surface element with unit normal vector $\boldsymbol{n}$
$V, V_{\mathbb{E}}$	Volume of the body, volume of the displacement fluid
$\boldsymbol{v}_a, \boldsymbol{v}_f, \boldsymbol{v}_r$	Absolute velocity, fixed body velocity, relative velocity
$\boldsymbol{x}$	Position vector
(x_C, y_C)	Position of the center of the two-dimensional area of domain $\mathcal{D}$

Greek Symbols

α	Free surface inclination of the fluid in a container with constant acceleration
Δ	LAPLACE operator, $\Delta = \nabla^2$
ΔM	Mass increment
ΔV	Volume increment
ε	Relative density difference: $\varepsilon = (\rho_2 - \rho_1)/\rho_2$
ζ_S	z-coordinate of the center of gravity of an element V of air

$\eta = a/A$	Ratio of two areas
$\delta\phi$	Infinitesimal rotation angle of a ship
ρ_F, ρ_f, ρ_G	Density of a fluid,—gas
ψ	Body force potential
λ	Slope of the water line of a floating body
$\sigma\boldsymbol{n}$	Normal stress of the surface traction $\boldsymbol{tn}$
τ	Shear traction, shear stress
$\boldsymbol{\omega}$	Angular velocity of a non-inertial frame relative to another (inertial) frame
$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$	Centripetal acceleration
$\dot{\boldsymbol{\omega}} \times \boldsymbol{r}$	EULER acceleration

2.1 Some Basic Concepts

Hydrostatics is the science of the **mechanical equilibria** of fluids and gases; when applied exclusively to gases it is also called **aerostatics**. In this chapter our intention is to describe its fundamental rules from a common viewpoint. Before turning to that, however, let us note a few typical features of proper fluids and proper gases. Because of their relatively large density (as compared to gases) proper fluids are nearly density preserving. Both gases and liquids have the tendency to fill the space, which is at their disposal. However, whereas gases do this without any restriction, liquids climb in a capillary and are capable to form droplets, if the spatial dimensions of the liquid masses are sufficiently small, **Fig. 2.1**a, b. These properties of liquids can be traced back to **surface tension**, which, at last, is due to the interactions of the liquid molecules in the immediate vicinity of the boundary of the liquid. Figure 2.1c displays the distribution of the attractive or repelling interaction force, to which two molecules at distance d are exposed; d_0 corresponds approximately to the diameter of the molecules. For gases, the mean diameter of two molecules is usually several mean molecule diameters (approximately $10d_0$ or more); the molecules are separated such that only very weak mutual cohesive interaction forces are effective. Only occasionally two gas molecules approach each other so closely that they repel each other. In these cases the interaction force is very large and of short duration. This justifies a simplified interacting force model, in which the interaction force for gases is restricted to ideal impacts. This also explains, why their densities are smaller than for liquids, and why no surface tension exists in gases.

In solid and fluid bodies the molecules are so densely packed together as is permitted by the repulsive forces. In solid bodies the bonds between the single molecules at a fixed position are permanent, and the molecules oscillate about their equilibrium positions (point $\mathcal{P}$ in Fig. 2.1c). At melting the density of almost all substances falls by a few percents (exception: water), and it is somewhat paradoxical that such a small reduction of the molecular distance gives rise to such large changes of the mobility of the material. One assumes that the molecules combine to clusters and that these clusters move relative to one another, whereby the arrangement of the molecules continuously changes.

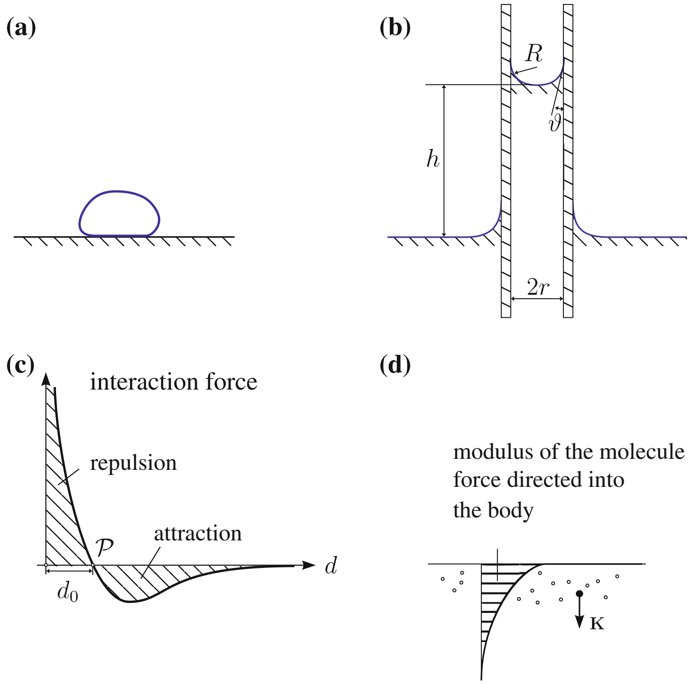

Fig. 2.1 Explaining surface stresses. (**a**) *Small droplet of a fluid of which the shape dependence is due to the surface stresses*. (**b**) *Because of the surface tension a liquid rises in a capillary tube to a height given by the weight of the fluid in the capillary tube and the surface tension acting at the meniscus*. (**c**) *Interaction force between two molecules plotted against the distance of the molecules*. (**d**) *Distribution of the force acting on a molecule as a function of the distance from the bounding surface*

The molecular origin of the surface stresses (tension) lies in the fact that the intermolecular interaction force at larger molecular distances corresponds effectively to an attractive force. In a molecule close to the bounding free surface the interacting force with the neighbouring molecules is reduced to a force directed *into* the fluid, Fig. 2.1d. In other words, a free surface of a liquid has the tendency to diminish its size.

Finally, we remark that, in spite of the above applied molecular structure of liquids and gases, these bodies will be viewed as **continua**, in which each geometric point is occupied by mass and thus can be identified as a material point. If we adopt this viewpoint in the ensuing developments, it is meaningful to define specific quantities of physical variables. In this spirit we call ρ **mass density** or simply **density** for brevity, and regard it as the limit

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V}$$

of the mass ΔM of a fluid element volume ΔV, when the latter becomes vanishingly small. The existence of such limits follows from the continuity assumption and will subsequently tacitly be assumed to hold. Finally, we also will often use the term **fluid particle**. By this we shall not mean the above mentioned molecules, but small elements equipped with fluid mass, which are considered to be marked by some means.

2.2 Fluid Pressure

In order to describe the mechanical equilibrium (**Fig. 2.2**) an arbitrary material volume V with boundary ∂V is cut out of a body. 'Material' means that the boundary ∂V is occupied at all times by the same fluid particles; this means in particular, that no fluid particles cross the boundary ∂V. According to the fundamental laws of mechanics this isolated volume is in mechanical equilibrium, if the body forces (which are acting at the volume element) and the surface forces (acting on the surface elements) reduce to the null force and a null moment (cutting principle). If one denotes by $\boldsymbol{k}$ the **specific body force** (per unit mass) and by $\boldsymbol{t_n}$ the **surface force** (force per unit surface area), also called **traction**, then the mechanical **forces** applied to the body

$$\boldsymbol{K}_V = \int_V \rho \boldsymbol{k}\, \mathrm{d}V, \qquad \boldsymbol{K}_O = \int_{\partial V} \boldsymbol{t_n}\, \mathrm{d}A \tag{2.1}$$

and the mechanical **moments** relative to the origin of the coordinate system

$$\boldsymbol{M}_V = \int_V \boldsymbol{x} \times \rho \boldsymbol{k}\, \mathrm{d}V, \qquad \boldsymbol{M}_O = \int_{\partial V} \boldsymbol{x} \times \boldsymbol{t_n}\, \mathrm{d}A \tag{2.2}$$

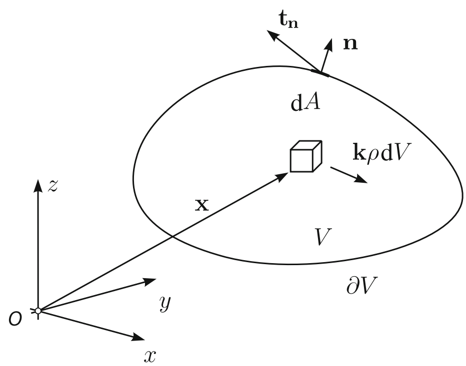

Fig. 2.2 Material volume. *Material volume V with boundary ∂V, the specific body force $\rho \boldsymbol{k}\, \mathrm{d}V$ and the surface force $\boldsymbol{t_n}\, \mathrm{d}A$. O is the origin of the Cartesian coordinates x, y, z*

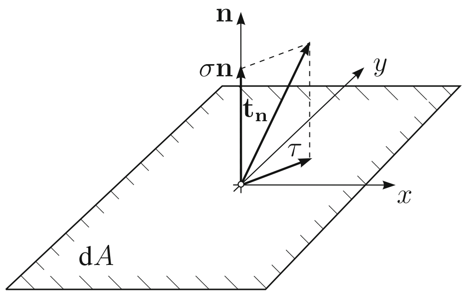

Fig. 2.3 Stress vector. *Surface element* dA *with associated stress vector* $\boldsymbol{t_n}$, *its component normal to the element*, $\sigma\boldsymbol{n}$ *and the shear stress component* $\boldsymbol{\tau}$ *tangential to the surface*

are the total volume and surface forces and the total volume and surface moments with respect to the origin of the coordinates, respectively. $\boldsymbol{x}$ is the **position vector** from the coordinate origin to the volume element dV and $\times$ denotes the vector product in three dimensions. $\boldsymbol{K}_V$ and $\boldsymbol{M}_V$ are therefore the volume forces and volume moments, integrated over the volume of their infinitesimal counterparts, and $\boldsymbol{K}_O$ and $\boldsymbol{M}_O$ are the corresponding surface forces and surface moments. Equilibrium prevails, if

$$\boldsymbol{K}_V + \boldsymbol{K}_O = 0, \qquad \boldsymbol{M}_V + \boldsymbol{M}_O = 0. \tag{2.3}$$

These equations express the force and moment conditions of the mechanical equilibrium.[1]

In the above considerations the stress vector $\boldsymbol{t_n}$ has tacitly been assumed as the cutting force per unit surface. This definition implies that the stress in the interior of a body depends on the position $\mathcal{P}$ as well as on the exterior unit normal vector $\boldsymbol{n}$ of the surface element on which the stress vector is acting. Consequently, for two surface elements with unit normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ in the same point $\mathcal{P}$, one must expect two stress vectors $\boldsymbol{t}_{\boldsymbol{n}_1}$ and $\boldsymbol{t}_{\boldsymbol{n}_2}$, which are generally distinct. The state of stress in the interior of a body is therefore not described by a spatially dependent vector, but rather by a **tensor**, i.e., by a quantity, which depends upon the position $\mathcal{P}$ and the orientation $\boldsymbol{n}$.

The stress vector or the specific surface force $\boldsymbol{t_n}$ can vectorially be decomposed for each surface element dA into a contribution perpendicular to the surface element, $\sigma\boldsymbol{n}$, and a contribution $\boldsymbol{\tau}$, parallel to the surface element, **Fig. 2.3**,

$$\boldsymbol{t_n} = \sigma\boldsymbol{n} + \boldsymbol{\tau}. \tag{2.4}$$

The first contribution on the right-hand side of (2.4), $\sigma\boldsymbol{n}$ is called the **normal stress**; σ can be positive or negative and is then called **tensile stress** or **pressure**. The second

[1]It is assumed that the reader is familiar with the concepts of the mechanics of rigid bodies as commonly taught in elementary mechanics courses at universities in compendia treating 'statics, dynamics and strength of materials'.

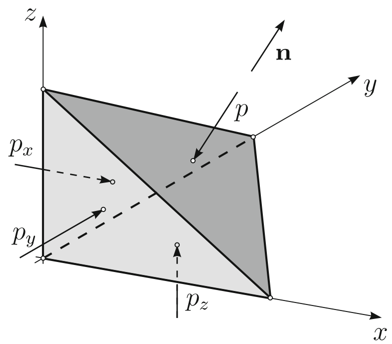

Fig. 2.4 Pressure forces. *Infinitesimal tetrahedron with three side surfaces normal to the coordinate axes, onto which the pressures p_x, p_y, p_z act. Perpendicular to the inclined surface with exterior unit normal vector $\boldsymbol{n}$, the pressure p applies*

quantity is called **shear stress** or **shear traction**.[2] Within the surface element it can have an arbitrary orientation. In Chap. 1, a fluid (gas or liquid) was defined as a material, which under the action of shear stresses is continuously deformed. This statement can now be made more precise by the statement that in a liquid at rest or in a gas at rest, no shear stresses can be withheld.: i.e., $\boldsymbol{\tau} = \boldsymbol{0}$. Furthermore, and excluding side effects such as surface tension, the normal stress is always negative, hence a pressure. This statement may even serve as a definition of a fluid. According to this definition a material is a fluid, if in equilibrium

$$\boldsymbol{t}_{\boldsymbol{n}} = -p\boldsymbol{n} \tag{2.5}$$

holds. The scalar p is called **hydrostatic pressure**, **fluid pressure** or simply **pressure**.

According to the concepts introduced here to for, it is possible that the pressure may depend on the position as well as on the orientation of the surface element at which it applies: $p = p(\boldsymbol{x}, \boldsymbol{n})$. However, the following statement holds: *The hydrostatic pressure p only depends on the position, but not on the orientation of the surface element at which it applies*: $p = p(\boldsymbol{x})$.

To prove this statement, consider the fluid element of **Fig. 2.4**, which has the form of a tetrahedron with three edges parallel to the coordinate axes. If its inclined side surface is denoted by $\mathrm{d}A$, its external unit normal vector by $\boldsymbol{n} = (n_x, n_y, n_z)$ and the remaining side surfaces by $\mathrm{d}A_x$, $\mathrm{d}A_y$, $\mathrm{d}A_z$, then simple geometric considerations show that

$$\mathrm{d}A_x = n_x\,\mathrm{d}A, \quad \mathrm{d}A_y = n_y\,\mathrm{d}A, \quad \mathrm{d}A_z = n_z\,\mathrm{d}A. \tag{2.6}$$

If, moreover, p, p_x, p_y, p_z are the pressures acting on the four bounding surface elements of the infinitesimal tetrahedron, then the resulting cutting forces

$$p\,\mathrm{d}A, \quad p_x n_x\,\mathrm{d}A, \quad p_y n_y\,\mathrm{d}A, \quad p_z n_z\,\mathrm{d}A \tag{2.7}$$

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[2] *Traction* is equivalent to *vector*. At a surface element the traction is decomposed into normal traction and shear traction.

Table 2.1 Units for pressure

Denotation	
Pascal	$1\,\mathrm{Pa} = 1\,\mathrm{N\,m^{-2}} = 1\,\mathrm{kg\,m^{-1}s^{-2}}$
Bar	$1\,\mathrm{bar} = 10^5\,\mathrm{Pa}$
Technical atmosphere	$1\,\mathrm{at} = 1\,\mathrm{kp\,cm^{-2}} = 0.981\,\mathrm{bar}$
Physical atmosphere	$1\,\mathrm{atm} = 760\,\mathrm{Torr} = 760\,\mathrm{mmHg} = 1.013\,\mathrm{bar}$
Torr	$1\,\mathrm{Torr} = 133.3\,\mathrm{mbar} = 13.3\,\mathrm{kPa}$

are of second order small, because they are proportional to the areal increment $\mathrm{d}A$. Body forces such as the weight of the tetrahedral element are of third order small and can be ignored in comparison to the pressure contributions. An example for this omission is the pressure difference between the element $\mathrm{d}A$ and a parallel element through the point $\mathcal{P}$. The three components of the force balance for the element yields

$$p_x n_x \,\mathrm{d}A - p n_x \,\mathrm{d}A = 0, \ldots, \tag{2.8}$$

in which dots indicate cyclic replacements of the indices x, y, z and the omitted statements represent the equilibrium conditions in the y- and z-directions. Therefore, (2.8) implies

$$p_x = p_y = p_z = p. \tag{2.9}$$

The hydrostatic pressure is therefore independent of the orientation of the surface element and a scalar valued function of space. The stress state in a fluid or gas at rest is therefore known once the pressure $p(\boldsymbol{x})$ is known in dependence of the space variable.

The physical dimension for the pressure is force per unit area, viz., $[p] = [FL^{-2}]$; in the International System of Units it is measured in Pascal [Pa], whereby $1\,\mathrm{Pa} = 1\,\mathrm{N\,m^{-2}}$. Other, less common and older units are collected in **Table 2.1**.

2.3 Fundamental Equation of Hydrostatics

The basis for the derivation of the fundamental equation of hydrostatics has already been laid down in the last subsection, the force and moment conditions (2.3) were written down for the forces applied on the body with volume V and boundary ∂V. Since according to the cutting principle the force and moment condition can be applied to any volume V with boundary ∂V, we consider in **Fig. 2.5** a fluid element with the shape of a cube with side lengths $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$. In this volume element also forces which are of third order small are now taken into account. Such a force is the specific body force $\boldsymbol{k} = (k_x, k_y, k_z)$.To third order accuracy $\boldsymbol{k}$ may be assumed to

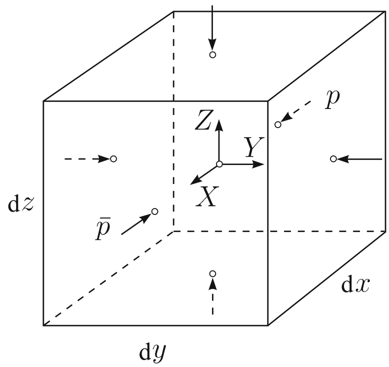

Fig. 2.5 Pressure and body force. *Elementary cube with side lengths* $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$, *loaded by the body forces* $X = \rho k_x$, $Y = \rho k_y$, $Z = \rho k_z$ *and the pressure* $p(\boldsymbol{x})$

apply at the center of gravity of the elementary cube; it consists in most cases just of the specific gravity force and gives rise to the components

$$(\rho k_x, \rho k_y, \rho k_z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{2.10}$$

Moreover, the differences of the wetting forces at opposite side areas must now be accounted for. For example, at the sides perpendicular to the x-axis one has the cutting forces

$$p\,\mathrm{d}y\mathrm{d}z, \qquad \bar{p}\,\mathrm{d}y\mathrm{d}z = \left(p + \frac{\partial p}{\partial x}\mathrm{d}x\right)\mathrm{d}y\mathrm{d}z. \tag{2.11}$$

Force equilibrium in the x-, y-, z-directions yields

$$\begin{aligned}
\rho k_x\,\mathrm{d}x\mathrm{d}y\mathrm{d}z - \left(p + \frac{\partial p}{\partial x}\mathrm{d}x\right)\mathrm{d}y\mathrm{d}z + p\,\mathrm{d}y\mathrm{d}z &= 0,\\
\rho k_y\,\mathrm{d}x\mathrm{d}y\mathrm{d}z - \left(p + \frac{\partial p}{\partial y}\mathrm{d}y\right)\mathrm{d}y\mathrm{d}z + p\,\mathrm{d}z\mathrm{d}x &= 0,\\
\rho k_z\,\mathrm{d}x\mathrm{d}y\mathrm{d}z - \left(p + \frac{\partial p}{\partial z}\mathrm{d}z\right)\mathrm{d}x\mathrm{d}y + p\,\mathrm{d}x\mathrm{d}y &= 0,
\end{aligned} \tag{2.12}$$

from which one easily deduces

$$\frac{\partial p}{\partial x} = \rho k_x, \qquad \frac{\partial p}{\partial y} = \rho k_y, \qquad \frac{\partial p}{\partial z} = \rho k_z, \tag{2.13}$$

equations, which connect the fluid pressure with the specific body force. Equations (2.13) can be summarized by the vector equation

$$\operatorname{grad} p = \rho \boldsymbol{k}, \tag{2.14}$$

an equation which is known as the **fundamental hydrostatic equation**. According to this equation the largest pressure rise points into the direction of the specific body force. Before proceeding with the general theory, let us add a few comments, related to the derivation of (2.14).

- By using the partial derivative in the derivation of (2.11) it was tacitly assumed that the pressure field is differentiable. Indeed, the expression for $\bar{p}$ in $(2.11)_2$ is a truncated Taylor series expansion of order 1 in the x-direction.
- The equilibrium condition for the forces must also be complemented by the equilibrium condition for the moments, and this can be done relative to the center of the cube. The moment of the specific body force, which does not exactly attack at the center, is of fourth order small and can be ignored. The same is true for the moments of the surface forces. The reader may become convinced about this by considering the pressure forces with points of attack, which do not lie in the centers of the surface rectangles. The only terms, which then arise in the momentum condition, are of fourth order small. It follows that the moment condition is automatically satisfied with the satisfaction of the fundamental hydrostatic equation.

Finally, we remark that the fundamental hydrodynamic equations can equally be derived with the above employed procedure. This will be done in the next chapter; here, we only note that owing to the **principle of d'Alembert** one simply needs to complement the specific body force by D'ALEMBERT's **inertial force**.[3] The latter can be written in the form $-\rho \boldsymbol{b}$, where $\boldsymbol{b}$ is the acceleration vector. Thus, Eq. (2.14) assumes the more general form

$$\operatorname{grad} \ p = \rho(\boldsymbol{k} - \boldsymbol{b}) \tag{2.15}$$

and holds in this form for moving accelerating fluids, provided the state of stress can also be reduced to a pure pressure.

The fundamental hydrostatic equations (2.14) can elegantly be derived, if the Gauss law is used in Eqs. (2.1) and (2.2). Indeed, with

$$\begin{aligned}
\boldsymbol{K}_O &= \int_{\partial V} \boldsymbol{t}_{\boldsymbol{n}} \, \mathrm{d}A = -\int_{\partial V} p\boldsymbol{n} \, \mathrm{d}A = -\int_V \operatorname{grad} p \, \mathrm{d}V, \\
\boldsymbol{M}_O &= \int_{\partial V} \boldsymbol{x} \times \boldsymbol{t}_{\boldsymbol{n}} \, \mathrm{d}A = -\int_{\partial V} \boldsymbol{x} \times p\boldsymbol{n} \, \mathrm{d}A = -\int_V \boldsymbol{x} \times \operatorname{grad} p \, \mathrm{d}V,
\end{aligned} \tag{2.16}$$

[3] JEAN LE ROND D'ALEMBERT (1717–1783), mathematician, physicist and philosopher. In mechanics he is best known for his principle, according to which mass times acceleration can be replaced by the inertial force as stated above.

Eq. (2.3) take the forms

$$\int_V (\rho \boldsymbol{k} - \operatorname{grad} p)\, \mathrm{d}V = \boldsymbol{0}\,,$$
$$\int_V \big(\boldsymbol{x} \times (\rho \boldsymbol{k} - \operatorname{grad} p)\big)\, \mathrm{d}V = \boldsymbol{0} \tag{2.17}$$

and are satisfied for an arbitrary, in particular an infinitely small volume, provided the integrand functions in (2.17) vanish. This requirement implies (2.14). On the other hand, the fundamental hydrostatic equation (2.14) implies that equations (2.17) are automatically satisfied. The derivation also implies that satisfaction of $(2.17)_1$ automatically leads to satisfaction of $(2.17)_2$: 'Equilibrium of the forces guarantees equilibrium of the moments'.

Forming the curl of both sides of (2.14) yields

$$\operatorname{curl}(\rho \boldsymbol{k}) = 0. \tag{2.18}$$

Thus, the liquid or gas is only at rest if the body force $\rho \boldsymbol{k}$ is **irrotational** or free of vorticity. If one introduces with

$$-\operatorname{grad} \Psi = \rho \boldsymbol{k}$$

the potential of the body force, Ψ, Eq. (2.14) takes the form

$$\operatorname{grad}(p + \Psi) = 0. \tag{2.19}$$

This equation expresses the fact that the sum $p + \Psi$ must have a constant value, c,

$$p + \Psi = c. \tag{2.20}$$

Equipotential surfaces for Ψ are the same surfaces as those of constant pressure.

For the gravity field we have $\boldsymbol{k} = -g\boldsymbol{e}_z$, where g is the gravity constant and $\boldsymbol{e}_z$ the unit vector against gravity, see **Fig. 2.6**. Equation (2.14) then yields

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho g, \tag{2.21}$$

so that the pressure is only a function of $z : p = p(z)$. The third equation of (2.21) is thus only integrable, if the density ρ is equally at most a function of $z : \rho = \rho(z)$, so that

$$p(z) = p_0 - \int_{z_0}^{z} (\rho g)(\bar{z})\, \mathrm{d}\bar{z}, \tag{2.22}$$

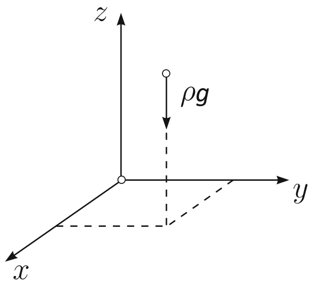

Fig. 2.6 Gravity field for a constant $\boldsymbol{g}$ *Gravity field with body force $-\rho\boldsymbol{g}$ in the negative z-direction. The coordinate system is Cartesian and $\boldsymbol{g}$ is constant*

which simplifies for a density preserving fluid to

$$p(z) = p_0 - \rho g(z - z_0). \tag{2.23}$$

Surfaces of constant pressure are the planes $z = \text{const.}$

In general, one needs for a fluid for unique determination of the pressure not only a functional equation but also a relation between density, pressure and temperature, $p = p(\rho, T)$; this is the **thermal equation of state**. If the variation of the temperature is small and can be ignored, then $p = p(\rho)$. These are material equations and identify different sorts of fluids. Two special cases are

- the *density preserving fluid* with $\rho = \text{const.}$,
- the *barotropic* or *elastic fluid* with $p = p(\rho)$.

Here, the first can be regarded as a special case of the second. For these two cases some basic problems of hydrostatics will now be analyzed.

2.4 Pressure Distribution in a Density Preserving Heavy Fluid

According to Eq. (2.23), the fluid pressure in a density preserving fluid at equilibrium is a linear function of the vertical coordinate. If, for instance a vessel of arbitrary shape, **Fig. 2.7**, is filled with a density preserving fluid and inserted in a gas (air), which can be regarded as incompressible, then a wall element at depth z is subject to two pressures. From the inside the force $p(z)\,\mathrm{d}A$ is acting, whilst $p_0\,\mathrm{d}A$ is acting from the outside, where p_0 is the atmospheric pressure, which may in many situations be assumed to be constant, because the density of the gas and therefore also the pressure variation in the vertical direction are negligibly small. The resulting force onto the wall element has the direction of the exterior unit vector and the modulus

$$\mathrm{d}P = (p - p_0)\,\mathrm{d}A = \rho g z\,\mathrm{d}A. \tag{2.24}$$

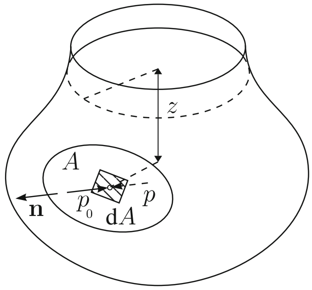

Fig. 2.7 Force onto a wall of a container. *Arbitrary vessel with surface element* dA *and exterior unit normal vector* $\boldsymbol{n}$ *of the wall of the vessel. The pressure inside is given by* p, *that outside is given by* p_0

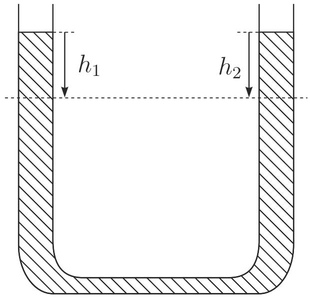

Fig. 2.8 Communicating vessel. *In a communicating vessel the free surfaces of its arms are at the same height*

The resultant of the elementary pressure forces acting on a given partial area A is obtained by reducing all these forces to a single force and possibly a moment in an arbitrary point.

In a communicating vessel, **Fig. 2.8**, the pressures at depth h_1 in the left arm and h_2 in the right arm can be computed with the aid of (2.23); because the two reference points are at the same height, their pressures are the same so,

$$\rho g h_1 = \rho g h_2 \quad \Rightarrow \quad h_1 = h_2. \tag{2.25}$$

In other words: a density preserving fluid in two communicating vessels is only in equilibrium, if both free surfaces are at the same level. Incidentally, this argument can easily be extended to a whole series of communicating vessels.

Equation (2.24) also explains the so-called **Pascal paradox**.[4] To explain it, consider **Fig. 2.10**, in which three vessels, symmetric to their mid plumb line and with the same basal surface but different volumes are shown. The resultant force of the fluid pressure exerted on the basal surface is in the three cases the same, namely a single force acting on the symmetry axis, $p = \rho g h A$, directed downward. For vessels with *freely movable* base, the force $F = P$ would have to be applied to hold the bottom surface at rest, irrespective of the total weight of the fluid contained in the vessel. If on the other hand, each arrangement is weighted, the three arrangements obviously

[4]Blaise Pascal (1623–1662), French mathematician and physicist. He is known for his paradox and the 'Pascal triangle'. For his short biography see **Fig. 2.9**

have different weights. The reasons are the on-loading (Fig. 2.10d) and de-loading (Fig. 2.10e, f) effects of the fluid pressure on the respective walls.

Fig. 2.9 BLAISE PASCAL (19. June 1623–19. Aug. 1662)

BLAISE PASCAL was a French mathematician, physicist, literary man and Christian philosopher. His earliest work was in the natural and applied sciences where he made important contributions to the study of fluids and clarified the concepts of pressure and vacuum by generalizing work of EVANGELISTA TORRICELLI (1608–1647). In 1642, still as a teenager he started pioneering work on calculating machines called Pascalins and finished several of these. He is known in hydrostatics for his paradox, see Fig. 2.10, and in mathematics via his 'Traité du triangle arithmétique' (1653) for the PASCAL Triangle, a tabular presentation for the binomial coefficients

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

The text is based on http://www.wikipedia.org

In **hydraulic presses** the hydrostatic pressure is used to lift large weights with small forces. **Figure. 2.11** explains the principle of such a press machine. The forces F_1 and F_2, which are applied to the platforms A_1 and A_2, respectively, generate immediately below the platforms the pressures $p_1 = F_1/A_1$ and $p_2 = F_2/A_2$. The oil in the communicating vessel with density ρ is in equilibrium, provided

$$\frac{F_1}{A_1} = \frac{F_2}{A_2} + \rho g h, \tag{2.26}$$

where h is the level difference between the platforms. In the practice of hydraulic presses we have $\rho g h \ll F_2/A_2$, so that with sufficient accuracy we have

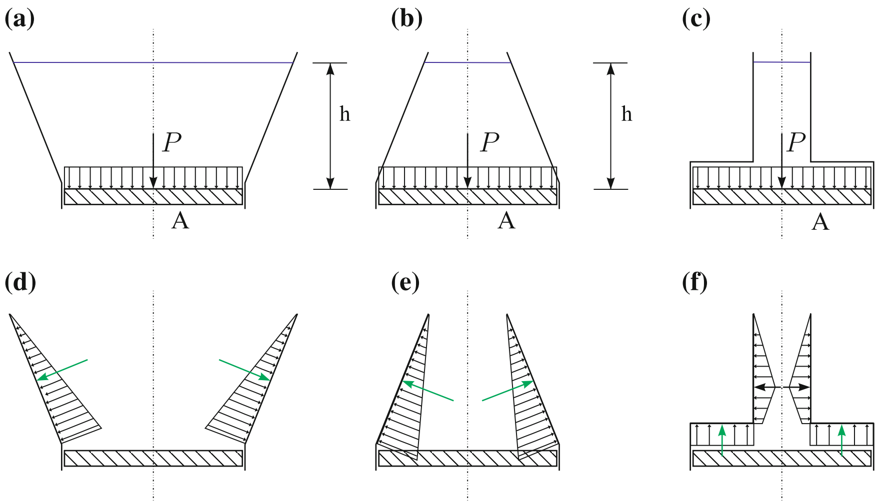

Fig. 2.10 Pascal paradox. *Three vessels, symmetric to their mid plumb line and with identical basal surfaces A but different volumes. In panels* (**a**), (**b**) *and* (**c**) *the pressures are shown which are exerted on the basal surface, all equal, whilst panels* (**d**), (**e**) *and* (**f**) *show the pressure distributions on the walls*

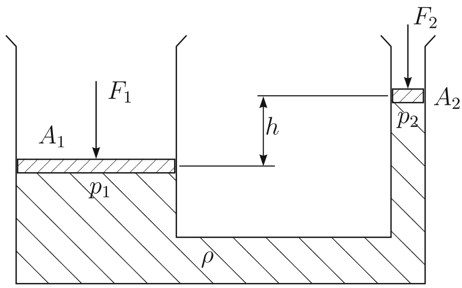

Fig. 2.11 Hydraulic press. *A load F_1 put on a large platform A_1 can be lifted with a small force F_2 on a small platform A_2*

$$\frac{F_1}{A_1} \cong \frac{F_2}{A_2} \quad \text{or} \quad \frac{A_1}{A_2} \cong \frac{F_1}{F_2} \quad \text{or} \quad F_1 \cong \frac{A_1}{A_2} F_2. \tag{2.27}$$

With $A_1 \gg A_2$ the hydraulic press allows to lift a large force F_1 with the application of a small force F_2. We mention, the above calculation assumes a quasi-static lift operation, in which accelerations can be ignored; needless to say that F_1 and F_2 contain the weights of the platforms.

U-tubes or **U-shaped manometers** are essentially also communicating vessels. The arrangement of **Fig.** 2.12 allows determination of the gas pressure at the connecting point of the manometer with the gas vessel by measuring the level difference Δh of the free surfaces in the two arms of the manometer with density ρ_F. Indeed,

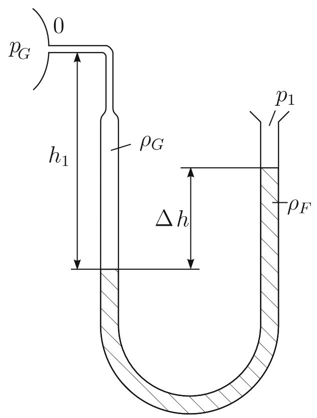

Fig. 2.12 U-tube manometer. *The pressure p_G in the gas-container can be determined by measuring the displacement Δh of the manometer fluid with density ρ_F and the external pressure p_1*

if one measures in the two arms the pressure at the lower of the menisci, once from the gas container and once from the right arm, one obtains, see Fig. 2.12,

$$p_G + \rho_G g h_1 \quad \text{and} \quad p_1 + \rho_F g \Delta h. \tag{2.28}$$

The two expressions must be equal, so that

$$p_G = p_1 + \rho_F g \Delta h \left(1 - \frac{\rho_G h_1}{\rho_F \Delta h}\right). \tag{2.29}$$

Since $\rho_G/\rho_F \ll 1$, this yields approximately

$$p_G - p_1 = \rho_F g \Delta h. \tag{2.30}$$

The pressure difference between the interior of the gas container and its exterior (the atmosphere) is approximately equal to the weight of the displaced manometer fluid.

To *measure small pressure differences* the type of manometer sketched in **Fig. 2.13** can be used. It is a communicating vessel that is filled with two immiscible fluids of different densities. For the determination of the pressure, it is necessary that the meniscus, i.e., interface of the two fluids is positioned in the thin tube-like part of the U-tube, and that it does not leave this part during the experiment. Let us assume that the two cylindrical containers possess the same cross section A and that the connecting tube has a cross sectional area given by a. Moreover, it is assumed that the two fluids with pressures p_1 and p_2 and position of the meniscus at $\mathcal{M}_0$ are in equilibrium (panel (a) in Fig. 2.13). For this arrangement we then have

$$p_1 + \rho_1 g h_1 = p_2 + \rho_2 g h_2. \tag{2.31}$$

With appropriate valves it can be achieved that always the same position $\mathcal{M}_0$ defines this equilibrium. If then the pressure p_2 is increased by Δp, the position $\mathcal{M}_0$

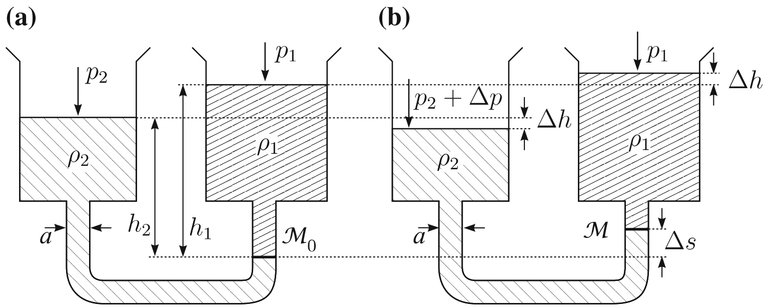

Fig. 2.13 U-tube manometer for the measurement of small pressure differences. *The manometer is filled with fluids of different density* $\rho_1 < \rho_2$. *In panel* (**a**) *it is shown when the two fluids are in equilibrium. Panel* (**b**) *shows the displaced configuration when* Δp *is applied in the left container*

of the meniscus will be displaced by the amount Δs into position $\mathcal{M}$. Because of the supposed density preserving of the two fluids, the free surfaces in the two cylinders will also be shifted by the amount Δh. Mass balance also requests that

$$A\Delta h = a\Delta s. \tag{2.32}$$

Finally, the pressure at the new position of the meniscus $\mathcal{M}$ can be computed for the left and right arms and must be equal to one another. This yields

$$p_1 + \rho_1 g(h_1 + \Delta h - \Delta s) = p_2 + \Delta p + \rho_2 g(h_2 - \Delta h - \Delta s). \tag{2.33}$$

With the aid of (2.31) this equation reduces to

$$\rho_1 g(\Delta h - \Delta s) = \Delta p - \rho_2 g(\Delta h + \Delta s), \tag{2.34}$$

which, owing to (2.32), can be transformed into

$$\Delta s = \frac{\Delta p}{\rho_2 g}\left(\frac{\rho_2 - \rho_1}{\rho_2} + \frac{\rho_2 + \rho_1}{\rho_2}\frac{a}{A}\right)^{-1} = \left(\frac{\Delta p}{\rho_2 g}\right)\underbrace{\frac{1}{\epsilon(1-\eta) + 2\eta}}_{[(1)]} \tag{2.35}$$

with

$$\epsilon := \frac{\rho_2 - \rho_1}{\rho_2} \quad \text{and} \quad \eta = \frac{a}{A}.$$

It is seen that Δs is independent of the absolute pressures p_1 and p_2. The quantities ϵ and η are generally small, implying that the factor [(1)] in (2.35) is large relative to '1'. Consequently, even small pressure differences Δp may generate large displacements Δs. If one chooses $\eta \ll \epsilon$, then the displacement Δs is essentially determined by

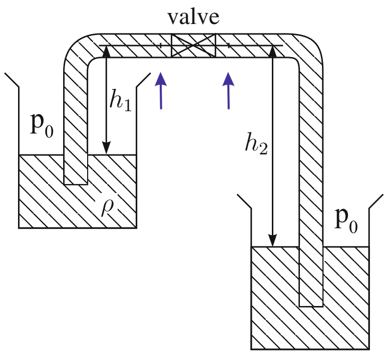

Fig. 2.14 Hydraulic heaver. *The fluid in the hydraulic heaver can be at rest in the displayed position if the valve is closed; else, a volume flow sets in from the upper to the lower container*

the relative density difference. If on the other hand, $\epsilon \ll \eta$, then the relative density difference does not influence Δs but rather the ratio of the areas, η.

In a **hydraulic heaver** two containers, filled with a density preserving fluid and free surface at different levels, may be connected with a U-tube as shown in **Fig. 2.14**. It is assumed that initially the U-tube is equally filled with the fluid and that the valve is closed. With the notations of Fig. 2.14 the pressures left and right of the valve are given by, see (2.23),

$$p_1 = p_0 - \rho g h_1, \qquad p_2 = p_0 - \rho g h_2, \tag{2.36}$$

so that the pressure difference is given by

$$p_1 - p_2 = \rho g (h_2 - h_1) > 0. \tag{2.37}$$

This says, there is a pressure difference (gradient) from the container with higher free surface to that with lower free surface, which is reduced after opening the valve by fluid flow into the container with the lower free surface.

For the determination of the *fluid pressure onto plane* (not necessarily vertical) *walls*, it is advantageous to use a coordinate system with origin, which is the intersecting point between the plane free water surface and the direction of steepest descent, see **Fig. 2.15**. If the vertical coordinate through this point is denoted by z, then the resulting pressure force, applied on a partial surface A can be written as

$$F = \int_A (p - p_0)\, \mathrm{d}A = \rho\, g \sin\alpha \int_A y\, \mathrm{d}A = \rho\, g \sin\alpha\, y_C A, \tag{2.38}$$

in which

$$y_C = \frac{1}{A} \int_A y\, \mathrm{d}A \tag{2.39}$$

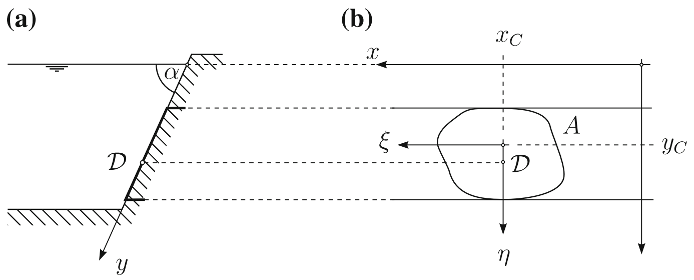

Fig. 2.15 Fluid pressure onto a plane wall inclined by the angle α. (**a**) *View of a container with plane wall, onto which the resulting pressure force, applied onto domain $\mathcal{D}$ (shown as heavy line) is to be determined.* (**b**) *Normal view onto the inclined wall with domain $\mathcal{D}$ of area A, for which the resulting force is to be determined. The x-coordinate lies in the water line, whereas the y-coordinate is in the direction of steepest descent. The origin of the (ξ, η) coordinates is in the center of the area A*

denotes the y-coordinate of the center of the area A. The total pressure force acting on area A equals the product of the area with its resultant pressure force acting on the center of A. In order to find the **center of pressure** $\mathcal{D}$ with the coordinates $(x_\mathcal{D}, y_\mathcal{D})$ one computes the balance of the static moment of the total force F and equates this to the sum of the static moments of the infinitesimal pressure forces $\mathrm{d}p = p\,\mathrm{d}A$ with respect to the coordinate axes; this argument leads to the two equations

$$\begin{aligned} x_\mathcal{D} F &= \int\limits_A x(\rho\, g \sin\alpha\, y)\,\mathrm{d}A = \rho\, g \sin\alpha \int\limits_A x y\,\mathrm{d}A, \\ y_\mathcal{D} F &= \int\limits_A y(\rho\, g \sin\alpha\, y)\,\mathrm{d}A = \rho\, g \sin\alpha \int\limits_A y^2\,\mathrm{d}A. \end{aligned} \tag{2.40}$$

The two integrals on the far right,

$$I_x = \int\limits_A y^2\,\mathrm{d}A, \qquad C_{xy} = -\int\limits_A x y\,\mathrm{d}A, \tag{2.41}$$

only depend on the form and size of the area A, and are called **moments of inertia** of the area with respect to the coordinates x and y. With these denotations and with (2.38) one obtains

$$x_\mathcal{D} = \frac{C_{xy}}{y_C A}, \qquad y_\mathcal{D} = \frac{I_x}{y_C A}. \tag{2.42}$$

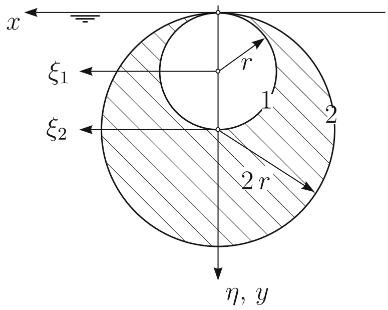

Fig. 2.16 Center of pressure. *Computation of the center of pressure for a moon-like plate*

If one, finally, introduces the moments of inertia with respect to the coordinates through the center of the area A, I_ξ and $C_{\xi\eta}$, and uses the so-called STEINER formulas

$$I_x = I_\xi + y_C^2 A, \qquad C_{xy} = C_{\xi\eta} - y_C x_C A, \tag{2.43}$$

one obtains

$$x_{\mathcal{D}} = x_C - \frac{C_{\xi\eta}}{y_C A}, \qquad y_{\mathcal{D}} = y_C + \frac{I_\xi}{y_C A}. \tag{2.44}$$

Because I_ξ is always positive the center of pressure $y_{\mathcal{D}}$ always lies below the center of the area, y_C. Moreover, if A is symmetric with respect to the η-axis, then $C_{\xi\eta} = 0$ and, consequently, $x_{\mathcal{D}} = x_C$.

Let A be a rectangle with side lengths a and b of which the η-axis agrees with the symmetry axis of the rectangle. Then, $F = \rho g a b y_s$ and $I_\xi = ab^3/12$, so that

$$x_{\mathcal{D}} = x_C, \qquad y_{\mathcal{D}} = y_C + \frac{b^2}{12 y_C}. \tag{2.45}$$

Finally, let us compute the center of pressure for the shaded area of **Fig. 2.16**, formed by two circles, of which the highest points touch the free surface, and lying in a vertical plane. The example illustrates how one can avoid the complicated computations of the moment of inertia I_ξ by employing the moment of inertia for the centers of the circles and the shifting rules of STEINER. It is clear that the center of pressure must lie on the symmetry line (the η- and y-axes). In these coordinates $C_{\xi\eta} = 0$. If one composes the moon-like area by circles and determines pressure centers of the circles individually, then simple computations suffice. Let the subscripts 1 and 2 stand for the small and large circles, respectively. With reference to Fig. 2.16 the resultant pressure forces onto the individual circular disks are (see formulas (2.40))

$$F_1 = \rho g r^3 \pi, \qquad F_2 = 8 \rho g r^3 \pi. \tag{2.46}$$

With the moments of inertia relative to the centers of the circles given by[5]

$$I_{\xi_1} = \frac{1}{4}\pi r^4, \qquad I_{\xi_2} = 4\pi r^4, \tag{2.47}$$

one obtains with the aid of (2.44)$_2$

$$y_{\mathcal{D}_1} = r + \frac{\pi r^4}{4}\frac{1}{\pi r^3} = \frac{5}{4}r, \qquad y_{\mathcal{D}_2} = \frac{5}{2}r. \tag{2.48}$$

To obtain the pressure center of the composed disk (large circle minus small circle) one now expresses the following statement:

Static moment of the difference pressure force $F_2 - F_1 = 7\rho g r^3\pi$ with respect to the x-axis	=	Difference of the static moments of the pressure forces onto the individual disks with respect to the x-axis

or

$$\left\{(7\rho g r^3\pi) y_{\mathcal{D}}\right\} = \left\{-(\rho g r^3\pi)\frac{5}{4}r + (8\rho g r^3\pi)\frac{5}{2}r\right\},$$

which implies

$$y_{\mathcal{D}} = \frac{75}{28}r = 2.68r. \tag{2.49}$$

Incidentally, the center of mass of the moon-like disk is at $y_C = \frac{7}{3}r = 2.33r$ and thus, as expected, above the pressure center.

Evaluation of the resulting pressure force on curved walls is similarly conducted. In such computations the pressure, which is perpendicular to the areal increment dA must be decomposed into components parallel to the Cartesian coordinates and only

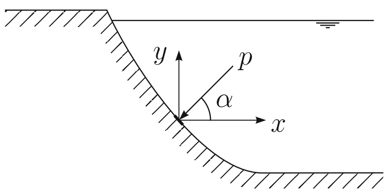

Fig. 2.17 Fluid pressure acting on curved surface

[5] Note I_{ξ_1} for a circle is half of the polar moment of inertia, which is given by $I_0 = \int_A r^2 r \mathrm{d}r \mathrm{d}\phi = 2\pi \int_A r^3 \mathrm{d}r = \frac{1}{2}\pi r^4$.

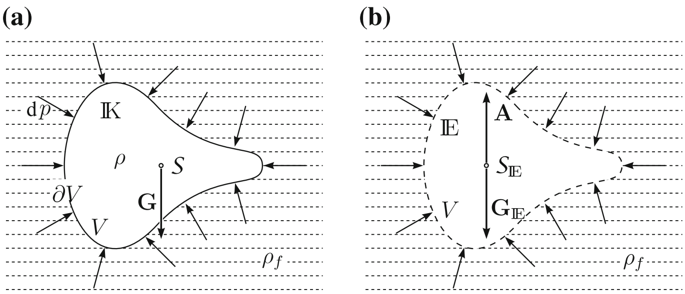

Fig. 2.18 ARCHIMEDES' principle. (**a**) *A rigid body* $\mathbb{K}$ *with volume* V *and boundary* ∂V, *completely immersed in a density preserving fluid loaded by the pressures* p *and gravity force* $\boldsymbol{G}$ *with point of attack* $\mathcal{S}$. (**b**) *Replacement of the original body by the body* $\mathbb{E}$ *with center of gravity* $\mathcal{S}_{\mathbb{E}}$, *consisting of the fluid and loaded by the external pressures and the gravity force* $\boldsymbol{G}_{\mathbb{E}}$. *The resultant force of the pressures applied at the surface can be summarized as the buoyancy force* $\boldsymbol{A}$ *which equilibrates the weight* $\boldsymbol{G}_{\mathbb{E}}$. *So,* $|A| = |G|$ *and* $\boldsymbol{A}$ *and* $\boldsymbol{G}_{\mathbb{E}}$ *are in opposite directions*

afterward the resultants of the components are computed. With reference to **Fig. 2.17** this yields

$$F_x = \int\limits_A p(y)\cos\alpha\,\mathrm{d}A, \qquad F_y = \int\limits_A p(y)\sin\alpha\,\mathrm{d}A. \tag{2.50}$$

Often these integrations can be avoided, e.g. by applying the ARCHIMEDEan principle (see below).

2.5 Hydrostatic Buoyancy of Floating Bodies

Consider an arbitrary, rigid body $\mathbb{K}$ completely immersed in an incompressible fluid of density ρ_f, **Fig. 2.18**a, and floating. Among the external forces acting on this body are (i) the gravity forces, applied in each volume element, but which can be summed to the resultant weight and whose point of attack is the gravity center $\mathcal{S}$ and (ii) the elementary pressures acting on the body surface. Using the methods introduced in the last subsection these pressures can be composed to a single force. This computation will be performed below; however, here we wish to first demonstrate the result using a trick, which is attributed to ARCHMEDES.[6]

[6] ARCHMEDES (∼287 B.C.–212 B.C.) ancient Greek mathematician, physicist and engineer, founder of hydrostatic and geometric statics. For his short biography see **Fig. 2.19**.

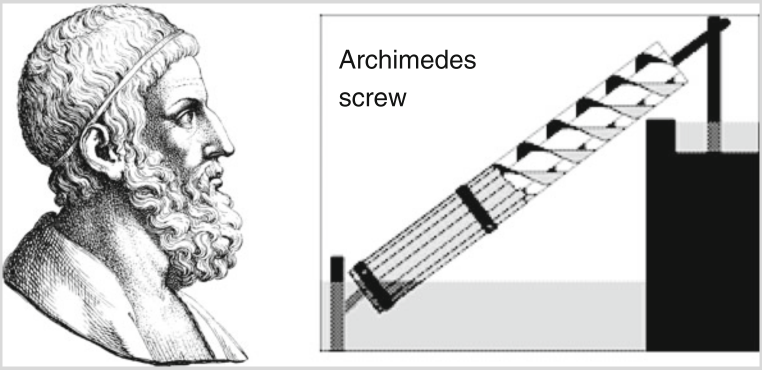

Fig. 2.19 ARCHIMEDES (∼287 BC–∼212 BC)

Detail of an antique engraved portrait of ARCHIMEDES and his screw
Famous ancient Greek mathematician and inventor, ARCHIMEDES lived ca. 290–280 BC in Syracuse (Sicily). The truth of this statement is seriously questioned in the modern historical literature [1]. The most commonly related story about ARCHIMEDES tells how he invented a method for measuring the volume of an object with an irregular shape. According to VITRUVIUS, a new crown in the shape of a laurel wreath had been made for King HIERON II, and ARCHIMEDES was asked to determine whether it was of solid gold, or whether silver had been added by a dishonest goldsmith. ARCHIMEDES had to solve the problem without damaging the crown, so he could not melt it down in order to measure its density as a cube, which would have been the simplest solution. While taking a bath, he noticed that the level of the water rose as he got in. He realized that this effect could be used to determine the volume of the crown, and therefore its density after weighing it. The density of the crown would be lower if cheaper and less dense metals had been added. He then went to the streets naked, so excited by his discovery that he had forgotten to dress, exclamating 'Eureka!' 'I have found it!'. The story about the golden crown does not appear in the known works of ARCHIMEDES and its truth is seriously questioned in the modern literature [1], but in his treatise *On Floating Bodies* he gives the principle known in hydrostatics as *ARCHIMEDES' Principle*. This states that a body immersed in a fluid experiences a buoyant force equal to the weight of the displaced fluid.

A large part of ARCHIMEDES' work in engineering arose from fulfilling the needs of his home city of Syracuse. The Greek writer ATHENAEUS of Naucratis described how King HIERON II commissioned ARCHIMEDES to design a huge ship, the *Syracusia*, which could be used for luxury travel, carrying supplies, and as a naval warship. The *Syracusia* is said to have been the largest ship built in classical antiquity. According to ATHENAEUS, it was capable of carrying 600 people and included garden decorations, a gymnasium and a temple dedicated to the goddess Aphrodite among its facilities. Since a ship of this size would leak a considerable amount of water through the hull, the ARCHIMEDES screw was purportedly developed in order to remove the bilge water. It was turned by hand, and could also be used to transfer water from a low-lying body of water into irrigation canals. The ARCHIMEDES screw described in Roman times by VITRUVIUS may have been an improvement on a screw pump that was used to irrigate the Hanging Gardens of Babylon.

The text is based on http://www.wikipedia.org and http://www.brown.edu

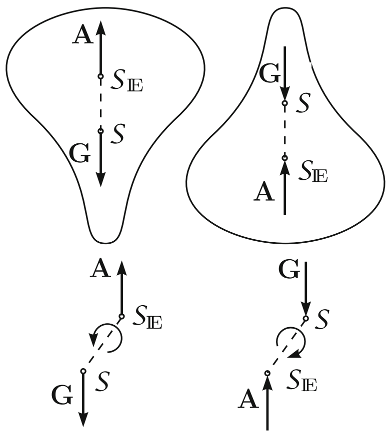

Fig. 2.20 Equilibrium positions. *Defining stable and unstable equilibrium configurations: If $\mathcal{S}_{\mathbb{E}}$ lies above $\mathcal{S}$ (panel at the left) the equilibrium configuration is stable, else it is called unstable (panel at the right)*

Because the body is floating in the fluid one may cut it out in imagination and replace it by any other body, hence e.g. the fluid itself, Fig. 2.18b. This replaced body $\mathbb{E}$ has the same volume V and the same weight $\boldsymbol{G}_{\mathbb{E}} = \rho_f g V \boldsymbol{e}_z$ in a center of gravity which may differ from the center of gravity $\mathcal{S}$ of $\mathbb{K}$. Since the fluid replacement body is at rest, the pressure distributions, which are the same for $\mathbb{K}$ and $\mathbb{E}$ can be reduced to a single force pointing upward, the so-called **buoyancy force** $\boldsymbol{A}$ with the modulus

$$A = \rho_f g V. \tag{2.51}$$

This argument shows that the pressures acting on the body $\mathbb{K}$ can be reduced to a force with vertical line of action. Moreover, it is now clear that a rigid body immersed in a density preserving fluid and floating in there, can only be in equilibrium if (i) $\boldsymbol{G} = -\boldsymbol{A}$ and (ii) if the centers of gravity $\mathcal{S}$ of $\mathbb{K}$ and $\mathcal{S}_{\mathbb{E}}$ of $\mathbb{E}$ lie on a vertical line. Such an **equilibrium configuration** is **stable**, if $\mathcal{S}_{\mathbb{E}}$ lies above $\mathcal{S}$ and it is unstable, when $\mathcal{S}_{\mathbb{E}}$ lies below $\mathcal{S}$, see **Fig. 2.20**. Indeed, a small deviation from the equilibrium configuration (by a small rotation) causes a moment, driving the body back into the equilibrium in the first case, but further away from it in the second case. Incidentally, it is quite obvious that for $\mathcal{S}_{\mathbb{E}} = \mathcal{S}$ every orientation of the body is an equilibrium configuration; in this case the equilibrium configurations are called **indifferent**.

The ARCHIMEDEan buoyancy force formula (2.51) can quite easily be derived, if the Gauss law is used. With reference to **Fig. 2.21**, we may write

$$\begin{aligned} \boldsymbol{A} &= -\int_{\partial V} p\boldsymbol{n}\,\mathrm{d}A \overset{\text{(Gauss)}}{=} -\int_V \operatorname{grad} p\,\mathrm{d}V \\ &= -\int_V \operatorname{grad}(p_0 - \rho_f g z)\,\mathrm{d}V = \rho_f g \boldsymbol{e}_z \int_V \mathrm{d}V = \rho_f g V \boldsymbol{e}_z, \end{aligned} \tag{2.52}$$

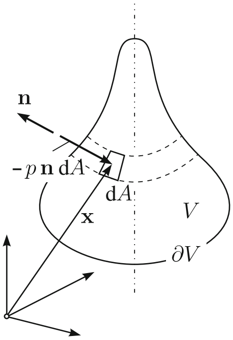

Fig. 2.21 Explaining the derivation of the ARCHIMEDEan buoyancy force formula

in which $\boldsymbol{e}_z$ is a unit vector in the z-direction. The point of attack of the buoyancy force can correspondingly be derived from the moment condition, formulated with respect to the origin of the coordinate system. Accordingly, this can be expressed as

Static moment of the buoyancy force with respect to the origin of the coordinate system	=	Sum of all static moments of the infinitesimal pressures at the body surface with respect to the origin of the coordinate system.

If one denotes by $\boldsymbol{x}_A$ the position vector of the point of attack of $\boldsymbol{A}$, the above statement reads

$$\begin{aligned}
\boldsymbol{x}_A \times \rho_f g V \boldsymbol{e}_z &= -\int\limits_{\partial V} \boldsymbol{x} \times p\boldsymbol{n}\,\mathrm{d}A \overset{\text{(Gauss)}}{=} -\int\limits_V \boldsymbol{x} \times \operatorname{grad} p\,\mathrm{d}V \\
&= -\int\limits_V \boldsymbol{x} \times \operatorname{grad}(p_0 - \rho_f g z)\,\mathrm{d}V \\
&= \int\limits_V \boldsymbol{x} \times \rho_f g \boldsymbol{e}_z\,\mathrm{d}V = \rho_f g \Bigg[\int\limits_V \boldsymbol{x}\,\mathrm{d}V\Bigg] \times \boldsymbol{e}_z,
\end{aligned}$$

which, after division by $\rho_f g V$, may also be written as

$$\Bigg(\boldsymbol{x}_A - \underbrace{\frac{1}{V}\int\limits_V \boldsymbol{x}\,\mathrm{d}V}_{\boldsymbol{x}_{\mathbb{E}}}\Bigg) \times \boldsymbol{e}_z = (\boldsymbol{x}_A - \boldsymbol{x}_{\mathbb{E}}) \times \boldsymbol{e}_z = \boldsymbol{0}. \tag{2.53}$$

The second term on the left can be interpreted as the center of gravity, $\boldsymbol{x}_{\mathbb{E}}$, of the replaced fluid as has been identified in (2.53). Because the coordinate system can be arbitrarily selected with respect to the orientation of gravity, it is concluded that (2.53) holds for all $\boldsymbol{e}_z$, so that $\boldsymbol{x}_A = \boldsymbol{x}_{\mathbb{E}}$ by necessity. The point of attack of the

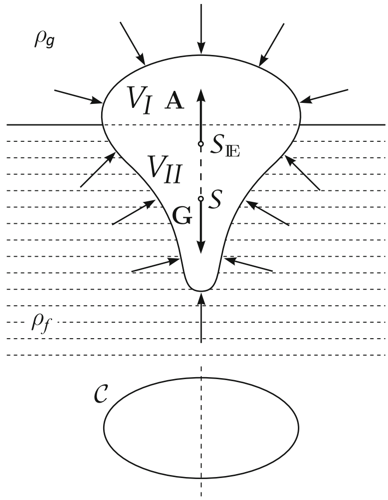

Fig. 2.22 Equilibrium position of a rigid body, partly immersed into a fluid. *The water line $\mathcal{C}$ is defined as the intersection between the body and the water surface. The center of gravity $\mathcal{S}_{\mathbb{E}}$ can be computed from the density ρ_f of the displaced fluid*

resultant buoyancy force is therefore the center of gravity of the replaced fluid body. This corroborates the result, obtained previously in a different way.

For a body, which floats at the surface as shown in **Fig. 2.22**, curve $\mathcal{C}$, along which the body intersects the water surface, is referred to as **water line**. On the body surface above this water line the pressure is that of the air which is generally assumed to be constant. The buoyancy force can be calculated as for a fully submerged body. In this process one must, however, account for the fact that the replaced fluid is composed of two different media. It consists below the water line of a fluid with density ρ_f, above it of a gas with density ρ_g. Often the mass of the gas can be ignored ($\rho_g = 0$); in this case the buoyancy force is again given by (2.51), but V is the volume of the submerged part of the body. It agrees in this approximation with the volume of the displaced fluid.

The first equilibrium condition, namely $\boldsymbol{A} = -\boldsymbol{G}$ determines the depth of immersion; the second, according to which $\mathcal{S}$ and $\mathcal{S}_{\mathbb{E}}$ must both lie on a vertical line is no longer as easy to determine as with the completely submersed body, because the column of the displaced fluid is now variable and dependent upon the depth of immersion and orientation of the body. For the same reason, the stability analysis of an equilibrium position is also more complicated, as one can see from the photo of **Fig. 2.23**, in which equilibria of various cylindrical corks with different aspect ratios (side length to beam length), floating on still water are shown. Depending on the aspect ratio the stable equilibrium position may have horizontal, inclined or vertical cylinder axis.

In the subsequent paragraphs the **stability of the equilibrium position** of partly submerged bodies is analyzed, if these bodies are long in the direction perpendicular to the drawing plane and possess in their equilibrium position a symmetry plane as shown in the left panel of **Fig. 2.24**a. If this body is rotated by a small angle $\delta\phi$ about the intersection of the symmetry plane and the water surface, then the displaced

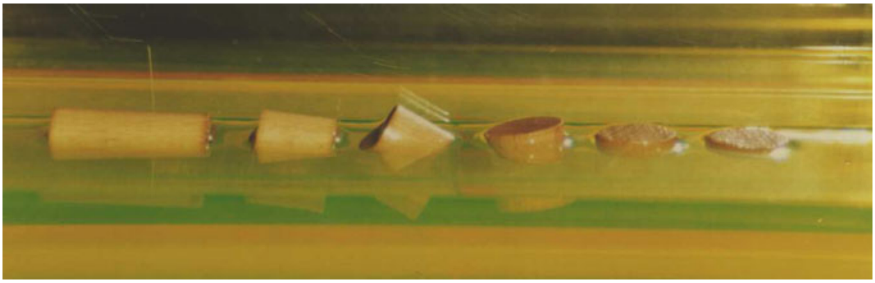

Fig. 2.23 Equilibrium positions of cylindrical corks with different aspect ratios. *A circular disk with large diameter relative to its thickness ($d \gg h$) floats stably with vertical cylinder axis. If the height is larger than the diameter of the disk ($h \gg d$) the stable equilibrium position has horizontal cylinder axis. If $h \cong d$ the stable equilibrium has inclined cylinder axis. For details see main text*

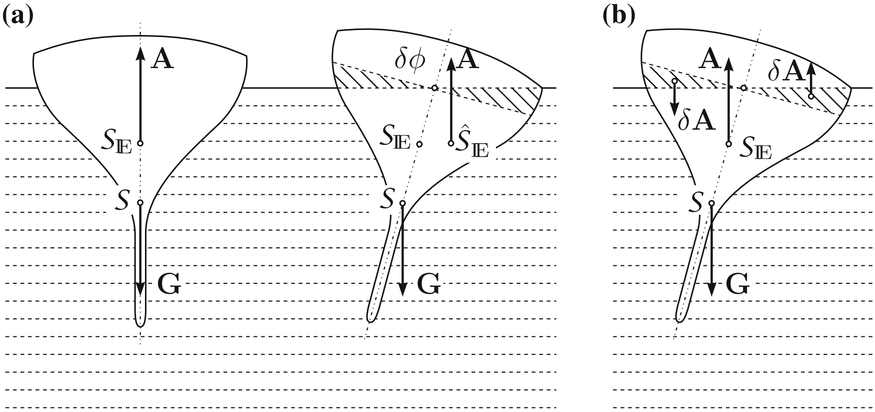

Fig. 2.24 Explaining the stability of equilibrium positions. (**a**) *Floating body, shown in its equilibrium position and a position that is rotated out of the equilibrium by an angle $\delta\phi$ with gravity force $\boldsymbol{G}$ and point of attack $\mathcal{S}$ and the centers $\mathcal{S}_{\mathbb{E}}$ and $\hat{\mathcal{S}}_{\mathbb{E}}$, respectively of the buoyancy force of the replaced body in vertical and rotated positions.* (**b**) *Same as in* (**a**) *but with buoyancy force now split into the buoyancy acting in $\mathcal{S}_{\mathbb{E}}$ and the additional incremental buoyancy forces $\delta\boldsymbol{A}$ and $-\delta\boldsymbol{A}$ due to the rotation of the body out of the equilibrium*

fluid experiences an increase of the buoyancy force on that side at which the body is inclined toward the water; it experiences an equal reduction of the buoyancy on the other side. As a consequence the center of gravity moves from the position $\mathcal{S}_{\mathbb{E}}$ to a position $\hat{\mathcal{S}}_{\mathbb{E}}$, see Fig. 2.24a.

If the center of gravity $\mathcal{S}$ of the body lies below $\mathcal{S}_{\mathbb{E}}$ as it is the case for yachts, then the body weight $\boldsymbol{G}$ and the displaced buoyancy force $\boldsymbol{A}$ build an erecting couple, rotating the body back to its normal floating position, which is thus stable. This can also be seen as follows, Fig. 2.24b: The buoyancy of the slightly rotated body can be composed of the force $\boldsymbol{A}$ in $\mathcal{S}_{\mathbb{E}}$ plus an infinitesimal couple ($\delta\boldsymbol{A}$, $-\delta\boldsymbol{A}$), applied at the centers of the 'triangular' shaded positions of the immersed and emerged volume

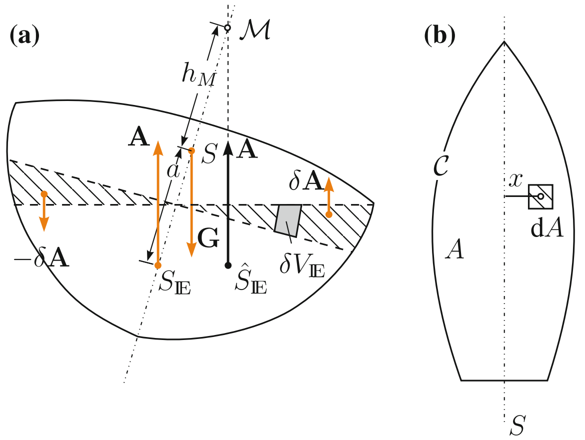

Fig. 2.25 Evaluation of the metacentric height.
(**a**) *Floating body rotated out of its equilibrium subjected to the buoyancy force* $\boldsymbol{A}$ *in* $\mathcal{S}_{\mathbb{E}}$, *and displaced* $\boldsymbol{A}$ *in* $\hat{\mathcal{S}}_{\mathbb{E}}$. *The point of intersection of the line of action of the displaced buoyancy force and the symmetry plane of the boat defines the metacenter* $\mathcal{M}$.
(**b**) *Explaining the evaluation of the areal moment of inertia for the area defined by* $\mathcal{C}$

increments. These displacements correspond to an additional buoyancy $\delta\boldsymbol{A}$ to the right of the body symmetry line, and a reduced buoyancy $-\delta\boldsymbol{A}$ to the left of it. In this way one obtains two couples $(\boldsymbol{G}, \boldsymbol{A})$ and $(\delta\boldsymbol{A}, -\delta\boldsymbol{A})$ of which in Fig. 2.24 both are working against an increase of $\delta\phi$. For most boats (e.g. dinghy, jolly-boat) the center of gravity $\mathcal{S}$ of the body lies above $\mathcal{S}_{\mathbb{E}}$. If, in this case $(\boldsymbol{G}, \boldsymbol{A})$ is again the couple formed with the buoyancy in $\mathcal{S}_{\mathbb{E}}$ and $(\delta\boldsymbol{A}, -\delta\boldsymbol{A})$, that describing the effect of the displaced fluid, then only this second couple has the tendency to reduce the angle $\delta\phi$ and erect the boat. The stability now depends on the fact which of the two couples has the larger modulus. If one counts 'erecting' moments positive and if one denotes by a the distance between $\mathcal{S}$ and $\mathcal{S}_{\mathbb{E}}$, then the moment of the pair $(\boldsymbol{G}, \boldsymbol{A})$ and an infinitesimal rotation $\delta\phi$ is given by

$$\delta M_1 = Aa\delta\phi = -\rho_f g a V_{\mathbb{E}} \delta\phi, \tag{2.54}$$

where $V_{\mathbb{E}}$ is the volume of the displaced fluid. With the aid of **Fig. 2.25** the contribution of the infinitesimal replaced fluid can be represented by differential volume elements $\delta V_{\mathbb{E}} = x\delta\phi\,\mathrm{d}A$, of which the buoyancy is given by $\rho_f g x \delta\phi\,\mathrm{d}A$ and gives rise to the 'erecting' moment

$$\delta M_2 = \int_A \rho_f g x^2 \mathrm{d}\phi\,\mathrm{d}A = \rho_f g \delta\phi \int_A x^2\,\mathrm{d}A, \tag{2.55}$$

in which A is the area of the water surface that is enclosed by the water line $\mathcal{C}$, and the integral on the far right is the areal moment of inertia of the area enclosed by $\mathcal{C}$ with respect to the symmetry line $\mathcal{S}$. If one defines

$$I_S = \int_A x^2\,\mathrm{d}A,$$

one obtains, by adding (2.54) and (2.55)

$$\delta M = \delta M_1 + \delta M_2 = \rho_f g (I_S - aV_{\mathbb{E}})\delta\phi. \tag{2.56}$$

The condition that $\delta M > 0$ and that, therefore, the boat returns to the normal equilibrium position, is

$$I_s - aV_{\mathbb{E}} \geqslant 0 \quad \text{or} \quad h_M = \frac{I_s}{V_{\mathbb{E}}} - a \geqslant 0. \tag{2.57}$$

The quantity h_M is a length; it is called the **metacentric height** and the intersection of the symmetry line with the vertical through $\hat{\mathcal{S}}_{\mathbb{E}}$ is called the **metacenter** $\mathcal{M}$. With the definition of h_M, (2.56) takes the form

$$\delta M = \left(\rho_f g V_{\mathbb{E}}\right) h_M \delta\phi = G h_M \delta\phi. \tag{2.58}$$

The stability condition (2.57) can thus also be so interpreted that one considers the couple, consisting of $\boldsymbol{G}$ and the buoyancy force $\boldsymbol{A}$ in $\mathcal{S}_{\mathbb{E}}$ and directly writes for the 'erecting' moment $\delta M = G h_M \delta\phi$. The metacentric height is therefore the distance between $\mathcal{S}$ of $\mathbb{K}$ and $\mathcal{M}$ (which is the intersection point between the line of action of the displaced buoyancy force and the symmetry line of the boat). Incidentally the higher the metacenter lies above $\mathcal{S}$, the larger is the 'erecting' moment. Practically one preferably does not make the metacenter too large in order to reduce the frequency of the oscillation of the ship in heavy sea.

As an example consider the **floating behavior of a beam**[7] of quadratic cross section with side length a and density ρ_K. Assume its length ℓ to be large in comparison to a, $\ell \gg a$, so that no stable equilibrium configuration exists with lengthwise inclination as for a cork in Fig. 2.23. The emerging mathematical problem is then plane. If the ratio of the densities of the beam, ρ_K and the fluid ρ_f is denoted by $D = \rho_K/\rho_f$, the volume of the displaced fluid is given by

$$V_{\mathbb{E}} = Da^2. \tag{2.59}$$

Apart from the trivially symmetric, and for the other values of $D \in [0, 1]$ non-symmetric, represented equilibrium positions in panel (a) of **Fig. 2.26** (top), all those equilibrium positions, which are sketched in panels (b) and (c), are possible. Concentrating for the moment on the water line in panel (b) (solid, bold), rising from the lower left to the upper right with parameterization

$$y = \lambda x + aD \tag{2.60}$$

[7]The computations for this example are a bit involved. In a first reading a reader may wish to simply concentrate on the physical implications of the computations.

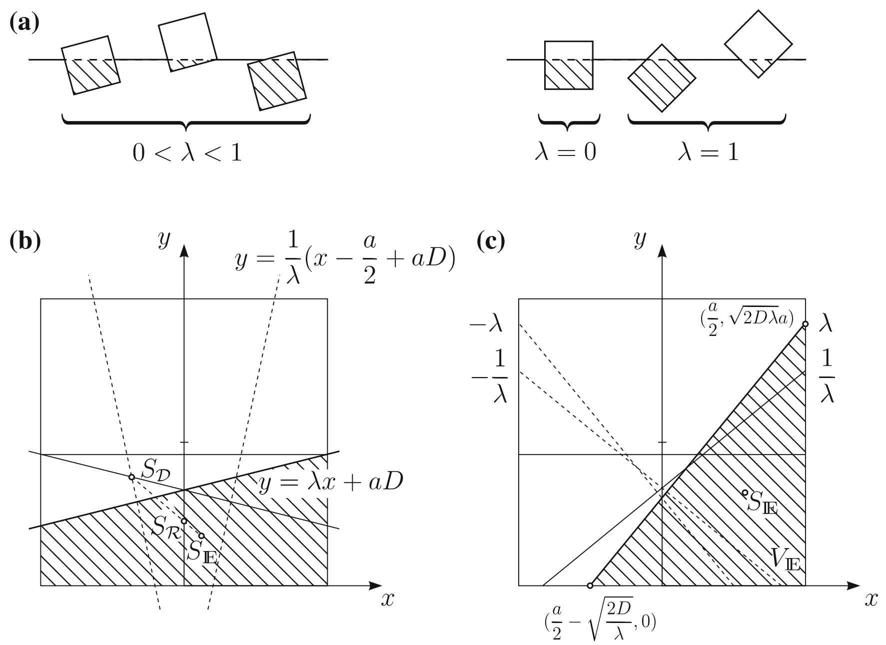

Fig. 2.26 Floating configurations of a beam with quadratic cross sections. (**a**) *Possible equilibrium positions at different values of the inclination of the water line.* (**b**) *Four different physically equivalent equilibrium positions (solid and dashed). For one case the wetted area is shown dashed.* (**c**) *Four possible additional equilibrium positions, if the wetted area is a triangle*

and yet still unknown positive slope λ, this water line intersects the vertical sides of the square, if

$$\lambda \leqslant \begin{cases} 2D, & \text{for } 0 \leqslant D \leqslant 0.5, \\ 2(1-D), & \text{for } 0.5 \leqslant D \leqslant 1. \end{cases} \tag{2.61}$$

The coordinates of the center of gravity $\mathcal{S}_{\mathbb{E}}$ of the trapezoidal cross section of the displaced fluid can most easily be computed from those of the rectangle $\mathcal{S}_R$, and the triangle $\mathcal{S}_D$, both of which are indicated in Fig. 2.26b. Applying the static moments with respect to the vertical and horizontal axes shows that

$$\left(x_{\mathcal{S}_{\mathbb{E}}}, y_{\mathcal{S}_{\mathbb{E}}}\right) = \left(\frac{\lambda a}{12D}, \frac{\lambda^2 a}{24D} + \frac{aD}{2}\right). \tag{2.62}$$

To fix a value for the water-line slope λ we employ the condition that $\mathcal{S}_{\mathbb{E}}$ lies vertically below $\mathcal{S}$. To this end we write with

$$y = -\frac{1}{\lambda}\left(x - \frac{\lambda a}{12D}\right) + \frac{\lambda^2 a}{24D} + \frac{aD}{2} \tag{2.63}$$

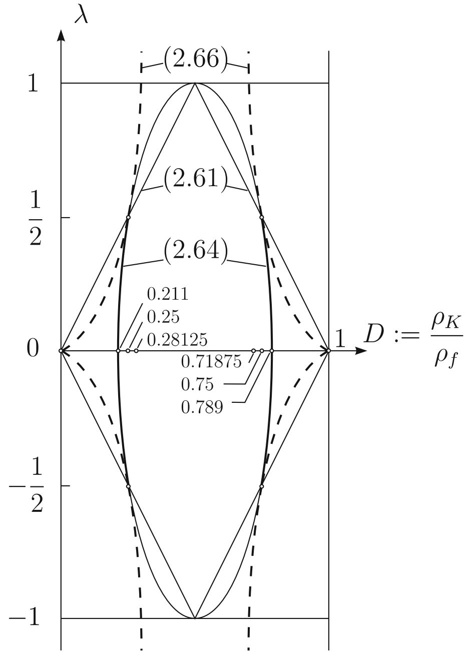

Fig. 2.27 Water line slopes for beams with quadratic cross section. *Slope λ of the water line as a function of the density ratio $D = \rho_K/\rho_f$. The solid, respectively dashed, segments of ellipses are defined in equation* (2.64) *and* (2.66). *Equation* (2.61) *constrains the domain of validity of the two curved segments*

that straight line, which is perpendicular to the water line (2.60), and is identically satisfied for $x = x_{S_{\mathbb{E}}}$, $y = y_{S_{\mathbb{E}}}$ as given in (2.62). We now also request that the point $\mathcal{S}$ lies on the line (2.63), i.e., (2.63) is identically satisfied for $(x, y) = (0, a/2)$, which leads to the equation

$$\left(\frac{D - \frac{1}{2}}{1/\sqrt{12}}\right)^2 + \lambda^2 = 1. \tag{2.64}$$

In a Cartesian coordinate system with D- and λ-axes it represents an ellipse with center $(\frac{1}{2}, 0)$ and semi axes $a = 0.2887$ and $b = 1$. Note also that, if λ is an admissible water line, so is $-\lambda$; this corresponds to a mirroring of the beam at the vertical symmetry line. If, moreover, the inequalities (2.61) are taken into account then only the bold segments in **Fig. 2.27** are admissible equilibrium configurations.

To each pair of equilibrium configurations (D, λ) and $(D, -\lambda)$ one also may construct a further pair, namely $(0, 1/\lambda)$, $(0, -1/\lambda)$ for the two dashed water lines, which so far were not discussed; they possess reciprocal slopes. These equilibria follow simply from geometric evidence of symmetry and do not require further proof.

At last the equilibrium configuration of Fig. 2.26c must be scrutinized with regard to its stability. The reader may show in an analogous way as above that the possible equilibrium positions for $\lambda > 0$ are in this case given by

$$
\begin{aligned}
&(\lambda-1)\left[3\sqrt{\lambda}-2\sqrt{2D}(\lambda+1)\right]=0, \qquad 0\leqslant D\leqslant 0.5,\\
&(\lambda-1)\left[3\sqrt{\lambda}-2\sqrt{2(1-D)}(\lambda+1)\right]=0,\ 0.5\leqslant D\leqslant 1.
\end{aligned}
\tag{2.65}
$$

They correspond to the trivial equilibrium with $\lambda = 1$ and to those having three and five wetted subsurface edges, respectively, Fig. 2.26a. From the formal invariance of (2.65) under the replacement $D \to (1-D)$ it follows that to each equilibrium configuration with three wetted edges there corresponds a configuration with five wetted edges and equal slope λ of the water line. Because (2.65) is also formally invariant under the replacement $\lambda \to 1/\lambda$, there are additional equilibrium configurations $(D, 1/\lambda)$, corresponding to those with (D, λ) with reciprocal slope. If one puts the expressions with brackets in (2.65) equal to zero, the following quadratic equations

$$
\begin{aligned}
&\lambda^2-2\beta\lambda+1=0,\ \beta=\tfrac{9}{16D}-1, \qquad D\leqslant 0.5,\\
&\lambda^2-2\hat{\beta}\lambda+1=0,\ \hat{\beta}=\tfrac{9}{16(1-D)}-1,\ D>0.5,
\end{aligned}
\tag{2.66}
$$

with positive discriminant and thus real λ are obtained, provided

$$
0\leqslant D\leqslant 0.28125, \quad \text{respectively}, \quad 0.71875\leqslant D\leqslant 1. \tag{2.67}
$$

The segments of the curves (2.66) are shown in Fig. 2.27 as dashed lines. They are drawn for $\lambda > 0$ and are mirrored at the horizontal D-coordinate for $\lambda < 0$. These curves smoothly continue the elliptic arches at $\lambda = \pm\frac{1}{2}$, and the admissible equilibria lie above and below, respectively, the straight lines, given by (2.61).

So far all possible equilibrium configurations have been given for a floating long beam with quadratic cross section. There remains to identify, which of these are, respectively, stable, indifferent and unstable.

In order to determine the stability, for instance, of the trivial position $\lambda = 0$, we compute the position of the metacenter. For reasons of symmetry we have $x_{\mathcal{M}} = 0$ and (2.57), $y_{\mathcal{M}} = I_s/V_{\mathbb{E}} - a$, or

$$
\mathcal{M}(\lambda)=\left(0, \frac{a}{12D}+\frac{\lambda^2}{24D}+\frac{aD}{2}\right). \tag{2.68}
$$

We may now request that for $\lambda \to 0$ the metacenter lies above the center of gravity of the body. This condition yields

$$
\lim_{\lambda\to 0}\left(\frac{a}{12D}+\frac{\lambda^2}{24D}+\frac{aD}{2}\right)>\frac{a}{2}, \tag{2.69}
$$

from which the stability condition

$$
D<\frac{1}{2}-\frac{1}{2\sqrt{3}}\cong 0.211 \tag{2.70}
$$

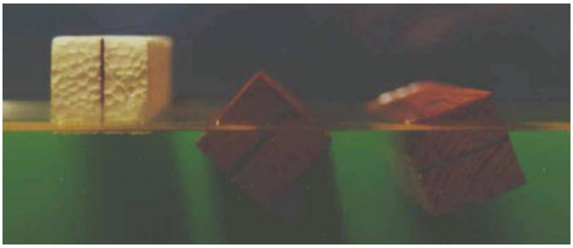

Fig. 2.28 Equilibrium configurations of a beam with quadratic cross section. *The three values of the density ratio $D = \rho/\rho_K$ correspond to the equilibria $\lambda = 0$, $\lambda = 1$ and $0 < \lambda < 1$*

follows. With increasing value of the density ratio above the value $D = 0.211$ list of the beam to the right or left occurs; it enhances this rotated position with increasing D, until the position with $\lambda = \pm 1$, $\diamond$, becomes stable for $D \in [0.28125, 0.1875]$. If D is further increased beyond the value $D = 0.71875$, the beam will again tilt from its preferred stable position $\diamond$ with $\lambda = \pm 1$ (which is unstable for $0.789 \leq D \leq 1$) into the stable position $\square$ for $\lambda = 0$; its proof is straightforward if not easy and is left to the reader.

The photo in **Fig. 2.28** and the figure legend corroborate this analysis.

2.6 Hydrostatics in an Accelerated Reference System

According to the considerations in Sect. 2.3 the fundamental hydrostatic equation establishes equilibrium between the pressure gradient and the specific body force, see Eq. (2.14). If the fluid is referred to a non-inertial reference system, then basically a dynamical system is in our hands, for which, using D'ALEMBERT's principle, the extended dynamical equation (2.15) is applicable with an acceleration vector $\boldsymbol{b}$, referred to the absolute inertial system (at rest), see **Fig. 2.29**. If the state of the motion of the accelerated reference system, whose origin in the absolute system is given by the vector $\boldsymbol{r}_0$ and if its velocity vector is $\dot{\boldsymbol{r}}_0$ and the angular velocity is $\boldsymbol{\omega}$, then according to the laws of kinematics of the relative motion, the temporal change

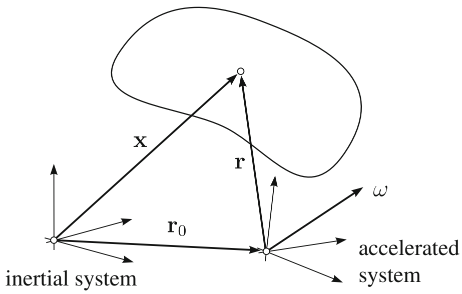

Fig. 2.29 Accelerated reference system. *Body referred to an inertial system and an accelerated system moving relative to the first system with velocity $\dot{\boldsymbol{r}}_0$ and angular velocity $\boldsymbol{\omega}$*

of a vector $\boldsymbol{a}$ (relative to the inertial system and denoted by $\mathrm{d}\boldsymbol{a}/\mathrm{d}t$) equals its relative temporal change (relative to the accelerated system and denoted by $\delta\boldsymbol{a}/\delta t$) plus the contribution due to the rotation of the accelerated system, viz.,

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \frac{\delta\boldsymbol{a}}{\delta t} + \boldsymbol{\omega} \times \boldsymbol{a}. \tag{2.71}$$

This formula is immediately understood, if a vector $\boldsymbol{a}$ is considered which is rigidly connected to the accelerated system and for which $\delta\boldsymbol{a}/\delta t = \mathbf{0}$, so that $\mathrm{d}\boldsymbol{a}/\mathrm{d}t = \boldsymbol{\omega} \times \boldsymbol{a}$. Application of the rule (2.71) to the position vector

$$\boldsymbol{x} = \boldsymbol{r}_0 + \boldsymbol{r} \tag{2.72}$$

yields the relation

$$\underbrace{\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}}_{\boldsymbol{v}_a} = \underbrace{\frac{\mathrm{d}\boldsymbol{r}_0}{\mathrm{d}t} + \boldsymbol{\omega} \times \boldsymbol{r}}_{\boldsymbol{v}_f} + \underbrace{\frac{\delta\boldsymbol{r}}{\delta t}}_{\boldsymbol{v}_r} = \boldsymbol{v}_f + \boldsymbol{v}_r \tag{2.73}$$

between the **absolute velocity** $\boldsymbol{v}_a$, the **fixed body velocity** $\boldsymbol{v}_f$[8] and the **relative velocity** $\boldsymbol{v}_r$; here, $\boldsymbol{v}_f$ is called '**fixed body velocity**', because its definition corresponds to that virtual velocity which obtains, if the body were fixed to the moving frame. Applying the rule (2.71) to (2.73), finally yields

$$\begin{aligned}
\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} &= \left(\frac{\delta}{\delta t} + \boldsymbol{\omega}\times\right)\left(\frac{\mathrm{d}\boldsymbol{r}_0}{\mathrm{d}t} + \boldsymbol{\omega} \times \boldsymbol{r} + \frac{\delta\boldsymbol{r}}{\delta t}\right) \\
&= \frac{\mathrm{d}^2\boldsymbol{r}_0}{\mathrm{d}t^2} + \frac{\delta\boldsymbol{\omega}}{\delta t} \times \boldsymbol{r} + \boldsymbol{\omega} \times \frac{\delta\boldsymbol{r}}{\delta t} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \frac{\delta^2\boldsymbol{r}}{\delta t^2} + \boldsymbol{\omega} \times \frac{\delta\boldsymbol{r}}{\delta t} \\
&= \underbrace{\frac{\delta^2\boldsymbol{r}}{\delta t^2}}_{\boldsymbol{b}_r} + \underbrace{2\boldsymbol{\omega} \times \frac{\delta\boldsymbol{r}}{\delta t}}_{\boldsymbol{b}_c} + \underbrace{\frac{\mathrm{d}^2\boldsymbol{r}_0}{\mathrm{d}t^2} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \frac{\delta\boldsymbol{\omega}}{\delta t} \times \boldsymbol{r}}_{\boldsymbol{b}_f}
\end{aligned}$$

or

$$\boldsymbol{b} = \boldsymbol{b}_r + \ddot{\boldsymbol{r}}_0 + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}. \tag{2.74}$$

Here,

$\boldsymbol{b} = \dfrac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}$ is the **absolute acceleration**,

$\boldsymbol{b}_r = \dfrac{\delta\boldsymbol{r}}{\delta t^2}$ is the **relative acceleration**,

[8]In the English literature $\boldsymbol{v}_f$ is not separately defined; it is that rigid body velocity, which agrees in each body point with the velocity of the same geometric point that performs the motion of the accelerated reference frame.

$$\boldsymbol{b}_c = 2\boldsymbol{\omega} \times \frac{\delta \boldsymbol{r}}{\delta t} = 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}} \text{ is the C}\textsc{oriolis}\ \textbf{acceleration},$$

$$\boldsymbol{b}_f = \frac{\mathrm{d}^2 \boldsymbol{r}_0}{\mathrm{d}t^2} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \frac{\delta \boldsymbol{\omega}}{\delta t} \times \boldsymbol{r} \text{ is the `}\textbf{guiding}\text{' }\textbf{acceleration}.$$

Incidentally, except for $\ddot{\boldsymbol{r}}_0$ dots represent in (2.74), differentiations with respect to time relative to the accelerated frame of reference, and

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \qquad \text{is called } \textbf{centripetal acceleration},$$
$$\dot{\boldsymbol{\omega}} \times \boldsymbol{r} \qquad \text{is called E}\textsc{uler}\ \textbf{acceleration}.$$

If the fluid particle is at rest in the accelerated frame, then $\boldsymbol{b}_r = \mathbf{0}$ and $\dot{\boldsymbol{r}} = \mathbf{0}$; so, Eq. (2.74) is simplified to

$$\boldsymbol{b} = \ddot{\boldsymbol{r}}_0 + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}$$

and (2.15) takes the form

$$\operatorname{grad} \ p = \rho [\boldsymbol{k} - \ddot{\boldsymbol{r}}_0 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) - \dot{\boldsymbol{\omega}} \times \boldsymbol{r}] . \tag{2.75}$$

This is the **hydrostatic equation relative to an accelerated frame of reference** whose motion is given by $\boldsymbol{r}_0(t)$ and $\boldsymbol{\omega}(t)$.

For a **first special case** of a permanently rotating system we have $\ddot{\boldsymbol{r}}_0 = \mathbf{0}$, $\dot{\boldsymbol{\omega}} = \mathbf{0}$, $\boldsymbol{\omega} = \omega \boldsymbol{e}_z$. The acceleration in (2.74) is then given by, see **Fig. 2.30**,

$$\boldsymbol{b} = \omega \boldsymbol{e}_z \times [\omega \boldsymbol{e}_z \times (r\boldsymbol{e}_r + z\boldsymbol{e}_z)] = -\omega^2 r \boldsymbol{e}_r, \tag{2.76}$$

in which $\boldsymbol{e}_r$ and $\boldsymbol{e}_z$ are unit vectors in the radial and axial directions. The fundamental hydrostatic equation (2.75) reduces to

$$\operatorname{grad} p = -\rho g \boldsymbol{e}_z + \rho r \omega^2 \boldsymbol{e}_r \tag{2.77}$$

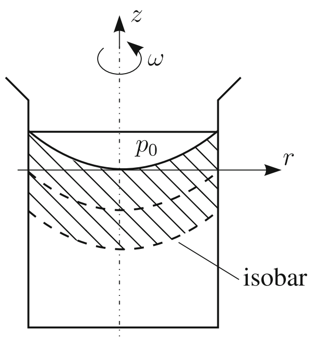

Fig. 2.30 Hydrostatics in a rotating system. *The free surface and the isobaric surfaces in a rotating system are circular paraboloids*

and for a density preserving fluid integrates to

$$p - p_0 = -\rho g z + \frac{\rho\omega^2}{2} r^2, \tag{2.78}$$

in which p_0 is the constant of integration, i.e., the pressure at $r = z = 0$. According to (2.77) the pressure gradient has a positive radial component with modulus $\rho r \omega^2$, which can be identified with the centrifugal force; moreover, it has also a vertical component due to gravity and of size ρg and acting downward. If p_0 is chosen to be the atmospheric pressure at the free surface, then $p = p_0$ and

$$z = \frac{\omega^2}{2g} r^2. \tag{2.79}$$

According to this equation the free surface is a circular paraboloid. The surfaces of constant pressure, the **isobaric surfaces**

$$z = \frac{\omega^2}{2g} r^2 - \frac{p_1 - p_0}{\rho g}, \qquad p_1 = \text{const.} \tag{2.80}$$

are surfaces, congruent to (2.79), but vertically displaced by $(p_1 - p_0)/(\rho g)$.

In the above computations the fluid has been assumed to be density preserving, but the hydrostatic equation (2.75) is not restricted to this case. To clarify, which density distributions $\rho(r, z)$ of a non-density preserving fluid are possible, we form on both sides of (2.77) the 'curl' and obtain this way

$$\text{curl grad}\, p = \mathbf{0} \quad \longrightarrow \quad \frac{\partial \rho}{\partial r} = -\frac{\omega^2}{g} r \frac{\partial \rho}{\partial z}. \tag{2.81}$$

This result implies that in a permanently rotating but only axially stratified fluid, a relative state of equilibrium is not possible. Because of the axial stratification the fluid builds also a radial density variation. If one considers for this a fluid with thermal equation of state of the form $\rho = \rho(T)$, then (2.81) can be expressed as the integrability condition for the temperature:

$$\frac{\partial T}{\partial r} = -\frac{\omega^2}{g} r \frac{\partial T}{\partial z}. \tag{2.82}$$

If the fluid is at rest in the rotating coordinate system, then the temperature must also obey the steady state heat equation in cylindrical coordinates

$$\Delta T = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = 0, \tag{2.83}$$

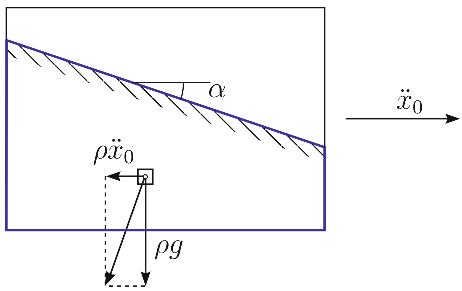

Fig. 2.31 Hydrostatics in the translatorically accelerated system. *For constant acceleration* $\ddot{x}_0 =$ const. *the free surface is inclined by the amount* $\tan\alpha\ddot{x}_0/g$

an equation, which shall be taken here for granted without proof. The only simultaneous solutions of (2.82) and (2.83), as can be shown, are uniform temperatures, $T =$ const.: there cannot be a stratification. If, however, the centripetal accelerations are ignored in (2.82), then $\partial T/\partial r = 0$, and (2.83) implies therefore $T = A + \Delta z$. In a permanently, slowly rotating incompressible fluid, which is thermally stratified in the z-direction, equilibria are possible provided the axial temperature is linear.

Finally, we remark that the ARCHIMEDEan principle also remains valid for hydrostatic conditions of permanently rotating density preserving fluids. Indeed, if one replaces the body with volume V, immersed in the fluid and at rest, by the displaced fluid, this fluid is trivially at rest. The pressure forces, applied to the boundary of the volume V are, thus, in equilibrium with the weight $\rho g V$ and the radial centrifugal force $\rho V \omega^2 r_{\mathbb{E}}$, where $r_{\mathbb{E}}$ denotes the radial distance of the center of gravity of the displaced fluid from the rotation axis.

We leave it to the reader to show that this argument can also be applied to the even more general equation (2.75). Further, he/she may convince him/herself how such arguments explain the functioning of a centrifugal pump.

As a **second special case** of the fundamental hydrostatic equation (2.75) let us evaluate the position of the free surface of the water in a translatorically accelerated container, **Fig. 2.31**. Equation (2.75) reduces in this case to

$$\text{grad}\, p = -\rho(g\boldsymbol{e}_z + \ddot{\boldsymbol{r}}_0), \tag{2.84}$$

which for constant ρ and constant acceleration $\ddot{\boldsymbol{r}}_0 = (\ddot{x}_0, \ddot{y}_0, \ddot{z}_0)$ can be integrated to

$$p - p_0 = -\rho(\ddot{x}_0 x + \ddot{y}_0 y + \ddot{z}_0 z) - \rho g z, \tag{2.85}$$

in which p_0 denotes the constant pressure on the free surface. The geometry of the free surface is therefore given by

$$z = -\frac{1}{g + \ddot{z}_0}\left(\ddot{x}_0 x + \ddot{y}_0 y\right),$$

or if $\ddot{y}_0 = \ddot{z}_0 = 0$

$$z = -x \tan\alpha, \qquad \tan\alpha = \frac{\ddot{x}_0}{g}. \tag{2.86}$$

For a co-moving observer the fluid is at rest; he experiences as body force a downward acting gravity force against ρg and a horizontal volume force of size $\rho \ddot{x}_0$ acting against the acceleration $\ddot{x}_0$.

2.7 Pressure Distribution in the Still Atmosphere

Let us return to the fundamental hydrostatic equation (2.14) and determine the pressure distribution in the atmosphere subject to the gravity field. Assume that the Earth has spherical symmetry with radius r_0 and that the gravity force changes with distance from the Earth's center according to NEWTON's gravity law

$$g(r) = g_0 \frac{r_0^2}{r^2}, \tag{2.87}$$

where g_0 denotes the acceleration due to gravity on the free surface. The pressure gradient has for spherical symmetry only a radial component, so that the hydrostatic equation takes the form

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -g_0 \rho \frac{r_0^2}{r^2} \tag{2.88}$$

and becomes integrable, if this equation is complemented by an additional equation, in which pressure and density are related to one another. If this relation is given by the **equation of state for ideal gases**

$$p = \rho R T, \tag{2.89}$$

in which R is the **specific gas constant** and T the **absolute temperature**, then with the assumption of a constant temperature, i.e., an **isothermal atmosphere**, the pressure would be determined by

$$p = p_0 \exp\left(\frac{g_0 r_0^2}{RT}\left(\frac{1}{r} - \frac{1}{r_0}\right)\right), \tag{2.90}$$

in which p_0, r_0, g_0 are corresponding values at the Earth's surface. With the aid of (2.89) the variable ρ can equally be evaluated.

In lieu of an isothermal atmosphere one assumes as a rule a **polytrope atmosphere**, which is given by

$$\rho = \rho_0 \left(\frac{p}{p_0}\right)^n \tag{2.91}$$

with **polytrope exponent** $n > 0$. Eliminating between (2.91) and (2.89) the pressure, one obtains for the temperature the formula

$$T = \frac{p_0}{R\rho_0^{1/n}}\rho^{(1-n)/n}. \tag{2.92}$$

From this formula one concludes for $0 < n < 1$ that the temperature rises with increasing density and remains bounded for all densities $\rho \geqslant 0$. By contrast, for $n > 1$, the exponent of ρ in (2.92) is negative so that the temperature falls with increasing density; for $\rho \to 0$ we then have $T \to \infty$. For $n = 1$ the polytropic atmosphere is as well isothermal. In the lower atmosphere, the so-called **troposphere**—these are the lowest 8–12 km—the exponent $n = 0.8$ is an adequate approximation to a realistic density distribution.

Substitution of (2.91) into (2.88) yields the separable differential equation

$$\frac{\mathrm{d}p}{p^n} = -\frac{g_0\rho_0}{p_0^n}\frac{r_0^2}{r^2}\mathrm{d}r \tag{2.93}$$

for the pressure as a function of the distance from the center of the Earth. (2.93) requires separate integration for $n = 1$ and $n \neq 1$. The result is

$$\frac{p}{p_0} = \begin{cases} \exp\left(-\dfrac{\rho_0 g_0 r_0^2}{p_0}\left(\dfrac{1}{r_0} - \dfrac{1}{r}\right)\right), & n = 1, \\ \left\{1 - (1-n)\dfrac{\rho_0 g_0 r_0^2}{p_0}\left(\dfrac{1}{r_0} - \dfrac{1}{r}\right)\right\}^{1/(1-n)}, & n \neq 1. \end{cases} \tag{2.94}$$

The solution for $n = 1$ has been given already in (2.90). Ordinarily one is interested in values of r, which deviate only slightly from r_0; so, when using TAYLOR series expansion we have

$$\begin{aligned} \frac{1}{r_0} - \frac{1}{r} &= \frac{1}{r_0}\left(1 - \frac{r_0}{r}\right) = \frac{1}{r_0}\left(1 - \frac{r_0}{r_0 + z}\right) \\ &= \frac{1}{r_0}\left(1 - \frac{1}{1 + \dfrac{z}{r_0}}\right) = \frac{1}{r_0}\left(\frac{z}{r_0} + \mathcal{O}\left(\left(\frac{z}{r_0}\right)^2\right)\right). \end{aligned} \tag{2.95}$$

If this expression is substituted into $(2.94)_2$ for $n \neq 1$, one obtains

$$\frac{p}{p_0} \cong \left(1 - (1-n)\frac{z}{H}\right)^{1/(1-n)}, \quad H = \frac{p_0}{\rho_0 g_0}. \tag{2.96}$$

H denotes an upper bound for the thickness of the troposphere for which (2.96) delivers reliable values for the pressure. With

$$\rho_0 = 1.293 \text{ kg m}^{-3}, \quad p_0 = 1.013 \cdot 10^5 \text{ kg s}^{-2}\text{m}^{-2}, \quad g_0 = 9.81 \text{ m s}^{-2}$$

one obtains $H \cong 8 \times 10^3$ m, which roughly is indeed the thickness of the troposphere or somewhat smaller. If relation (2.96) is being expanded in a TAYLOR series for small values of z/H, then the linear approximation

$$\frac{p}{p_0} \cong 1 - \frac{z}{H} = 1 - \frac{\rho_0 g_0}{p_0} z, \qquad \frac{z}{H} \ll 1 \tag{2.97}$$

is obtained, valid for the lower troposphere. This linear dependence in the z-coordinate corresponds to that of a density preserving fluid.

If, alternatively, the argument of the exponential function in (2.90) is linearized in z/H, one obtains the **barometric height formula**

$$\frac{p}{p_0} = \exp\left(-\frac{\rho_0 g_0}{p_0} z\right) = \exp\left(-\frac{z}{H}\right), \quad n = 1, \tag{2.98}$$

which, after further TAYLOR series expansion for $z/H \ll 1$ yields again (2.97).

To close this exposition on the hydrostatic equation, let us ask, under which conditions an atmospheric layer of air at rest remains stable. To this end, we consider a small element of air, isolated from the ambient air, and assume that its density variations can be ignored across its vertical extent. This element of air is comparable to a balloon, which floats at its height, i.e., its weight and buoyancy forces are equilibrated. For the ensuing analysis we assume linear density distribution of the atmosphere with height. If such an element is vertically displaced by the distance ζ, its buoyancy changes, because it is now positioned in an environment of different density. If one assumes that this density varies negligibly in the extent of the particle motion, then, locally, the buoyancy can be evaluated with the aid of the ARCHIMEDEan principle; one obtains $\rho(\zeta_s)gV$ where ζ_s is the z-coordinate of the center of gravity of the element of air with volume V. The element is, thus, subjected to the upward pointing vertical force

$$[\rho(\zeta_s) - \rho(0)]gV \cong \frac{\mathrm{d}\rho}{\mathrm{d}\zeta}(0) gV\zeta_s, \tag{2.99}$$

in which TAYLOR series expansion has been used to obtain the expression on the far right: $\rho(\zeta_s) = \rho(0) + (\mathrm{d}\rho/\mathrm{d}\zeta)\zeta_s + \cdots$. If one regards the particle of air (or the balloon) as a rigid body or as a mass point, then

$$\rho(0)\dot{\zeta}_s V \tag{2.100}$$

is its momentum. NEWTON's second law, thus leads to the equation of motion

$$\rho_0 \ddot{\zeta}_s V = \frac{\mathrm{d}\rho}{\mathrm{d}\zeta}(0) g \zeta_s V \quad \Longrightarrow \quad \ddot{\zeta}_s - \frac{\frac{\mathrm{d}\rho}{\mathrm{d}\zeta}(0) g}{\rho(0)} \zeta_s = \ddot{\zeta}_s + N^2 \zeta_s = 0. \tag{2.101}$$

The factor

$$N^2 = -\frac{\frac{\mathrm{d}\rho}{\mathrm{d}\zeta}(0) g}{\rho(0)}$$

has the dimension of a squared frequency. N is called **buoyancy frequency** or BRUNT- VÄISÄLÄ **frequency** and is a measure for the stability of the air particle in its layer. Indeed, Eq. (2.101) describes for $N^2 > 0$ a harmonic motion

$$\zeta_s = A \sin(Nt) + B \cos(Nt), \qquad N^2 > 0 \tag{2.102}$$

with constants of integration A and B. If the particle is displaced out of its equilibrium position, it will harmonically oscillate about its equilibrium position. This oscillation will in reality be damped, an effect, which has been ignored in the above analysis. Thus, the air particle will in reality return to its original equilibrium position. We say in this case that *the air is stably stratified*. If, on the contrary, $N^2 < 0$, the density of the air grows with height. The solution of (2.101) is then

$$\zeta_s = A \exp(|N|t) + B \exp(-|N|t), \quad N^2 < 0 \tag{2.103}$$

and possesses exponential character. An initial perturbation will move the particle with growing time farther and farther away from its initial position. The fluid particle can no longer return to its initial position. For $N^2 < 0$ *the mass of the air is therefore unstably stratified*.

The reader may show that the air in a homogeneous layer is in an indifferent state of equilibrium.

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