Chapter 2
Static Cascade Models

> *Happy families are all alike; every unhappy family is unhappy in its own way* (Leo Tolstoy, Anna Karenina, Part 1, Chap. 1. First published 1874–1877).

Abstract Network effects such as default contagion and liquidity hoarding are transmitted between banks by direct contact through their interbank exposures. During asset fire sales, shocks are transmitted indirectly from a bank selling assets to other banks via the impact on the price of their common assets. Banks maintain safety buffers in normal times, but these may be weakened or fail during a crisis. Asset prices that are relatively stable in normal times may collapse during a crisis. Banks react to such stresses by making large adjustments to their balance sheets. Such adjustments send further shocks to their counterparties both directly through their exposures and indirectly via asset price impact, creating a cascade. All these cascade mechanisms can be modelled mathematically starting from a common framework. In such models, the eventual extent of a crisis is a fixed point or equilibrium of a cascade mapping. Towards the end of the chapter, a proposal is made that the properties of cascade mappings can be most clearly understood when implemented on very large random financial networks.

Keywords Cascade mechanism · Default and liquidity buffers · Fixed point equations · Cascade equilibrium · Asset fire sales · Random financial network

If one takes the Anna Karenina Principle seriously, one imagines that stable banking systems must all be alike, while every type of financial instability has its own characteristics. This chapter will explore some of the different ways financial instability can propagate through the system. Paradoxically, we will find that while such channels are definitely distinct, they retain common features that we can exploit in the mathematical models developed in this book.

Contagion, meaning the transmission of a disease by direct or indirect contact, is an appropriate term for the damaging effects that can be transmitted through the network of banks (and possibly other financial entities) linked by the contracts they

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exchange. This chapter will develop various mathematical frameworks for contagion, or cascading instability, both direct and indirect, that can arise in hypothetical financial networks. The essential picture will be to treat banks and their behaviour under stress as determined by their balance sheets, and to concentrate on damaging shocks that can be transmitted through interbank links. In addition to direct bank-to-bank effects, we will find that indirect effects can also be included by extending the definition of node to include non-banks.

The cascade models of this chapter all follow a common script:

1. At the onset of the crisis, the system that was previously in a normal or quiescent state is hit by a damaging shock. This shock impacts banks' balance sheets sufficiently hard that one or more fail or become stressed;
2. Failed or stressed banks transmit balance sheet shocks to their network counterparties;
3. Thereafter, the network undergoes a sequence of updates as banks respond to the balance sheet shocks they receive, thereby inflicting further shocks to their counterparties.
4. Eventually, the system settles down into a new post-crisis equilibrium.

The precise form of updating, called the *cascade mechanism*, amounts to a set of behavioural rules that banks are assumed to follow. Each updating step can be thought of as leading to a *cascade mapping* of the system state into its new state. In our models, the cascade mapping is always monotonically increasing in the damage it causes, which is sufficient to guarantee that its iterations converge to a fixed point called the *cascade equilibrium*. The total damage inflicted during the crisis on both the financial system and the economy at large is determined from quantitative risk measures computed in the equilibrium.

This script is sufficiently general to cover a wide variety of economic narratives. For example, the stylized facts of the 2007–2009 US financial crisis can be mapped schematically to this script: (1) prior to the active phase of the crisis, the year long collapse of the US real estate market acted as a non contagious *correlated asset shock* that brought the whole financial system to an unhealthy, susceptible state; (2) the September 2008 collapse of Lehman Bros provided the trigger for the contagious phase of the crisis; (3) during the subsequent months, rounds of the crisis lead to the defaults of other financial institutions, fire sales in the CDO markets, a freezing of the repo market, liquidity hoarding and other elements that can be viewed collectively as a cascade mechanism; (4) by Spring 2009, the contagion had slowed down, leaving the US financial system close to a new, more quiescent cascade equilibrium.

Obviously the cascade mechanism that underlay the most contagious period of this crisis was an extremely complex interweaving of different effects. In this chapter, we separate out three different contagion channels for mathematical study: default cascades, liquidity cascades and asset fire sales. Later, one can investigate more complex models of higher dimensional cascades that combine two or more different contagion transmission mechanisms, each with some of these basic characteristics.

The basic cascade models of this chapter concern a financial system assumed to consist of N "banks", labelled by $v \in \{1, 2, \dots, N\} := [N]$ (which may include

Table 2.1 An over-simplified bank balance sheet

Assets	Liabilities
Interbank assets $\bar{\mathrm{Z}}$	Interbank debt $\bar{\mathrm{X}}$
External fixed assets $\bar{\mathrm{Y}}^F$	External debt $\bar{\mathrm{D}}$
External liquid assets $\bar{\mathrm{Y}}^L$	Equity $\bar{\mathrm{E}}$

non-regulated, non-deposit taking, leveraged institutions such as hedge funds, or other regulated financial institutions such as insurance companies). Their balance sheets can be characterized schematically as in Table 2.1.

At the outset, the entries in these banks' balance sheets refer to *nominal values* of assets and liabilities, and give the aggregated values of contracts, valued as if all banks are solvent. Nominal values, denoted by upper case letters with bars, can also be considered *book values* or *face values*. Assets and liabilities are also decomposed into interbank and external quantities depending on whether the loan or debt counterparty is a bank or not. Banks and institutions such as foreign banks that are not part of the system under analysis are deemed to be part of the exterior, and their exposures are included as part of the external debts and assets. It is also convenient to separate *fixed assets*, which comprises the assets such as the bank's loan portfolio that cannot be sold without high liquidation costs, from *liquid assets*, such as cash and cash equivalents.

Definition 2.1 The *nominal value of assets* of bank v at any time consists of *nominal external assets*, both fixed and liquid, denoted by $\bar{\mathrm{Y}}_v = \bar{\mathrm{Y}}_v^F + \bar{\mathrm{Y}}_v^L$, plus *nominal interbank assets* $\bar{\mathrm{Z}}_v$. The *nominal value of liabilities* of the bank includes *nominal external debt* $\bar{\mathrm{D}}_v$ and *nominal interbank debt* $\bar{\mathrm{X}}_v$. The bank's *nominal equity* is defined by $\bar{\mathrm{E}}_v = \bar{\mathrm{Y}}_v + \bar{\mathrm{Z}}_v - \bar{\mathrm{D}}_v - \bar{\mathrm{X}}_v$. The *nominal exposure* of bank w to bank v, that is the amount v owes w, is denoted by $\bar{\Omega}_{vw}$. We define interbank loan fractions to be $\bar{\Pi}_{vw} = \bar{\Omega}_{vw}/\bar{\mathrm{X}}_v$ as long as $\bar{\mathrm{X}}_v > 0$. Interbank assets and liabilities satisfy the constraints:

$$\bar{\mathrm{Z}}_v = \sum_w \bar{\Omega}_{wv}, \quad \bar{\mathrm{X}}_v = \sum_w \bar{\Omega}_{vw}, \quad \sum_v \bar{\mathrm{Z}}_v = \sum_v \bar{\mathrm{X}}_v, \quad \bar{\Omega}_{vv} = 0.$$

The combined balance sheets of all banks in the network are shown in Table 2.2.

Such a schematic financial system can be pictured as a network of N nodes representing banks, connected by $|\{(v, w) : \bar{\Omega}_{vw} > 0\}|$ directed edges that point from debtor banks to creditor banks (a direction we sometimes call "downstream").

Economic cascade models invoke the notion of limited liability, and define a firm to be defaulted when its mark-to-market equity is non-positive, meaning its aggregated assets are insufficient to pay its aggregated debt. Analogously, we regard a bank without liquid assets available to pay demand depositors as subject to *liquidity stress*.

Definition 2.2 A *defaulted bank* is a bank with $\mathrm{E} = 0$. A *solvent bank* is a bank with $\mathrm{E} > 0$. A *stressed bank* is a bank with $\mathrm{Y}^L = 0$.

Table 2.2 The first N rows of this table represent different banks' liabilities and the first N columns represent their assets

	1	2	$\cdots$	N	$\bar{X}$	$\bar{D}$	$\bar{E}$
1	0	$\bar{\Pi}_{12}\bar{X}_1$	$\cdots$	$\bar{\Pi}_{1N}\bar{X}_1$	$\bar{X}_1$	$\bar{D}_1$	$\bar{E}_1$
2	$\bar{\Pi}_{21}\bar{X}_2$	0	$\cdots$	$\bar{\Pi}_{2N}\bar{X}_2$	$\bar{X}_2$	$\bar{D}_2$	$\bar{E}_2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	$\bar{\Pi}_{N1}\bar{X}_N$	$\bar{\Pi}_{N2}\bar{X}_N$	$\cdots$	0	$\bar{X}_N$	$\bar{D}_N$	$\bar{E}_N$
$\bar{Z}$	$\bar{Z}_1$	$\bar{Z}_2$	$\cdots$	$\bar{Z}_N$			
$\bar{Y}^F$	$\bar{Y}^F_1$	$\bar{Y}^F_2$	$\cdots$	$\bar{Y}^F_N$			
$\bar{Y}^L$	$\bar{Y}^L_1$	$\bar{Y}^L_2$	$\cdots$	$\bar{Y}^L_N$			

The matrix of interbank exposures contains the values $\bar{\Omega}_{vw} = \bar{\Pi}_{vw}\bar{X}_v$

When a bank v is known to be insolvent or defaulted, creditors of v will naturally mark down their exposure to less than their nominal values. Similarly, the response of a bank to liquidity stress on its liability side will naturally include reducing their interbank lending. We denote by symbols Z, Y^F, Y^L, X, D, E, Ω without upper bars, the changing *actual* or *mark-to-market* values of balance sheets that typically decrease during the steps of the cascade.

With these common definitions, we are now in a position to discuss models of the three basic contagion channels: default contagion, liquidity contagion and asset fire sales. These models will be called "static" because they describe cascades whose end result is a deterministic function of the initial balance sheets and exposures. In typical applications, static cascades proceed from a *random* initial configuration $\bar{Z}$, $\bar{Y}^F$, $\bar{Y}^L$, $\bar{X}$, $\bar{D}$, $\bar{E}$, $\bar{\Omega}$ through a *cascade mechanism* that generates a series of deterministic steps until a steady state or *cascade equilibrium* is reached. Of course as a deterministic function of a random variable, the cascade equilibrium is a random variable, and one is interested to compute a variety of *systemic risk measures* defined as certain expectations over the cascade equilibrium.

2.1 Default Cascades

Basic default cascades depend on $\bar{Y}$ but not on $\bar{Y}^F$ and $\bar{Y}^L$ separately. In such models, the triggering event can be taken to be an initial shock that leaves some banks with nonpositive nominal equity $\bar{E} = 0$ and therefore insolvent. As the cascade progresses, the market value of equity of all banks generally decreases, potentially leading to secondary defaulted banks. The relative claims by creditors in the event a debtor bank defaults are determined by the nominal amounts $\bar{Y}$, $\bar{Z}$, $\bar{D}$, $\bar{X}$, $\bar{\Omega}$. The rule by which defaulted claims are valued distinguishes the two different approaches we now examine.

2.1.1 The Eisenberg–Noe 2001 Model

A slightly extended version of the famous model of default contagion introduced by Eisenberg and Noe [35] makes two additional assumptions that determine a precise default resolution mechanism:

Assumption 2.3

1. External debt is senior to interbank debt and all interbank debt is of equal seniority;
2. There are no losses due to bankruptcy charges.

The original model discussed only the case when external assets $\bar{\mathrm{Y}}$ exceed external debt $\bar{\mathrm{D}}$ for all banks. As discussed in [37], the justification for making this restriction was based on a faulty argument, and therefore we treat only the more general case.

In tandem with the limited liability assumption, the first assumption means that on default a bank's equity is valued at zero, and none of its interbank debt is paid before its external debt is paid in full. A variant of this assumption is if some or all of the external debt has the same seniority as interbank debt. This can be incorporated without additional modelling complexity by adding a fictitious bank labelled by $v = 0$ that lends to but does not borrow from other banks and can never default.

The no-bankruptcy costs assumption is somewhat optimistic in the context of systemic risk and has the strong consequence that when the system is viewed as a whole, no system-wide equity is lost during the crisis. That is, the system equity, defined as total assets minus total liabilities, is independent of the payments within the interbank sector:

$$\bar{\mathrm{E}}_{sys} = \sum_v (\bar{\mathrm{Y}}_v + \bar{\mathrm{Z}}_v - \bar{\mathrm{D}}_v - \bar{\mathrm{X}}_v) = \sum_v (\bar{\mathrm{Y}}_v - \bar{\mathrm{D}}_v).$$

Let us suppose the banking network, previously in equilibrium with no defaulted banks, experiences a catastrophic event, such as the discovery of a major fraud in a bank or a system wide event, whereby the nominal assets of some banks suddenly contract. If one or more banks are then found to be in a state of *primary default*, they are assumed to be quickly liquidated, and any proceeds go to pay off these banks' creditors, in order of seniority. We let $p_v^{(n)}, v \in [N]$ denote the (mark-to-market) amount available to pay v's *internal debt* at the end of the nth step of the cascade, and $\mathrm{p}^{(n)} = [p_1^{(n)}, \dots, p_N^{(n)}]'$. Similarly, we let $q_v^{(n)}, v \in [N]$ denote the (mark-to-market) amount available to pay v's *total debt* and $\mathrm{q}^{(n)} = [q_1^{(n)}, \dots, q_N^{(n)}]'$. We draw the reader's attention to the vector and matrix notation described in Appendix A.1.

By Assumption 2.3, the value $p_v^{(n)}$ is split amongst the creditor banks of v in proportion to the fractions $\bar{\Pi}_{vw} = \bar{\Omega}_{vw}/\bar{\mathrm{X}}_v$ (when $\bar{\mathrm{X}}_v = 0$, we define $\bar{\Pi}_{vw} = 0$ for all w). Therefore, at step $n \geq 1$ of the cascade, every bank w values its interbank assets as $\mathrm{Z}_w^{(n)} = \sum_v \bar{\Pi}_{vw} p_v^{(n-1)}$. Since by assumption there are no bankruptcy charges, we find for all $v \in [N]$:

$$q_v^{(n)} = \min\left(\bar{\mathrm{Y}}_v + \sum_w \bar{\Pi}_{wv} p_w^{(n-1)}, \bar{\mathrm{D}}_v + \bar{\mathrm{X}}_v\right), \tag{2.1}$$

$$p_v^{(n)} = (q_v^{(n)} - \bar{\mathrm{D}}_v)^+. \tag{2.2}$$

This can be written compactly in terms of p alone:

$$\mathrm{p}^{(\mathrm{n})} = \mathrm{F}^{(EN)}(\mathrm{p}^{(\mathrm{n}-1)}) \tag{2.3}$$

where $\mathrm{F}^{(EN)} = [F_1^{(EN)}, \ldots, F_N^{(EN)}]'$ with

$$F_v^{(EN)}(\mathrm{p}) = \min\left(\bar{\mathrm{X}}_v, \max\left(\bar{\mathrm{Y}}_v + \sum_w \bar{\Pi}_{wv} p_w - \bar{\mathrm{D}}_v, 0\right)\right). \tag{2.4}$$

Now we let $\mathrm{p} = [p_1, \ldots, p_N]'$ denote the vector of banks' internal debt values at the end of the cascade. This *clearing vector* satisfies the *clearing condition* or *fixed point condition*

$$\mathrm{p} = \mathrm{F}^{(EN)}(\mathrm{p}) := \min(\bar{\mathrm{X}}, \max(\bar{\mathrm{Y}} + \bar{\Pi}' \cdot \mathrm{p} - \bar{\mathrm{D}}, 0)). \tag{2.5}$$

The main theorem of Eisenberg and Noe is the following:

Theorem 2.1 *Corresponding to every financial system* $(\bar{\mathrm{Y}}, \bar{\mathrm{Z}}, \bar{\mathrm{D}}, \bar{\mathrm{X}}, \bar{\Omega})$ *satisfying Assumption 2.3 there exists a greatest and a least clearing vector* p^+ *and* p^-.

Proof of Theorem 2.1: The result follows immediately from the Knaster–Tarski Fixed Point Theorem[1] once we verify certain characteristics of the mapping $\mathrm{F}^{(EN)}$. We note that $\mathrm{F}^{(EN)}$ maps the hyperinterval $[\mathbf{0}, \bar{\mathrm{X}}] := \{x \in \mathbb{R}^N : 0 \le x_v \le \bar{\mathrm{X}}_v\}$ into itself. We also note that it is monotonic: $x \le y$ implies $\mathrm{F}^{(EN)}(x) \le \mathrm{F}^{(EN)}(y)$. Finally, note that $[\mathbf{0}, \bar{\mathrm{X}}]$ is a complete lattice. We therefore conclude that the set of clearing vectors, being the fixed points of the mapping $\mathrm{F}^{(EN)}$, is a complete lattice, hence nonempty, and with maximum and minimum elements p^+ and p^-. □

Finding natural necessary and sufficient conditions on E–N networks to ensure the uniqueness of the clearing vector proves to be more challenging. Figure 2.1 shows an example of non-uniqueness of the clearing vector.

We present here a complete characterization of the clearing vector set in the E–N model without the condition $\bar{\mathrm{Y}} \ge \bar{\mathrm{D}}$. First, we identify groups of banks called *in-subgraphs* (these are essentially the same as the *surplus sets* in [35]), that do not lend outside their group.

Definition 2.4

1. *In-subgraph*: any subgraph $\mathscr{M} \subset \mathscr{N}$ (possibly $\mathscr{M} = \mathscr{N}$) with no out-links to its complement $\mathscr{M}^c$.

[1] A statement and proof of this result can be found at http://en.wikipedia.org/wiki/Knaster-Tarski-theorem.

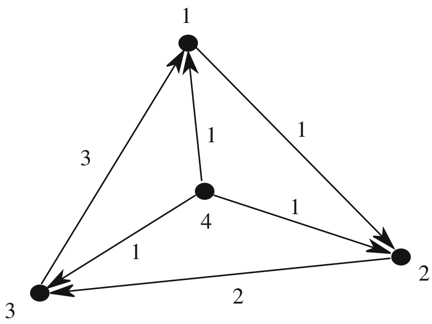

Fig. 2.1 This $N = 4$ bank network with $\bar{\mathrm{Y}} - \bar{\mathrm{D}} = 0$ has multiple clearing vectors $\mathrm{p} = \lambda[1, 1, 1, 0]$ with $\lambda \in [0, 1]$

2. *Irreducible in-subgraph*: an in-subgraph $\mathscr{M} \subset \mathscr{N}$ with at least 2 nodes that does not contain a smaller in-subgraph.

The exposure matrix $\bar{\Pi}'$ is always a substochastic matrix, and when projected onto any irreducible in-subgraph $\mathscr{M}$, is always an irreducible substochastic matrix, where we recall the standard definitions:

Definition 2.5

1. A *stochastic (substochastic) matrix* has non-negative entries and columns that sum to 1 (respectively, ≤ 1).
2. An *irreducible* substochastic matrix Γ has the property that for every $v, w \in [N]$ there is $k \ge 1$ such that $(\Gamma^k)_{vw} > 0$.

Having identified all the irreducible in-subgraphs of a given network, we can simplify the following discussion by removing all singleton *in-banks* (i.e. that do not lend to other banks). Any such bank v will have $p_v = \bar{\mathrm{X}}_v = 0$ and its state has no effect on other banks. We then consider the exposure matrix $\bar{\Pi}'$ restricted to the *reduced network* $\tilde{\mathscr{N}}$ without such in-banks, which may then be strictly substochastic in some columns.

The theorem that characterizes the set of clearing vectors in the E–N model is:

Theorem 2.2 *Let the reduced network $\tilde{\mathscr{N}}$ have exactly $K \ge 0$ non-overlapping irreducible in-graphs $\mathscr{M}_1, \mathscr{M}_2, \ldots, \mathscr{M}_K$, and a decomposition $\tilde{\mathscr{N}} = \mathscr{M}_0 \cup \left(\cup_{k=1}^{K} \mathscr{M}_k\right)$. Let the possible fixed points be written in block form $\mathrm{p}^* = [\mathrm{p}_0^*, \mathrm{p}_1^*, \ldots, \mathrm{p}_K^*]'$. Then:*

1. *In case $K = 0$, the clearing vector $\mathrm{p}^* = \mathrm{p}_0^*$ is unique;*
2. *In case there are exactly $K \ge 1$ (non-overlapping) irreducible in-graphs $\mathscr{M}_1, \mathscr{M}_2, \ldots, \mathscr{M}_K$, then the multiplicity of the clearing vectors is characterized as follows: p_0^* is unique, and each p_k^* is either unique or of the form $\mathrm{p}_k^* = \alpha_k v_k$ where the vector v_k is the unique normalized 1-eigenvector of the matrix $P_k \cdot \bar{\Pi}' \cdot P_k{}'$ projected onto $\mathscr{M}_k$ and $\alpha_k \in [0, \bar{\alpha}_k]$ for some $\bar{\alpha}_k$. The precise form of each p_k^* is shown in the proof.*

This theorem demonstrates that non-uniqueness in this model is a highly non-generic property: only very special arrangements of the network lead to multiplicity of solutions.

The proof of the theorem involves extending the following lemma to the most general kind of E–N fixed point equation.

Lemma 2.3 *Consider the system*

$$\mathrm{p} = \min(\mathrm{X}, \max(\mathrm{Y} + \Gamma \cdot \mathrm{p}, 0)) \tag{2.6}$$

where Γ is substochastic and irreducible, X *is a positive vector and* Y *is arbitrary.*

1. *If Γ is strictly substochastic, then there is a unique fixed point.*
2. *If Γ is stochastic, with y the unique eigenvector such that $\Gamma \cdot y = y$ and $\mathbf{1} \cdot y = 1$ where $\mathbf{1} = [1, \ldots, 1]$, then one of two possible cases holds:*
   a. *If $\mathbf{1} \cdot \mathrm{Y} = 0$, there is a one-parameter family of solutions that have the form $\mathrm{p}^* = \lambda y$, $\lambda \in [0, \lambda_{max}]$.*
   b. *If $\mathbf{1} \cdot \mathrm{Y} \neq 0$, there is a unique fixed point, of which at least one component is either 0 or X.*

Proof of Lemma: Note that for irreducible substochastic matrices, the largest eigenvalue is simple and 1 if it is stochastic or less than one if it is strictly substochastic. Moreover, every submatrix obtained by deleting one or more nodes has largest eigenvalue less than one. Thus in Part 1, uniqueness is guaranteed because $I - \Gamma$ has largest eigenvalue strictly less than one and hence an explicit inverse given by the convergent matrix power series $(I - \Gamma)^{-1} = \sum_{k=0}^{\infty} \Gamma^k$.

Under the conditions of Part 2, by dotting (2.6) with the vector $\mathbf{1} = [1, \ldots, 1]$ and noting that $\mathbf{1} \cdot \Gamma = \mathbf{1}$, we see that the system $\mathrm{p} = \mathrm{Y} + \Gamma \cdot \mathrm{p}$ has a solution if and only if $\mathbf{1} \cdot \mathrm{Y} = 0$. If $\mathbf{1} \cdot \mathrm{Y} = 0$ one can check that case 2(a) of the lemma holds. If $\mathbf{1} \cdot \mathrm{Y} \neq 0$, at least one component of any fixed point of (2.6) must be X or 0. Substituting in this component value, and reducing the system by one dimension now leads to a new fixed point equation of the same form (2.6) but where the submatrix matrix $\tilde{\Gamma}$ has largest eigenvalue less than one. Such systems have a unique fixed point by part 1. □

Proof of Theorem 2.2: We must now deal with the system (2.6) when $\Gamma = \bar{\Pi}'$ is not irreducible. In general, it is easy to see that $\mathscr{N}$, itself an in-subgraph, contains within it a maximal number of irreducible non-overlapping in-subgraphs $\mathscr{M}_1, \mathscr{M}_2, \ldots, \mathscr{M}_K$, $K \geq 0$ plus the possibility of additional single non-lending nodes with $\bar{\mathrm{X}}_v = 0$. As discussed above, as a first step we can eliminate all non-lending nodes and consider the reduced piece-wise linear fixed point problem on the subgraph $\tilde{\mathscr{N}}$. The case when $\tilde{\mathscr{N}}$ has no irreducible in-subgraphs, i.e. $K = 0$, has a unique clearing vector because then, by Part 1 of the Lemma, $\bar{\Pi}'$ must be substochastic with largest eigenvalue less than one.

If $K > 0$, we decompose into $\tilde{\mathscr{N}} = \mathscr{M}_0 \cup \left(\cup_{k=1}^{K} \mathscr{M}_k\right)$, and after reordering nodes write the matrix $\bar{\Pi}'$ in block form with respect to this decomposition:

$$\bar{\Pi}' = \begin{pmatrix} A & 0 & \cdots & 0 \\ B_1 & \Pi_1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ B_K & 0 & 0 & \Pi_K \end{pmatrix}$$

with the column sums less than or equal to one. We shall characterize the possible fixed points written in block form $\mathrm{p} = [\mathrm{p}_0^*, \mathrm{p}_1^*, \ldots, \mathrm{p}_K^*]'$. We note that A is strictly substochastic with largest eigenvalue less than 1. Therefore the fixed point equation for p_0^* is a closed equation that has a unique solution by part 1 of Lemma 2.3. Each of the remaining pieces of p^*, p_k^*, $k \geq 1$, is a solution of a piecewise linear equation in the following form:

$$\mathrm{p}_k^* = \min\left(\bar{X}_k, \max\left(B_k \cdot \mathrm{p}_0^* + \bar{\mathrm{Y}}_k - \bar{\mathrm{D}}_k + \Pi_k \cdot \mathrm{p}_k^*, 0\right)\right).$$

Now we note that each Π_k is an irreducible stochastic matrix, and by Part 2 of Lemma 2.3, p_k^* is unique if $\mathbf{1}_k \cdot (B_k \cdot \mathrm{p}_0^* + \bar{\mathrm{Y}}_k - \bar{\mathrm{D}}_k) \neq 0$ and a point in a one-dimensional interval if $\mathbf{1}_k \cdot (B_k \cdot \mathrm{p}_0^* + \bar{\mathrm{Y}}_k - \bar{\mathrm{D}}_k) = 0$. $\square$

2.1.2 Reduced Form E–N Cascade Mechanism

In models such as this, cascades of defaults arise when primary defaults trigger further losses to the remaining banks. Theorem 2.1 proves the existence of an "equilibrium" clearing vector, which is usually unique, that gives the end result of cascades in the E–N framework. Sometimes different balance sheet specifications lead to identical cascades, and we can characterize the cascade mechanism and resultant clearing vectors in terms of a reduced set of balance sheet data. It turns out that the most important information to track is something we call the *default buffer*, which extends the notion of equity. We assume the initial default buffer $\Delta_v^{(0)}$ of bank v is its nominal equity, but possibly negative:

$$\Delta_v^{(0)} = \bar{\Delta}_v := \bar{\mathrm{Y}}_v + \sum_w \bar{\Omega}_{wv} - \bar{\mathrm{D}}_v - \bar{\mathrm{X}}_v. \tag{2.7}$$

As before, define $p_v^{(n)}$ to be the amount available to pay $\bar{\mathrm{X}}_v$ at the end of cascade step n, initialized to $p_v^{(0)} = \bar{\mathrm{X}}_v$ at $n = 0$. Introduce the normalized *threshold function* h that maps the extended real line $[-\infty, \infty]$ to the unit interval $[0, 1]$:

$$h(x) = (x+1)^+ - x^+ = \max(0, \min(x+1, 1)). \tag{2.8}$$

Then an important but straightforward calculation shows that Eqs. (2.1) and (2.2) for $n > 0$ the nth step of E–N cascade are expressible in terms of the default buffers $\Delta^{(n-1)}$:

$$p_v^{(n)} = \bar{\mathrm{X}}_v \, h(\Delta_v^{(n-1)}/\bar{\mathrm{X}}_v) \tag{2.9}$$
$$q_v^{(n)} = (\bar{\mathrm{D}}_v^- + \bar{\mathrm{X}}_v) \, h(\Delta_v^{(n-1)}/(\bar{\mathrm{D}}_v + \bar{\mathrm{X}}_v)) \tag{2.10}$$

whereas the default buffers themselves satisfy a set of closed equations

$$\Delta^{(n)} = \mathrm{F}^{(EN)}(\Delta^{(n-1)}); \quad F_v^{(EN)}(\Delta) := \bar{\Delta}_v - \sum_w \bar{\Omega}_{wv}(1 - h(\Delta_w/\bar{\mathrm{X}}_w)). \tag{2.11}$$

Thus the cascade mapping boils down to a vector-valued function $\Delta^{(n-1)} \mapsto \Delta^{(n)} = \mathrm{F}^{(EN)}(\Delta^{(n-1)} \| \bar{\Delta}, \bar{\Omega})$ that depends parametrically only on the initial default buffers $\bar{\Delta}$ and the interbank exposures $\bar{\Omega}$. The mark-to-market equity is the positive part of the default buffer, $\mathrm{E}_v^{(n)} = (\Delta_v^{(n)})^+$, and default of bank v occurs at the first step that $\Delta_v^{(n)} \le 0$. As $n \to \infty$, the monotone decreasing sequence $\Delta^{(n)}$ converges to the maximal fixed point of $\Delta = \mathrm{F}^{(EN)}(\Delta)$ which is a well-defined function of the reduced balance sheet data

$$\Delta^+ = G^+(\bar{\Delta}, \bar{\Omega}),$$

The corresponding maximal clearing vectors $\mathrm{p}^+, \mathrm{q}^+$ are given by

$$p_v^+ = \bar{\mathrm{X}}_v \, h(\Delta_v^+/\bar{\mathrm{X}}_v),$$
$$q_v^+ = (\bar{\mathrm{Y}}_v + \bar{\mathrm{X}}_v) \, h(\Delta_v^+/(\bar{\mathrm{Y}}_v + \bar{\mathrm{X}}_v)).$$

If instead of starting the cascade at the optimistic initial values $\Delta_v^{(0)} = \bar{\Delta}_v$, we had begun with the most pessimistic values $\Delta_v^{(0)} \le \bar{\Delta}_v - \bar{\mathrm{Z}}_v$, we would obtain a monotone *increasing* sequence $\Delta^{(n)}$ that converges to the minimal fixed point $\Delta^- := G^-(\bar{\Delta}, \bar{\Omega})$.

The scaled variable $\Delta/\bar{\mathrm{X}}$, or alternatively $\Delta/(\bar{\mathrm{D}} + \bar{\mathrm{X}})$, has the interpretation of a bank's *distance-to-default*, and the threshold function h determines both the fractional recovery on interbank debt and on total debt when Δ is negative. Other possible threshold functions h are an important characteristic of different cascade models.

Some simple systemic risk measures of the total damage caused by the crisis are computable in terms of the cascade equilibrium $(\Delta^+, \mathrm{p}^+, \mathrm{q}^+)$:

1. Default probability: $\mathrm{DP} = \frac{1}{N} \sum_v \mathbf{1}(\Delta_v^+ \le 0)$;
2. Default cascade impact on the financial sector: $\mathrm{DCI}^{(1)} = \sum_v (\bar{\mathrm{X}}_v - p_v^+)$;
3. Default cascade impact on the entire economy: $\mathrm{DCI}^{(2)} = \sum_v (\bar{\mathrm{D}}_v + \bar{\mathrm{X}}_v - q_v^+)$.

2.1.3 The Gai–Kapadia 2010 Default Model

The threshold function $h(x)$ for the E–N 2001 model encodes a *soft* type of default in which the interbank debt of a defaulted bank with $\Delta/\bar{X} = x \sim 0$ recovers almost all its value. In their 2010 paper [44], Gai and Kapadia offer a model with *hard defaults*: interbank debt on defaulted banks recovers zero value. They justify their zero recovery assumption with the statement[2]: "This assumption is likely to be realistic in the midst of a crisis: in the immediate aftermath of a default, the recovery rate and the timing of recovery will be highly uncertain and banks' funders are likely to assume the worst-case scenario."

The G–K cascade mechanism boils down to the following assumptions:

Assumption 2.6

1. At step 0 of the cascade, one or more banks experience asset shocks that make their *default buffers* $\Delta_v^{(0)} = \bar{\Delta}_v \le 0$ go negative.
2. A defaulted bank v's interbank liabilities recover zero value and thus a default shock of magnitude $\bar{\Omega}_{vw}$ is sent to each of v's creditor banks w.
3. At each step $n \ge 0$ of the crisis, bank v marks to zero any interbank asset $\bar{\Omega}_{wv}$ from a newly defaulted counterparty bank w.

This cascade mechanism turns out to be precisely of the E–N type, but with a *zero-recovery* threshold function

$$\tilde{h}(x) = \mathbf{1}(x > 0), \tag{2.12}$$

and exactly as in Sect. 2.1.2 it defines the sequence of vectors $\mathrm{p}^{(n)}$ and buffers $\Delta^{(n)}$ satisfying Eqs. (2.9) and (2.11), with h replaced by $\tilde{h}$. That is,

$$p_v^{(n)} = \bar{\mathrm{X}}_v \tilde{h}(\Delta_v^{(n-1)}), \tag{2.13}$$

$$\Delta_v^{(n)} = \bar{\Delta}_v - \sum_w \bar{\Omega}_{wv}(1 - \tilde{h}(\Delta_w^{(n-1)})). \tag{2.14}$$

The clearing vector condition is now

$$\mathrm{p} = \bar{\mathrm{X}}\tilde{h}(\Delta)$$

where Δ is any fixed point of

$$\Delta = \mathrm{F}^{(GK)}(\Delta); \quad F_v^{(GK)}(\Delta) := \bar{\Delta}_v - \sum_w \bar{\Omega}_{wv}(1 - \tilde{h}(\Delta_w)). \tag{2.15}$$

The existence of fixed points $\Delta = \mathrm{F}^{(GK)}(\Delta)$ follows as in the E–N model by repeating the proof of Theorem 2.1. On the other hand, uniqueness of fixed points has not been carefully studied for this model, and appears to be considerably more complicated. Like the E–N model, there are examples in which non-uniqueness arises associated to irreducible in-graphs, but now the multiplicity of clearing vectors is discrete. For example, the $N = 4$ bank network with $\bar{\Omega}$ as in Fig. 2.1, and with $\bar{\mathrm{Y}} = [0, 0, 0, 4]$, $\bar{\mathrm{D}} = [1, 1, 1, 0]$ has exactly two clearing vectors of the form $\mathrm{p} =$

---

[2] Reference [44], footnote 9.

$\lambda[1, 1, 1, 0]'$ with $\lambda = 0$ or $\lambda = 1$. As in the E–N model in case of non-uniqueness, our cascades typically start from $p^{(0)} = \bar{X}$ and therefore reach the maximal clearing vector p^+ with default buffers Δ^+.

As before, the simplest systemic risk measure is default probability, given in terms of the fixed point Δ^+ by

$$\mathrm{DP} = \frac{1}{N} \sum_v \mathbf{1}(\Delta_v^+ \leq 0).$$

The zero-recovery assumption implies there is a large cost to the financial system for banks that default given by *default cascade impact*:

$$\mathrm{DCI} = \sum_v \bar{X}_v \, \mathbf{1}(\Delta_v^+ \leq 0). \tag{2.16}$$

Recall that bankruptcy charges are ruled out in the E–N 2001 model and are maximal in the G–K model. We can interpolate between these two extreme cases with a single parameter $\tau \in [0, 1]$ that represents the fraction of interbank debts that are paid as bankruptcy charges at the time any bank defaults. The cascade mapping is again given by Eqs. (2.9) and (2.11), now with h replaced by the interpolated threshold function

$$h^{(\tau)}(x) = (1 - \tau) h\left(\frac{x}{1 - \tau}\right) + \tau \tilde{h}(x). \tag{2.17}$$

The simple static default cascades just investigated can be summarized as follows:

1. They are characterized by shocks that are transmitted *downstream* from defaulting banks to the asset side of their creditor banks' balance sheets;
2. At each cascade step, banks update their default buffers by determining the current amount lost given default of its counterparties;
3. Different default recovery assumptions arise through the choice of a threshold function h, $\tilde{h}$ or $h^{(\tau)}$.

2.2 Liquidity Cascades

A funding liquidity cascade is a systemic phenomenon that occurs when stressed banks hoard liquidity, that is they curtail lending to each other on a large scale. In such a cascade, shocks are transmitted *upstream*, from creditor banks to their debtors as they act to reduce their interbank lending. A fundamental treatment of the connection between funding liquidity and market liquidity by Brunnermeier and Pedersen [21] proposes a picture of how the funding constraints on a bank impact the liquidity of its market portfolio, that its external assets. One finds the idea that when a bank's capital is reduced to below a threshold where a funding liquidity constraint becomes binding, that bank will reduce its assets by a discontinuous amount, and experience a discontinuous increase in its margin and collateral requirements. If, as is natural, we assume that at this threshold the bank will also reduce its interbank lending by a discontinuous amount, then this picture provides the seed of a cascade mechanism that is transmitted through the interbank network, from creditors to debtors. It turns out that our schematic default cascade models, when shocks are reversed and reinterpreted, become basic models of funding liquidity cascades.

The first paper to introduce a network model of funding liquidity cascades is a companion paper [43] to the default model by Gai–Kapadia [44]. We now describe the G–K liquidity cascade model and two variations, all based on banks that have stylized balance sheets given as in Table 2.1.

2.2.1 Gai–Kapadia 2010 Liquidity Cascade Model

This systemic risk model aims to account for the observation that starting in August 2007 and continuing until after September 2008, interbank lending froze around the world as banks hoarded cash and curtailed lending to other banks. As Gai and Kapadia explain, during the build up of credit risk prior to 2007, some banks that held insufficiently liquid assets began to face funding liquidity difficulties. Such banks moved to more liquid positions by hoarding liquidity, in some cases reducing their interbank lending almost entirely.

What would a counterparty bank do when impacted by such a hoarding bank? Of course they might seek funding elsewhere, but in a climate of uncertainty they might themselves elect to become liquidity hoarders, thereby propagating further liquidity shocks.

The following liquidity cascade model assumes that prior to the crisis, banks hold assets and liabilities as shown in Fig. 2.2. On the asset side we have: $\bar{\mathrm{Y}}_v^F$ (external fixed assets, namely the bank book of loans to the economy at large), $\bar{\mathrm{Z}}_v$ (interbank assets assumed to be short term unsecured loans to other banks) and liquid assets $\bar{\mathrm{Y}}_v^L$. When non-zero, the liquid assets $\bar{\mathrm{Y}}_v^L$ are used as a *stress buffer* $\bar{\Sigma}_v$ from which to pay liabilities as they arise. In analogy to the default buffers Δ_v, Σ_v can become negative: such a bank is called a *stressed bank*. On the liability side we have as before external debt $\bar{\mathrm{D}}_v$, interbank debt $\bar{\mathrm{X}}_v$ and the default buffer $\bar{\Delta}_v$.

Assumption 2.7

1. At step 0 of the cascade, one or more banks experience funding liquidity shocks or *stress shocks* that make their *stress buffers* $\Sigma_v^{(0)} = \bar{\Sigma}_v$ go negative.
2. Banks respond at the moment they become stressed by preemptively hoarding a fixed fraction $\lambda \leq 1$ of interbank lending. This sends a stress shock of magnitude $\lambda\bar{\Omega}_{wv}$ to each of the debtor banks w of v. Stressed banks remain stressed for the duration of the crisis.
3. At each step $n \geq 0$ of the crisis, bank v pays any interbank liabilities $\lambda\bar{\Omega}_{vw}$ that have been recalled by newly stressed banks w.

The assumption of a fixed hoarding fraction λ across the network is clearly a gross oversimplification that can be refined later. The essential point is that under

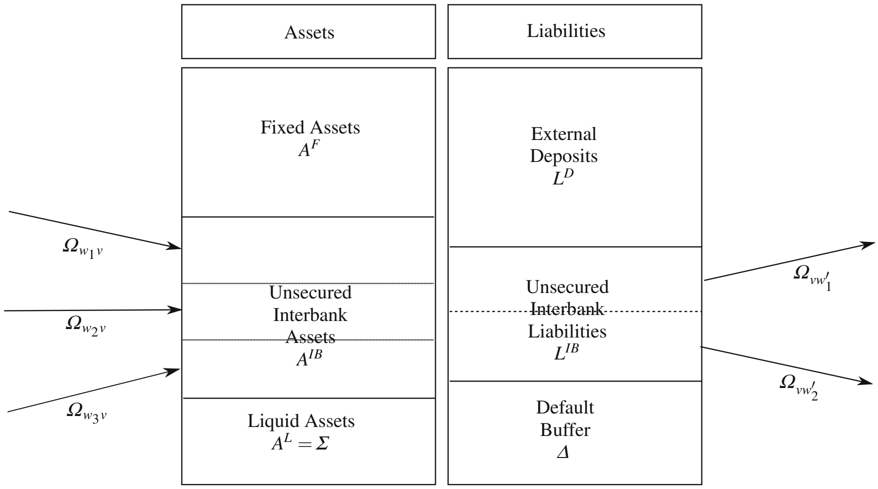

Fig. 2.2 The stylized balance sheet of a bank v with three debtor banks w_1, w_2, w_3 and two creditor banks w'_1, w'_2

stress, banks act preemptively to shrink their balance sheets by a fraction close to one. In order to disentangle default and stress, the third assumption implies that unstressed banks are always able to meet recalled interbank liabilities without negatively impacting their default buffers. In other words, this is a model of funding liquidity with zero market illiquidity effects.

These simple behavioural rules lead to a cascade mechanism (CM) that can be expressed succinctly as the recursive updating of the stress buffers of all banks starting from an initial state with $\Sigma_v^{(0)} = \bar{\Sigma}_v$. Given the collection of stress buffers $\Sigma_v^{(n-1)}$ at step $n-1$ of the cascade, the updated stress buffers are given by

$$\Sigma_v^{(n)} = \bar{\Sigma}_v - \lambda \sum_w \bar{\Omega}_{vw} \left(1 - \tilde{h}(\Sigma_w^{(n-1)})\right). \tag{2.18}$$

In this proposed cascade mechanism, the parameter λ represents the average strength of banks' collective stress response. Under our assumptions, the iterated cascade mapping converges to an equilibrium set of buffers given by the maximal fixed point Σ^+. Two simple systemic risk measures quantify the effect of the crisis:

1. Stress probability: $\mathrm{SP} = \frac{1}{N}\sum_v \mathbf{1}(\Sigma_v^+ \le 0)$;
2. Liquidity cascade impact on the financial system, that is, the total amount of interbank assets frozen during the crisis: $\mathrm{LCI} = \lambda \sum_v \bar{Z}_v \mathbf{1}(\Sigma_v^+ \le 0)$.

It should not be a surprise that (2.18) is identical in form to (2.14), but with shocks going upstream from creditors to debtors instead of downstream from debtor banks to creditors. The pair of models [43, 44] by Gai and Kapadia is a first instance of a formal symmetry of financial networks under interchange of assets and liabilities.

2.2.2 *The Liquidity Model of S.H. Lee 2013*

As a second illustration of how liquidity cascades are the mirror image of default cascades, we now show how a simple liquidity cascade model proposed by S.H. Lee [63] is formally identical to a version of the E–N cascade. This model is again based on banks with balance sheets as shown in Fig. 2.2. The essential assumption of the model is:

Assumption 2.8 Banks pay all withdrawals from external and interbank depositors first by selling liquid external and interbank assets in constant proportion, and then when these assets are depleted, from the illiquid external assets.

To put this model into an E–N form while preserving the labelling of Lee's paper, we introduce a fictitious "source" bank labelled by 0 that borrows from but does not lend to other banks and arbitrarily define $\bar{\mathrm{Z}}_0 = \bar{\mathrm{D}}_0 = 0$. We now label the interbank exposures as in [63]:

$$\bar{\Omega}_{vw} = \begin{cases} b_{vw} & v, w \neq 0 \\ q_w & v = 0, w \neq 0 \\ 0 & w = 0. \end{cases}$$

As before, let $\bar{\mathrm{Z}}_v = \sum_w \bar{\Omega}_{wv}$, $\bar{\mathrm{X}}_v = \sum_w \bar{\Omega}_{vw}$ and identify $\bar{\mathrm{Y}}^F_v = z_v$, $\bar{\mathrm{Y}}^L_v = q_v = \bar{\Omega}_{0v}$, $\bar{\mathrm{D}}_v = d_v$.

At time 0, each bank experiences deposit withdrawals (a liquidity shock) $\Delta d_v \geq 0$. These withdrawals are paid immediately by each bank v in order of seniority: first from the liquid interbank assets $\bar{\mathrm{Z}}_v$ (which now includes lending to the fictitious bank $\bar{\Omega}_{0v}$) until these are depleted, and then by selling fixed external assets $\bar{\mathrm{Y}}^F_v$. Let us now define the initial *stress buffer* to be $\Sigma^{(0)}_v = \bar{\Sigma}_v = -\Delta d_v$. Then at the nth step of the liquidity cascade, each buffer $\Sigma^{(n)}_v$, which is the negative of bank v's total liquidity needs $\ell^{(n)}_v$, will have accumulated shocks as follows

$$\Sigma^{(n)}_v = \bar{\Sigma}_v - \sum_w \bar{\Omega}_{vw} \left(1 - h\left(\Sigma^{(n-1)}_w / \bar{\mathrm{Z}}_w\right)\right).$$

This equation, being formally identical to (2.11), reveals that the Lee model is a special case of the E–N model provided $\bar{\mathrm{Z}}$ and $\bar{\mathrm{X}}$ are interchanged, and the exposures are reversed. However, in the Lee model, we begin with buffers $\Sigma^{(0)}_v \leq 0$ for all v except $v = 0$. For completeness, at step n we can define $p^{(n)}_v = \bar{\mathrm{Z}}_v h(\Sigma^{(n-1)}_v / \bar{\mathrm{Z}}_v)$, the amount of liquid and interbank assets remaining unsold, and $q^{(n)}_v = (\bar{\mathrm{Y}}^F_v + \bar{\mathrm{Z}}_v) h(\Sigma^{(n-1)}_v / (\bar{\mathrm{Y}}^F_v + \bar{\mathrm{Z}}_v))$, the total assets remaining unsold. We can call a bank *illiquid* when $\Sigma_v \leq -\bar{\mathrm{Z}}_v$. Two natural measures of systemic liquidity risk are determined by the maximal cascade fixed point Σ^+:

1. Illiquidity probability: $\mathrm{LP} = \frac{1}{N} \sum_v \mathbf{1}(\Sigma^+_v \leq -\bar{\mathrm{Z}}_v)$;
2. Liquidity cascade impact on the entire economy, that is, the total amount of fixed external assets sold during the crisis: $\mathrm{LCI} = \sum_v (-\Sigma^+_v - \bar{\mathrm{Z}}_v)^+$.

2.2.3 Generalized Liquidity Cascades

The liquidity cascade model of S.H. Lee supposes that deposit withdrawals are paid in equal proportion from interbank assets and liquid external assets. A reasonable alternative picture is that each bank keeps a first line reserve of liquid external assets (or simply "cash") $\bar{\mathrm{Y}}^L$ to absorb liquidity shocks. We now think of this as the stress buffer, labelled by Σ, to be kept positive during normal banking business. When the stress buffer goes zero or negative, the bank becomes *stressed* and must meet further withdrawals by liquidating first interbank assets $\bar{\mathrm{Z}}$, and finally illiquid fixed assets $\bar{\mathrm{Y}}^F$.

As for the Lee model, we may also add a fictitious sink bank $v = 0$ to represent external agents that borrow amounts $\bar{\Omega}_{0v}$ where in terms of liquidation priority, these external loans will be considered a component of a bank's interbank assets: $\bar{\mathrm{Z}}_v = \sum_{w=0}^{N} \bar{\Omega}_{wv}$.

Let us suppose that just prior to an initial withdrawal shock that hits any or all of the banks, the banks' balance sheets are given as in Fig. 2.2 by notional amounts $(\bar{\mathrm{Y}}^F, \bar{\mathrm{Z}}, \bar{\mathrm{Y}}^L, \bar{\mathrm{D}}, \bar{\mathrm{X}}, \bar{\mathrm{E}}, \bar{\Omega})$. At the onset of the liquidity crisis, all banks are hit by withdrawal shocks $\Delta \mathrm{D}_v$ that reduce the initial stress buffers $\Sigma_v^{(0)} = \bar{\mathrm{Y}}_v^L - \Delta \mathrm{D}_v$ of at least some banks to below zero, making them stressed. Stressed banks then liquidate assets first from $\bar{\mathrm{Z}}$, inflicting additional liquidity shocks to their debtor banks' liabilities, and then from $\bar{\mathrm{Y}}^F$. A stressed bank that has depleted all of $\bar{\mathrm{Z}}$ will be called *illiquid*, and must sell external fixed assets $\bar{\mathrm{Y}}^F$ to meet further liquidity shocks.

Let $p_v^{(n)}$ be the amount of bank v's interbank assets remaining unsold after n steps of the liquidity cascade, starting with $p_v^{(0)} = \bar{\mathrm{Z}}_v$. Illiquid banks have $p_v^{(n)} = 0$, stressed banks are those with $0 < p_v^{(n)} < \bar{\mathrm{Z}}_v$ while normal, unstressed banks have $p_v^{(n)} = \bar{\mathrm{Z}}_v$. If $\Sigma_v^{(n)}$ is the stress buffer after n steps and each stressed bank liquidates exactly enough additional interbank assets at each step to meet the additional liquidity shocks, the update rule is

$$p_v^{(n)} = \max\left(0, \min(\bar{\mathrm{Z}}_v, (\mathrm{D}_v - \Delta \mathrm{D}_v)) - \bar{\mathrm{Y}}^F + \sum_w \bar{\Omega}_{vw} \left(p_w^{(n-1)}/\bar{\mathrm{Z}}_w\right)\right). \quad (2.19)$$

We note that

$$\Sigma_v^{(n)} = \Sigma_v^{(0)} - \sum_w \bar{\Omega}_{vw}(1 - p_w^{(n-1)}/\bar{\mathrm{Z}}_w), \quad (2.20)$$

and that (2.19) can be written

$$p_v^{(n)} = \bar{\mathrm{Z}}_v h(\Sigma_v^{(n-1)}/\bar{\mathrm{Z}}_v)$$

with the threshold function h of (2.8) used before.

Comparison of these equations with (2.9) and (2.11) reveals that our model is precisely equivalent to the full E–N 2001 model, with the role of assets and liabilities, and stress and default buffers, interchanged: $\bar{\mathrm{Y}}^F \leftrightarrow \bar{\mathrm{D}}$, $\bar{\mathrm{Z}} \leftrightarrow \bar{\mathrm{X}}$, $\bar{\mathrm{Y}}^F \leftrightarrow \bar{\mathrm{E}}$, $\Delta \leftrightarrow \Sigma$. We recover the Lee model simply by taking $\bar{\mathrm{Y}}^L_v = 0$, which also has the effect of making all the banks initially stressed since the initial stress buffers are $\Sigma_v^{(0)} = -\Delta \mathrm{D}_v \leq 0$. We also recover the G–K 2010 liquidity model by replacing h by $\tilde{h}$.

To keep various cascade mechanisms separate, the funding liquidity cascade models we have just described neglect *market illiquidity*, which of course is the very important systemic effect that large scale selling of assets will drive asset prices down. The next type of cascade turns the focus on this effect.

2.3 Asset Fire Sales

Certainly one of the basic triggers of financial crises is when an important asset class held by many banks is beset by bad news, resulting in a major devaluation shock that hits these banks. We identify this as an *asset correlation shock* described in Sect. 1.4 in which the external assets Y_v held by many banks exhibit a sharp one-time decline. If this shock is sufficient to cause the default of some banks, we face the possibility of a pure default cascade of the same nature as we have described already.

Of a different nature are *asset fire sales*, in which banks under stress (of which there will be many during a crisis) react by selling external assets on a large scale, driving their prices down. As described in detail in the 2005 paper by Cifuentes et al. [26], an asset fire sale creates a negative feedback loop in the financial network. Large scale selling of an asset class by banks leads to strong downward pressure on the asset price, which leads to market-to-market losses by all banks holding that asset, to which they respond by selling this and other assets.

Of course, small and medium scale versions of such selling spirals are an everyday occurrence in financial markets, sometimes leading to an asset correlation shock. In the present context, we will focus on large scale selling spirals that form during and as a result of the crisis and are essential amplifiers of financial distress. Our aim in this section is to provide a stylized modelling framework that highlights the network cascade aspects of the fire sale mechanism.

2.3.1 Fire Sales of One Asset

The basic network picture of asset fire sales is most clearly explained by the CFS model of Cifuentes et al. [26]. The baseline CFS model consists of a network of N banks with balance sheets with the same components ($\bar{\mathrm{Y}}^F$, $\bar{\mathrm{Z}}$, $\bar{\mathrm{Y}}^L$, $\bar{\mathrm{D}}$, $\bar{\mathrm{X}}$, $\bar{\Omega}$) as shown in Fig. 2.2 for funding liquidity cascade models. Since liquidity and solvency are both considered in this model, it can be regarded as a generalization of the Eisenberg–Noe model. In the one asset model, all banks hold their fixed assets $\bar{\mathrm{Y}}^F_v$ in the same security, which we might view as the market portfolio. We set the initial price of the asset to be $\bar{p} = p^{(0)} = 1$ so that each bank v holds $s_v^{(0)} = \bar{\mathrm{Y}}^F_v$ units.

The essential new feature is to include a *capital adequacy ratio* (CAR) as a regulatory constraint: For some fixed regulatory value r^* (say 7 %), the bank must maintain the lower bound[3]

$$\frac{\Delta_v}{\mathrm{Y}_v^F + \mathrm{Z}_v} \geq r^* . \tag{2.21}$$

As soon as this condition is violated, any bank is compelled to restore the condition by selling fixed illiquid assets (but, in contrast to the Gai–Kapadia and Lee liquidity cascade models, not interbank assets[4]). This then triggers a downward impact on asset prices.

The detailed assumptions of [26] are as follows:

Assumption 2.9

1. A bank with $r^*\mathrm{Z}_v \leq \Delta_v < r^*(\mathrm{Y}_v^F + \mathrm{Z}_v)$ is called *non-compliant* but solvent, and must sell fixed assets, but not interbank assets, to restore the CAR condition.[5]
2. A bank with $r^*\mathrm{Z}_v > \Delta_v$ is *insolvent*, and must be fully liquidated. The picture is that even by selling all of Y^F, v cannot achieve the CAR condition, and hence must be terminated, even if its default buffer is still positive.
3. In the event of insolvency, the defaulted interbank assets are distributed at face value proportionally among the bank's creditors, and the bank ceases to function. External deposits have equal seniority to interbank debt and thus defaulted liabilities are valued in proportion to $\bar{\Pi}_{vw} = \bar{\Omega}_{vw}/(\bar{\mathrm{X}}_v + \bar{\mathrm{D}}_v)$.
4. The asset price when sold is determined by an inelastic supply curve and a downward sloping inverse demand function $d^{(-1)}(\cdot)$. That is, the asset price is $p = d^{(-1)}(s)$ when $s = \sum_v s_v$ is the aggregated amount sold. For the inverse demand function, [26] works with the family of exponential functions $d^{(-1)}(s) = e^{-\alpha s}$ for a specific value of α.

The crisis unfolds starting at step $n = 0$ from an initial balance sheet configuration $(\bar{\mathrm{Y}}^F, \bar{\mathrm{Z}}, \bar{\mathrm{Y}}^L, \bar{\mathrm{X}}, \bar{\Omega})$ with default buffers

$$\Delta^{(0)} = \bar{\Delta} = \bar{\mathrm{Y}}^F + \bar{\mathrm{Z}} + \bar{\mathrm{Y}}^L - \bar{\mathrm{X}} \tag{2.22}$$

in which at least one bank is found to be in violation of its CAR bound. In view of the equal seniority assumption on the debt, we adopt here the usual trick of replacing external debt by interbank debt owed to a non-borrowing fictitious bank $v = 0$ that has lent $\bar{\mathrm{D}}_w := \bar{\Omega}_{w0}$ to each bank w. Recall we set $p^{(0)} = 1$ so the number of fixed assets held initially is $s_v^{(0)} = \bar{\mathrm{Y}}_v^F$. Then, the following recursive steps for the balance sheets $(\mathrm{Y}^{F(n)}, \mathrm{Z}^{(n)}, \mathrm{Y}^{L(n)}, \Delta^{(n)})$ of each bank and the asset price $p^{(n)}$ for $n = 1, 2, \ldots$ are consistent with the underlying model assumptions:

[3] We deviate from [26] at this point by omitting liquid assets $\bar{\mathrm{Y}}^L$ from the denominator of the CAR.

[4] It is interesting that these modelling frameworks make essentially contradictory assumptions at this point. The assumption of [26] removes the need to consider how interbank assets are liquidated.

[5] As [26] explains, "interbank loans normally cannot be expected to be recalled early in the event of default of the lender."

1. Each bank v adjusts its fixed asset holdings by selling[6]

$$\delta s_v = \min\left(s_v^{(n-1)}, \max(0, s_v^{(n-1)} + \mathrm{Z}_v^{(n-1)}\right)/p^{(n-1)} - \Delta_v^{(n-1)}/(r^* p^{(n-1)})) \quad (2.23)$$

   units at the price $p^{(n-1)}$. Note that $\delta s_v = 0$ for a compliant bank, and $\delta s_v = s_v^{(n-1)}$ for an insolvent bank. When $\delta s_v > 0$, the sale of the fixed asset increases the value of the bank's liquid assets to $\mathrm{Y}_v^{L(n)} = \mathrm{Y}_v^{L(n-1)} + \delta s_v p^{(n-1)}$ and the number of shares held becomes $s_v^{(n)} = s_v^{(n-1)} - \delta s_v$.
2. In case $s_v^{(n)} = 0$, the insolvent bank v must be liquidated in the manner described above and the mark-to-market value of its debt adjusted: $\mathrm{X}_v^{(n)} = \bar{\mathrm{X}}_v + \min(0, \Delta_v^{(n-1)})$.
3. After all banks have completed their asset sales, the market price moves downwards according to the aggregate amount sold. Thus $p^{(n)} = d^{(-1)}(\sum_v (s_v^{(0)} - s_v^{(n)}))$ and the interbank assets are updated to account for new default losses, $\mathrm{Z}_v^{(n)} = \sum_w \bar{\Pi}_{wv} \mathrm{X}_w^{(n)}$.
4. The updated default buffer of bank v becomes

$$\Delta_v^{(n)} = s_v^{(n)} p^{(n)} + \mathrm{Y}_v^{L(n)} + \sum_w \bar{\Pi}_{wv} \mathrm{X}_w^{(n)} - \bar{\mathrm{X}}_v. \quad (2.24)$$

Just as the E–N framework could be simplified into a reduced form cascade mapping by focussing on the default buffers $\Delta_v^{(n)}$, it turns out the above recursion simplifies in a very similar way if we focus on the pairs $\Delta_v^{(n)}, s_v^{(n)}$. One key fact to recognize is that once a bank becomes noncompliant, it sells the minimal amount needed to become compliant if the asset price is $p^{(n-1)}$. However, since this asset price immediately drops, it does not actually become compliant. Nor can an insolvent bank recover. Having seen this, one can easily verify that the result of the n-th step of the CFS cascade is given by

$$\mathrm{X}_v^{(n)} = \bar{\mathrm{X}}_v\, h\left(\Delta_v^{(n-1)}/\bar{\mathrm{X}}_v\right) \quad (2.25)$$

$$\Delta_v^{(n)} = \bar{\Delta}_v - \sum_{m=1}^{n} (p^{(m-1)} - p^{(m)})\, s_v^{(m)} - \sum_w \bar{\Omega}_{wv}\left(1 - h\left(\Delta_w^{(n-1)}/\bar{\mathrm{X}}_w\right)\right) \quad (2.26)$$

$$s_v^{(n)} = \max\left(0, \min\left(s_v^{(n-1)}\right), \frac{1}{p^{(n-1)}}\left[\frac{\Delta_v^{(n-1)}}{r^*} - \sum_w \bar{\Omega}_{wv} h\left(\Delta_w^{(n-1)}/\bar{\mathrm{X}}_w\right)\right]\right) \quad (2.27)$$

$$p^{(n)} = d^{(-1)}\left(\sum_v (s_v^{(0)} - s_v^{(n)}\right). \quad (2.28)$$

The third of these equations corresponds to the trichotomy of possibilities of the bank being compliant, noncompliant but solvent, and insolvent.

[6]This is a slight modification of [26] who assume these units are sold at an n dependent equilibrium price somewhat lower than $p^{(n-1)}$.

By comparison to the E–N cascade dynamics given by (2.9) and (2.11), we note that the key effect of the fire sale on the cascade is to provide additional shocks that further reduce the default buffers, thereby amplifying the contagion. Structurally, the fire sale cascade mechanism can be expressed as a monotonically decreasing mapping $(\Delta^{(n)}, p^{(n)}) = \mathbf{F}^{CFS}(\Delta^{(n-1)}, p^{(n-1)})$ from $\mathbb{R}^{N+1}$ to itself, which leads as usual to a cascade equilibrium. In case the price impact is omitted by assuming $d^{(-1)}(\cdot) = 1$, one reproduces the E–N model.

An interesting special case of the model emerges if we set the compliancy parameter $r^* = 0$ and take Eq. (2.27) to mean

$$s_v^{(n)} = s_v^{(0)} \, \mathbf{1}\left(\Delta_v^{(n-1)} > 0\right). \tag{2.29}$$

We then interpret the breach of this regulatory condition as the insolvency of the bank in two senses: first, the bank has insufficient assets to repay its liabilities; second, the bank has insufficient equity to support its assets and thus needs to liquidate all fixed assets. Thus the bank must sell all fixed assets at this moment. However, as in the E–N model, the bank continues to lose money after it defaults, further eroding the value of $\mathrm{X}_v^{(n)}$. Thus, when $r^* = 0$ this model looks very similar to the E–N 2001 model, albeit with a G–K-like condition to determine the amount of fixed assets sold.

The simplification $r^* = 0$ also provides a simpler setting to address a different question: how do fire sales create contagion when banks hold different portfolios of a multiplicity of assets. As it turns out, this variation has been studied already, in a paper [22] we shall now describe.

2.3.2 *Fire Sales of Many Assets*

A model due to Caccioli et al. [22] addresses the question: How do fire sales create contagion when banks hold different but overlapping portfolios of a multiplicity of assets? Their paper is a variation of the CFS approach in which N banks have balance sheets $(\bar{\mathrm{Y}}^F, \bar{\mathrm{Y}}^L, \bar{\mathrm{D}})$ with the interbank sector set to zero $\bar{\mathrm{Z}}_v = \bar{\mathrm{X}}_v = 0$ for simplicity, and the fixed assets are portfolios of M non-bank assets labelled by $a \in \{1, 2, \ldots, M\} := [M]$. One can describe their model in terms of a bipartite graph with nodes of two colours, "blue" nodes that represent banks and "red" nodes that represent non-bank assets, and links connecting banks to assets when the bank has a significant holding of that asset. Figure 2.3 shows a typical network with 5 banks and 4 assets.

Again, banks are constrained by the regulatory CAR condition which says

$$\frac{\Delta_v}{\mathrm{Y}_v^F} \ge r^*. \tag{2.30}$$

The paper [22] deals only with the case $r^* = 0$, but here we consider the general case with $r^* \ge 0$. The assumptions for the many asset fire sale are slight modifications

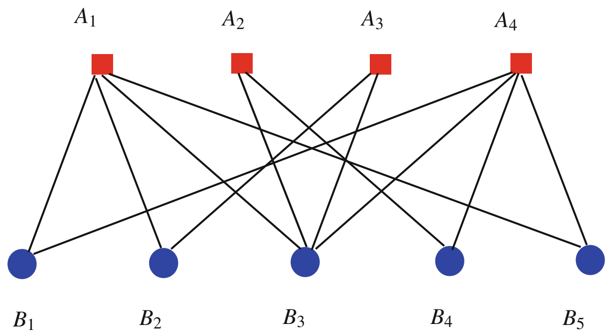

Fig. 2.3 A bipartite graph with 5 banks (*blue nodes*) co-owning 4 assets (*red nodes*)

of Assumption 2.9, but with some additional points: The solvency condition is now $\Delta_v > 0$, and ownership of insolvent banks goes to external debtholders; there is an inverse demand function $d_a^{(-1)}$ for each asset; a portfolio policy must be specified for each bank that determines its portfolio weights during rebalancing.

The cascade picture that arises from these assumptions is that non-compliant banks that fail the CAR condition (2.30) are forced to liquidate some fixed assets, and while insolvent banks sell all their remaining fixed assets to pay their external creditors. At any cascade step, forced asset sales by a bank create shocks transmitted along each link to its assets that drive down the prices of these assets. Each asset price drop creates mark-to-market shocks acting in the reverse direction, from the asset to the banks that hold it. As usual, the iterated cascade steps converge monotonically to a cascade equilibrium.

Without loss of generality, we assume the initial share prices are $\bar{p}_a = 1$. The initial external fixed asset value of bank v is then $\bar{\mathrm{Y}}_v^F = \sum_a \bar{s}_{av}$ where $s_{av}^{(0)} = \bar{s}_{av}$ denotes the initial number of shares of asset a owned by v. If after n cascade steps each bank v holds $s_{av}^{(n)}$ shares of asset a, each asset share price will have been driven down to the value

$$p_a^{(n)} = d_a^{(-1)}\left(\sum_v \left(s_{av}^{(0)} - s_{av}^{(n)}\right)\right) \tag{2.31}$$

determined by the total amount of selling and its inverse demand function. Various portfolio selection rules can be proposed to determine in which proportion banks choose to sell assets during the cascade. For illustrative purposes, we assume that non-compliant banks follow a *fixed-mix trading strategy* that keeps the *number of shares* in different assets in proportion to the initial ratios

$$\bar{\phi}_{av} := \frac{\bar{s}_{av}}{\sum_b \bar{s}_{bv}}. \tag{2.32}$$

Based on these assumptions, recursively for $n = 1, 2, \ldots$ the balance sheets of each bank $(\mathrm{Y}_v^{F(n)}, \mathrm{Y}_v^{L(n)}, \Delta_v^{(n)})$, the portfolio allocations $s_{av}^{(n)}$ and asset prices $p_a^{(n)}$ are updated according to the following steps, starting from the initial values $\mathrm{Y}^{F(0)} = \bar{\mathrm{Y}}^F, \mathrm{Y}^{L(0)} = \bar{\mathrm{Y}}^L, \Delta^{(0)} = \bar{\mathrm{Y}}^F + \bar{\mathrm{Y}}^L - \bar{\mathrm{D}}$:

1. Each bank v adjusts its fixed asset holdings by selling

$$\delta s_{av} = s_{av}^{(n-1)} \min\left(1, \max\left(0, 1 - \Delta_v^{(n-1)}\right) / \left(r^* \mathrm{Y}_v^{F(n-1)}\right)\right) \tag{2.33}$$

   units at the price $p^{(n-1)}$. Note that $\delta s_{av} = 0$ for a compliant bank, and $\delta s_v = s_{av}^{(n-1)}$ for an insolvent bank. When $\delta s_v > 0$, the sale of the fixed assets increases the value of the bank's liquid assets to $\mathrm{Y}_v^{L(n)} = \mathrm{Y}_v^{L(n-1)} + \sum_a \delta s_{av} p_a^{(n-1)}$ and the number of shares held becomes $s_{av}^{(n)} = s_{av}^{(n-1)} - \delta s_{av}$.
2. After all banks have completed their asset sales, the market prices move downwards according to the aggregate amount sold: $p_a^{(n)} = d_a^{(-1)}(\sum_v (s_{av}^{(0)} - s_{av}^{(n)}))$.
3. The updated default buffer and fixed assets of bank v decrease:

$$\Delta_v^{(n)} = \Delta_v^{(n-1)} - \sum_a s_{av}^{(n)} (p_a^{(n-1)} - p_a^{(n)}), \tag{2.34}$$

$$\mathrm{Y}^{F(n)} = \sum_a s_{av}^{(n)} p_a^{(n)}. \tag{2.35}$$

We observe the formal similarity between this model and the original E–N 2001 model, in the sense that the cascade mechanism can be expressed as a cascade mapping

$$(\Delta^{(n)}, p^{(n)}) = \mathbf{F}^{CSMF}(\Delta^{(n-1)}, p^{(n-1)}) \tag{2.36}$$

from $\mathbb{R}^{N+M}$ to itself. This means asset prices $p_a^{(n)}$ behave as if they were "asset buffers" attached to the red nodes, in analogy to the buffers $\Delta_v^{(n)}$ attached to blue nodes. In addition, the dynamics depends explicitly on the initial balance sheets only through $\bar{\Delta}$ and the fixed-mix ratios $\bar{\phi}_{av}$ that are analogous to the edge weights $\bar{\Omega}_{wv}$. The inverse demand functions d_a^{-1} play a role similar to the $h, \tilde{h}$ threshold functions in our default cascade models. Finally, the cascade mapping from step $n-1$ to n is monotonic and bounded below, and thus its iterates converge to a fixed point.

2.4 Random Financial Networks

This chapter has explored stylistic features of various types of shocks that can be transmitted through the interbank exposures (or, in one case, through bank-to-asset exposures), and how they might potentially cascade into large scale disruptions of the sort seen during the 2007–2008 financial crisis. We have seen how to build these shock channels into a variety of different financial network models, all of which boil down to a monotone cascade mapping whose iterates converge to a cascade equilibrium.

The real world financial systems in most countries are of course far from behaving like these models. Bank balance sheets are hugely complex. Interbank exposure data are never publicly available, and in many countries nonexistent even for central regulators. Sometimes, the only way to infer exposures is indirectly, for example, through bank payment system data as done in [42]. Interbank exposures are of a diversity of types and known to change rapidly day to day. In a large jurisdiction like the US, the banking sector is highly heterogeneous, and the systemic impact due to the idiosyncrasies of individual banks will likely overwhelm anything one might predict from their average properties.

Nonetheless, a large and rapidly growing web of economics and complex systems research continues to address the real world implications of theoretical network cascades. The conceptual tools that we will explore in the remainder of this book come partly from the experience gained by modelling large complex networks that arise in other areas, such as the world wide web, Facebook and power grids.

The central theme of this book is that something fundamental about financial systems can be learned by studying very large stochastic networks. There are at least three important reasons why stochastic network models are a good way to approach studying real world financial systems. The first and most fundamental reason comes from over a century of statistical mechanics theory, which has discovered that the macroscopic properties of matter, for example crystals, are determined by the ensemble averages of the deterministic dynamics of constituent microscopic particles. Even a completely known deterministic system, if it is large enough, can be well described by the average properties of the system. From this fact we can expect that for large N, an E–N model with fully specified parameters will behave as if it were a stochastic model with averaged characteristics.

The second important reason is that in a concrete sense, true financial networks are stochastic at any moment in time. The balance sheets of banks, between reporting dates, are not observed even in principle. Moreover, they change so quickly that last week's values, if they were known, will have only limited correlation with this week's values. This fact is especially true for the interbank exposures that provide the main channels for transmitting network cascades: even in jurisdictions where they are reported to central regulators, they comprise a diversity of different securities, including derivatives and swaps whose mark-to-market valuations fluctuate dramatically on intraday time-scales.

A third important reason is that a hypothetical financial system, with all balance sheets completely known, will be hit constantly by random shocks from the outside, stochastic world. A deterministic system, subjected to a generic random shock, becomes after only one cascade step a fully stochastic system.

For all these reasons, and more, the next chapters will consider the stochastic nature of financial networks, and the extent to which the large scale properties of cascades might be predictable from their local stochastic properties. From now on in this book, our various contagion channels will usually take their dynamics within the framework of "random financial networks", defined provisionally as follows:

Definition 2.10 A *random financial network* or RFN is a random object representing the possible states of the financial network at an instant in time. It consists of three layers of mathematical structure. At the base structural level, the *skeleton* is a random directed graph $(\mathcal{N}, \mathcal{E})$ whose nodes $\mathcal{N}$ represent "banks" and whose directed edges or links $\mathcal{E}$ represent the presence of a non-negligible "interbank exposure" between a debtor bank and its creditor bank. Conditioned on a realization of the skeleton, the second structural layer is a collection of random *balance sheets*, one for each bank. In our simple models this is usually a coarse-grained description, listing for example the amounts $(\bar{\mathrm{Y}}_v, \bar{\mathrm{Z}}_v, \bar{\mathrm{D}}_v, \bar{\mathrm{X}}_v)$ for each $v \in \mathcal{N}$ as in Sect. 2.1.1. Finally, conditioned on a realization of the skeleton and balance sheets, the third level is a collection of positive random variables $\bar{\Omega}_\ell$ for each link $\ell = (w, v) \in \mathcal{E}$ that represent the *exposure* of w to v, that is, what v owes w. The interbank assets and liabilities are constrained to equal the aggregated exposures:

$$\bar{\mathrm{Z}}_v = \sum_w \bar{\Omega}_{wv}, \quad \bar{\mathrm{X}}_v = \sum_w \bar{\Omega}_{vw}. \tag{2.37}$$

Typically in cascade scenarios, we consider the RFN at the instant a crisis triggering event unexpectedly occurs. We will combine the choice of RFN with the choice of a cascade mechanism (CM) such as the ones described in this chapter to describe what happens next. Depending on the cascade mechanism, only reduced balance sheet information in the form of buffer random variables $\bar{\Delta}_v$ and exposures $\bar{\Omega}_\ell$ is needed to follow the progress of the cascade. In that case, we can work with a minimal parametrization of the RFN by the quadruple $(\mathcal{N}, \mathcal{E}, \bar{\Delta}, \bar{\Omega})$.

This schematic definition will prove to be acceptable for the description of simple contagion models. But more than that, it scales conceptually to much more complex settings. Nodes may have additional attributes or "types" beyond their connectivity to represent a more diverse class of economic entities. Links might have extended meaning where the random variables $\bar{\Omega}_\ell$ take vector values representing different categories of exposures. We also recognize that even a very simple RFN is a complicated random variable of enormous dimension. Before proceeding to any analysis of network dynamics, the distributions of these collections of random variables must be fully specified. We will proceed stepwise, first focussing in the next chapter on characterizing possible models for the skeleton. In subsequent chapters we will consider how to specify the random structure of balance sheets and interbank exposures. Our optimistic view that something meaningful can be learned about systemic risk in the real world through the study of schematic or stylized RFNs is derived from our collective experience in other fields of complex disordered stochastic systems, rooted in the theory of statistical mechanics.

2.5 Bibliographic Notes

A seminal paper in 2000 by Allen and Gale [6] relates the structure of a stylized network of four banks to its resilience to default contagion, concluding that the completely connected network is the most resilient. The Eisenberg–Noe model [35] was originally intended to describe clearing mechanisms in payment systems or listed exchanges, but has since become a paradigm for modelling general financial systemic risk. The original paper dealt only with the case $\bar{\mathrm{Y}}_v \geq \bar{\mathrm{D}}_v$, and some of its main results were specific to this case. In two papers, [38, 39], Elsinger et al. applied this framework to the Austrian and UK networks and concluded that in the early 2000s these systems were rather resilient. Elsinger [37], Gourieroux et al. [47] and Elliot et al. [36] have studied model properties including uniqueness of clearing vectors in a more general setting that includes interbank equity crossholdings and multiple debt seniority layers. Rogers and Veraart [76] have explored the E–N model to understand when banks should cooperate to rescue a failing bank. Other authors have worked with models similar to the E–N framework aiming to understand the dependence between network connectivity and systemic resilience. In a simulation based study, Nier et al. [74] showed that the resiliency of finite size networks can depend non-monotonically on key parameters. Glasserman and Young [45] and Acemoglu et al. [1] also show that while for small initial shocks, connectivity improves stability, this relation can be reversed for large shocks. Upper [82] gives an in-depth survey of 15 different papers that use simulation techniques to determine the possibility of contagion in interbank markets prior to 2007, concluding that none of them foresaw any indication of the upcoming crisis. Under the zero recovery assumption as in the G–K model, Amini et al. [7] were able to prove asymptotic results on the cascade equilibrium in large random networks.

A fundamental treatment of the connection between funding liquidity and market liquidity by Brunnermeier and Pedersen [21] proposes a picture of how the funding constraints on a bank impact and are impacted by the liquidity of its market portfolio, that is its external assets. Funding liquidity cascades are a contagious, interbank version of the classic problem of bank runs that was studied by Diamond and Dybvig [33]. Despite a widespread opinion that funding liquidity cascades may be more important in real world crises, they have been less studied than the classic default cascade. Krishnamurthy [62] has investigated the systemic feedback due to both funding liquidity hoarding triggered by uncertainty, and asset fire sales. A paper by Minca and Amini [66] describes a number of network models for different channels for contagion, including funding liquidity cascades similar to those described in Sect. 2.2.

The justification for studying financial contagion on large random financial networks has been discussed by Gai and Kapadia [44], Amini et al. [7, 66] and others.

Some recent papers have attempted to construct cascade mechanisms that effectively combine two or more contagion channels. The double cascade model for funding illiquidity and insolvency proposed by Hurd et al. [55] unifies the assumptions of the Gai–Kapadia default model and the Gai–Kapadia liquidity model. Bookstaber [18] has developed an agent-based cascade modelling framework that incorporates versions of most of the contagion channels discussed in this chapter.

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