Chapter 2
Algebraic Structures. Spaces. Reference Frames

Mathematical structures are usually defined by using sets of axioms. The definition of the axioms must meet the following three rules:

1. The sets of elements to which the axioms apply must not be empty.
2. The sets of elements to which the axioms apply must not be trivial; in other words, elements which do not fulfill the axioms must exist.
3. The axioms must be independent, i.e. none of the axioms should be obtained from the other axioms.

2.1 Sets

The definitions and the mathematical concepts are based on set theory. Moreover, the methods of mathematical thinking are combinations of arguments of mathematical logic and of set theory. To help the reader, some definitions and elementary results of the set theory are briefly presented (Gellert et al. 1980; Kaufmann and Precigout 1973). Georg Cantor (1845–1918) is the founder of the set theory. He gave the following definition: a set is obtained when several objects specified by human perception or by thought are included into a single entity; these objects are called the elements of the set. This definition, although imprecise and prone to induce some contradictions, has the advantage of an intuitive image. If an object x is an element of the set S, one writes $x \in S$. If S contains two distinct elements a and b, then S is called *unordered pair* and it is denoted $S = \{a, b\}$. A subset T of a set S is any set which contains only elements belonging to S. This is denoted $T \subseteq S$. It is said that the set T is included in the set S. The empty set is a set without elements. The set defined by a sentence $H(x)$ is denoted by $\{x \mid H(x)\}$ (It is read "the set of all x so that $H(x)$").

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The sets whose elements are sets are called *families* (or systems, or classes) of sets. An important class is the set of all subsets of a given set S; this is called the power set of S (or the set of the parts of S) and is denoted $\mathbf{P}(S)$.

All the systems of axioms of the set theory have in common the following four principles.

- The *principle of extensionality,* which say that two sets S and T, having the same elements, are identical (it is written $S = T$).
- The *principle of construction* indicates different specific types of sentences that are used for defining the sets. Usually it requires that those sentences must contain only symbols of objects, logic symbols and the symbol $\in$.
- The *principle of the existence of infinite sets* must be understood as such. Although it is difficult to motivate it in connection with reality, without this principle an important part of mathematics (including the differential and the integral calculus) would lose its meaning.
- The fourth principle is usually called the *axiom of choice*: If S is a class of non-empty sets, then there is a set A that has precisely one element in common with each set of S.

For the construction of new sets, starting from given sets, operations with sets are used. The main operations are: union, intersection and difference, which are defined in Table 2.1.

Sets whose intersection is the empty set are called *disjoint sets*. If S is a subset of U, then U-S is called the *complement (or relative complement)* of S in U.

The main properties of operations with sets, which are often used in practice, are commutativity, associativity, distributivity and idempotency. For convenience, these well-known properties are reminded in Table 2.2.

Table 2.1 Operations with sets S and T

Operation name	Notation	Definition
Intersection	$S \cap T$	$\{x \vert x \in S \text{ and } x \in T\}$
Union	$S \cup T$	$\{x \vert x \in S \text{ or } x \in T\}$
Difference	$S \backslash T$	$\{x \vert x \in S \text{ and } x \notin T\}$

Table 2.2 Properties of operations with sets

Name	Explanation
Commutativity	$S \cap T = T \cap S \qquad S \cup T = T \cup S$
Associativity	$S \cap (T \cap R) = (S \cap T) \cap R \qquad S \cup (T \cup R) = (S \cup T) \cup R$
Distributivity	$S \cap (T \cup R) = (S \cap T) \cup (S \cap R)$ $S \cup (T \cap R) = (S \cup T) \cap (S \cup R)$
Idempotency	$S \cap S = S \qquad S \cup S = S$

If S and T are subsets of U and their complements in U are S' and T', respectively, then the following relations occur:

$$(S \cap T)' = S' \cup T' \qquad (S \cup T)' = S' \cap T' \tag{2.1}$$

These are the *laws of De Morgan*, often used in applications.

2.2 Relations

An *ordered pair* (a, b) is intuitively defined as a juxtaposition of two objects a and b so that a can be distinguished as the first element of the ordered pair and b as the second element. A rigorous definition will be given later.

A relation R on a set S is a set of ordered pairs of elements of S. If $(a, b) \in R$, it is said that R takes place for the ordered pair (a, b). Sometimes this is denoted aRb. For example, in the set S of all people alive at some moment, we can define the relationship "A is the parent of B".

The set $\{x \in S|(x, y) \in R \text{ for at least one } y \text{ from } S\}$ is called the support of R. The set $\{y \in S|(x, y) \in R \text{ for at least one } x \text{ from } S\}$ is called the set of values or the codomain (or the range, or the image) of R. These sets will be denoted by Sup R and Ran R, respectively. The set Dom R = Sup R is called the domain of R. Of course, Dom $R \subseteq S$.

There are many relations that are commonly used in mathematics as well as in other areas, of which the exact sciences come on the first place. It is found, through a systematic analysis, that the relations have certain common properties, which are listed in Table 2.3.

Table 2.3 Attributes for a relationship R on a set S

Attribute	Definition
Reflexive	xRx takes place for all $x \in S$
Non-reflexive	Does not exist $x \in S$ so that xRx takes place
Symmetric	For every $x, y \in S$, from xRy it comes yRx
Asymmetric	Does not exist elements $x, y \in S$ with xRy and yRx
Antisymmetric	For every $x, y \in S$: if xRy and yRx, then $x = y$
Transitive	For every $x, y, z \in S$: if xRy and yRx, then xRz
Connex	For every $x, y \in S$: if $x \neq y$, then xRy or yRx
Left-unique (injective)	For every $x, y, z \in S$: if xRz and yRz, then $x = y$
Right-unique (univalent, right defined)	For every $x, y, z \in S$: if xRy and xRz, then $y = z$
Biunivocal (one-to-one)	If left-unique and right-unique

2.2.1 Equivalence Relations

An *equivalence relation* on a set S is a reflexive, symmetric and transitive relation which has S as support. For example, the relation of parallelism between two straight lines d si d', which is noted $d||d'$, is an equivalence relation.

An equivalence relation R on S induces a partition of S into classes, which are composed of those elements between which the equivalence relation is defined. A partition of a set S is a nonempty family $\mathbf{P}$ of non-vide subsets of S, called *partition classes*, with the following two properties:

- (i) two distinct classes are disjoint
- (ii) any element of S belongs to a class.

The next theorem is called the *principle of identification.*

Theorem 2.1 *If R is an equivalence relation on a set S, then there is a partition $\mathbf{P}$ of S so that the elements $(a, b) \in S$ are in the same class of $\mathbf{P}$, if, and only if aRb. Conversely, if $\mathbf{P}$ is a particular partition of S, then the relation $\{(a, b)|\, \text{there is a class } C \in P \text{ with } a, b \in C\}$ is an equivalence relation.*

2.2.2 Ordering

A relation R on the set S is called *partial order relation* if R is reflexive, transitive and antisymmetric. If, in addition, R is concave, the relation is called *total order relation* or *linear order relation*. For example, the relation $a \leq b$ is a partial order relation, actually a total order relation, on the set of real numbers.

An *ordered set* is defined as a pair (S, R), where R is a partial order relation on the set S. Often, for brevity, the ordered set is simply denoted S. On an ordered set it can be defined an upper bound and a maximal element. A lemma commonly used in mathematics (*Kuratowski-Zorn lemma*) states that, if a totally ordered set (S, R) has a upper bound on S, then S has a maximal element. One can prove that this lemma is equivalent to the axiom of choice.

The notion of ordered pair (a, b) can be rigorously defined as follows: $(a, b) \equiv \{\{a\}, \{a, b\}\}$. This definition specifies the difference between the positions of the two elements. Ordered pairs have the following fundamental property: $(a_1, a_2) = (b_1, b_2)$ if and only if $a_1 = b_1$ and $a_2 = b_2$.

Consider two sets S and T. The *Cartesian product* $S \times S$ of these sets (noted $S \times T$) is the set of all ordered pairs (a, b) with $a \in S$ and $b \in T$. The Cartesian product $S \times S$ is shortly noted S^2. The Cartesian product $S^2 \times S$ is noted S^3 and this system of notation can be generalized. The elements of S^n are called *n-tuple* of elements of S. For example the 3-tuple, $((a, b), c)$ also called *triplet* is simply denoted (a, b, c). Another example: the set of complex numbers can be considered as the Cartesian product $R \times R = R^2$ of the set of real numbers with itself.

A relation with *n arguments* or *n-ary* on S is defined as a subset of S^n. Relations with two arguments are called *binary relations*. Relations with n arguments are called *predicates*. For example, the relationship "point A lies between the points B and C" is a relation with three arguments for the points on a straight line.

2.3 Functions and Maps

A *function* on a set S with values in T is a right-unique relation with the support S and the set of values T. The term *map* (or *mapping*) is used to mean a function, sometimes with a specific property of particular importance. If the support is the entire set S, it is said that the *map* is of *S in T*. If set of values of a map is the entire set T, then it is said that the map is of *S on T*, or *surjective*. Examples of functions are the functions defined on the set of real numbers R with values in the same set (called *real functions*). The functions of n real variable are functions defined on R^n with values in R. The maps of the set of natural numbers N in itself are called *arithmetic functions*.

The functions on S with values in T are subsets of $S \times T$. In some branches of mathematics (e.g. complex analysis) functions are not defined as right-unique. The most commonly used functions are maps of a set S into another set T. The set of all maps of S in T is denoted T^S.

One can define functions or maps whose arguments are functions or maps. Such functions are called *operations* (e.g. maps of S^2 in S), *functionals* (functions defined on a set of functions with values in the set of real numbers), *operators* (functions defined on a set of functions with values in another set of functions), *functors* and *morphisms* (maps that preserve in some sense the algebraic structures).

If F is a function on S with values in T and if $(x, y) \in F$, then y is called the *image* of x by F or the *value* of F in x. This is denoted in various ways, such as, $y = x^F, y = xF, y = F(x)$ or $y = F_x$. The set $F^{-1}(y) = \{x \in S \mid F(x) = y\}$ is called the *inverse image* (or *preimage*) of y.

A function on S with values in T is called *injective*, *injection*, *invertible*, *one-to-one* or *biunivocal* if it is a left-unique relation. In this case, any element from the domain of the values has a unique image and the set $\{(y, x) \in T \times S | (x, y) \in F\}$ is a function on T with values in S, which is noted F^{-1} and is called the inverse function of F. If F is an injective function, then F^{-1} is a function if and only if F is surjective. Such a function is called bijective. The inverse of a bijective function is itself bijective.

2.4 Groups

An operation on a set S is an application which associates each ordered pair (a, b) of elements of S a third element c in the same set S. For example, ordinary addition and multiplication are operations on sets of integer, rational, real or complex numbers, respectively.

An operation (denoted by $\otimes$) on a set S is called *associative* if $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ for any elements a, b and c of S. The operation is called commutative if $a \otimes b = b \otimes a$. An element e of the set S is called neutral element of the operation $\otimes$ if $a \otimes e = e \otimes a = a$ for any element a of S. The neutral element, if any, is unique. An element a' of S is called the *inverse* of the element a of S if $a' \otimes a = a \otimes a' = e$. Sometimes the notation a^{-1} is used to designate the inverse of the element a.

In case of operations similar with ordinary multiplication (called multiplicative-like operations), the neutral element is called *identity element.* In case of operations similar with ordinary addition (called additive-like operations), the neutral element is called *zero element* and the inverse element is called *additive inverse element* (or *opposite element*).

A group is a set G for which the following conditions are fulfilled:

1. An operation is defined on G;
2. That operation is associative;
3. The set G has a neutral element;
4. Any element of G has an inverse in G.

If the operation is commutative, the group is called *commutative group* (or *Abelian group*).

A group is *finite* or *infinite* as the set of its elements is finite or infinite. The number of elements is the *order of the group*.

A subset H of a group G is called subgroup if H is a group for the group operation defined on G. All groups with one element are called *trivial subgroups* and all subgroups of a group G different from G are called *proper subgroups*.

2.4.1 Homeomorphism

An mapping f of a group $(G, \otimes)$ in a group $(G', \circ)$ is called *homeomorphism* if the relation $f(a \otimes b) = f(a) \circ f(b)$ occurs for any elements $a, b \in G$. The left-side product is taken in G and the right-side product is taken in G'. The image of G by f is a subgroup of G'. If a surjective homeomorphism of G on G' exists, then G' is the homeomorphic image of G. An homeomorphism may apply distinct elements of G on the same element of G'. Homeomorphisms are not necessarily injective.

The definition of homeomorphism suggests that it preserves in a certain sense the structure of the original group. However, in general the image is "smaller" than the original group. The set of the elements of G applied on the neutral element of the image is a measure of the narrowing of G. These elements form a subgroup of G called the *kernel of the homeomorphism.*

2.4.2 Isomorphism

A bijective homeomorphism is called *isomorphism*. If f is an isomorphism of G on G', then its image is G'. If there is an isomorphism from G to G', then it is said that the groups G and G' are isomorphic.

The isomorphism is an *equivalence relation* between groups, so that the class of all groups is divided into isomorphism classes. Isomorphic groups have the same structure and the calculations follow the same laws, even if the elements are of different nature and operations are defined in different ways.

2.4.3 Automorphism

An automorphism is an isomorphism of the group G on itself. The composition of two automorphisms of a group G is also an automorphism of G. If f is an automorphism, then f^{-1} is also an automorphism. The automorphisms of G form a group for the operation of composition of functions. This group is called the *group of the automorphisms* on G.

2.5 Fields

A field is a set K of elements that meet the following conditions (axioms):

1. On K two operations are defined (they will be referred to as addition and multiplication).
2. The addition determines on K an Abelian group, with 0 being the neutral element.
3. The multiplication determines on the nonzero elements of K an Abelian group.
4. Multiplication is distributive in relation to addition; therefore, for any elements a, b, c of K the relation $a(b+c) = ab + ca$ is true.

Examples of fields (for which the two operations are the ordinary addition and multiplication, respectively) are the sets of rational numbers, real numbers and complex numbers. Intuitively, it can be said that a field is a set in which ordinary arithmetic operations can be performed. The fields are finite and infinite, according to their number of elements.

A subset P of a field K is called subfield if it satisfies the axioms of the field for the operations defined in K. Obviously, the sum and product of the elements of P should belong to P, as well as their inverse elements and their opposite (additive inverse) elements of P. The field K is called the *extension of the field P*.

Any field K can be considered as a *vector space*. For this, any subfield P can be considered a *set of scalars*. The addition is defined on the elements of the field as the operation of addition in the vector space, while the multiplication on the elements of the subfield is defined as the operation of multiplication by scalars. If the field K has a finite dimension n, the elements $\beta_1, \beta_2, \ldots, \beta_n$ can be found, so that any $\beta \in K$ is expressed by the unique form $\beta = c_1\beta_1 + c_2\beta_2 + \cdots + c_n\beta_n$, where $c_1, c_2, \ldots, c_n$ are elements of P. The elements $\beta_1, \beta_2, \ldots, \beta_n$ form a *basis* of K on P.

Given two fields K_1 and K_2, a bijective mapping f from K_1 to K_2, with the properties that $f(a+b) = f(a) + f(b)$ and $f(a \cdot b) = f(a) \cdot f(b)$, for any elements $a, b \in K_1$, is called *isomorphism* of K_1 on K_2. In this case, K_1 and K_2 are called isomorphic. An isomorphism of the field K on itself is called *automorphism*.

2.6 Spaces

2.6.1 Linear Spaces

There are several ways to define and to present the main properties of the linear spaces. Here the guide is (Beju et al. 1976).

Two sets are given: a set V on which the operation + is defined and a set K on which two operations are defined, namely $\oplus$ and $\circ$. The three operations are called *internal composition laws*. It is assumed that that $(V, +)$ is an Abelian group and that $(K, \oplus, \circ)$ is a commutative field. An operation called "product" (denoted $*$) of the elements of the group V with elements of the field K, is defined as follows:

$$K \times V \ni (\alpha, \mathbf{x}) \to \alpha * \mathbf{x} \in V \tag{2.2}$$

The operation * is called *external composition law* on V, because it attaches the element $\alpha * \mathbf{x}$ of V to the couple consisting of the element α of K and the element $\mathbf{x}$ of V, respectively. Denote by $\mathbf{x}, \mathbf{y}$ two arbitrary elements of V, by α, β two arbitrary elements of K and by 1 the identity element of the filed K. It is assumed that the following four axioms are fulfilled:

$$\begin{aligned} \alpha * (\mathbf{x} + \mathbf{y}) &= \alpha * \mathbf{x} + \alpha * \mathbf{y} \\ (\alpha \oplus \beta) * \mathbf{x} &= \alpha * \mathbf{x} + \beta * \mathbf{x} \\ (\alpha \circ \beta) * \mathbf{x} &= \alpha * (\beta * \mathbf{x}) \\ 1 * \mathbf{x} &= \mathbf{x} \end{aligned} \tag{2.3a–d}$$

Note that Eq. (2.3a–d) regulates how the external composition law * operates together with the internal composition laws defined in V and K. Equation (2.3a) shows how the "product" * with elements of K behaves in relation to the addition of elements of V. Equation (2.3b) and (2.3c) show how the "product" * behaves in relation to the addition and multiplication of the elements of K. Equation (2.3d) defines the effect of the identity element of the field K in relation to the elements of the group V. This equation ensures that the set of the values of the external composition law equals the set V (with other words, the product with elements of the field K is surjective).

If the axioms (2.3a–d) are fulfilled, it is said that V is a linear space on the field K. It is denoted V/K. In practice, two particular cases are more important, namely when K is the field of real numbers and the field of complex numbers, respectively. It is said that V/K represents a real or a complex linear space, respectively.

For convenience, it is customary to denote by 0 and 1, the zero element and the identity element of the field K, respectively. By $\mathbf{0}$ is usually denoted the zero element of the group V.

2.6.1.1 Vectors and Scalars

If the space V/K is linear, the elements of the group V are called *vectors* and the elements of the field K are called *scalars*. For this reason, V/K is called *vector space*. Therefore, the operation * can be assimilated to the product between vectors and scalars. In this case, the zero element $\mathbf{0}$ is called *zero vector.*

2.6.1.2 Linear Subspace

Consider the linear space V/K and a subgroup U of V. Also, α and $\mathbf{x}$ are arbitrary elements of K and U, respectively. In case that $\alpha * \mathbf{x} \in U$, then U is a linear space on K, denoted U/K, and it is called *linear subspace* of V/K. The product with scalars of the elements of U is the product with scalars from V/K.

2.6.1.3 Linear Independence

Assume a linear space V/K and a number n of elements of V, i.e. $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in V$. The vectors $\mathbf{x}_i$ $(i = 1, \ldots, n)$ are called *linearly independent* if the relation

$$\alpha_1 \mathbf{x}_1 + \cdots \alpha_n \mathbf{x}_n = 0 \tag{2.4}$$

with $\alpha_i \in K(i = 1, \ldots, n)$, implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Otherwise, these vectors are called *linearly dependent.* An equivalent wording is: the system of vectors

$\{\mathbf{x}_i, i = 1, \ldots, n)\}$ is linearly independent (respectively, linearly dependent). In a linearly dependent system of vectors, one of the vectors (in general, not anyone) can be expressed as a linear combination of the other vectors.

2.6.1.4 Dimension of a Linear Space

The linear spaces can be divided into two types. Thus, if in a linear space the number of linearly independent vectors is infinite, the space is called of *infinite dimension*. These spaces are studied by the *functional analysis*.

A linear space V/K is called of *finite dimension* if there is a finite upper bound for the number of its linearly independent vectors. In other words, there is a natural number n, so that there are n linearly independent vectors, but, at the same time, any m vectors ($m > n$) are mandatory linearly dependent. The number n is called the *dimension of the linear space*.

2.6.1.5 Basis for a Linear Space

A maximal system of n linearly independent vectors constitutes a *basis* of V/K. In another formulation, if n linearly independent vectors can be found in a linear space, any $n+1$ vectors being linearly dependent, then the linear space has the dimension n (it is said that the space is n-dimensional), the n linearly independent vectors being its basis.

2.6.1.6 An Important Example of Linear Space

Assume a commutative field K. A set K^n is built, consisting of sets of n elements taken from K. Using the Cartesian product definition, the set K^n has the form:

$$K^n \equiv \underbrace{K \times K \times \cdots \times K}_{\text{n } times} \tag{2.5}$$

Two operations are defined: the "addition" of elements of K^n and the "multiplication" of elements of K with other elements of K^n:

$$\begin{aligned} (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) &\equiv (x_1 \oplus y_1, x_2 \oplus y_2, \ldots, x_n \oplus y_n) \\ \lambda * (x_1, x_2, \ldots, x_n) &\equiv (\lambda \circ x_1, \lambda \circ x_2, \ldots, \lambda \circ x_n) \end{aligned} \tag{2.6, 7}$$

The set K^n, for which the operations (2.6,7) are defined, has the structure of a linear space on the field K. This linear space, which is denoted K^n/K, is called n-dimensional Cartesian space, because its dimension is n. A basis of K^n/K is made up of the vectors

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \mathbf{e}_n = (0, 0, \ldots, 1) \tag{2.8}$$

Thus, any vector $\mathbf{x} \in K^n/K, x \equiv (x^1, x^2, \ldots, x^n)$ can be written as follows:

$$\mathbf{x} = (x^1, 0, \ldots, 0) + (0, x^2, \ldots, 0) + \cdots + (0, 0, \ldots, x^n) \tag{2.9}$$

which is equivalent with the more compact form:

$$\mathbf{x} = x^1 e_1 + x^2 e_2 + \cdots + x^n e_n \equiv x^i \mathbf{e}_i \tag{2.10}$$

The second equality in Eq. (2.10) represents the *Einstein's summation convention*, which is a compact way of writing some sums. From Eq. (2.10) it follows that any $n + 1$ vectors are linearly dependent.

2.6.1.7 Comments on the Axioms of Linear Spaces

Note that the set of axioms (2.3a–d) follows the rules presented in the beginning of this chapter. To show that the first rule is fulfilled, it must be proven that there are elements that meet the set of axioms. This can be easily proved by example (K^n/K is such an example of linear space). To show that the second rule, concerning the properties of the sets of axioms properly constructed, is checked, a counter-example will be given. Assume an additive Abelian group V and a field K. The following law of external composition with elements of V and K is defined:

$$\alpha * \mathbf{x} = \mathbf{0}, \quad (\forall \mathbf{x} \in V, \forall \alpha \in K) \tag{2.11}$$

Further assume two arbitrary elements $\mathbf{x}, \mathbf{y} \in V$ and two arbitrary elements $\alpha, \beta \in K$. Using the definition of the law of composition Eq. (2.11) it can be written:

$$\begin{aligned} \alpha * (\mathbf{x} + \mathbf{y}) &= \alpha * \mathbf{x} + \alpha * \mathbf{y} = \mathbf{0} \\ (\alpha \oplus \beta) * \mathbf{x} &= \alpha * \mathbf{x} + \beta * \mathbf{x} = \mathbf{0} \\ (\alpha \circ \beta) * \mathbf{x} &= \alpha * (\beta * \mathbf{x}) = \mathbf{0} \end{aligned} \tag{2.12a–c}$$

The three Eqs. (2.12a–c) lead to the following conclusions. The first conclusion is that the first three axioms (2.3a–c) of the definition of linear space are met. A second conclusion comes from the fact that the axiom Eq. (2.3d) is not met; therefore the second condition of the correct way of construction of the system of axioms is satisfied, because it was shown that there are elements that do not check the entire set of axioms. The third conclusion, which derives from the first two, is that axiom (2.3d) is independent of the other three axioms.

It may be shown that the axiom (2.3b) is independent, by using the following example, where V and K are an Abelian group and an arbitrary field, respectively. The product of the elements of V with scalars of K is defined as follows:

$$\alpha * \mathbf{x} = \mathbf{x}, \quad (\forall \mathbf{x} \in V, \forall \alpha \in K) \tag{2.13}$$

Assume two elements $\mathbf{x}, \mathbf{y} \in V$ and two elements $\alpha, \beta \in K$. Using the definition of the law of composition (2.12), it can be written:

$$\begin{aligned} \alpha * (\mathbf{x} + \mathbf{y}) &= \alpha * \mathbf{x} + \alpha * \mathbf{y} = \mathbf{x} + \mathbf{y} \\ (\alpha \circ \beta) * \mathbf{x} &= \alpha * (\beta * \mathbf{x}) = \mathbf{x} \\ 1 * \mathbf{x} &= \mathbf{x} \end{aligned} \tag{2.14}$$

It is found that the axioms (2.3a), (2.3c) and (2.3d) are checked. Instead, the axiom (2.3b) is not checked, because it is seen that

$$\begin{aligned} (\alpha \oplus \beta) * \mathbf{x} &= \mathbf{x}, \\ \alpha * \mathbf{x} + \beta * \mathbf{x} &= \mathbf{x} + \mathbf{x} \end{aligned} \tag{2.15}$$

Hence the axiom (2.3b) is independent of the other three axioms. The third rule of defining in a correct way a system of axioms asks the independence of the axioms (2.3a) and (2.3c). This can be done quite easily using procedures similar to those above.

2.6.1.8 Properties of Vector Spaces

Using the axioms of linear spaces, several important properties can be formulated. Demonstrations can be found in Beju et al. (1976, pp. 107–108).

1. For any $\mathbf{x} \in V$, it can be checked that $0 * \mathbf{x} = \mathbf{0}$.
2. For any $\alpha \in K$, it can be checked that $\alpha * \mathbf{0} = \mathbf{0}$.
3. In any linear space, $\alpha * \mathbf{x} = 0$ if and only if $\alpha = 0$ or $\mathbf{x} = \mathbf{0}$.
4. In any linear space, the axiom of the commutativity of the group (V, +) is a consequence of the axioms of the linear space.
5. Any vector system containing the zero vector is linearly dependent.
6. Any system of linearly independent vectors does not contain the zero vector.
7. A system consisting of a single vector is linearly independent if and only if $\mathbf{x} \neq \mathbf{0}$.

2.6.1.9 Coordinates in Linear Spaces

Assume an ordered basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mathrm{n}$ of an n-dimensional linear space V/K and an arbitrary element $\mathbf{x} \in V$. Note that the following $n+1$ vectors: $\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, are linearly dependent. It follows that some scalars $\alpha_0, \alpha_1, \ldots, \alpha_n \in K$ exist, which are not all null, so that

$$\alpha_0 \mathbf{x} + \alpha_1 \mathbf{e}_1 + \cdots + \alpha_n \mathbf{e}_n = 0 \tag{2.16}$$

It is mandatory that $\alpha_0 \neq 0$, because otherwise it would follow that the n vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_\mathrm{n}$ are not linearly independent. Thus, the relation (2.16) can be written as:

$$\mathbf{x} = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \cdots + \lambda_n \mathbf{e}_n \tag{2.17}$$

It is easily shown that the scalars $\lambda_i (i = 1, \ldots, n)$ are uniquely determined (Beju et al. 1976, p. 110). Therefore, a set of scalars $\lambda_i (i = 1, \ldots, n)$ is associated to any vector $\mathbf{x}$. This set of scalars is unique in a given ordered basis. They constitute the *coordinates* of $\mathbf{x}$ in that basis.

For this reason, an ordered basis in a linear space is called *coordinate system*.

2.6.1.10 Isomorphism of Linear Spaces

The above allow a one-to-one correspondence between the set of vectors and the set of rows (or columns) formed by using coordinates of vectors. By this correspondence, the operations of addition of vectors and the product of a vector by a scalar may be associated with operations in K^n/K by using the rows (or columns) of coordinates.

An important consequence of this observation is that, whatever the nature of the elements of a n-dimensional linear space V/K (these elements being functions, arrays, physical quantities, etc.), that space does not differ fundamentally (in terms of its operations) from the space K^n/K. This observation is more rigorously stated as follows:

Definition 2.1 Two vector spaces V/K and W/K are called isomorphic if there is a mapping (function) $f : V \to W$ which has the properties:

(a) the function f is bijective;
(b) for any elements $\mathbf{x}, \mathbf{y} \in V$ and $\lambda \in K$, the following relations are true

$$f(x+y) = f(x) + f(y), \qquad f(\lambda x) = \lambda f(x) \tag{2.18}$$

All linear spaces of dimension n, whatever their nature, are isomorphic among themselves and isomorphic with the space K^n/K, which can be imagined as a space of rows (or columns). In general, the group operations of different vector spaces differ between them. However, due to the isomorphism, they can be marked with the same sign (for example, +).

2.6.1.11 Scalar Product in Linear Spaces

Assume some elements $\mathbf{x}, \mathbf{y}, \mathbf{z}$ of the linear space V/K and the arbitrary element $\lambda \in K$. Also, it is denoted by $\overline{c}$ the complex number conjugate of c. The scalar product in the space V/K is defined as a mapping f that associates certain elements $\mathbf{x}, \mathbf{y}$ of the vector product $V \times V$ with an element, denoted $<\mathbf{x}, \mathbf{y}>$, of the field K, namely

$$f(x, y) \rightarrow \; <x, y> \tag{2.19}$$

and fulfills the following four axioms:

$$\begin{aligned}
&<\mathbf{x}, \mathbf{y}> \; = \; <\overline{\mathbf{y}, \mathbf{x}}> \\
&<\mathbf{x} + \mathbf{y}, \mathbf{z}> \; = \; <\mathbf{x}, \mathbf{z}> + <\mathbf{y}, \mathbf{z}> \\
&<\lambda \cdot \mathbf{x}, \mathbf{y}> \; = \lambda \cdot <\mathbf{x}, \mathbf{y}> \\
&\begin{cases} <\mathbf{x}, \mathbf{x}> \; > 0, & \text{if } \mathbf{x} \neq 0 \\ <\mathbf{x}, \mathbf{x}> \; = 0 & \text{if } \mathbf{x} = 0 \end{cases}
\end{aligned} \tag{2.20a–d}$$

Note that, in the particular case of real linear spaces ($K \equiv R$), the axiom (2.20a) becomes $<\mathbf{x}, \mathbf{y}> \; = \; <\mathbf{y}, \mathbf{x}>$, i.e. the scalar product is symmetric in its arguments. Using the definition of the scalar product it can be shown that the following general propositions take place:

$$\begin{aligned}
&<\mathbf{x}, \lambda \cdot \mathbf{y}> \; = \overline{\lambda} \cdot <\mathbf{x}, \mathbf{y}>, \quad <\mathbf{x}, \mathbf{y} + \mathbf{z}> \; = \; <\mathbf{x}, \mathbf{y}> + <\mathbf{x}, \mathbf{z}> \\
&<\mathbf{0}, \mathbf{x}> \; = 0, \quad (<\mathbf{x}, \mathbf{y}>)^2 \leq \; <\mathbf{x}, \mathbf{x}> \cdot <\mathbf{y}, \mathbf{y}>
\end{aligned} \tag{2.21a–d}$$

Here $\overline{\lambda}$ is the conjugate complex number of the number λ. In the case of real linear spaces ($K \equiv R$), $\overline{\lambda} = \lambda$. The proposition (2.21d) is the *Cauchy–Bunyakovsky–Schwarz inequality*.

2.6.1.12 Norm and Distance in Linear Spaces

The scalar product can be used to introduce the notions of norm and distance on a linear space. The norm of an element $\mathbf{x}$ of V/K, usually denoted by $\|\mathbf{x}\|$, is a function from the set V/K in the set of real numbers, defined by

$$\|\mathbf{x}\| \equiv (<\mathbf{x}, \mathbf{x}>)^{1/2} \tag{2.22}$$

The norm on a linear space fulfills the standard axioms of the norm-like functions

$$\begin{cases} \|\mathbf{x}\| \geq 0 & \text{if} \quad \mathbf{x} \neq \mathbf{0} \\ \|\mathbf{x}\| = 0 & \text{if} \quad \mathbf{x} = \mathbf{0} \end{cases}$$
$$\lambda \cdot \|\mathbf{x}\| = |\lambda| \cdot \|\mathbf{x}\| \tag{2.23a–c}$$
$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|,$$

Relation (2.23c) is the *triangle inequality*.

A vector $\mathbf{u} \in E$ is called *unit vector* or *normalized vector*, or *versor*, if $\|\mathbf{u}\| = 1$. Any vector $\mathbf{x} \neq \mathbf{0}$ can be normalized by dividing it by $\|\mathbf{x}\|$.

The distance between the elements $\mathbf{x}$ and $\mathbf{y}$ of V/K, which is usually denoted by $d(\mathbf{x}, \mathbf{y})$, is a function from the set $V \times V$ in the set of real numbers, defined by:

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \tag{2.24}$$

The distance between two elements in a linear space fulfills all standard axioms of the distance-like functions, i.e.:

$$\begin{cases} d(\mathbf{x}, \mathbf{y}) \geq 0 & \text{if} \quad \mathbf{x} \neq \mathbf{y} \\ d(\mathbf{x}, \mathbf{y}) = 0 & \text{if} \quad \mathbf{x} = \mathbf{y} \end{cases}$$
$$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x}), \qquad d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \geq d(\mathbf{x}, \mathbf{z}) \tag{2.25a–c}$$

It is seen from property (2.25b) that the distance is a symmetric function in its arguments.

2.6.2 Unitary and Euclidean Spaces

The concepts of scalar product, norm and distance, previously defined for linear spaces, have properties similar with the analogous notions of the Euclidean geometry. This observation allows an extension of the formal analogy, by introducing new definitions.

The complex linear spaces in which a scalar product has been defined are called *unitary spaces* while the real linear spaces in which a scalar product has been defined are called *Euclidean spaces*. The dimension of the unitary (Euclidean) space is given by the dimension of the linear space V/K. The Euclidean spaces are usually denoted by E, while the scalar product in Euclidean spaces is usually written in the more compact form $\mathbf{x} \cdot \mathbf{y}$, commonly used in geometry. Several notions of geometric inspiration will be introduced and discussed further.

2.6.2.1 Orthogonal Vectors

In the Euclidean space E, the scalar product can be used to introduce the notion of angle between two vectors. Assume $\mathbf{x}$ and $\mathbf{y}$ are two non-zero vectors. The angle $\theta(\mathbf{x},\mathbf{y})$ between these vectors is defined by the relation:

$$\cos\theta(\mathbf{x},\mathbf{y}) = \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|}, \qquad (0 \le \theta \le \pi) \tag{2.26}$$

By using the Cauchy–Bunyakovsky–Schwarz inequality and the properties of the cosine function it can be shown that the angle $\theta(x, y)$ is well defined, i.e. it has values in the range [0,1].

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if their scalar product is zero, i.e.:

$$\mathbf{x}\cdot\mathbf{y} = 0 \tag{2.27}$$

By using relations (2.26) and (2.27), it can be easily deduced that the angle between two orthogonal vectors equals $\pi/2$.

A set of nonzero vectors $\mathbf{x}_i(i = 1,\ldots,n)$ is called *orthogonal system* if $\mathbf{x}_i\cdot\mathbf{x}_j \neq 0$ only for $i = j, (i, j = 1,\ldots,n)$. The systems of orthogonal vectors have several important properties that result from the following two theorems (for demonstration, see Beju et al. 1976, p. 117):

Theorem 2.2 *An orthogonal system of vectors* $\{\mathbf{x}_i, i = 1,\ldots,n\}$ *is linearly independent.*

Theorem 2.3 *In any Euclidean space of finite dimension there are orthogonal bases.*

2.6.2.2 Orthogonalization Process

Assume a basis $\{\mathbf{e}_i, i = 1,\ldots,n\}$ of a n-dimensional Euclidean space E. Obviously, $\mathbf{e}_i \neq 0\,(i = 1,\ldots,n)$. Starting from this basis, an orthogonal basis (denoted $\mathbf{f}_1, \mathbf{f}_2,\ldots,\mathbf{f}_n$) can be obtained by using the following procedure, called *orthogonalization* or *orthogonalization process.*

First, denote $\mathbf{f}_1 = \mathbf{e}_1$. Then, define $\mathbf{f}_2 = \mathbf{e}_2 + \alpha\cdot\mathbf{f}_1$ and determine the scalar α, by requiring that $\mathbf{f}_1$ and $\mathbf{f}_2$ are orthogonal (i.e. $\mathbf{f}_2\cdot\mathbf{f}_1 = 0$). It is found that $\alpha = -\mathbf{e}_2\cdot\mathbf{f}_1/\|\mathbf{f}_1\|^2$. Next, the mathematical induction is used. It starts from the premise that a set of k non-zero vectors, orthogonal two by two, denoted $\mathbf{f}_1, \mathbf{f}_2,\ldots,\mathbf{f}_n$, has been built by using the relation

$$\mathbf{f}_k = \mathbf{e}_k + \alpha_1 \cdot \mathbf{f}_1 + \cdots + \alpha_{k-1} \cdot \mathbf{f}_{k-1} \qquad (k = 1, \ldots, m-1) \tag{2.28}$$

In this case, any vector $\mathbf{f}_m$ will be given by a relation similar with equality (2.28), i.e.

$$\mathbf{f}_m = \mathbf{e}_m + \alpha_1 \cdot \mathbf{f}_1 + \cdots + \alpha_{m-1} \cdot \mathbf{f}_{m-1} \tag{2.29}$$

The scalar coefficients $\alpha_1, \ldots, \alpha_{m-1}$ will be determined by using the orthogonality conditions $\mathbf{f}_m \cdot \mathbf{f}_k = 0 (k = 1, \ldots, m-1)$. In the generic case, it is found that $\mathbf{e}_m \cdot \mathbf{f}_k + \alpha_k \cdot \mathbf{f}_k \cdot \mathbf{f}_k = 0$, from which the coefficient α_k is determined.

A orthogonal system of normalized vectors is called *orthonormal system*. From Theorem 2.3 it follows that any n-dimensional Euclidean space has *orthonormal bases*. Note that in an orthonormal basis, the scalar product has the simple form:

$$\mathbf{x} \cdot \mathbf{y} = \sum_i x_i y_i \equiv x_i y_i \tag{2.30}$$

where $x_i, y_i (i = 1, \ldots, n)$ are the components of the vectors $\mathbf{x}$ and $\mathbf{y}$, respectively, on the orthonormal basis $\{\mathbf{e}_i\}$.

2.6.3 Affine Spaces

Geometry was the first branch of mathematics where sets of vectors have been attached to sets of points. With the generalization of the notion of vector for linear spaces, this observation is valid in the case of several disciplines of physics, such as mechanics, electrodynamics and, as will see in the next chapters, thermodynamics. For example, the force vector can be attached to a material point and a velocity vector field can be attached to a continuous domain of points of the physical space (i.e. a three-dimensional body).

The notion of affine space allows processing within the same mathematical structure of two distinct categories of elements, some of which are described as "points" and the other are called "vectors." The two categories of elements include geometric points and geometric vectors as particular cases.

The set of points will be denoted by M. The linear space of the vectors will be denoted V. Each *ordered pair* of points of M (for example, P, Q) is associated with a vector of V (for example $\mathbf{x}$) by defining an *association law*. In this case, the first point, P, is called the origin, or initial point, of the vector $\mathbf{PQ} \equiv \mathbf{x}$, and the second point, Q, is called its terminal point. With this notation, it is obvious that $\mathbf{PQ} \in V$.

For the set M, associated with the linear space V, to constitute an affine space, the following axioms should be fulfilled:

(a) For any point $\mathrm{P} \in M$ and any vector $\mathbf{x} \in V$, there is a point, and only one, $\mathrm{Q} \in M$, so that $\mathbf{PQ} = \mathbf{x}$.

(b) If $\mathbf{PQ} = \mathbf{x}$ and $\mathbf{QR} = \mathbf{y}$, then $\mathbf{PR} = \mathbf{x} + \mathbf{y}$.
Note that the ordered pair of overlaid points $(\mathrm{P,P}) \in M$ is associated with the null vector of V, i.e. $\mathbf{PP} = \mathbf{0}$. Also, if $\mathbf{PQ} = \mathbf{x}$, then it can be deduced that $\mathbf{QP} = -\mathbf{x}$ (Beju et al. 1976, p. 113).

If the linear space V is real or complex, finite or infinite dimensional, then it is said that the affine space is, respectively, real or complex, finite or infinite dimensional. The dimension of the affine space M is equal to the dimension of the linear space V.

2.6.3.1 Coordinates in Affine Spaces

Assume the n-dimensional affine space M. Assume a point O in M and a basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in the linear space V. The assembly consisting of the point O and the basis constitutes a system of affine coordinates in M. The point O is the origin of the coordinate system and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the vectors of the basis.

Assume a point P of M. The ordered pair $(\mathrm{O,P})$ is associated with the vector $\mathbf{OP}$, which is called the *position vector* of P in relation with the origin O. The coordinates of $\mathbf{OP}$ in the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are denoted $(x_1, x_2, \ldots, x_n)$. In this case, the following relation can be written:

$$\mathbf{OP} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n \equiv x_i\mathbf{e}_i \tag{2.31}$$

The coefficients $(x_1, x_2, \ldots, x_n)$ represent the *affine coordinates* of the point P. Note that these coordinates depend on the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of the linear space V and the origin O (element of M).

Assume the point $\mathrm{Q} \in M$, of affine coordinates $y_1, y_2, \ldots, y_n$. The affine coordinates of the vector $\mathbf{PQ}$ are given by:

$$\mathbf{PQ} = \mathbf{PO} + \mathbf{OQ} = \mathbf{OQ} - \mathbf{OP} = (y_1 - x_1)\mathbf{e}_1 + \cdots + (y_n - x_n)\mathbf{e}_n \tag{2.32}$$

Relation (2.32) was obtained by considering the axiom (b) of the affine spaces and the fact that $\mathbf{PO} = -\mathbf{OP}$. The conclusion is that the affine coordinates of the vector $\mathbf{PQ}$ can be written using only the affine coordinates of the points P and Q.

2.6.3.2 Distance in Affine Spaces

Consider the case of an affine space which is defined by using a linear space on which a scalar product has been introduced. In this case, the distance between two points A and B of the affine space can be defined by the relationship

$$d(\mathrm{A}, \mathrm{B}) = (\mathbf{AB} \cdot \mathbf{AB})^{1/2} \tag{2.33}$$

By choosing an affine coordinate system consisting of a point O and an orthonormal basis, one obtains an image of the affine space similar to that of the space in Euclidean geometry.

2.6.3.3 Connection Between Affine Spaces and Linear Spaces

Between affine spaces and linear spaces there is a structural link. Thus, any affine space M can be regarded as a linear space V. To prove this, it takes an arbitrary point $\mathrm{O} \in M$. Next, a position vector **OP** is attached to any point $\mathrm{P} \in M$. Then, considering the axioms (a) and (b), it is concluded that the set of position vectors coincides with V. Conversely, any linear space V can be regarded as an affine space M. To prove this, the elements of V are regarded as points of M and to the ordered pair of points (a, b) it is attached the vector $\mathbf{b} - \mathbf{a} \in V$. This interpretation is encountered when making reference to the linear space of position vectors, defining, with its help, the vectors-oriented segments connecting the ends of two vectors of position.

To remove ambiguities, it is important to specify which of the analyzed elements have the role of points and vectors, respectively.

2.7 Equivalence Classes for Reference Frame Transformation

Practice shows that the same physical quantity may be perceived and described differently by different observers. The objective (scientific) description of the quantity must be done using the intrinsic properties of that quantity, which do not change when passing from the description of one observer to another description, of a different observer. More generally, the objective character of scientific knowledge can only be ensured by consistent usage of the invariant aspects of the phenomena, which do not depend on how the phenomena are perceived by particular observers.

In this section it is exemplified how the issues which are invariant in respect to the observer can be described by using mathematical methods. The simplest example refers to a fundamental concept in physics, that of distance. At the end of the section some useful generalizations are presented.

2.7.1 Intrinsic Distance

It is accepted that the mathematical structure of the physical space is that of the Euclidean space, denoted E^3. Further, by *observer* it is understood a right-handed orthonormal frame (for definition, see Sect. 3.1.1.1) equipped with a procedure for measuring distances and time.

Assume two observers, denoted $ox_1x_2x_3$ and $OX_1X_2X_3$, respectively. Assume two distinct points, R and S, of coordinates $\mathbf{y} \equiv (y_1, y_2, y_3)$ and $\mathbf{z} \equiv (z_1, z_2, z_3)$, respectively, in respect with the first observer, and of coordinates $\mathbf{Y} \equiv (Y_1, Y_2, Y_3)$ and $\mathbf{Z} \equiv (Z_1, Z_2, Z_3)$, respectively, in respect with the second observer. The distances between the points R and S, determined by the two observers, are given by, respectively:

$$\begin{aligned} d^2(\mathbf{y}, \mathbf{z}) &= (z_1 - y_1)^2 + (z_2 - y_2)^2 + (z_3 - y_3)^2 \\ D^2(\mathbf{Y}, \mathbf{Z}) &= (Z_1 - Y_1)^2 + (Z_2 - Y_2)^2 + (Z_3 - Y_3)^2 \end{aligned} \tag{2.34a, b}$$

It is clear that, generally, the functions $d(\mathbf{y}, \mathbf{z})$ and $D(\mathbf{Y}, \mathbf{Z})$ are different. Imposing further restrictions makes possible that these functions always lead to the same numerical value, once the points R and S are given. That numerical function, which does not depend on the procedures adopted by observers for measuring the distance, but only on the position of the two points, is called *intrinsic distance*.

Of interest are the conditions that must be met in order to obtain intrinsic distances from the relationship of type (2.34a, b). Denote by P a certain point, of coordinates $\mathbf{x} \equiv (x_1, x_2, x_3)$ in respect to the first observer, and of coordinates $\mathbf{X} \equiv (X_1, X_2, X_3)$ in respect to the second observer. Denote by f the function that makes the connection between coordinates $\{x_i\}$ and $\{X_i\}$ of the point P, as determined by the two observers. Obviously, this function performs a biunivocal correspondence between E^3 and E^3. In vector notation, it be written:

$$\mathbf{X} = f(\mathbf{x}) \tag{2.35}$$

In case that the coordinates of the position vectors $\mathbf{x}$ and $\mathbf{X}$ of point P are used, the relationship (2.35) becomes

$$X_K = f_K(x_1, x_2, x_3) = f_K(x_i) \qquad (K, i = 1, 2, 3) \tag{2.36}$$

The condition of existence of an intrinsic distance in space E^3 is equivalent to the equality

$$d(\mathbf{y}, \mathbf{z}) = D(\mathbf{Y}, \mathbf{Z}) \qquad \forall R, S \in E^3 \tag{2.37}$$

The constraint (2.37) strongly restricts the set of functions f that fulfill the relationship (2.36). Now, it is considered that, for the first observer, the three points P, R and S are collinear. This is equivalent with writing

$$d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) = d(\mathbf{x}, \mathbf{z}) \tag{2.38}$$

Restriction (2.37) leads to the following equality

$$D(\mathbf{X}, \mathbf{Y}) + D(\mathbf{Y}, \mathbf{Z}) = D(\mathbf{X}, \mathbf{Z}) \tag{2.39}$$

which is equivalent to saying that the points P, R and S are collinear from the view-point of the second observer. Therefore, the bijective function f transforms collinear points into collinear points, or, in other words, it transforms straight lines into straight lines. It follows that the function f must be a linear function, according to a theorem of affine geometry (see Mihaileanu 1971). It can be concluded that the relationships between the components of the position vectors $\mathbf{x}$ and $\mathbf{X}$ of the point P in the frames of the two observers must have the following form:

$$X_K = Q_{Kk} x_k + B_K \qquad (K = 1, 2, 3) \tag{2.40}$$

where Q_{Kk} and B_K $(K, k = 1, 2, 3)$ are constants. Denote by $\mathbf{Q}$ the matrix of the components Q_{Kk} and by $\mathbf{B}$ the column matrix of the components B_K. The position vectors $\mathbf{x}$ and $\mathbf{X}$ will be considered of column matrix type. In these conditions, Eq. (2.40) can be written under the following matrix form, which emphasizes the character of linear transformations

$$\mathbf{X} = \mathbf{Q} \cdot \mathbf{x} + \mathbf{B} \tag{2.41}$$

Classical mechanics, for example, is based on such relationships for changing the coordinates of a point from a reference frame into another. The linearity of the relationship (2.41) is a consequence of the premise that an intrinsic distance exists in the physical space modeled by E^3.

2.7.2 *Orthogonal Transformations*

It was shown that the distance between two points in the physical space E^3 is invariant for the coordinate transformations (2.41). Therefore, these so-called *orthogonal transformations* are important, they being able to underpin an objective description of the physical phenomena.

The orthogonal transformations are of two types, as seen below. Assume two points $\mathbf{y}, \mathbf{z} \in E^3$. By using relations (2.34) and (2.37) it is obtained:

$$(Z_K - Y_K)(Z_K - Y_K) = (z_k - y_k)(z_k - y_k) \tag{2.42}$$

Then, using the relation (2.40), it can be written:

$$Z_K - Y_K = Q_{Kk}(z_k - y_k) \qquad (K = 1, 2, 3) \tag{2.43}$$

Therefore, the Eq. (2.42) becomes

$$Q_{Kk}(z_k - y_k)Q_{Kj}(z_j - y_j) = (z_k - y_k)(z_j - y_j) \tag{2.44}$$

By the reversal of the summing order, from (2.44) it is obtained:

$$(Q_{Kk}Q_{Kj} - \delta_{kj})(z_k - y_k)(z_j - y_j) = 0 \tag{2.45}$$

In (2.45) the symbol δ_{ij} has been used, which is known as *Kronecker's symbol*, defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{2.46}$$

The equality (2.45) should take place regardless of the elements $\mathbf{y}, \mathbf{z} \in E^3$. Therefore, the following relation should be true:

$$Q_{Kk}Q_{Kj} = \delta_{kj} \qquad (k, j = 1, 2, 3) \tag{2.47}$$

Relation (2.47) can be more compactly rewritten in the following matrix form:

$$\mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{E} \tag{2.48}$$

In the equality (2.48) the common notation has been used: $\mathbf{Q}^T$ is the transposed matrix of the matrix $\mathbf{Q}$, and $\mathbf{E}$ is the unit matrix. All these matrices are of order three. A consequence of the relationship (2.48) is that

$$\det(\mathbf{Q}^T \cdot \mathbf{Q}) = 1 \tag{2.49}$$

or, in other words, that

$$(\det \mathbf{Q})^2 = 1 \tag{2.50}$$

The previous relationships allow some comments about important properties of the matrix $\mathbf{Q}$. First, physical quantities are generally characterized by their physical dimension (e.g., dimension of length (L), mass (M), time (T)). In some cases, the physical dimension may be a more complex expression, of the form $L^a M^b T^c$, where a, b, c are real numbers. However, from the relation (2.41) it is found that the matrix $\mathbf{Q}$ has dimensionless components from physical point of view. Second, the matrix $\mathbf{Q}$ is non-singular. Then, from relations (2.48) and (2.50) it is found that the matrix $\mathbf{Q}$ is invertible. The inverse of the matrix $\mathbf{Q}$ is:

$$\mathbf{Q}^{-1} = \mathbf{Q}^{T} \tag{2.51}$$

Using again the eq. (2.48) it is deducted that:

$$\mathbf{Q} \cdot \mathbf{Q}^{T} = \mathbf{E} \tag{2.52}$$

or, in unfolded writing:

$$Q_{Kk}Q_{Lk} = \delta_{KL} \qquad (K, L = 1, 2, 3) \tag{2.53}$$

The matrices that check the equalities (2.51) are called *orthogonal matrices*. This explains why the linear relationships (2.41), where the matrix Q appears, are called orthogonal transformations. It is also observed that if in (2.41) an orthogonal matrix Q is included, the relationship (2.37) is checked unconditionally. This is justified by the next calculations, if the relationship (2.47) is also taken into account:

$$\begin{aligned} D^2(\mathbf{Y}, \mathbf{Z}) &= (Z_K - Y_K)(Z_K - Y_K) = Q_{Kk}(z_k - y_k)Q_{Kj}(z_j - y_j) \\ &= Q_{Kk}Q_{Kj}(z_k - y_k)(z_j - y_j) = \delta_{jk}(z_k - y_k)(z_j - y_j) \\ &= (z_k - y_k)(z_k - y_k) = d^2(\mathbf{y}, \mathbf{z}) \end{aligned} \tag{2.54}$$

Relation (2.50) can be rewritten in the following form, which is useful for the classification of the orthogonal matrices and transformations:

$$\det \mathbf{Q} = \pm 1 \tag{2.55}$$

An orthogonal transformation is said to be proper transformation, if $\det \mathbf{Q} = 1$. In the opposite case ($\det \mathbf{Q} = -1$), the transformation is called improper. Two reference frames (or two observers) are said to belong to the same class, if the orthogonal transformation that turns one into the other is a proper orthogonal transformation. Otherwise, it says that those reference frames (or observers) belong to different classes. From Eq. (2.55) one sees that there are only two classes of reference frames. The reference frames in the same class turns one into another by using a proper orthogonal transformation. The reference frames in different classes turns one into another by using an improper orthogonal transformation.

2.7.3 Classes of Physical Quantities

The importance of the transformations (2.40) or (2.41) between two reference frames $\mathrm{o}x_1x_2x_3$ and $\mathrm{O}X_1X_2X_3$, respectively, is that they lead to an intrinsic definition of the length. This has consequences on the shape and size of bodies.

The procedure developed previously for the intrinsic definition of the length can be extended to other types of physical quantities. Several types of physical quantities will be defined and classified in the following, according with their behavior when the transformation (2.40) occurs.

2.7.3.1 Scalars. Pseudoscalars. Vectors. Pseudovectors

In the following, only quantities with physical dimension will be considered. Therefore, the existence of this property will not be reminded.

A quantity characterized in any reference frame by a single real number invariant to the transformation (2.41) is called *scalar*. It is called *pseudoscalar*, a quantity characterized in any reference frame by a real number, which, when the transformation (2.41) occurs, is changed according to the rule:

$$\Omega = (\det Q)\omega \tag{2.56}$$

where ω and Ω is the value of the pseudoscalar in the reference frames $\mathrm{o}x_1x_2x_3$ and $\mathrm{O}X_1X_2X_3$, respectively.

A quantity characterized in any reference frame by a triplet of real numbers is called a *vector* if, when the transformation (2.41) occurs, the following relation is fulfilled between the triplets (v_1, v_2, v_3) and (V_1, V_2, V_3) which characterize the quantity in the two reference frames:

$$V_K = Q_{Kk}v_k \qquad (K = 1, 2, 3) \tag{2.57}$$

The three numbers represent the *vector components* in that reference frame. The vector thus defined is called a *polar vector*. A quantity characterized in any reference frame by a triplet of real numbers is called *pseudovector* if, when the transformation (2.41) occurs, the following relation is fulfilled between the triplets (v_1, v_2, v_3) and (V_1, V_2, V_3) which characterize the quantity in the two reference frames:

$$V_K = (\det Q)Q_{Kk}v_k \qquad (K = 1, 2, 3) \tag{2.58}$$

The three numbers represent the components of the pseudovector in that reference frame. The pseudovector is also called *axial vector*.

2.7.3.2 The Importance of Physical Dimension

The quantities previously defined (scalar, pseudoscalar, vector, pseudovector) need not be associated with a physical dimension, as long as they are maintained at an abstract level. Sometimes, using these parameters in practice requires specification of physical dimension. In this way it is avoided the composition of quantities that have obviously different physical significance (such as velocities and accelerations) although they are of the same type (i.e. vectors, in this particular case). As already mentioned, the quantities Q_{Kk} are dimensionless. By assuming the homogeneity of the transformations (2.56) and (2.58), it is seen that the physical dimension of the quantities is kept when the reference frame is changed.

2.7.3.3 Examples of Vectors

Next, some examples of mechanical vectors are shown. The theory can easily be generalized and used in other areas of physics.

Assume a reference frame in the three-dimensional space and the points P^0 (of coordinates $\left(x_1^0, x_2^0, x_3^0\right)$ in that reference frame) and P (of coordinates (x_1, x_2, x_3) in the same reference frame). Assume the quantity $\mathbf{r}\left(\mathrm{P}^0, \mathrm{P}\right)$, with physical dimension of length, characterized in the same reference frame by the triplet $(x_1 - x_1^0, x_2 - x_2^0, x_3 - x_3^0)$. If a change of reference frame is made, the new coordinates of the two points, P and P^0, are given by

$$\begin{aligned} X_K &= Q_{Kk} x_k + B_K \\ X_K^0 &= Q_{Kk} x_k^0 + B_K \qquad (K = 1, 2, 3) \end{aligned} \tag{2.59}$$

By subtraction of the two relationships (2.59) is obtained:

$$X_K - X_K^0 = Q_{Kk}(x_k - x_k^0) \qquad (K = 1, 2, 3) \tag{2.60}$$

Relation (2.60), which has the same form as the relationship (2.40), reveals that $\mathbf{r}\left(\mathrm{P}^0, \mathrm{P}\right)$ is a vector. It is called the *position vector* of point P in relation with the point P^0.

Further, it is considered that the coordinates of point P are differentiable functions of a real parameter t: $x_k = x_k(t)$ $(k = 1, 2, 3)$. In this case, a change of reference frame transforms these coordinates into:

$$X_K(t) = Q_{Kk} x_k(t) + B_K \qquad (K = 1, 2, 3) \tag{2.61}$$

The new coordinates are also differentiable functions of the real parameter t. Assume that the point P^0 is fixed. In this case, the velocity vector of the point P in relation with point P^0 can be obtained by using the vector $\mathbf{r}\left(\mathrm{P}^0, \mathrm{P}\right)$, as follows:

$$\mathbf{v}(\mathrm{P}^0, \mathrm{P}) = \frac{d}{dt} \mathbf{r}(\mathrm{P}^0, \mathrm{P}) \tag{2.62}$$

Its physical dimension is length divided by time. The components of this vector are denoted $(\dot{x}_1, \dot{x}_2, \dot{x}_3)$. Using (2.60), it is found that

$$\dot{X}_K(t) = Q_{Kk}\dot{x}_k(t) \qquad (K = 1, 2, 3) \tag{2.63}$$

Relation (2.63) shows that the velocity of a point is a vector, because the components of the velocity of point P do not depend on the coordinates of the point P^0. Thus:

$$\mathbf{v}(P^0, P) = \mathbf{v}(P) \tag{2.64}$$

Assume the case that the functions $x_k = x_k(t)$ $(k = 1, 2, 3)$ have second order derivatives in relation with the parameter t. Repeating the above process, it can be shown that the vector acceleration of the point P, denoted $\mathbf{a}(P)$, is given by

$$\mathbf{a}(P) = \frac{d}{dt}\mathbf{v}(P) \tag{2.65}$$

This vector has physical dimension of length divided by squared time and its components are $(\ddot{x}_1, \ddot{x}_2, \ddot{x}_3)$.

2.7.3.4 The Existence Theorem

Denotes by a_1, a_2, a_3 the triplet of real numbers associated (in a reference frame) with a quantity $\mathbf{a}$. In the same reference frame, the components of a vector (or pseudovector) $\mathbf{b}$ are denoted by b_1, b_2, b_3. The next theorem (given without demonstration) states the necessary and sufficient conditions for a quantity to be vector (or pseudovector).

Theorem 2.4 *(a) If, whatever the vector* $\boldsymbol{b}$*, the sum* $a_k b_k$ *is a scalar (or a pseudoscalar), then* $\boldsymbol{a}$ *is a vector (or a pseudovector). (b) If, whatever the vector* $\boldsymbol{b}$*, the sum* $a_k b_k$ *is a pseudoscalar (or a scalar) then* $\boldsymbol{a}$ *is a pseudovector (or a vector).*

2.7.3.5 Change of Components at Reference Frame Transformations

A vector or a pseudovector is completely determined by its components in a certain reference frame. Its components can then be determined in any other reference frame, by the transformation relationships presented below.

The components of the vector $\mathbf{v}$ can be obtained from relationships (2.57) and (2.58), using the identities (2.47) and (2.53). The procedure is as follows. Multiplying the Eq. (2.58) with Q_{Kj} and by summing up, it is found

$$Q_{Kj}V_K = (\det Q)Q_{Kk}Q_{Kj}v_k = (\det Q)\delta_{kj}v_k = (\det Q)v_j \tag{2.66}$$

Then, taking into consideration the relationship (2.55), it is obtained:

$$v_k = (\det Q)Q_{Kk}V_K \qquad (k = 1, 2, 3) \tag{2.67}$$

Similarly, from the relationship (2.57), it is obtained

$$v_k = Q_{Kk}V_K \qquad (k = 1, 2, 3) \tag{2.68}$$

Relations (2.57) and (2.58) allow the calculation of the new components of the vector, in case that the old components are known. The old components can be calculated as function of the new ones by using the relationships (2.68) and (2.67).

2.7.4 *Operations with Scalars and Vectors*

The quantities between which a relation exists must have the same physical dimension. The quantities subjected to operations may have the same physical dimension or may have different physical dimensions.

2.7.4.1 Relation of Equality

The relationship of equality makes sense only between quantities with the same physical dimension.

It is said that two scalars (or pseudoscalars) are equal if, in the same reference frame, they are characterized by the same real number. It is said that two vectors (or pseudovectors) are equal if, in the same reference frame, their components are equal.

If two vectors (or pseudovectors) have equal components in a reference frame, then they will have equal components in any other reference frame. To prove this, consider two vectors **u** and **v** which, in two different reference frames, have the components $u_k, v_k \ (k = 1, 2, 3)$ and respectively $U_K, V_K \ (K = 1, 2, 3)$. Using Eq. (2.57), it is found that

$$U_K = Q_{Kk}u_k, \qquad V_K = Q_{Kk}v_k \qquad (K = 1, 2, 3) \tag{2.69}$$

If the two vectors are equal in the first reference frame, then $u_k = v_k \ (k = 1, 2, 3)$. From Eq. (2.69) it is deduced that the equality $U_K = V_K \ (K = 1, 2, 3)$ takes place in the second reference frame.

Similar sentences can be formulated in the case of pseudovectors.

2.7.4.2 Operation of Addition

The addition makes sense only between quantities with the same physical dimension.

The addition of scalars (pseudoscalars) can be done in a given reference frame. It consists in the sum of the numerical values of the scalars (pseudoscalars). Assume two vectors (pseudovectors) $\mathbf{u}$ and $\mathbf{v}$, of components u_k and v_k $(k = 1, 2, 3)$ in a given reference frame. The sum of the two vectors (pseudovectors) is a vector (pseudovector), denoted $\mathbf{w}$, whose components are equal to the sum of the numerical values of the components of the two vectors. The addition is written as follows:

$$\mathbf{w} = \mathbf{u} + \mathbf{v} \tag{2.70}$$

To prove that the addition of two vectors (pseudovectors) produces a vector (pseudovector), consider that $w_k = u_k + v_k$ $(k = 1, 2, 3)$ and respectively $W_K = U_K + V_K$ $(K = 1, 2, 3)$, are the components of $\mathbf{w}$ in two reference frames connected by the relation (2.40). In case of vectors, the transformation relationship (2.57) is applied to the components of the vectors $\mathbf{u}$ and $\mathbf{v}$. It is found that:

$$W_K = U_K + V_K = Q_{Kk}(u_k + v_k) = Q_{Kk}w_k \tag{2.71}$$

In the case that $\mathbf{u}$ and $\mathbf{v}$ are pseudovectors, from (2.58) it is obtained

$$W_K = U_K + V_K = (\det Q)Q_{Kk}(u_k + v_k) = (\det Q)Q_{Kk}w_k \tag{2.72}$$

Consequently, by changing the reference frames, the components of the sum of two vectors (pseudovectors) transforms itself as a vector (pseudovector).

2.7.4.3 Scalar Product

The set of all vectors and pseudovectors, regardless of their physical size, will be denoted by V. Assume that $\mathbf{u}$ and $\mathbf{v}$ are elements of V. Their components, in a certain reference frame, will be denoted u_k, v_k $(k = 1, 2, 3)$. The *scalar product* (or *inner product*, or *dot product*) of $\mathbf{u}$ and $\mathbf{v}$ is defined as the number $u_k v_k$. This product has physical dimension and is denoted $\mathbf{u} \cdot \mathbf{v}$.

If $\mathbf{u}$ and $\mathbf{v}$ are vectors, their scalar product is a scalar. This can be easily shown. Denote with U_K and $V_K (K = 1, 2, 3)$ the components of $\mathbf{u}$ and $\mathbf{v}$ in a reference frame. By using relations (2.47) and (2.57), it is obtained

$$U_K V_K = Q_{Ki} Q_{Kj} u_i v_j = \delta_{ij} u_i v_j = u_k v_k \tag{2.73}$$

i.e. the quantity $u_k v_k$ is invariant to the transformation (2.40); thus it is a scalar.

In case **u** and **v** are pseudovectors, their scalar product is a scalar. This can be shown using the previous procedure, together with relation (2.58):

$$U_K V_K = (\det Q)^2 Q_{Ki} Q_{Kj} u_i v_j = \delta_{ij} u_i v_j = u_k v_k \tag{2.74}$$

The above results can be applied in the case $\mathbf{v} \equiv \mathbf{u}$. Obviously, the scalar product $\mathbf{u} \cdot \mathbf{u}$ is a scalar, being a quantity invariant to orthogonal transformations. This observation allows the introduction of a quantity which characterizes intrinsically the vector (pseudovector) **u**. This quantity is the module of **u**, defined as $|\mathbf{u}| \equiv (\mathbf{u} \cdot \mathbf{u})^{1/2}$. The module is a scalar with physical dimension equal to the physical dimension of the vector (pseudovector) **u**. The unit vector (or the unit pseudovector) attached to **u** is defined as $\mathbf{w} \equiv \mathbf{u}/|\mathbf{u}|$. This quantity, which has the module equal to the unity, is a dimensionless vector (pseudovector). The vectorial (pseudovectorial) character of this quantity it is easily shown, using relations (2.57) and (2.58):

$$\frac{U_K}{|\mathbf{u}|} = \frac{U_K}{|\mathbf{U}|} = Q_{Kk} \frac{u_k}{|\mathbf{u}|}, \quad \frac{U_K}{|\mathbf{u}|} = \frac{U_K}{|\mathbf{U}|} = (\det Q) Q_{Kk} \frac{u_k}{|\mathbf{u}|} \tag{2.75}$$

The vector (pseudovector) **u** can be represented geometrically through the oriented segment $\mathbf{P'P}$. In this case, $|\mathbf{u}| = |\mathbf{P'P}|$, i.e. the vector's module is numerically equal to the length of the segment $\mathbf{P'P}$. Indeed, using Eq. (2.74), it is obtained by simple processing:

$$|\mathbf{u}| = (u_k u_k)^{1/2} = [(x_k - x_k^0)(x_k - x_k^0)]^{1/2} = |\mathbf{P'P}| \tag{2.76}$$

This relationship is used to justify why the notion of length of vector (pseudovector) is sometimes used instead of the notion of module.

If **u** is a vector and **v** is a pseudovector, their scalar product is a pseudoscalar. The demonstration involves using relationships (2.57) and (2.58):

$$U_K V_K = (\det Q) Q_{Ki} Q_{Kj} u_i v_j = (\det Q) \delta_{ij} u_i v_j = (\det Q) u_k v_k \tag{2.77}$$

Therefore, the quantity $u_k v_k$ is transformed, indeed, as a pseudoscalar.

2.7.4.4 Vector Product

Assume two elements $\mathbf{u}, \mathbf{v} \in V$. In a reference frame, they have the components u_k and $v_k (k = 1, 2, 3)$, respectively. The *vector product* (or *cross product*) of **u** and **v** is defined as that quantity of components

$$w_i = \in_{ijk} u_j v_k \quad (i = 1, 2, 3) \tag{2.78}$$

Here $\in_{ijk}$ $(i, j, k = 1, 2, 3)$ is the *symbol of Levi-Civita*, which has only six nonzero components, i.e.:

$$\in_{ijk} = \in_{jki} = \in_{kij} = -\in_{jik} = -\in_{ikj} = -\in_{kji} = 1 \qquad (i, j, k = 1, 2, 3) \tag{2.78'}$$

The vector product is usually written as

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} \tag{2.79}$$

and it has a physical dimension equal to the product of the physical dimensions of $\mathbf{u}$ and $\mathbf{v}$.

Note that if $\mathbf{u}$ and $\mathbf{v}$ are vectors, their vector product is a pseudovector. This is easy to prove. The components of $\mathbf{u}$ and $\mathbf{v}$ in a different reference frame are denoted by U_K and $V_K (K = 1, 2, 3)$. Using relations (2.57), the following compact form of the vector product in the same reference frame is found

$$W_I = \in_{IJK} U_j V_K = \in_{IJK} Q_{Jj} Q_{Kk} u_j v_k \tag{2.80}$$

By multiplication with Q_{Ii} and summing, it is obtained

$$Q_{Ii} W_I = \in_{IJK} Q_{Ii} Q_{Jj} Q_{Kk} u_j v_k = \in_{ijk} (\det Q) u_j v_k = (\det Q) w_i \tag{2.81}$$

In other words

$$w_i = (\det Q) Q_{Ii} W_I \tag{2.82}$$

which shows that the result of the vector product is a pseudovector.

The vector product of the pseudovectors $\mathbf{u}$ and $\mathbf{v}$ is a pseudovector. The demonstration starts by observing that

$$W_I = \in_{IJK} U_J V_K = \in_{IJK} (\det Q)^2 Q_{Jj} Q_{Kk} u_j v_k = \in_{IJK} Q_{Jj} Q_{Kk} u_j v_k \tag{2.83}$$

The calculus continues in a way similar to the case previously presented.

If one of $\mathbf{u}$ and $\mathbf{v}$ is vector and the other one is pseudovector, then their vector product is a vector. The demonstration starts by observing that

$$W_I = \in_{IJK} U_J V_K = \in_{IJK} (\det Q) Q_{Jj} Q_{Kk} u_j v_k \tag{2.84}$$

By multiplication with Q_{Ii} and summing, it is obtained

$$Q_{Ii} W_I = \in_{IJK} (\det Q) Q_{Ii} Q_{Jj} Q_{Kk} u_j v_k = \in_{ijk} (\det Q)^2 u_j v_k = w_i \tag{2.85}$$

which shows that the result of the vector product acts as a vector.

2.7.4.5 Composed Operations

External Product

Assume that S is the set of scalars and pseudoscalars, regardless of their physical dimension. Assume an element $\lambda \in S$ and an element $\mathbf{u} \in V$, having the components u_k $(k = 1, 2, 3)$ in a given reference frame. The external product of $\mathbf{u}$ and λ is defined as a quantity having the components λu_k $(k = 1, 2, 3)$ in that reference frame. The external product has the physical dimension given by the product of the physical dimensions of $\mathbf{u}$ and λ, and it is denoted as follows:

$$\mathbf{w} = \lambda \mathbf{u} \tag{2.86}$$

The following sentences can be easily verified. If $\mathbf{u}$ is vector and λ is scalar, then $\mathbf{w}$ is vector. If $\mathbf{u}$ is pseudovector and λ is scalar, then $\mathbf{w}$ is pseudovector. If $\mathbf{u}$ is vector and λ is pseudoscalar, then $\mathbf{w}$ is pseudovector. If $\mathbf{u}$ is pseudovector and λ is pseudoscalar, then $\mathbf{w}$ is vector.

Now, it is assumed that λ and μ are scalars with the same physical dimension and $\mathbf{u}$ and $\mathbf{v}$ are vectors with the same physical dimension. In this case, the quantity $(\lambda\,\mathbf{u} + \lambda\,\mathbf{v})$ is a vector. Its physical dimension is different from that of $\mathbf{u}$ and $\mathbf{v}$. Therefore, the external product $\lambda\,\mathbf{u}$ is not closed, i.e. if this operation is applied on a vector $\mathbf{u}$, the result is an element with different physical dimension than that of the vector $\mathbf{u}$. Hence the important conclusion is outlined that, in general, the set of vectors with the same physical dimension, does not form a linear space. In case that K is a field of dimensionless scalars, then the set of vectors with the same physical dimension forms a linear space over K, because the external product is closed.

Mixed Product

Consider three elements $\mathbf{u}, \mathbf{v}, \mathbf{w}$ of V. The *mixed product* (or *scalar triple product*, or *box product*) of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is defined as being the quantity resulting from the composite operation $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

The following sentences be can demonstrated. If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors, then the mixed product is a pseudoscalar. It can be shown, indeed, that in fact the operation is a scalar product between a vector ($\mathbf{u}$) and a pseudovector ($\mathbf{v} \times \mathbf{w}$). If one of the elements $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ is pseudovector and the other two elements are vectors, their mixed product is a scalar. If two elements of the mixed product are pseudovectors, and the third element is a vector, then their mixed product is a pseudoscalar. If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are pseudovectors, their mixed product is a scalar.

Vector Triple Product

The vector triple product of three vectors, $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, is defined as being given by the composite operation $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$.

The following properties can be easily demonstrated. If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors, then the vector triple product is a vector. If one of $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ is pseudovector and the other two elements are vectors, then the vector triple product is pseudovector. If two elements are pseudovectors and the third is a vector, then the vector triple product is a vector. If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are pseudovectors, then the vector triple product is a pseudovector.

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