

# An Elementary Fact About Unlinked Braid Closures

J. Elisenda Grigsby and Stephan M. Wehrli

**Abstract** Let  $n \in \mathbb{Z}^+$ . We provide two short proofs of the following classical fact, one using Khovanov homology and one using Heegaard–Floer homology: if the closure of an  $n$ -strand braid  $\sigma$  is the  $n$ -component unlink, then  $\sigma$  is the trivial braid.

**Keywords** Braids · Khovanov homology · Heegaard–floer homology

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Let  $\mathcal{B}_n$  denote the  $n$ -strand braid group,  $\mathbb{1}_n \in \mathcal{B}_n$  the  $n$ -strand trivial braid, and  $U_n$  the  $n$ -component unlink in  $S^3$ . Denote by  $\widehat{\sigma}$  the closure of  $\sigma \in \mathcal{B}_n$ , considered as a link in  $S^3$ . The following fact first appears in the literature in [4, Thm. 4.1]. It can also be obtained as an immediate corollary of [3, Thm.1]:

**Proposition 1** *Let  $\sigma \in \mathcal{B}_n$ . If  $\widehat{\sigma} = U_n$ , then  $\sigma = \mathbb{1}_n$ .*

The primary purpose of this note is to provide two short proofs of Proposition 1, one using Khovanov homology and one using Heegaard–Floer homology. Although the classical proof contained in [4] is straightforward, we hope these new proofs will also be of interest, since they suggest ways in which algebraic properties of link homology theories can give information about braid dynamics.

The key geometric idea underlying both proofs is the following pair of simple but powerful observations, from [8]:

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J.E. Grigsby (✉)

Mathematics Department, Boston College, 522 Maloney Hall, Chestnut Hill, MA 02467, USA  
e-mail: grigsbyj@bc.edu

S.M. Wehrli

Department of Mathematics, Syracuse University, 215 Carnegie, Syracuse, NY 13244, USA  
e-mail: smwehrli@syr.edu

- A self-diffeomorphism of a surface with non-empty boundary that fixes the boundary pointwise is isotopic to the identity rel boundary iff it is both right- and left-veering (cf. [9] for a definition);
- If the Heegaard–Floer contact invariant (resp., Plamenevskaya’s transverse invariant) is nonzero, then every open book supporting the contact structure (resp., every braid closure representing the transverse link) is right-veering [9, 16] (resp., [1]).

The two proofs of Proposition 1 we present are formally analogous. The proof involving Khovanov homology (Sect. 1) requires little more than the two facts above, while the Heegaard–Floer homology proof (Sect. 2) involves applying the facts above to fibered links in connected sums of copies of  $S^1 \times S^2$ , using ideas of Birman–Hilden [2]. Recall that the preimage of the braid axis in the double branched cover of a braid closure in  $S^3$  is a fibered link. When the closure of the braid is the unlink, one obtains a fibered link in a connected sum of copies of  $S^1 \times S^2$ . The existence of a nontrivial  $n$ -strand braid whose closure is  $U_n$  would imply the existence of a non-trivial fibered link of minimal complexity (i.e., maximal Euler characteristic) in  $\#^{n-1}(S^1 \times S^2)$ .

More precisely, let  $Y_n$  denote  $\#^n(S^1 \times S^2)$ . For  $L$  a fibered link with fiber  $F$ , we will abuse terminology and refer to  $\chi(F)$  as the *Euler characteristic of  $L$* .

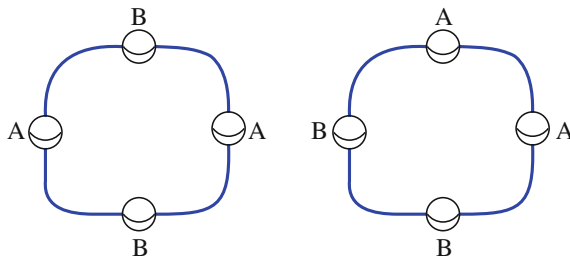
Define

$$\mathcal{L}_n := \{\ell \in \mathbb{Z}^+ \mid \ell \leq (n+1) \text{ and } \ell \equiv (n+1) \bmod 2\}.$$

Note that for each  $\ell \in \mathcal{L}_n$ , it is straightforward to construct a fibered  $\ell$ -component link,  $\mathbf{L}_\ell \subset Y_n$ , of Euler characteristic  $1 - n$ . See Fig. 1. The monodromy of  $\mathbf{L}_\ell$  is trivial, and the pair  $(Y_n, \mathbf{L}_\ell)$  is well-defined up to diffeomorphism.

The following result appears in [14].

**Proposition 2** [14, Prf. of Thm. 1.3] *Let  $L_\ell \subset Y_n$  be a fibered,  $\ell$ -component link with  $\ell \in \mathcal{L}_n$  and Euler characteristic  $1 - n$ . Then the pair  $(Y_n, L_\ell)$  is diffeomorphic to the pair  $(Y_n, \mathbf{L}_\ell)$ .*



**Fig. 1** Kirby diagrams of the links  $\mathbf{L}_1$  (left) and  $\mathbf{L}_3$  (right) in  $Y_2 := \#^2 S^1 \times S^2$ . The  $S^2$ ’s (boundaries of the feet of 4-dimensional 1-handles) are identified as labeled, via a reflection in the plane perpendicular to the *straight line* joining their centers. The fibered link in each case is drawn in *blue*. To construct  $\mathbf{L}_\ell \in Y_n$  in general, arrange  $n$  pairs of  $S^2$ ’s along an unknot in  $S^3$  so that attaching 2-dimensional one-handles to the disk bounded by the unknot, via the chosen configuration, forms an oriented surface with  $\ell$  boundary components

It is clear (cf. Lemma 1) that if  $\ell \notin \mathcal{L}_n$ , then an  $\ell$ -component link cannot have Euler characteristic  $1 - n$ . It is also clear (cf. Lemma 2) that  $1 - n$  is the maximal possible Euler characteristic among all fibered links in  $Y_n$ . Informally, Proposition 2 therefore says that for allowable  $\ell$ , maximal Euler characteristic fibered  $\ell$ -component links in  $\#^n(S^1 \times S^2)$  are unique up to diffeomorphism. After the first version of this note appeared, it was pointed out in [5, Cor. 1.3] that Proposition 2 implies Proposition 1, by the main result of [2].

In Sect. 2.1 we present an alternative proof of Proposition 2 using Heegaard–Floer homology. We thank John Baldwin for pointing out that this proof of Proposition 2 implies:

**Corollary 1** *If  $Y \not\cong Y_n$  is a closed, oriented 3-manifold with the same Heegaard–Floer module structure as  $Y_n$ , then  $Y$  contains no fibered links of Euler characteristic  $1 - n$ .*

There is a unique maximal Euler characteristic fibered link in  $S^3$  (namely, the unknot) whose corresponding open book supports the standard tight contact structure. Ken Baker (cf. [11]) asked the following interesting question:

*Question 1* Fix a contact structure,  $\xi$ , on a 3-manifold,  $Y$ , and let

$$\overline{\chi}_\xi := \max\{\chi(L) \mid L \text{ is a fibered link whose open book supports } \xi\}.$$

Up to diffeomorphism, are there finitely many fibered links  $L$  supporting  $\xi$  with  $\chi(L) = \overline{\chi}_\xi$ ?

Proposition 2 tells us that for the standard tight contact structure on  $Y_n$  the answer is yes.

## 1 Khovanov Homology Proof of Proposition 1

*Proof* (Proposition 1) Choose a diagram,  $D(\widehat{\sigma})$ , for  $\widehat{\sigma}$  obtained as the closure of a diagram for  $\sigma$ , and mark the  $n$  points on the diagram corresponding to the intersection with the closure arc. Recall that the ( $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ ) Khovanov homology,  $\text{Kh}(\widehat{\sigma})$ , of  $\widehat{\sigma}$  is an invariant of the isotopy class of  $\widehat{\sigma} \subset S^3$  that takes the form of a bigraded vector space over  $\mathbb{F}$ . Since we have also chosen a basepoint on each of the  $n$  link components, [7, Prop.1] tells us that  $\text{Kh}(\widehat{\sigma})$  inherits the structure of a module over the ring

$$\mathcal{A}_n := \mathbb{F}[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$$

as follows.

Associated to the diagram of  $\widehat{\sigma}$  is a cube of resolutions whose vertices are in one-to-one correspondence with complete resolutions (i.e., Kauffman states) of the diagram. The basis elements (generators) of the underlying vector space of the Khovanov chain

complex,  $\text{CKh}(D(\widehat{\sigma}))$ , are, in turn, in one-to-one correspondence with markings of the components of each resolution with either a 1 or an  $x$  (i.e., *enhanced* Kauffman states).

Let  $\mathcal{I}_{\text{braid}}$  be the unique “braid-like” complete resolution of  $D(\widehat{\sigma})$ , and denote by  $\Psi^+$  (resp.,  $\Psi^-$ ) the basis element  $1 \otimes \dots \otimes 1$  (resp.,  $x \otimes \dots \otimes x$ ) in the vector space associated to  $\mathcal{I}_{\text{braid}}$ .  $\Psi^-$  is a cycle, hence represents an element in  $\text{Kh}(\widehat{\sigma})$ . Indeed,  $[\Psi^-] \in \text{Kh}(\widehat{\sigma})$  is precisely Plamenevskaya’s invariant [17] of the transverse isotopy class of the transverse link represented by  $\widehat{\sigma}$ .

We are now ready to understand the  $\mathcal{A}_n$  structure induced by the  $n$  points  $p_1, \dots, p_n$ . For each complete resolution,  $\mathcal{I}$ , choose a numbering of its  $\ell_{\mathcal{I}}$  connected components, and let  $v_1 \otimes \dots \otimes v_{\ell_{\mathcal{I}}}$  represent the Khovanov generator whose  $j$ th component in  $\mathcal{I}$  is marked with  $v_j \in \{1, x\}$ . Suppose  $p_i$  lies on the  $k$ th component of  $\mathcal{I}$ . Then the action of  $x_i \in \mathcal{A}_n$  is the  $\mathbb{F}$ -linear extension of the assignment:

$$x_i \cdot (v_1 \otimes \dots \otimes v_k \otimes \dots \otimes v_{\ell_{\mathcal{I}}}) := v_1 \otimes \dots \otimes x \otimes \dots \otimes v_{\ell_{\mathcal{I}}}$$

if  $v_k = 1$  and 0 otherwise.

It is straightforward to check that the Khovanov differential commutes with the action of  $\mathcal{A}_n$ , and it is shown in [7] (see also [12, 13]) that the homotopy equivalences associated to Reidemeister moves respect the  $\mathcal{A}_n$ -module structure, and moving a basepoint past a crossing yields a homotopic map. The homology,  $\text{Kh}(\widehat{\sigma})$ , therefore inherits the structure of an  $\mathcal{A}_n$ -module, and this  $\mathcal{A}_n$ -module structure is an invariant of the link.

With these preliminaries in place, assume that  $\widehat{\sigma} = U_n$ . A quick calculation using the standard diagram of  $U_n$  tells us that  $\text{Kh}(U_n) \cong \mathcal{A}_n$  as an  $\mathcal{A}_n$ -module. Let  $\theta \in \text{CKh}(D(\widehat{\sigma}))$  be a cycle representing the homology class  $1 \in \text{Kh}(U_n) \cong \mathcal{A}_n$ .

We now claim that when  $\theta$  is expressed as a linear combination of the standard Khovanov generators, the coefficient of  $\Psi^+$  must be 1. To see this, note that  $x_1 \cdots x_n(\theta)$  represents the non-zero homology class  $x_1 \cdots x_n \in \text{Kh}(\widehat{\sigma})$ , but if  $v$  is any basis element not equal to  $\Psi^+$ , then  $x_1 \cdots x_n(v) = 0$ . We see this immediately for  $v \neq \Psi^+ \in \mathcal{I}_{\text{braid}}$ , and any complete resolution  $\mathcal{I} \neq \mathcal{I}_{\text{braid}}$  contains at least one connected component intersecting the closure arc more than once, hence containing at least two basepoints  $p_i, p_j, i \neq j$ . We conclude that any basis element  $v$  associated to  $\mathcal{I} \neq \mathcal{I}_{\text{braid}}$  satisfies  $x_i x_j(v) = 0$ , hence also satisfies  $x_1 \cdots x_n(v) = 0$ .

The arguments in the previous paragraph imply that  $x_1 \cdots x_n(\theta) = x_1 \cdots x_n(\Psi^+) = \Psi^-$ , so  $[\Psi^-] = x_1 \cdots x_n \in \text{Kh}(\widehat{\sigma})$ . In particular,  $[\Psi^-] \neq 0$ .

But [1, Prop. 3.1] then implies that  $\sigma$  is *right-veering*.

Repeat the argument above on  $m(\sigma)$ , the mirror of  $\sigma$ , to conclude that  $\sigma$  is also *left-veering*. Since the only braid which is both left- and right-veering is the identity braid (cf. [1, Lem. 3.1]),  $\sigma = \mathbb{1}_n$ , as desired.

## 2 Fibred Links in $\#^n(S^1 \times S^2)$

Recall that  $\mathcal{L}_n := \{\ell \in \mathbb{Z}^+ \mid \ell \leq (n+1) \text{ and } \ell \equiv (n+1) \pmod{2}\}$ .

**Lemma 1** *If an  $\ell$ -component link  $L$  has Euler characteristic  $1-n$ , then  $\ell \in \mathcal{L}_n$ .*

*Proof* Let  $S$  denote the fiber surface of  $L$ ,  $\chi(S)$  its Euler characteristic, and  $g(S)$  its genus. Then  $\chi(S) = 1-n = (2-2g(S)) - \ell$ . Since  $g(S) \in \mathbb{Z}^{\geq 0}$ , we obtain  $\ell \equiv (n+1) \pmod{2}$  and  $\ell \leq n+1$ .

**Lemma 2** *If  $L \subset Y_n$  is a fibered link, then  $\chi(L) \leq 1-n$ .*

*Proof* Suppose  $L$  has  $\ell$  components, and let  $S$  denote the fiber surface of  $L$ , and  $h$  its monodromy.  $H_1(S)$  is free of rank  $1 - \chi(S) = 2g(S) + (\ell - 1)$ . Viewing  $Y_n - L$  as the mapping torus of  $h$  (cf. Sect. 2.1), we obtain a corresponding presentation of  $H_1(Y_n) \cong \mathbb{Z}^n$  with  $1 - \chi(L)$  generators, hence  $1 - \chi(L) \geq n$ .

### 2.1 Heegaard–Floer Homology Proof of Proposition 2

We begin with some background on Heegaard–Floer homology.

#### 2.1.1 Heegaard–Floer Module

Recall that in [15], Ozsváth–Szabó associate to a closed, oriented 3-manifold  $Y$  a graded vector space (for simplicity we work over  $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ ),  $\widehat{HF}(Y)$ , which splits over  $\text{Spin}^c(Y)$ , the set of  $\text{spin}^c$  structures on  $Y$ :

$$\widehat{HF}(Y) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y)} \widehat{HF}(Y, \mathfrak{s})$$

For appropriate choices of symplectic and almost complex structures,  $\widehat{HF}(Y)$  is the Lagrangian Floer homology of a natural pair of Lagrangian tori,  $\mathbb{T}_\alpha$  and  $\mathbb{T}_\beta$ , in the  $g$ -fold symmetric product of a pointed Heegaard surface,  $(\Sigma, w)$ , for  $Y$ .

$\widehat{HF}(Y)$  can be given the structure of a module over  $\Lambda^*(H_1(Y; \mathbb{F}))$ , as described in [15, Sect. 4.2.5]. Explicitly, let

$$(\Sigma, \alpha = \{\alpha_1, \dots, \alpha_g\}, \beta = \{\beta_1, \dots, \beta_g\}, z)$$

be a pointed, genus  $g$  Heegaard splitting of  $Y$ , and consider  $\zeta \in H_1(Y; \mathbb{F})$ . Ozsváth–Szabó define an associated chain map,

$$A_\zeta : \widehat{CF}(\Sigma, \alpha, \beta, z) \rightarrow \widehat{CF}(\Sigma, \alpha, \beta, z),$$

on the Heegaard–Floer chain complex as follows ([15, Rmk. 4.20]). Let  $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  be generators of the chain complex. Recall that  $\pi_2(\mathbf{x}, \mathbf{y})$  denotes the set of domains in  $\Sigma$  representing topological Whitney disks connecting  $\mathbf{x}$  to  $\mathbf{y}$ , in the sense of [15, Sect. 2.4]. If  $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ , we follow the notation in [14, Sect. 2.1], letting  $\partial_\alpha \phi := (\partial \phi) \cap \mathbb{T}_\alpha$ , regarded as a 1-chain with boundary  $\mathbf{y} - \mathbf{x}$ .

Choose an immersed curve,

$$\gamma_\zeta \subset \Sigma - \{\alpha_i \cap \beta_j\}_{i,j \in \{1, \dots, g\}},$$

representing  $\zeta \in H_1(Y; \mathbb{F})$  and define

$$a(\gamma_\zeta, \phi) := \# \widehat{\mathcal{M}}(\phi)(\gamma_\zeta \cdot \partial_\alpha \phi),$$

where  $\gamma_\zeta \cdot \partial_\alpha \phi$  is the algebraic intersection number of  $\gamma_\zeta$  and  $\partial_\alpha \phi$ . Then the chain map associated to  $\zeta$  is given by:

$$A_\zeta(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\left\{ \phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \mu(\phi)=1, n_w(\phi)=0 \right\}} a(\gamma_\zeta, \phi) \cdot \mathbf{y}.$$

The map  $A_\zeta$  is well-defined (independent of the choice of  $\gamma$ ) up to chain homotopy (cf. [14, Lem. 2.4]).

### 2.1.2 Heegaard–Floer Contact Invariant

We now recall the definition of the Heegaard–Floer contact invariant [16], following the alternative construction given in [10]. Let  $\xi$  be a contact structure on a closed, connected, oriented 3-manifold  $Y$ . Then Giroux tells us [6] that there exists some fibered link  $L$  whose corresponding open book supports  $\xi$ . One can then build a Heegaard diagram for  $-Y$  ( $Y$  with the opposite orientation) using

- a choice of *basis*,  $\{a_1, \dots, a_n\}$ , for a page  $S$  (of Euler characteristic  $1 - n$ ) of the open book [10, Sect. 3.1], and
- the data of the monodromy,  $h$ , of the open book.

Honda–Kazez–Matić then identify a distinguished cycle in the corresponding chain complex,  $\widehat{CF}(-Y)$ , and prove both that the class it represents in  $\widehat{HF}(-Y)$  is invariant of the choices used in its construction and that it agrees with the contact invariant defined in [16].

We will need the following, which appears in [8] and follows immediately from [16, Thm. 1.4] and [9, Thm. 1.1]:

**Lemma 3** *If  $L \subset Y$  is a fibered link whose monodromy,  $h$ , is not right-veering, then the Heegaard–Floer contact invariant associated to the contact structure supported by  $L$  vanishes.*

We now proceed to the proof.

*Proof* (Proposition 2) Let  $L_\ell \subset Y_n$  be an  $\ell$ -component fibered link of Euler characteristic  $1 - n$ . Construct a corresponding Heegaard diagram for  $-Y_n$  as in [10, Sect. 3].

The module structure on  $\widehat{HF}(-Y_n)$  has been computed in [15, Lemma. 9.1]. Explicitly,  $\widehat{HF}(-Y_n) \cong \mathcal{A}_n$  as a module over

$$\Lambda^*(H_1(-Y_n; \mathbb{F})) \cong \mathcal{A}_n := \mathbb{F}[\zeta_1, \dots, \zeta_n]/(\zeta_1^2, \dots, \zeta_n^2).$$

In particular,  $\zeta_1 \cdots \zeta_n \neq 0 \in \widehat{HF}(-Y_n)$ .

We can understand the module action explicitly in our setting as follows. All of our notation matches [10]. Examine the Honda-Kazez-Matić pointed Heegaard diagram

$$(\Sigma = S_{1/2} \cup -S_0, \{\beta_1, \dots, \beta_n\}, \{\alpha_1, \dots, \alpha_n\}, z)$$

associated to the fibered link,  $L_\ell$ . In particular, choose a small perturbation,  $b_i \subset S$ , of each arc  $a_i \subset S$ , as described in [10, Sect. 3.1]. Subject to the identifications  $S \cong S_{1/2} \cong S_0$ , form:

$$\begin{aligned} \alpha_i &:= (a_i \subset S_{1/2}) \cup (a_i \subset -S_0) \\ \beta_i &:= (b_i \subset S_{1/2}) \cup (h(b_i) \subset -S_0). \end{aligned}$$

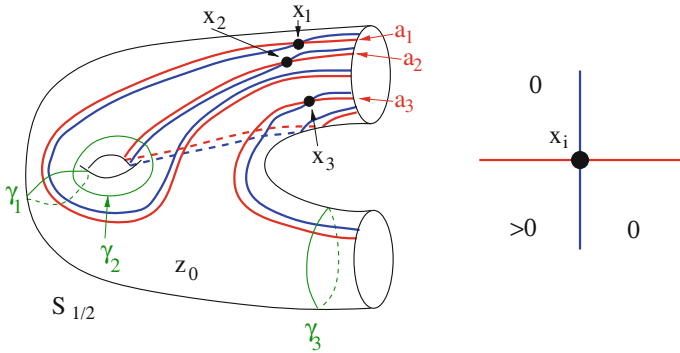
By construction,  $|S_{1/2} \cap (\alpha_i \cap \beta_j)| = \delta_{ij}$ . Let  $x_i$  denote the unique intersection point in  $(S_{1/2} \cap \alpha_i \cap \beta_i)$ , and let  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \text{Sym}^n(\Sigma)$ . Honda-Kazez-Matić prove that  $\mathbf{x}$  is a cycle in the Heegaard-Floer chain complex and that it represents the Heegaard-Floer contact class  $c(\xi_{(S,h)}) \in \widehat{HF}(-Y_n)$  associated to the contact structure  $\xi_{(S,h)}$  compatible with the open book  $(S, h)$ .

Now choose a *dual basis*,  $\{\gamma_1, \dots, \gamma_n\}$ , of simple closed curves on  $S_{1/2}$  satisfying  $|a_i \cap \gamma_j| = \delta_{ij}$ . The set of homology classes,  $\{[\gamma_1], \dots, [\gamma_n]\}$ , obtained by viewing the  $\gamma_i$  as 1-cycles in  $-Y_n$ , forms a basis for  $H_1(-Y_n; \mathbb{F})$ . Hence, for each  $i \in \{1, \dots, n\}$ , the corresponding map on homology induced by the chain map  $A_{[\gamma_i]}$  can be identified with  $\zeta_i \in \mathcal{A}_n$ .

Let  $\theta \in \widehat{CF}(-Y_n)$  be any cycle representing  $1 \in \widehat{HF}(-Y_n)$ . Since  $\zeta_1 \cdots \zeta_n \neq 0 \in \widehat{HF}(-Y_n)$ , we know that there exists at least one generator  $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  satisfying

$$\langle A_{[\gamma_1]} \cdots A_{[\gamma_n]} \cdot \theta, \mathbf{y} \rangle \equiv 1 \pmod{2}.$$

Associated to such a generator  $\mathbf{y}$  is an odd number of corresponding Maslov index  $n$  domains in  $\pi_2(\theta, \mathbf{y})$ , each of which can be realized as the sum of  $n$  of the Maslov index 1 domains contributing to the chain maps  $A_{[\gamma_1]}, \dots, A_{[\gamma_n]}$ . Consider the local multiplicity of such a Maslov index  $n$  domain,  $\psi$ , in the *four* regions adjacent to one of the constituent intersection points,  $x_i$ , of the distinguished cycle  $\mathbf{x} = (x_1, \dots, x_n)$  representing the contact class. We know (see Fig. 2) that the local multiplicity of  $\psi$  in the two regions adjacent to  $x_i$  that contain the basepoint,  $z_0$ , must be 0 and also that



**Fig. 2** The contact class on (half of) the Honda-Kazez-Matić Heegaard diagram associated to a fibered link  $L_2 \subset Y_3$ . The *right-hand picture* is a close-up of one of the constituent *intersection points* of the contact class and restrictions on the local multiplicities of the Maslov index  $n$  domain  $\psi$ . The NW, SE domains must have multiplicity 0 since they contain the basepoint  $z_0$ . One of the other two domains must have positive multiplicity, since it is the unique domain intersecting  $\gamma_i$

the local multiplicity in the region adjacent to the unique intersection point between  $\gamma_i$  and  $a_i$  must be nonzero (hence positive, since  $\psi$  is a sum of domains representing holomorphic disks). Since the fourth region must have non-negative multiplicity, we conclude that  $x_i$  must be a corner, of multiplicity at least one, in the boundary of  $\psi$ , implying that  $x_i$  must be a constituent intersection point of the generator  $\mathbf{y}$ .

Since the above argument holds for each of the  $x_i$ , we conclude that, in fact,  $\mathbf{y}$  is actually the distinguished contact class,  $\mathbf{x}$ , and it follows that (working mod 2)  $A_{[\gamma_1]} \cdots A_{[\gamma_n]} \cdot \theta = \mathbf{x}$ . Therefore,

$$[A_{[\gamma_1]} \cdots A_{[\gamma_n]} \cdot \theta] = [\mathbf{x}] = \zeta_1 \cdots \zeta_n \neq 0 \in \widehat{HF}(-Y_n),$$

so the Heegaard–Floer contact invariant associated to the contact structure supported by  $L_\ell$  is nonzero. By Lemma 3, the monodromy,  $h$ , of  $L_\ell$  is right-veering.

Now consider the *mirror* of  $L$ , i.e., the fibered link  $L \subset -Y_n$  with monodromy  $h^{-1}$ . By running the same argument above, we conclude that the contact invariant associated to the contact structure supported by the mirror of  $L$  is also nonzero. Hence,  $h^{-1}$  is right-veering, implying that  $h$  is left-veering.

But if  $h$  is both right- and left-veering, then  $h$  is isotopic to the identity mapping class, and hence  $(Y_n, L_\ell)$  is diffeomorphic as a pair to  $(Y_n, \mathbf{L}_\ell)$ .

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