Chapter 2
Approximation by Max-Product Bernstein Operators

Section 2.1 of this chapter contains general results of approximation obtained by applying Theorem 1.1.2, Jackson-type estimates for some particular classes of functions and results of shape preserving.

In Section 2.2 we improve for strictly positive functions the estimates in approximation by max-product Bernstein operators.

Section 2.3 contains saturation results and Section 2.4 contains very strong localization results for these operators.

Section 2.5 studies the iterations and the fixed points of the max-product Bernstein operators, while Section 2.6 contains applications to approximation of fuzzy numbers and explicit estimates for the approximation in the L^1-norm.

In Section 2.7 we present the approximation and shape preserving properties for two kinds of bivariate max-product Berstein operators.

Section 2.8 gives applications to image processing including some by graphics illustrating them.

In Section 2.9 we show how new max-product type operators can be constructed in such a way that the positivity of the approximated function can be dropped and also we introduce and study new approximation operators called of the sum-max type.

2.1 Estimates for Positive Functions

In this section we study the approximation properties for the max-product operator $B_n^{(M)}$ introduced by formula (1.20) in Subsection 1.1.2, at the point (i).

Firstly, by using Theorem 1.1.2, we obtain the order of approximation $\mathcal{O}(\omega_1(f; 1/\sqrt{n}))$. Then, one proves by a counterexample that in a sense, for arbitrary f this order of approximation cannot be improved. However, for subclasses of functions f including, for example, that of concave functions, we find the

B. Bede et al., *Approximation by Max-Product Type Operators*,
DOI 10.1007/978-3-319-34189-7_2

Jackson-type order of approximation $\mathcal{O}(\omega_1(f;1/n))$, which for many functions f is essentially better than the order of approximation obtained by the linear Bernstein operators. Finally, some shape preserving properties are presented and comparisons between the max-product and the linear Bernstein operators are discussed.

Since it is easy to check that $B_n^{(M)}(f)(0) - f(0) = B_n^{(M)}(f)(1) - f(1) = 0$ for all n, notice that in the notations, proofs, and statements of the all approximation results, that is in the next Lemmas 2.1.1–2.1.3, Theorem 2.1.5, Lemmas 2.1.6–2.1.8, Corollaries 2.1.10, 2.1.11, in fact we always may suppose that $0 < x < 1$. For the proofs of the main results we need some notations and auxiliary results, as follows.

For each $k, j \in \{0, 1, 2, \ldots n\}$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$, let us denote

$$M_{k,n,j}(x) = \frac{p_{n,k}(x)\left|\frac{k}{n} - x\right|}{p_{n,j}(x)}, m_{k,n,j}(x) = \frac{p_{n,k}(x)}{p_{n,j}(x)}.$$

It is clear that if $k \geq j+1$ then

$$M_{k,n,j}(x) = \frac{p_{n,k}(x)(\frac{k}{n} - x)}{p_{n,j}(x)}$$

and if $k \leq j-1$ then

$$M_{k,n,j}(x) = \frac{p_{n,k}(x)(x - \frac{k}{n})}{p_{n,j}(x)}.$$

Also, for each $k, j \in \{0, 1, 2, \ldots n\}$, $k \geq j+2$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$ let us denote

$$\overline{M}_{k,n,j}(x) = \frac{p_{n,k}(x)(\frac{k}{n+1} - x)}{p_{n,j}(x)}$$

and for each $k, j \in \{0, 1, 2, \ldots n\}$, $k \leq j-2$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$ let us denote

$$\underline{M}_{k,n,j}(x) = \frac{p_{n,k}(x)(x - \frac{k}{n+1})}{p_{n,j}(x)}.$$

Lemma 2.1.1 (Bede–Coroianu–Gal [21]). *Let* $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$.

(i) For all $k, j \in \{0, 1, 2, \ldots n\}$, $k \geq j+2$ *we have*

$$\overline{M}_{k,n,j}(x) \leq M_{k,n,j}(x) \leq 3\overline{M}_{k,n,j}(x).$$

(ii) For all $k, j \in \{0, 1, 2, \ldots n\}$, $k \leq j-2$ *we have*

$$M_{k,n,j}(x) \leq \underline{M}_{k,n,j}(x) \leq 6M_{k,n,j}(x).$$

Proof. (i) The inequality $\overline{M}_{k,n,j}(x) \le M_{k,n,j}(x)$ is immediate.
On the other hand,

$$\frac{M_{k,n,j}(x)}{\overline{M}_{k,n,j}(x)} = \frac{\frac{k}{n} - x}{\frac{k}{n+1} - x} \le \frac{\frac{k}{n} - \frac{j}{n+1}}{\frac{k}{n+1} - \frac{j+1}{n+1}}$$
$$\le \frac{kn + k - nj}{n(k - j - 1)} = \frac{k - j}{k - j - 1} + \frac{k}{n(k - j - 1)} \le 3,$$

which proves (i).

(ii) The inequality $M_{k,n,j}(x) \le \underline{M}_{k,n,j}(x)$ is immediate.
On the other hand,

$$\frac{\underline{M}_{k,n,j}(x)}{M_{k,n,j}(x)} = \frac{x - \frac{k}{n+1}}{x - \frac{k}{n}} \le \frac{\frac{j+1}{n+1} - \frac{k}{n+1}}{\frac{j}{n+1} - \frac{k}{n}}$$
$$= \frac{(n+1)(j+1-k)}{nj - nk - k} \le \frac{(n+1)(j+1-k)}{nj - nk - n} = \frac{n+1}{n} \cdot \frac{j+1-k}{j-k-1} \le 2 \cdot \frac{j+1-k}{j-k-1}$$
$$= 2\left(1 + \frac{2}{j-k-1}\right) \le 6,$$

which proves (ii) and the lemma. □

Lemma 2.1.2 (Bede–Coroianu–Gal [21]). *For all* $k, j \in \{0, 1, 2, ..n\}$ *and* $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$ *we have*

$$m_{k,n,j}(x) \le 1.$$

Proof. We have two cases: 1) $k \ge j$ and 2) $k \le j$.

Case 1). Since clearly the function $h(x) = \frac{1-x}{x}$ is nonincreasing on $[j/(n+1), (j+1)/(n+1)]$, it follows

$$\frac{m_{k,n,j}(x)}{m_{k+1,n,j}(x)} = \frac{k+1}{n-k} \cdot \frac{1-x}{x} \ge \frac{k+1}{n-k} \cdot \frac{1 - \frac{j+1}{n+1}}{\frac{j+1}{n+1}} = \frac{k+1}{n-k} \cdot \frac{n-j}{j+1} \ge 1,$$

which implies $m_{j,n,j}(x) \ge m_{j+1,n,j}(x) \ge m_{j+2,n,j}(x) \ge \cdots \ge m_{n,n,j}(x)$.

Case 2). We get

$$\frac{m_{k,n,j}(x)}{m_{k-1,n,j}(x)} = \frac{n-k+1}{k} \cdot \frac{x}{1-x} \ge \frac{n-k+1}{k} \cdot \frac{\frac{j}{n+1}}{1 - \frac{j}{n+1}}$$
$$= \frac{n-k+1}{k} \cdot \frac{j}{n+1-j} \ge 1,$$

which immediately implies

$$m_{j,n,j}(x) \geq m_{j-1,n,j}(x) \geq m_{j-2,n,j}(x) \geq \cdots \geq m_{0,n,j}(x).$$

Since $m_{j,n,j}(x) = 1$, the conclusion of the lemma is immediate. □

Lemma 2.1.3 (Bede–Coroianu–Gal [21]). *Let* $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$.

(i) If $k \in \{j+2, j+3, \ldots, n-1\}$ *is such that* $k - \sqrt{k+1} \geq j$*, then* $\overline{M}_{k,n,j}(x) \geq \overline{M}_{k+1,n,j}(x)$.

(ii) If $k \in \{1, 2, \ldots j-2\}$ *is such that* $k + \sqrt{k} \leq j$*, then* $\underline{M}_{k,n,j}(x) \geq \underline{M}_{k-1,n,j}(x)$.

Proof. (i) We observe that

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)} = \frac{k+1}{n-k} \cdot \frac{1-x}{x} \cdot \frac{\frac{k}{n+1} - x}{\frac{k+1}{n+1} - x}.$$

Since the function $g(x) = \frac{1-x}{x} \frac{\frac{k}{n+1}-x}{\frac{k+1}{n+1}-x}$ clearly is nonincreasing on $(0, 1]$, it follows that $g(x) \geq g(\frac{j+1}{n+1}) = \frac{n-j}{j+1} \cdot \frac{k-j-1}{k-j}$ for all $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. Then, since the condition $k - \sqrt{k+1} \geq j$ implies $(k+1)(k-j-1) \geq (j+1)(k-j)$, we obtain

$$\frac{\overline{M}_{k,n,j}(x)}{\overline{M}_{k+1,n,j}(x)} \geq \frac{k+1}{n-k} \cdot \frac{n-j}{j+1} \cdot \frac{k-j-1}{k-j} \geq 1.$$

(ii) We observe that

$$\frac{\underline{M}_{k,n,j}(x)}{\underline{M}_{k-1,n,j}(x)} = \frac{n-k+1}{k} \cdot \frac{x}{1-x} \cdot \frac{x - \frac{k}{n+1}}{x - \frac{k-1}{n+1}}.$$

Since the function $h(x) = \frac{x}{1-x} \cdot \frac{x-\frac{k}{n+1}}{x-\frac{k-1}{n+1}}$ is nondecreasing on $[0, 1)$, it follows that $h(x) \geq h(\frac{j}{n+1}) = \frac{j}{n+1-j} \cdot \frac{j-k}{j-k+1}$ for all $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. Then, since the condition $k + \sqrt{k} \leq j$ implies $j(j-k) \geq k(j-k+1)$, we obtain

$$\frac{\underline{M}_{k,n,j}(x)}{\underline{M}_{k-1,n,j}(x)} \geq \frac{n-k+1}{k} \cdot \frac{j}{n+1-j} \cdot \frac{j-k}{j-k+1} \geq 1,$$

which proves the lemma. □

Also, a key result in the proof of the first main result is the following.

Lemma 2.1.4 (Bede–Coroianu–Gal [21]). *We have*

$$\bigvee_{k=0}^{n} p_{n,k}(x) = p_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right], j = 0, 1, \ldots, n,$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Proof. First we show that for fixed $n \in \mathbb{N}$ and $0 \le k < k+1 \le n$ we have

$$0 \le p_{n,k+1}(x) \le p_{n,k}(x), \text{ if and only if } x \in [0, (k+1)/(n+1)].$$

Indeed, the inequality one reduces to

$$0 \le \binom{n}{k+1} x^{k+1} (1-x)^{n-(k+1)} \le \binom{n}{k} x^k (1-x)^{n-k},$$

which after simplifications is equivalent to

$$0 \le x \left[\binom{n}{k+1} + \binom{n}{k}\right] \le \binom{n}{k}.$$

But since $\binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1}$, the above inequality immediately becomes equivalent to

$$0 \le x \le \frac{k+1}{n+1}.$$

By taking $k = 0, 1, ..,$ in the inequality just proved above, we get

$$p_{n,1}(x) \le p_{n,0}(x), \text{ if and only if } x \in [0, 1/(n+1)],$$
$$p_{n,2}(x) \le p_{n,1}(x), \text{ if and only if } x \in [0, 2/(n+1)],$$
$$p_{n,3}(x) \le p_{n,2}(x), \text{ if and only if } x \in [0, 3/(n+1)],$$

so on,

$$p_{n,k+1}(x) \le p_{n,k}(x), \text{ if and only if } x \in [0, (k+1)/(n+1)],$$

so on,

$$p_{n,n-2}(x) \le p_{n,n-3}(x), \text{ if and only if } x \in [0, (n-2)/(n+1)],$$
$$p_{n,n-1}(x) \le p_{n,n-2}(x), \text{ if and only if } x \in [0, (n-1)/(n+1)],$$
$$p_{n,n}(x) \le p_{n,n-1}(x), \text{ if and only if } x \in [0, n/(n+1)].$$

From all these inequalities, reasoning by recurrence we easily obtain:

$$\text{if } x \in [0, 1/(n+1)], \text{ then } p_{n,k}(x) \le p_{n,0}(x), \text{ for all } k = 0, 1, \ldots, n,$$
$$\text{if } x \in [1/(n+1), 2/(n+1)], \text{ then } p_{n,k}(x) \le p_{n,1}(x), \text{ for all } k = 0, 1, \ldots, n,$$
$$\text{if } x \in [2/(n+1), 3/(n+1)], \text{ then } p_{n,k}(x) \le p_{n,2}(x), \text{ for all } k = 0, 1, \ldots, n,$$

and so on finally

$$\text{if } x \in [n/(n+1), 1], \text{ then } p_{n,k}(x) \le p_{n,n}(x), \text{ for all } k = 0, 1, \ldots, n,$$

which proves the lemma. □

The first main result of this section is the following.

Theorem 2.1.5 (Bede–Coroianu–Gal [21]). *If $f : [0,1] \to \mathbb{R}_+$ is continuous, then we have the estimate*

$$|B_n^{(M)}(f)(x) - f(x)| \le 12\omega_1\left(f; \frac{1}{\sqrt{n+1}}\right), \textit{for all } n \in \mathbb{N}, x \in [0,1],$$

where

$$\omega_1(f;\delta) = \sup\{|f(x) - f(y)|; x, y \in [0,1], |x - y| \le \delta\}.$$

Proof. It is easy to check that the max-product Bernstein operators fulfil the conditions in Theorem 1.1.2 and we have

$$|B_n^{(M)}(f)(x) - f(x)| \le \left(1 + \frac{1}{\delta_n} B_n^{(M)}(\varphi_x)(x)\right)\omega_1(f;\delta_n), \tag{2.1}$$

where $\varphi_x(t) = |t - x|$. So, it is enough to estimate

$$E_n(x) := B_n^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x)\left|\frac{k}{n} - x\right|}{\bigvee_{k=0}^{n} p_{n,k}(x)}.$$

Let $x \in [j/(n+1), (j+1)/(n+1)]$, where $j \in \{0, \ldots, n\}$ is fixed, arbitrary. By Lemma 2.1.4 we easily obtain

$$E_n(x) = \max_{k=0,\ldots,n}\{M_{k,n,j}(x)\}, x \in [j/(n+1), (j+1)/(n+1)].$$

In all what follows we may suppose that $j \in \{1, \ldots, n\}$, because for $j = 0$ simple calculation shows that in this case we get $E_n(x) \le \frac{1}{n}$, for all $x \in [0, 1/(n+1)]$. So it remains to obtain an upper estimate for each $M_{k,n,j}(x)$ when $j = 1,\dots,n$ is fixed, $x \in [j/(n+1),(j+1)/(n+1)]$ and $k = 0,\dots,n$. In fact we will prove that

$$M_{k,n,j}(x) \le \frac{6}{\sqrt{n+1}}, \text{ for all } x \in [j/(n+1),(j+1)/(n+1)], k = 0,\dots,n, \tag{2.2}$$

which immediately implies that

$$E_n(x) \le \frac{6}{\sqrt{n+1}}, \text{ for all } x \in [0,1], n \in \mathbb{N},$$

and taking $\delta_n = \frac{6}{\sqrt{n+1}}$ in (2.1) we immediately obtain the estimate in the statement.

In order to prove (2.2), we distinguish the following cases:

1) $k \in \{j-1, j, j+1\}$; 2) $k \ge j+2$ and 3) $k \le j-2$.

Case 1). If $k = j$, then $M_{j,n,j}(x) = \left|\frac{j}{n} - x\right|$. Since $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$, it easily follows that $M_{j,n,j}(x) \le \frac{1}{n+1}$.

If $k = j+1$, then $M_{j+1,n,j}(x) = m_{j+1,n,j}(x)(\frac{j+1}{n} - x)$. Since by Lemma 2.1.2 we have $m_{j+1,n,j}(x) \le 1$, we obtain $M_{j+1,n,j}(x) \le \frac{j+1}{n} - x \le \frac{j+1}{n} - \frac{j}{n+1} = \frac{n+j+1}{n(n+1)} \le \frac{3}{n+1}$.

If $k = j-1$, then $M_{j-1,n,j}(x) = m_{j-1,n,j}(x)(x - \frac{j-1}{n}) \le \frac{j+1}{n+1} - \frac{j-1}{n} = \frac{2n-j+1}{n(n+1)} \le \frac{2}{n+1}$.

Case 2). Subcase a). Suppose first that $k - \sqrt{k+1} < j$. We get

$$\overline{M}_{k,n,j}(x) = m_{k,n,j}(x)(\frac{k}{n+1} - x) \le \frac{k}{n+1} - x \le \frac{k}{n+1} - \frac{j}{n+1} \le$$
$$\frac{k}{n+1} - \frac{k - \sqrt{k+1}}{n+1} = \frac{\sqrt{k+1}}{n+1} \le \frac{1}{\sqrt{n+1}}.$$

Subcase b). Suppose now that $k - \sqrt{k+1} \ge j$. Since the function $g(x) = x - \sqrt{x+1}$ is nondecreasing on the interval $[0,\infty)$ it follows that there exists $\overline{k} \in \{0,1,2,\dots n\}$, of maximum value, such that $\overline{k} - \sqrt{\overline{k}+1} < j$. Then for $k_1 = \overline{k} + 1$ we get $k_1 - \sqrt{k_1+1} \ge j$ and

$$\overline{M}_{\overline{k}+1,n,j}(x) = m_{\overline{k}+1,n,j}(x)(\frac{\overline{k}+1}{n+1} - x) \le \frac{\overline{k}+1}{n+1} - x$$
$$\le \frac{\overline{k}+1}{n+1} - \frac{j}{n+1} \le \frac{\overline{k}+1}{n+1} - \frac{\overline{k} - \sqrt{\overline{k}+1}}{n+1}$$
$$= \frac{\sqrt{\overline{k}+1} + 1}{n+1} \le \frac{2}{\sqrt{n+1}}.$$

Also, we have $k_1 \geq j+2$. Indeed, this is a consequence of the fact that g is nondecreasing on the interval $[0,\infty)$ and because it is easy to see that $g(j+1) < j$. By Lemma 2.1.3, (i) it follows that $\overline{M}_{\overline{k}+1,n,j}(x) \geq \overline{M}_{\overline{k}+2,n,j}(x) \geq \cdots \geq \overline{M}_{n,n,j}(x)$. We thus obtain $\overline{M}_{k,n,j}(x) \leq \frac{2}{\sqrt{n+1}}$ for any $k \in \{\overline{k}+1, \overline{k}+2, \ldots n\}$.

Therefore, in both subcases, by Lemma 2.1.1, (i) too, we get $M_{k,n,j}(x) \leq \frac{6}{\sqrt{n+1}}$.

Case 3). Subcase a). Suppose first that $k+\sqrt{k} \geq j$. Then we obtain

$$\underline{M}_{k,n,j}(x) = m_{k,n,j}(x)(x - \frac{k}{n+1}) \leq \frac{j+1}{n+1} - \frac{k}{n+1}$$
$$\leq \frac{k+\sqrt{k}+1}{n+1} - \frac{k}{n+1} = \frac{\sqrt{k}+1}{n+1} \leq \frac{\sqrt{n}+1}{n+1} \leq \frac{2}{\sqrt{n+1}}.$$

Subcase b). Suppose now that $k+\sqrt{k} < j$. Let $\tilde{k} \in \{0,1,2,\ldots n\}$ be the minimum value such that $\tilde{k}+\sqrt{\tilde{k}} \geq j$. Then $k_2 = \tilde{k}-1$ satisfies $k_2+\sqrt{k_2} < j$ and

$$\underline{M}_{\tilde{k}-1,n,j}(x) = m_{\tilde{k}-1,n,j}(x)(x - \frac{\tilde{k}-1}{n+1}) \leq \frac{j+1}{n+1} - \frac{\tilde{k}-1}{n+1}$$
$$\leq \frac{\tilde{k}+\sqrt{\tilde{k}}+1}{n+1} - \frac{\tilde{k}-1}{n+1} = \frac{\sqrt{\tilde{k}}+2}{n+1} \leq \frac{3}{\sqrt{n+1}}.$$

Also, because in this case we have $j \geq 2$ it is immediate that $k_2 \leq j-2$. By Lemma 2.1.3, (ii) it follows that $\underline{M}_{\tilde{k}-1,n,j}(x) \geq \underline{M}_{\tilde{k}-2,n,j}(x) \geq \cdots \geq \underline{M}_{0,n,j}(x)$. We obtain $\underline{M}_{k,n,j}(x) \leq \frac{3}{\sqrt{n+1}}$ for any $k \leq j-2$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$.

In both subcases, by Lemma 2.1.1, (ii) too, we get $M_{k,n,j}(x) \leq \frac{3}{\sqrt{n+1}}$.

In conclusion, collecting all the estimates in the above cases and subcases we easily get the relationship (2.2), which completes the proof. □

Remarks. 1) The order of approximation in terms of $\omega_1(f;\sqrt{n})$ in Theorem 2.1.5 cannot be improved, in the sense that the order of $\max_{x\in[0,1]}\{E_n(x)\}$ is exactly $\frac{1}{\sqrt{n}}$ (here $E_n(x)$ is defined in the proof of Theorem 2.1.5). Indeed, for $n \in \mathbb{N}$ let us take $j_n = [\frac{n}{2}]$, $k_n = j_n + [\sqrt{n}]$, $x_n = \frac{j_n+1}{n+1}$ and denote $\tilde{n} = n - [\frac{n}{2}]$. Then we can write

$$\overline{M}_{k_n,n,j_n}(x_n) = \frac{\binom{n}{k_n} x_n^{k_n}(1-x_n)^{n-k_n}}{\binom{n}{j_n} x_n^{j_n}(1-x_n)^{n-j_n}}(\frac{k_n}{n+1} - x_n)$$
$$= \frac{(\tilde{n}-[\sqrt{n}]+1)(\tilde{n}-[\sqrt{n}]+2)\ldots\tilde{n}}{([\frac{n}{2}]+1)([\frac{n}{2}]+2)\ldots([\frac{n}{2}]+[\sqrt{n}])}\left(\frac{[\frac{n}{2}]+1}{\tilde{n}}\right)^{[\sqrt{n}]} \cdot \frac{[\sqrt{n}]-1}{n+1}.$$

Since $2[\frac{n}{2}] \geq n-1$, we easily get $[\frac{n}{2}]+1 \geq \tilde{n}$, which implies $\left(\frac{[\frac{n}{2}]+1}{\tilde{n}}\right)^{[\sqrt{n}]} \geq 1$ for all $n \in \mathbb{N}$. On the other hand,

$$\frac{(\tilde{n}-[\sqrt{n}]+1)(\tilde{n}-[\sqrt{n}]+2)\ldots\tilde{n}}{([\frac{n}{2}]+1)([\frac{n}{2}]+2)\ldots([\frac{n}{2}]+[\sqrt{n}])} \geq \left(\frac{\tilde{n}-[\sqrt{n}]+1}{[\frac{n}{2}]+[\sqrt{n}]}\right)^{[\sqrt{n}]} \geq \left(\frac{\frac{n}{2}-\sqrt{n}+1}{\frac{n}{2}+\sqrt{n}}\right)^{\sqrt{n}}.$$

Because $\lim_{n\to\infty}\left(\frac{\frac{n}{2}-\sqrt{n}+1}{\frac{n}{2}+\sqrt{n}}\right)^{\sqrt{n}} = e^{-4}$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{(\tilde{n}-[\sqrt{n}]+1)(\tilde{n}-[\sqrt{n}]+2)\ldots\tilde{n}}{([\frac{n}{2}]+1)([\frac{n}{2}]+2)\ldots([\frac{n}{2}]+[\sqrt{n}])} \geq e^{-5},$$

for all $n \geq n_0$. It follows

$$\overline{M}_{k_n,n,j_n}(x_n) \geq \frac{e^{-5}([\sqrt{n}]-1)}{n+1} \geq \frac{e^{-5}}{6\sqrt{n}},$$

for all $n \geq \max\{n_0, 4\}$. Taking into account Lemma 2.1.1, (i) too, it follows that for all $n \geq \max\{n_0, 4\}$ we have $M_{k_n,n,j_n}(x_n) \geq \frac{e^{-5}}{6\sqrt{n}}$, which implies the desired conclusion.

2) With respect to the method of proof in Bede–Gal [30], the method in this section presents, at least, two advantages: it produces the explicit constant 12 in front of $\omega_1(f; 1/\sqrt{n+1})$ and its ideas can be easily used for other max-product Bernstein type operators too, a fact which will be seen in the next chapters.

In what follows we will prove that for large subclasses of functions f, the order of approximation $\omega_1(f; 1/\sqrt{n+1})$ in Theorem 2.1.5 can essentially be improved to $\omega_1(f; 1/n)$.

For this purpose, for any $k, j \in \{0, 1, ..n\}$, let us define the functions $f_{k,n,j}$: $[\frac{j}{n+1}, \frac{j+1}{n+1}] \to \mathbb{R}$,

$$f_{k,n,j}(x) = m_{k,n,j}(x) f\left(\frac{k}{n}\right) = \frac{\binom{n}{k}}{\binom{n}{j}}\left(\frac{x}{1-x}\right)^{k-j} f\left(\frac{k}{n}\right).$$

Then it is clear that for any $j \in \{0, 1, \ldots, n\}$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$ we can write

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x).$$

Also we need the following four auxiliary lemmas.

Lemma 2.1.6 (Bede–Coroianu–Gal [21]). *Let* $f : [0, 1] \to [0, \infty)$ *be such that*

$$B_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\} \text{ for all } x \in [j/(n+1), (j+1)/(n+1)].$$

Then

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le 2\omega_1\left(f; \frac{1}{n}\right), \text{ for all } x \in [j/(n+1), (j+1)/(n+1)].$$

Proof. We distinguish two cases:

Case (i). Let $x \in [j/(n+1), (j+1)/(n+1)]$ be fixed such that $B_n^{(M)}(f)(x) = f_{j,n,j}(x)$. Because by simple calculation we have $\frac{-1}{n+1} \le x - \frac{j}{n} \le \frac{1}{n+1}$ and $f_{j,n,j}(x) = f(\frac{j}{n})$, it follows that

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le \omega_1\left(f; \frac{1}{n+1}\right).$$

Case (ii). Let $x \in [j/(n+1), (j+1)/(n+1)]$ be such that $B_n^{(M)}(f)(x) = f_{j+1,n,j}(x)$. We have two subcases:

(ii_a) $B_n^{(M)}(f)(x) \le f(x)$, when evidently $f_{j,n,j}(x) \le f_{j+1,n,j}(x) \le f(x)$ and we immediately get

$$\begin{aligned}\left|B_n^{(M)}(f)(x) - f(x)\right| &= \left|f_{j+1,n,j}(x) - f(x)\right| \\ &= f(x) - f_{j+1,n,j}(x) \le f(x) - f(j/n) \le \omega_1\left(f; \frac{1}{n+1}\right).\end{aligned}$$

(ii_b) $B_n^{(M)}(f)(x) > f(x)$, when

$$\begin{aligned}\left|B_n^{(M)}(f)(x) - f(x)\right| &= f_{j+1,n,j}(x) - f(x) = m_{j+1,n,j}(x) f\left(\frac{j+1}{n}\right) - f(x) \\ &\le f\left(\frac{j+1}{n}\right) - f(x).\end{aligned}$$

Because $0 \le \frac{j+1}{n} - x \le \frac{j+1}{n} - \frac{j}{n+1} = \frac{j}{n(n+1)} + \frac{1}{n} < \frac{2}{n}$ it follows $f(\frac{j+1}{n}) - f(x) \le 2\omega_1\left(f; \frac{1}{n}\right)$, which proves the lemma. □

Lemma 2.1.7 (Bede–Coroianu–Gal [21]). *Let* $f : [0, 1] \to [0, \infty)$ *be such that*

$$B_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j-1,n,j}(x)\} \text{ for all } x \in [j/(n+1), (j+1)/(n+1)].$$

Then

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le 2\omega_1\left(f; \frac{1}{n}\right), \text{ for all } x \in [j/(n+1), (j+1)/(n+1)].$$

Proof. We distinguish two cases:

Case (i). $B_n^{(M)}(f)(x) = f_{j,n,j}(x)$, when as in Lemma 2.1.6 we get

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1\left(f; \frac{1}{n+1}\right).$$

Case (ii). $B_n^{(M)}(f)(x) = f_{j-1,n,j}(x)$, when we have two subcases:

(ii_a) $B_n^{(M)}(f)(x) \leq f(x)$, when as in the case of Lemma 2.1.6 we obtain

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1\left(f; \frac{1}{n+1}\right).$$

(ii_b) $B_n^{(M)}(f)(x) > f(x)$, when by using the same idea as in the subcase (ii_b) of Lemma 2.1.6 and taking into account that

$$0 \leq x - \frac{j-1}{n} \leq \frac{j+1}{n+1} - \frac{j-1}{n} = \frac{-j}{n(n+1)} + \frac{1}{n+1} + \frac{1}{n} < \frac{2}{n},$$

we obtain

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right),$$

which proves the lemma. □

Lemma 2.1.8 (Bede–Coroianu–Gal [21]). *Let* $f : [0,1] \to [0,\infty)$ *be such that*

$$B_n^{(M)}(f)(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x), f_{j+1,n,j}(x)\},$$

for all $x \in [j/(n+1),(j+1)/(n+1)]$. *Then*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right), \textit{ for all } x \in [j/(n+1),(j+1)/(n+1)].$$

Proof. Let $x \in [j/(n+1),(j+1)/(n+1)]$. If $B_n^{(M)}(f)(x) = f_{j,n,j}(x)$ or $B_n^{(M)}(f)(x) = f_{j+1,n,j}(x)$, then $B_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}$ and from Lemma 2.1.6 it follows

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right).$$

If $B_n^{(M)}(f)(x) = f_{j-1,n,j}(x)$, then $B_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j-1,n,j}(x)\}$ and from Lemma 2.1.7 we get

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right),$$

which ends the proof. □

Lemma 2.1.9 (see, e.g., Lorentz [114], p. 44, Bede–Coroianu–Gal [21]). *Let* $f : [0,1] \to [0,\infty)$ *be concave. Then the following two properties hold:*

(i) The function $g : (0,1] \to [0,\infty), g(x) = \frac{f(x)}{x}$ *is nonincreasing ;*
(ii) The function $h : [0,1) \to [0,\infty), h(x) = \frac{f(x)}{1-x}$ *is nondecreasing.*

Proof. (i) Let $x, y \in (0,1]$ be with $x \leq y$. Then

$$f(x) = f\left(\frac{x}{y}y + \frac{y-x}{y}0\right) \geq \frac{x}{y}f(y) + \frac{y-x}{y}f(0) \geq \frac{x}{y}f(y),$$

which implies $\frac{f(x)}{x} \geq \frac{f(y)}{y}$.
(ii) Let $x, y \in [0,1)$ be with $x \geq y$. Then

$$f(x) = f\left(\frac{1-x}{1-y}y + \frac{x-y}{1-y}1\right) \geq \frac{1-x}{1-y}f(y) + \frac{x-y}{1-y}f(1) \geq \frac{1-x}{1-y}f(y),$$

which implies $\frac{f(x)}{1-x} \geq \frac{f(y)}{1-y}$. □

Corollary 2.1.10 (Bede–Coroianu–Gal [21]). *Let* $f : [0,1] \to [0,\infty)$ *be a concave function. Then*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right), \textit{ for all } x \in [0,1].$$

Proof. Let $x \in [0,1]$ and $j \in \{0,1,\ldots n\}$ such that $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. Let $k \in \{0,1,\ldots n\}$ be with $k \geq j$. Then

$$\begin{aligned} f_{k+1,n,j}(x) &= \frac{\binom{n}{k+1}}{\binom{n}{j}}\left(\frac{x}{1-x}\right)^{k+1-j} f(\frac{k+1}{n}) \\ &= \frac{\binom{n}{k}}{\binom{n}{j}}\frac{n-k}{k+1}\left(\frac{x}{1-x}\right)^{k-j}\frac{x}{1-x}f(\frac{k+1}{n}). \end{aligned}$$

From Lemma 2.1.9, (i), we get $\frac{f(\frac{k+1}{n})}{\frac{k+1}{n}} \leq \frac{f(\frac{k}{n})}{\frac{k}{n}}$, that is $f(\frac{k+1}{n}) \leq \frac{k+1}{k}f(\frac{k}{n})$. Since $\frac{x}{1-x} \leq \frac{j+1}{n-j}$, we get

$$\begin{aligned} f_{k+1,n,j}(x) &\leq \frac{\binom{n}{k}}{\binom{n}{j}}\frac{n-k}{k+1}\left(\frac{x}{1-x}\right)^{k-j}\frac{j+1}{n-j}\cdot\frac{k+1}{k}f(\frac{k}{n}) \\ &= f_{k,n,j}(x)\frac{j+1}{k}\cdot\frac{n-k}{n-j}. \end{aligned}$$

It is immediate that for $k \geq j+1$ it follows $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$. Thus we obtain

$$f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq \cdots \geq f_{n,j,n}(x). \tag{2.3}$$

Now let $k \in \{0, 1, \ldots n\}$ be with $k \leq j$. Then

$$f_{k-1,n,j}(x) = \frac{\binom{n}{k-1}}{\binom{n}{j}} \left(\frac{x}{1-x}\right)^{k-1-j} f(\frac{k-1}{n})$$
$$= \frac{\binom{n}{k}}{\binom{n}{j}} \cdot \frac{k}{n-k+1} \left(\frac{x}{1-x}\right)^{k-j} \frac{1-x}{x} f(\frac{k-1}{n}).$$

From Lemma 2.1.9, (ii), we get $\frac{f(\frac{k}{n})}{1-\frac{k}{n}} \geq \frac{f(\frac{k-1}{n})}{1-\frac{k-1}{n}}$, that is $f(\frac{k}{n}) \geq \frac{n-k}{n-k+1} f(\frac{k-1}{n})$. Because $\frac{1-x}{x} \leq \frac{n+1-j}{j}$, we get

$$f_{k-1,n,j}(x) \leq \frac{\binom{n}{k}}{\binom{n}{j}} \frac{k}{n-k+1} \left(\frac{x}{1-x}\right)^{k-j} \frac{n+1-j}{j} \cdot \frac{n-k+1}{n-k} f(\frac{k}{n})$$
$$= f_{k,n,j}(x) \frac{k}{j} \cdot \frac{n+1-j}{n-k}.$$

For $k \leq j-1$ it is immediate that $f_{k,n,j}(x) \geq f_{k-1,n,j}(x)$, which implies

$$f_{j-1,n,j}(x) \geq f_{j-2,n,j}(x) \geq \cdots \geq f_{0,n,j}(x). \tag{2.4}$$

From (2.3) and (2.4) we obtain

$$B_n^{(M)}(f)(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x), f_{j+1,n,j}(x)\},$$

which combined with Lemma 2.1.8 implies

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1 \left(f; \frac{1}{n}\right)$$

and proves the corollary. □

Corollary 2.1.11 (Bede–Coroianu–Gal [21]).

(i) If $f : [0, 1] \to [0, \infty)$ is nondecreasing and such that the function $g : (0, 1] \to [0, \infty)$, $g(x) = \frac{f(x)}{x}$ is nonincreasing, then

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1 \left(f; \frac{1}{n}\right), \textit{ for all } x \in [0, 1].$$

(ii) If $f : [0, 1] \to [0, \infty)$ is nonincreasing and such that the function $h : [0, 1) \to [0, \infty)$, $h(x) = \frac{f(x)}{1-x}$ is nondecreasing, then

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1 \left(f; \frac{1}{n}\right), \textit{ for all } x \in [0, 1].$$

Proof. (i) Since f is nondecreasing it follows (see the proof of the next Theorem 2.1.15)

$$B_n^{(M)}(f)(x) = \bigvee_{k\geq j}^{n} f_{k,n,j}(x), \text{ for all } x \in [j/(n+1), (j+1)/(n+1)].$$

Following the proof of Corollary 2.1.10, we get

$$B_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}, \text{ for all } x \in [j/(n+1), (j+1)/(n+1)],$$

and from Lemma 2.1.6 we obtain

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right).$$

(ii) Since f is nonincreasing it follows (see the proof of the next Corollary 2.1.16)

$$B_n^{(M)}(f)(x) = \bigvee_{k\geq 0}^{j} f_{k,n,j}(x), \text{ for all } x \in [j/(n+1), (j+1)/(n+1)].$$

Following the proof of Corollary 2.1.10 we get

$$B_n^{(M)}(f)(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x)\},$$

and from Lemma 2.1.7, we obtain

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right).$$

□

Remark. By simple reasonings, it follows that if $f : [0,1] \to [0,\infty)$ is a convex, nondecreasing function satisfying $\frac{f(x)}{x} \geq f(1)$ for all $x \in [0,1]$, then the function $g : (0,1] \to [0,\infty), g(x) = \frac{f(x)}{x}$ is nonincreasing and as a consequence for f is valid the conclusion of Corollary 2.1.11, (i). Indeed, for simplicity let us suppose that $f \in C^1[0,1]$ and denote $F(x) = xf'(x) - f(x)$, $x \in [0,1]$. Then $g'(x) = \frac{F(x)}{x^2}$, for all $x \in (0,1]$. Since the inequality $\frac{f(x)}{x} \geq f(1)$ can be written as $\frac{f(1)-f(x)}{1-x} \leq f(1)$, for all $x \in [0,1)$, passing to limit with $x \to 1$ it follows $f'(1) \leq f(1)$, which implies (since f' is nondecreasing)

$$F(x) \leq xf'(1) - f(x) \leq xf'(1) - xf(1) = x[f'(1) - f(1)] \leq 0, \text{ for all } x \in (0,1],$$

This means that $g(x)$ is nonincreasing.

An example of function satisfying the above conditions is $f(x) = e^x$, $x \in [0,1]$.

Analogously, if $f : [0,1] \to [0,\infty)$ is a convex, nonincreasing function satisfying $\frac{f(x)}{1-x} \geq f(0)$, then for f is valid the conclusion of Corollary 2.1.11, (ii). An example of function satisfying these conditions is $f(x) = e^{-x}$, $x \in [0,1]$.

In what follows we will present some shape preserving properties, by proving that the max-product Bernstein operator preserves the monotonicity and the quasi-convexity. First we have the following simple result.

Lemma 2.1.12 (Bede–Coroianu–Gal [21]). *For any arbitrary function* $f : [0,1] \to \mathbb{R}_+$, $B_n^{(M)}(f)(x)$ *is positive, continuous on* $[0,1]$ *and satisfies* $B_n^{(M)}(f)(0) = f(0)$, $B_n^{(M)}(f)(1) = f(1)$.

Proof. Since $p_{n,k}(x) > 0$ for all $x \in (0,1)$, $n \in \mathbb{N}$, $k \in \{0,\dots,n\}$, it follows that the denominator $\bigvee_{k=0}^{n} p_{n,k}(x) > 0$ for all $x \in (0,1)$ and $n \in \mathbb{N}$. But the numerator is a maximum of continuous functions on $[0,1]$, so it is a continuous function on $[0,1]$ and this implies that $B_n^{(M)}(f)(x)$ is continuous on $(0,1)$. To prove now the continuity of $B_n^{(M)}(f)(x)$ at $x = 0$ and $x = 1$, we observe that $p_{n,k}(0) = 0$ for all $k \in \{1,2,\dots,n\}$, $p_{n,k}(0) = 1$ for $k = 0$ and $p_{n,k}(1) = 0$ for all $k \in \{0,1,\dots,n-1\}$, $p_{n,k}(1) = 1$ for $k = n$, which implies that $\bigvee_{k=0}^{n} p_{n,k}(x) = 1$ in the case of $x = 0$ and $x = 1$. The fact that $B_n^{(M)}(f)(x)$ coincides with $f(x)$ at $x = 0$ and $x = 1$ immediately follows from the above considerations, which proves the theorem. □

Remark. Note that because of the continuity of $B_n^{(M)}(f)(x)$ on $[0,1]$, it will suffice to prove the shape properties of $B_n^{(M)}(f)(x)$ on $(0,1)$ only. As a consequence, in the notations and proofs below we always may suppose that $0 < x < 1$.

As before, for any $k,j \in \{0,1,..n\}$, let us consider the functions $f_{k,n,j} : [\frac{j}{n+1}, \frac{j+1}{n+1}] \to \mathbb{R}$,

$$f_{k,n,j}(x) = m_{k,n,j}(x) f\left(\frac{k}{n}\right) = \frac{\binom{n}{k}}{\binom{n}{j}} \left(\frac{x}{1-x}\right)^{k-j} f\left(\frac{k}{n}\right).$$

For any $j \in \{0,1,\dots,n\}$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$ we can write

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x).$$

Lemma 2.1.13 (Bede–Coroianu–Gal [21]). *If* $f : [0,1] \to \mathbb{R}_+$ *is a nondecreasing function, then for any* $k,j \in \{0,1,\dots n\}, k \leq j$ *and* $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$ *we have* $f_{k,n,j}(x) \geq f_{k-1,n,j}(x)$.

Proof. Because $k \leq j$, by the proof of Lemma 2.1.2, case 2), it follows that $m_{k,n,j}(x) \geq m_{k-1,n,j}(x)$. From the monotonicity of f we get $f\left(\frac{k}{n}\right) \geq f\left(\frac{k-1}{n}\right)$. Thus we obtain

$$m_{k,n,j}(x)f\left(\frac{k}{n}\right) \geq m_{k-1,n,j}(x)f\left(\frac{k-1}{n}\right),$$

which proves the lemma. □

Corollary 2.1.14 (Bede–Coroianu–Gal [21]). *If $f : [0,1] \to \mathbb{R}_+$ is nonincreasing, then $f_{k,n,j}(x) \geq f_{k+1,n,j}(x)$ for any $k, j \in \{0, 1, \ldots n\}$, $k \geq j$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$.*

Proof. Because $k \geq j$, by the proof of Lemma 2.1.2, case 1), it follows that $m_{k,n,j}(x) \geq m_{k+1,n,j}(x)$. From the monotonicity of f we get $f\left(\frac{k}{n}\right) \geq f\left(\frac{k+1}{n}\right)$. Thus we obtain

$$m_{k,n,j}(x)f\left(\frac{k}{n}\right) \geq m_{k+1,n,j}(x)f\left(\frac{k+1}{n}\right),$$

which proves the corollary. □

Theorem 2.1.15 (Bede–Coroianu–Gal [21]). *If $f : [0,1] \to \mathbb{R}_+$ is nondecreasing, then $B_n^{(M)}(f)$ is nondecreasing.*

Proof. Because $B_n^{(M)}(f)$ is continuous on $[0,1]$, it suffices to prove that on each subinterval of the form $[\frac{j}{n+1}, \frac{j+1}{n+1}]$, with $j \in \{0, 1, \ldots n\}$, $B_n^{(M)}(f)$ is nondecreasing.

So let $j \in \{0, 1, \ldots n\}$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. Because f is nondecreasing, from Lemma 2.1.13 it follows that

$$f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq f_{j-2,n,j}(x) \geq \cdots \geq f_{0,n,j}(x).$$

But then it is immediate that

$$B_n^{(M)}(f)(x) = \bigvee_{k \geq j}^{n} f_{k,n,j}(x),$$

for all $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. Clearly that for $k \geq j$ the function $f_{k,n,j}$ is nondecreasing and since $B_n^{(M)}(f)$ is defined as the maximum of nondecreasing functions, it follows that it is nondecreasing. □

Corollary 2.1.16 (Bede–Coroianu–Gal [21]). *If $f : [0,1] \to \mathbb{R}_+$ is nonincreasing, then $B_n^{(M)}(f)$ is nonincreasing.*

Proof. Because $B_n^{(M)}(f)$ is continuous on $[0,1]$, it suffices to prove that on each subinterval of the form $[\frac{j}{n+1}, \frac{j+1}{n+1}]$, with $j \in \{0, 1, \ldots n\}$, $B_n^{(M)}(f)$ is nonincreasing.

So let $j \in \{0, 1, \ldots n\}$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. Because f is nonincreasing, from Corollary 2.1.14 it follows that

$$f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq f_{j+2,n,j}(x) \geq \cdots \geq f_{n,n,j}(x).$$

But then it is immediate that

$$B_n^{(M)}(f)(x) = \bigvee_{k \geq 0}^{j} f_{k,n,j}(x),$$

for all $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. Clearly that for $k \leq j$ the function $f_{k,n,j}$ is nonincreasing and since $B_n^{(M)}(f)$ is defined as the maximum of nonincreasing functions, it follows that it is nonincreasing. $\square$

In what follows, let us consider the following concept generalizing the monotonicity and convexity.

Definition 2.1.17. Let $f : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$. One says that the function $f : [0, 1] \to \mathbb{R}$ is quasiconvex on $[0, 1]$ if it satisfies the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}, \text{ for all } x, y, \lambda \in [0, 1].$$

(see, e.g., the book Gal [84], p. 4, (iv)).

Remark. By Popoviciu [128], the continuous function f is quasiconvex on $[0, 1]$ equivalently means that there exists a point $c \in [0, 1]$ such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, 1]$. The class of quasiconvex functions includes the class of nondecreasing functions and the class of nonincreasing functions. Also, it obviously includes the class of convex functions on $[0, 1]$.

Corollary 2.1.18 (Bede–Coroianu–Gal [21]). *If $f : [0, 1] \to \mathbb{R}_+$ is continuous and quasiconvex on $[0, 1]$ then for all $n \in \mathbb{N}$, $B_n^{(M)}(f)$ is quasiconvex on $[0, 1]$.*

Proof. If f is nonincreasing (or nondecreasing) on $[0, 1]$ (that is the point $c = 1$ (or $c = 0$) in the above Remark), then by the Corollary 2.1.16 (or Theorem 2.1.15, respectively) it follows that for all $n \in \mathbb{N}$, $B_n^{(M)}(f)$ is nonincreasing (or nondecreasing) on $[0, 1]$.

Suppose now that there exists $c \in (0, 1)$, such that f is nonincreasing on $[0, c]$ and nondecreasing on $[c, 1]$. Define the functions $F, G : [0, 1] \to \mathbb{R}_+$ by $F(x) = f(x)$ for all $x \in [0, c]$, $F(x) = f(c)$ for all $x \in [c, 1]$ and $G(x) = f(c)$ for all $x \in [0, c]$, $G(x) = f(x)$ for all $x \in [c, 1]$.

It is clear that F is nonincreasing and continuous on $[0, 1]$, G is nondecreasing and continuous on $[0, 1]$ and that $f(x) = \max\{F(x), G(x)\}$, for all $x \in [0, 1]$.

But it is easy to show that

$$B_n^{(M)}(f)(x) = \max\{B_n^{(M)}(F)(x), B_n^{(M)}(G)(x)\}, \text{ for all } x \in [0, 1],$$

where by the Corollary 2.1.16 and Theorem 2.1.15 , $B_n^{(M)}(F)(x)$ is nonincreasing and continuous on $[0,1]$ and $B_n^{(M)}(G)(x)$ is nondecreasing and continuous on $[0,1]$. We have two cases: 1) $B_n^{(M)}(F)(x)$ and $B_n^{(M)}(G)(x)$ do not intersect each other ; 2) $B_n^{(M)}(F)(x)$ and $B_n^{(M)}(G)(x)$ intersect each other.

Case 1). We have $\max\{B_n^{(M)}(F)(x), B_n^{(M)}(G)(x)\} = B_n^{(M)}(F)(x)$ for all $x \in [0,1]$ or $\max\{B_n^{(M)}(F)(x), B_n^{(M)}(G)(x)\} = B_n^{(M)}(G)(x)$ for all $x \in [0,1]$, which obviously proves that $B_n^{(M)}(f)(x)$ is quasiconvex on $[0,1]$.

Case 2). In this case it is clear that there exists a point $c' \in [0,1]$ such that $B_n^{(M)}(f)(x)$ is nonincreasing on $[0,c']$ and nondecreasing on $[c',1]$, which by the result in Popoviciu [128] implies that $B_n^{(M)}(f)(x)$ is quasiconvex on $[0,1]$ and proves the corollary. □

Remark. The preservation of the quasiconvexity by the linear Bernstein operators was proved in Paltanea [126].

It is of interest to exactly calculate $B_n^{(M)}(f)(x)$ for $f(x) = e_0(x) = 1$ and for $f(x) = e_1(x) = x$. In this sense we can state the following.

Lemma 2.1.19 (Bede–Coroianu–Gal [21]). *For all $x \in [0,1]$ and $n \in \mathbb{N}$ we have $B_n^{(M)}(e_0)(x) = 1$ and*

$$B_n^{(M)}(e_1)(x) = x \cdot \frac{p_{n-1,0}(x)}{p_{n,0}(x)} = \frac{x}{1-x}, \text{ if } x \in [0, 1/(n+1)],$$

$$B_n^{(M)}(e_1)(x) = x \cdot \frac{p_{n-1,0}(x)}{p_{n,1}(x)} = \frac{1}{n}, \text{ if } x \in [1/(n+1), 1/n],$$

$$B_n^{(M)}(e_1)(x) = x \cdot \frac{p_{n-1,1}(x)}{p_{n,1}(x)} = \frac{x}{1-x} \cdot \frac{n-1}{n}, \text{ if } x \in [1/n, 2/(n+1)],$$

$$B_n^{(M)}(e_1)(x) = x \cdot \frac{p_{n-1,1}(x)}{p_{n,2}(x)} = \frac{2}{n}, \text{ if } x \in [2/(n+1), 2/n],$$

$$B_n^{(M)}(e_1)(x) = x \cdot \frac{p_{n-1,2}(x)}{p_{n,2}(x)} = \frac{x}{1-x} \cdot \frac{n-2}{n}, \text{ if } x \in [2/n, 3/(n+1)],$$

$$B_n^{(M)}(e_1)(x) = x \cdot \frac{p_{n-1,2}(x)}{p_{n,3}(x)} = \frac{3}{n}, \text{ if } x \in [3/(n+1), 3/n],$$

and so on, in general we have

$$B_n^{(M)}(e_1)(x) = \frac{x}{1-x} \cdot \frac{n-j}{n}, \text{ if } x \in [j/n, (j+1)/(n+1)],$$

$$B_n^{(M)}(e_1)(x) = \frac{j+1}{n}, \text{ if } x \in [(j+1)/(n+1), (j+1)/n],$$

for $j \in \{0, 1, \ldots, n-1\}$.

Proof. The formula $B_n^{(M)}(e_0)(x) = 1$ is immediate by the definition of $B_n^{(M)}(f)(x)$.

To find the formula for $B_n^{(M)}(e_1)(x)$ we will use the explicit formula in Lemma 2.1.4, which says that

$$\bigvee_{k=0}^{n} p_{n,k}(x) = p_{n,j}(x), \text{ for all } x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right], j = 0, 1, \ldots, n,$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Indeed, since

$$\max_{k=0,\ldots,n} \left\{ p_{n,k}(x) \frac{k}{n} \right\} = \max_{k=1,\ldots,n} \left\{ p_{n,k}(x) \frac{k}{n} \right\} = x \cdot \max_{k=0,\ldots,n-1} \{p_{n-1,k}(x\},$$

this follows by applying Lemma 2.1.4 to both expressions $\max_{k=0,\ldots,n}\{p_{n,k}(x)\}$, $\max_{k=0,\ldots,n-1}\{p_{n-1,k}(x)\}$, taking into account that we get the following division of the interval $[0, 1]$

$$0 < \frac{1}{n+1} \le \frac{1}{n} \le \frac{2}{n+1} \le \frac{2}{n} \le \frac{3}{n+1} \le \frac{3}{n} \le \frac{4}{n+1} \le \frac{4}{n} \le \cdots,.$$

$\square$

Remarks. 1) The convexity of f on $[0, 1]$ is not preserved by $B_n^{(M)}(f)$ as can be seen from Lemma 2.1.19. Indeed, while $f(x) = e_1(x) = x$ is obviously convex on $[0, 1]$, it is easy to see that $B_n^{(M)}(e_1)$ is not convex on $[0, 1]$.

2) Also, if f is supposed to be starshaped on $[0, 1]$ (that is, $f(\lambda x) \le \lambda f(x)$ for all $x, \lambda \in [0, 1]$), then again by Lemma 2.1.19 it follows that $B_n^{(M)}(f)$ for $f(x) = e_1(x)$ is not starshaped on $[0, 1]$, although $e_1(x)$ obviously is starshaped on $[0, 1]$.

Despite the absence of the preservation of the convexity, we can prove the interesting property that for any arbitrary function f, the max-product Bernstein operator $B_n^{(M)}(f)$ is piecewise convex on $[0, 1]$. We present the following.

Theorem 2.1.20 (Bede–Coroianu–Gal [21]). *For any function $f : [0, 1] \to [0, \infty)$, $B_n^{(M)}(f)$ is convex on any interval of the form $[\frac{j}{n+1}, \frac{j+1}{n+1}]$, $j = 0, 1, \ldots, n$.*

Proof. For any $k, j \in \{0, 1, ..n\}$, let us consider the functions $f_{k,n,j} : [\frac{j}{n+1}, \frac{j+1}{n+1}] \to \mathbb{R}$,

$$f_{k,n,j}(x) = m_{k,n,j}(x) f(\frac{k}{n}) = \frac{\binom{n}{k}}{\binom{n}{j}} \left(\frac{x}{1-x}\right)^{k-j} f(\frac{k}{n}).$$

Clearly we have

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x),$$

for any $j \in \{0, 1, \ldots, n\}$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$.

We will prove that for any fixed j, each function $f_{k,n,j}(x)$ is convex on $[\frac{j}{n+1}, \frac{j+1}{n+1}]$, which will imply that $B_n^{(M)}(f)$ can be written as a maximum of some convex functions on $[\frac{j}{n+1}, \frac{j+1}{n+1}]$.

Since $f \geq 0$ it suffices to prove that the functions $g_{k,j} : [0,1] \to \mathbb{R}$, $g_{k,j}(x) = \left(\frac{x}{1-x}\right)^{k-j}$ are convex on $[\frac{j}{n+1}, \frac{j+1}{n+1}]$.

For $k = j$, $g_{j,j}$ is constant so is convex.

For $k = j+1$, we get $g_{j+1,j}(x) = \frac{x}{1-x}$ for any $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. Then $g''_{j+1,j}(x) = \frac{2}{(1-x)^3} > 0$ for any $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$.

For $k = j-1$ it follows $g_{j-1,j}(x) = \frac{1-x}{x}$ for any $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. Then $g''_{j-1,j}(x) = \frac{2}{x^3} > 0$ for any $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$.

If $k \geq j+2$, then $g''_{k,j}(x) = \frac{k-j}{(1-x)^4}\left(\frac{x}{1-x}\right)^{k-j-2}(k-j-1+2x) > 0$ for any $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$.

If $k \leq j-2$, then $g''_{k,j}(x) = \frac{k-j}{(1-x)^4}\left(\frac{x}{1-x}\right)^{k-j-2}(k-j-1+2x)$. Since $(k-j-1+2x) \leq k-j+1 \leq -1$ for any $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$, it follows that $(k-j)(k-j-1+2x) > 0$, which implies $g''_{k,j}(x) > 0$ for any $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$.

Since all the functions $g_{k,j}$ are convex on $[\frac{j}{n+1}, \frac{j+1}{n+1}]$, we get that $B_n^{(M)}(f)$ is convex on $[\frac{j}{n+1}, \frac{j+1}{n+1}]$ as maximum of these functions, which proves the theorem. □

Let us note that although $B_n^{(M)}(f)$ does not preserve the convexity too, by using $B_n^{(M)}(f)$ it easily can be constructed new nonlinear operators which converge to the function and preserve the convexity too.

Indeed, in this sense, for example we present the following.

Theorem 2.1.21 (Bede–Coroianu–Gal [21]). *For f belonging to the set*

$$S[0,1] = \{f : [0,1] \to \mathbb{R}; f \in C^1[0,1], f(0) = 0, f \textit{ is nondecreasing on } [0,1]\},$$

let us define the following sub-additive and positive homogenous operators (as function of f)

$$L_n(f)(x) = \int_0^x B_n^{(M)}(f')(t)dt, x \in [0,1], n \in \mathbb{N}.$$

If $f \in S[0,1]$ is convex, then $L_n(f)(x)$ is nondecreasing and convex on $[0,1]$. In addition, if f' is concave on $[0,1]$, then the order of approximation of f through $L_n(f)$ is $\omega_1(f'; 1/n)$.

Proof. Indeed, since f is convex, it follows that $f'(x)$ is nondecreasing on $[0,1]$, which by Theorem 2.1.15 implies that $B_n^{(M)}(f')(x)$ is nondecreasing and therefore we get the convexity of $L_n(f)(x)$ on $[0,1]$. The monotonicity of $L_n(f)(x)$ is immediate by $f' \geq 0$ on $[0,1]$ and by the relationship $L'_n(f)(x) = B_n^{(M)}(f')(x) \geq 0$ for all $x \in [0,1]$.

Also, writing $f(x) = \int_0^x f'(t)dt$ and supposing that f' is concave, by Corollary 2.1.10 we get that the order of approximation of f by $L_n(f)$ is $\omega_1(f'; 1/n)$. In addition, $L_n(f)(x)$ obviously is of C^1-class (which is not the case of original operator $B_n^{(M)}(f)(x)$) and $L_n'(f)(x)$ converges uniformly to f' on $[0, 1]$ with the same order of approximation $\omega_1(f'; 1/n)$. □

Remarks. 1) A simple example of function f verifying the statement of Theorem 2.1.21 is $f(x) = 1 - \cos x$, because in this case we easily get that $f(0) = 0$, $f'(x) = \sin x \geq 0, f''(x) = \cos x \geq 0$ and $f'''(x) = -\sin x \leq 0$, for all $x \in [0, 1]$.

2) In the definition of $L_n(f)(x)$ in the above Theorem 2.1.21, obviously that the values $f'(k/n)$ are involved. To involve values of f only but without to lose the properties mentioned in Theorem 2.1.21, we can replace there $f'(k/n)$ by, for example, $\frac{f((k+1)/n)-f(k/n)}{(k+1)/n-k/n} = n[f((k+1)/n) - f(k/n)]$ or by $\frac{f((k+1)/(n+1))-f(k/n)}{(k+1)/(n+1)-k/n}$.

At the end of this section we compare the max-product Bernstein operator $B_n^{(M)}(f)$, with the linear Bernstein operator $B_n(f)$ given by the formula (1.1). According to the considerations in Subsection 1.1.1, point (i), the best possible uniform approximation result is of the order $\omega_2^{\varphi}(f; 1/\sqrt{n})$ (see formula (1.2)).

Now, if f is, for example, a nondecreasing concave polygonal line on [0, 1], then by simple reasonings we get that $\omega_2^{\varphi}(f; \delta) \sim \delta$ for $\delta \leq 1$, which shows that the order of approximation obtained in this case by the linear Bernstein operator is exactly $\frac{1}{\sqrt{n}}$. On the other hand, since such of function f obviously is a Lipschitz function on [0, 1] (as having bounded all the derivative numbers) by Corollary 2.1.10 we get that the order of approximation by the max-product Bernstein operator is less than $\frac{1}{n}$, which is essentially better than $\frac{1}{\sqrt{n}}$. In a similar manner, by Corollary 2.1.11 and by the Remark after this corollary, we can produce many subclasses of functions for which the order of approximation given by the max-product Bernstein operator is essentially better than the order of approximation given by the linear Bernstein operator. In fact, the Corollaries 2.1.10 and 2.1.11 have no correspondent in the case of linear Bernstein operator. All these prove the advantages we may have in some cases, by using the max-product Bernstein operator. Intuitively, the max-product Bernstein operator has better approximation properties than its linear counterpart, for non-differentiable functions in a finite number of points (with the graphs having some "corners"), as for example, for functions defined as a maximum of a finite number of continuous functions on [0, 1].

On the other hand, in other cases (e.g., for differentiable functions) the linear Bernstein operator has better approximation properties than the max-product Bernstein operator, as can be seen from the formula for $B_n^{(M)}(e_1)(x)$ in Lemma 2.1.19. Indeed, by direct calculation it can be easily proved that $\|B_n^{(M)}(e_1) - e_1\| \sim \frac{1}{n}$, while it is well known that $\|B_n(e_1) - e_1\| = 0$.

Concerning now the shape preserving properties, it is clear that the linear Bernstein operator has better properties. However, for some particular classes of functions, the type of construction in Theorem 2.1.21, combined with Corollaries 2.1.10 and 2.1.11, can produce max-product Bernstein type operators with good preservation properties (e.g., preserving monotonicity and convexity) and giving in some cases (supposing, for example, that f' is a concave polygonal line) the same order of approximation as the linear Bernstein operator.

2.2 Improved Estimates for Strictly Positive Functions

In this section, the uniform estimate of the order $O[n\omega_1(f;1/n)^2 + \omega_1(f;1/n)]$ is achieved for strictly positive functions. In addition, near to the endpoints 0 and 1, the better pointwise estimate of the order $\omega_1(f, \sqrt{x(1-x)/n})$ is obtained. Finally, we prove that besides the preservation of quasiconvexity found in the previous section, the nonlinear max-product Bernstein operator preserves the quasiconcavity too. Note that because $B_n^{(M)}(f)$ is not linear, this is not a direct consequence of the preservation of quasiconvexity already proved in the previous section.

For any $k, j \in \{0, 1, \ldots, n\}$, let us consider the functions $f_{k,n,j} : [\frac{j}{n+1}, \frac{j+1}{n+1}] \to \mathbb{R}$, $f_{k,n,j}(x) = m_{k,n,j}(x) f(\frac{k}{n})$, where $m_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \cdot \left(\frac{x}{1-x}\right)^{k-j}$. It is easy to check that for any $k \geq j$, $f_{k,n,j}$ is nondecreasing and for any $k \leq j$, $f_{k,n,j}$ is nonincreasing.

We need the following results.

Lemma 2.2.1 (Bede–Coroianu–Gal [21]). *Let* $k, j \in \{0, 1, 2, ..n\}$ *and* $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. *The following assertions hold:*

(i) If $j \leq k \leq k+1 \leq n$, *then* $1 \geq m_{k,n,j}(x) \geq m_{k+1,n,j}(x)$;
(ii) If $0 \leq k \leq k+1 \leq j$, *then* $m_{k,n,j}(x) \leq m_{k+1,n,j}(x) \leq 1$.

Proof. (i) See the proof of Lemma 2.1.2, Case 1).
(ii) See the proof of Lemma 2.1.2, Case 2). □

Lemma 2.2.2 (see the relationship just before the Lemma 2.1.6). *Let* $x \in [0, 1]$ *and let* $j \in \{0, 1, \ldots, n\}$ *be such that* $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. *Then, one has*

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x).$$

Remark. By Theorem 2.1.15, Corollary 2.1.16 and by the monotonicity properties of the functions $f_{k,n,j}$ mentioned before Lemma 2.2.1, we get that for $j \in \{0, 1, \ldots, n\}$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$, $B_n^{(M)}(f)(x) = \bigvee_{k=j}^{n} f_{k,n,j}(x)$ for any nondecreasing function f and $B_n^{(M)}(f)(x) = \bigvee_{k=0}^{j} f_{k,n,j}(x)$ for any nonincreasing function f.

Definition 2.2.3. Let $f : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$. The function f is called quasiconcave, if $-f$ is quasiconvex as in Definition 2.1.17.

Remark. By Popovivicu [128] and from the above definition, we easily get that the continuous function f is quasiconcave on $[0, 1]$, equivalently means that there exists a point $c \in [0, 1]$ such that f is nondecreasing on $[0, c]$ and nonincreasing on $[c, 1]$.

For the beginning, we deal with the estimates in approximation of polygonal lines by the max-product Bernstein operator. Besides their itself importance, these results will be useful to obtain the main approximation results.

By Theorem 2.1.5 it was proved that for an arbitrary positive and continuous function on $[0, 1]$, the order of uniform approximation by the nonlinear Bernstein operator of max-product kind is, in general, $\omega_1(f, 1/\sqrt{n})$.

Firstly, below we show by an example that for the whole class of positive and continuous functions on $[0, 1]$, this is the best possible order of uniform approximation. More precisely, in what follows we give an example of simple monotone continuous polygonal line f, such that the order of approximation of f by the nonlinear Bernstein operator of max-product kind is exactly $\omega_1(f, 1/\sqrt{n})$.

Example. Let us consider the function $f : [0, 1] \to [0, \infty)$, $f(x) = 0$ if $x \in [0, 1/2]$ and $f(x) = x - 1/2$ if $x \in [1/2, 1]$. Then $B_n^{(M)}(f)(1/2) - f(1/2) = B_n^{(M)}(f)(1/2)$. It is easy to check that $1/2 \in [\frac{n_0}{n+1}, \frac{n_0+1}{n+1}]$ for all $n \in \mathbb{N}$, where $n_0 = [n/2]$. Then, since f is nondecreasing, we get (see the Remark after Lemma 2.2.2) $B_n^{(M)}(f)(1/2) = \bigvee_{k=n_0}^{n} f_{k,n,n_0}(1/2) = \bigvee_{k=n_0}^{n} \frac{\binom{n}{k}}{\binom{n}{n_0}} f(\frac{k}{n})$. Take $k_n = n_0 + [\sqrt{n}]$. This implies

$$\bigvee_{k=n_0}^{n} f_{k,n,j}(1/2)$$

$$\geq f_{k_n,n,n_0}(1/2) = \frac{\binom{n}{k_n}}{\binom{n}{n_0}} f(\frac{k_n}{n}) = \frac{\binom{n}{k_n}}{\binom{n}{n_0}}(\frac{k_n}{n} - \frac{1}{2}) \geq \frac{\binom{n}{k_n}}{\binom{n}{n_0}}(\frac{k_n}{n} - \frac{n_0 + 1}{n + 1})$$

$$\geq \frac{\binom{n}{k_n}}{\binom{n}{n_0}}(\frac{k_n}{n+1} - \frac{n_0 + 1}{n + 1}) = \frac{\binom{n}{k_n}}{\binom{n}{n_0}} \cdot \frac{[\sqrt{n}] - 1}{n + 1},$$

where for n sufficiently large we have $\frac{[\sqrt{n}]-1}{n+1} > 0$. Let us denote $n_1 = n - n_0$. We get

$$\frac{\binom{n}{k_n}}{\binom{n}{n_0}} = \frac{(n - k_n + 1)(n - k_n + 2) \ldots (n - n_0)}{(n_0 + 1)(n_0 + 2) \ldots k_n}$$

$$= \frac{(n_1 - [\sqrt{n}] + 1)(n_1 - [\sqrt{n}] + 2) \ldots n_1}{(n_0 + 1)(n_0 + 2) \ldots (n_0 + [\sqrt{n}])} \geq \left(\frac{n_1 - [\sqrt{n}] + 1}{n_0 + [\sqrt{n}]}\right)^{[\sqrt{n}]}$$

Since $n_0 \leq \frac{n}{2}$ and $n_1 \geq \frac{n}{2}$, we obtain

$$\left(\frac{n_1 - [\sqrt{n}] + 1}{n_0 + [\sqrt{n}]}\right)^{[\sqrt{n}]} \geq \left(\frac{\frac{n}{2} - \sqrt{n}}{\frac{n}{2} + \sqrt{n}}\right)^{\sqrt{n}}.$$

Since $\lim_{n\to\infty}\left(\frac{\frac{n}{2}-\sqrt{n}}{\frac{n}{2}+\sqrt{n}}\right)^{\sqrt{n}} = e^{-4}$, it follows that for sufficiently large n we have $\frac{\binom{n}{k_n}}{\binom{n}{n_0}} \geq e^{-5}$. This implies

$$B_n^{(M)}(f)(1/2) \geq e^{-5} \cdot \frac{[\sqrt{n}]-1}{n+1} \geq \frac{e^{-5}}{6\sqrt{n}}$$

for sufficiently large n. Taking into account that $\omega_1(f, 1/\sqrt{n}) = \frac{1}{\sqrt{n}}$ for all $n \geq 4$, we get

$$B_n^{(M)}(f)(1/2) = B_n^{(M)}(f)(1/2) - f(\frac{1}{2}) \geq \frac{e^{-5}}{6}\omega_1(f, 1/\sqrt{n})$$

for sufficiently large n, which proves the desired conclusion.

However, there exist subclasses of continuous functions such that the approximation order $\omega_1(f, 1/\sqrt{n})$ can be essentially improved to $\omega_1(f, 1/n)$.

In the same spirit of ideas, we will prove that for many types of continuous polygonal lines on the interval [0, 1], we have the order of approximation $O(1/n) \equiv O(\omega_1(f, 1/n))$.

In the next Propositions 2.2.5–2.2.8 and in Theorem 2.2.10, all the functions will be assumed to be continuous and strictly positive on [0, 1]. In addition, although will be not explicitly mentioned that in every proof, in all their proofs we may always assume that $B_n^{(M)}(f)(x) > f(x)$ and that $x \leq \frac{n}{n+1}$.

This fact can be summarized by the following.

Lemma 2.2.4 (Coroianu–Gal [52]). *Let* $f : [0, 1] \to \mathbb{R}_+$.

(i) If at a point $x \in [0, 1]$ *we have* $B_n^{(M)}(f)(x) \leq f(x)$, *then*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1(f, \frac{1}{n});$$

(ii) If $x \in [\frac{n}{n+1}, 1]$ *and* f *is nondecreasing on* $[0, 1]$, *then*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1(f, \frac{1}{n}).$$

Proof. (i) Indeed, if $B_n^{(M)}(f)(x) \leq f(x)$, then let $j \in [0, 1, \ldots, n\}$ be such that $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. This implies

$$\begin{aligned}
&\left|B_n^{(M)}(f)(x) - f(x)\right| \\
&= f(x) - B_n^{(M)}(f)(x) = f(x) - \bigvee_{k=0}^{n} f_{k,n,j}(x) \\
&\leq f(x) - f_{j,n,j}(x) = f(x) - f(\frac{j}{n})
\end{aligned}$$

and since $x, \frac{j}{n} \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$, we get

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1(f, \frac{1}{n+1}) \leq \omega_1(f, \frac{1}{n}).$$

(ii) Now, if $x \in [\frac{n}{n+1}, 1]$ and f is suppose nondecreasing, then by Theorem 2.1.15 it follows that $B_n^{(M)}(f)$ is nondecreasing and noting that $B_n^{(M)}(f)(1) = f(1)$, we get

$$\begin{aligned}&\left|B_n^{(M)}(f)(x) - f(x)\right|\\ &= B_n^{(M)}(f)(x) - f(x) \leq B_n^{(M)}(f)(1) - f(x)\\ &= f(1) - f(x) \leq \omega_1(f, \frac{1}{n+1}) \leq \omega_1(f, \frac{1}{n}).\end{aligned}$$

□

Remark. Notice that since $B_n^{(M)}(f)(0) - f(0) = B_n^{(M)}(f)(1) - f(1) = 0$, in all the approximation results we may assume that $x \in (0, 1)$.

Proposition 2.2.5 (Coroianu–Gal [52]). *Let us consider* $c \in [0, 1]$ *and the continuous nondecreasing function* $f : [0, 1] \to [0, \infty)$*, of the form*

$$f(x) = \begin{cases} 1; & x \in [0, c], \\ ax + b; & x \in [c, 1], \end{cases}$$

that is, $a \geq 0$ *and* $ac + b = 1$*. Then, for all* $n \in \mathbb{N}$ *and all* $x \in [0, 1]$ *we have the estimate*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \frac{(a+2)a}{n}.$$

Proof. Firstly, note that from the Lemma 2.2.4, if $B_n(M)(f)(x) \leq f(x)$ or $x > \frac{n}{n+1}$, then $|B_n^{(M)}(f)(x) - f(x)| \leq \omega_1(f; 1/n) \leq \frac{a}{n} \leq \frac{a(a+2)}{n}$ for all $n \in \mathbb{N}$. Therefore, in what follows we can suppose that $B_n^{(M)}(f)(x) > f(x)$ and $x \leq \frac{n}{n+1}$.

Let $x \in [0, 1]$ be fixed. We distinguish two cases: (i) $x \in [c, 1]$ and (ii) $x \in [0, c]$.

Case (i). Let $j \in \{0, 1, \ldots, n-1\}$ be such that $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. (The case $j = n$ can be excluded according to Lemma 2.2.4, (ii)). Since f is nondecreasing we get $B_n^{(M)}(f)(x) = \bigvee_{k=j}^{n} f_{k,n,j}(x)$.

Let us suppose that there exists $k \in \{j+1, \ldots, n\}$ such that $k \geq j + a$. Then, we have

$$\frac{f_{k+1,n,j}(x)}{f_{k,n,j}(x)} = \frac{n-k}{k+1} \cdot \frac{x}{1-x} \cdot \frac{f(\frac{k+1}{n})}{f(\frac{k}{n})}.$$

Since the function $g(y) = \frac{y}{1-y}$ is nondecreasing on $[\frac{j}{n+1}, \frac{j+1}{n+1}]$, it follows that $\frac{x}{1-x} \leq g(\frac{j+1}{n+1}) = \frac{j+1}{n-j}$ which combined with the fact that $\frac{k}{n} \geq c$ gives us

$$\frac{f_{k+1,n,j}(x)}{f_{k,n,j}(x)} \leq \frac{n-k}{k+1} \cdot \frac{j+1}{n-j} \cdot \frac{a\frac{k+1}{n} + b}{a\frac{k}{n} + b}$$
$$= \frac{n-k}{k+1} \cdot \frac{j+1}{n-j} \cdot \frac{a\frac{k}{n} + b + \frac{a}{n}}{a\frac{k}{n} + b}.$$

Clearly, the function $h(y) = \frac{ay+b+\frac{a}{n}}{ay+b}$ is nonincreasing and well defined on $[c, \frac{k}{n}]$. Indeed by the continuity of f it follows that $a\frac{k}{n} + b \geq ac + b = 1$. Since h is nonincreasing it follows that

$$\frac{a\frac{k}{n} + b + \frac{a}{n}}{a\frac{k}{n} + b} = h(\frac{k}{n}) \leq h(c) = \frac{ac + b + \frac{a}{n}}{ac + b} = 1 + \frac{a}{n}$$
$$= \frac{n+a}{n} \leq \frac{j+1+a}{j+1}.$$

This implies

$$\frac{f_{k+1,n,j}(x)}{f_{k,n,j}(x)} \leq \frac{n-k}{k+1} \cdot \frac{j+1}{n-j} \cdot \frac{j+1+a}{n+1}$$
$$\leq \frac{n-k}{n-j} \cdot \frac{j+1+a}{k+1}.$$

Since $k \geq j + a$ it immediately follows that $\frac{f_{k+1,n,j}(x)}{f_{k,n,j}(x)} \leq 1$.

Therefore, for $k \geq j + a$ we have $f_{k+1,n,j}(x) \leq f_{k,n,j}(x)$ and since there exists $k \in \{j + 1, \ldots, n\}$ such that $k \geq j + a$, then this implies $B_n^{(M)}(f)(x) = \bigvee_{k \in J(a)} f_{k,n,j}(x)$ where $J(a) = \{k \in \mathbb{N} : j \leq k \leq j + a\}$.

Note that if there not exist $k \in \{j + 1, \ldots, n\}$ with $k \geq j + a$ then $J(a) = \{j, j + 1, \ldots, n\}$.

Let $k_0 \in J(a)$ be such that $B_n^{(M)}(f)(x) = f_{k_0,n,j}(x)$. This implies

$$B_n^{(M)}(f)(x) - f(x)$$
$$= f_{k_0,n,j}(x) - f(x) = m_{k_0,n,j}(x) f(\frac{k_0}{n}) - f(x) \leq f(\frac{k_0}{n}) - f(x).$$

Since $\frac{k_0}{n} - x \leq \frac{j+a}{n} - \frac{j}{n+1} = \frac{a}{n} + \frac{j}{n(n+1)} \leq \frac{a+1}{n}$, we get

$$B_n^{(M)}(f)(x) - f(x) \leq \omega_1(f, \frac{1+a}{n}) \leq (a+2)\omega_1(f, \frac{1}{n}),$$

where we used the well-known inequality $\omega_1(f;\lambda\delta) \le (\lambda+1)\omega_1(f;\delta)$. Now, since clearly $\omega_1(f,\frac{1}{n}) \le \frac{a}{n}$, we get the desired conclusion in this case.

Case (ii). Taking into account the monotonicity of $B_n^{(M)}(f)$ and that in this case we have $f(x) = f(c)$, we obtain

$$\begin{aligned} & B_n^{(M)}(f)(x) - f(x) \\ & \le B_n^{(M)}(f)(c) - f(x) = B_n^{(M)}(f)(c) - f(c). \end{aligned}$$

Since for c we can apply the result in the above case (i), it follows that

$$B_n^{(M)}(f)(x) - f(x) \le \frac{(a+2)a}{n}$$

and the proposition is proved. □

Remark. Note that the conclusion of the above proposition does not depend on c.

Proposition 2.2.6 (Coroianu–Gal [52]). *Let us consider* $0 \le c_1 \le c_2 \le 1$ *and the nondecreasing continuous function* $f : [0,1] \to [0,\infty)$*, of the form*

$$f(x) = \begin{cases} 1; & x \in [0, c_1], \\ a_1 x + b_1; & x \in [c_1, c_2], \\ a_2 x + b_2; & x \in [c_2, 1], \end{cases}$$

that is, $a_1 \ge 0$*,* $a_1 c_1 + b_1 = 1$*,* $a_2 \ge 0$ *and* $a_2 c_2 + b_2 = a_1 c_2 + b_1$*. Then, for all* $x \in [0,1]$ *and* $n \in \mathbb{N}$ *we have the estimate*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le \frac{\left(\max\limits_{i\in\{1,2\}} (a_i+2)a_i\right)}{n}.$$

Proof. If there exists $x_0 \in [0, c_2]$ such that $a_2 x_0 + b_2 = 1$, then we introduce the functions:

$$g(x) = \begin{cases} 1; & x \in [0, c_1], \\ a_1 x + b_1; & x \in [c_1, 1]. \end{cases}$$

and

$$h(x) = \begin{cases} 1; & x \in [0, x_0], \\ a_2 x + b_2; & x \in [x_0, 1]. \end{cases} \tag{2.5}$$

If $a_2 x + b_2 > 1$ for all $x \in [0,1]$, then we take $h(x) = a_2 x + b_2$ for all $x \in [0,1]$. We distinguish three cases: (i) $x \in [c_2, 1]$; (ii) $x \in [c_1, c_2]$, and (iii) $x \in [0, c_1]$.

Case (i) Let $j \in \{0, 1, \ldots, n-1\}$ be such that $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. Then we have $B_n^{(M)}(f)(x) = \bigvee_{k=j}^{n} f_{k,n,j}(x) = \bigvee_{k=j}^{n} m_{k,n,j}(x)f(\frac{k}{n})$. Let $k_0 \in \{j, j+1, \ldots, n\}$ be such that $B_n^{(M)}(f)(x) = f_{k_0,n,j}(x)$. If $k_0 = j$, then $k_0/n \in [j/(n+1), (j+1)/(n+1)]$ and $B_n^{(M)}(f)(x) = f_{j,n,j}(x) = f(\frac{j}{n})$ and it is immediate that

$$
\begin{aligned}
B_n^{(M)}(f)(x) - f(x) &= f(j/n) - f(x) \le \omega_1(f, 1/n+1) \\
&\le \omega_1(f, 1/n) \le \frac{\max\{a_1, a_2\}}{n}.
\end{aligned}
$$

If $k_0 > j$, then it is immediate that $\frac{k_0}{n} \ge c_2$, which implies $f(k_0/n) = h(k_0/n)$ and therefore $f_{k_0,n,j}(x) = h_{k_0,n,j}(x)$, where by definition

$$h_{k,n,j}(x) = m_{k,n,j}(x)h(k/n).$$

We get $B_n^{(M)}(f)(x) = f_{k_0,n,j}(x) = h_{k_0,n,j}(x) \le B_n^{(M)}(h)(x)$ and because in this case $f(x) = h(x)$ it follows

$$B_n^{(M)}(f)(x) - f(x) = B_n^{(M)}(f)(x) - h(x) \le B_n^{(M)}(h)(x) - h(x).$$

If h is as in (2.5), then it satisfies the hypothesis of Proposition 2.2.5 and it follows that $B_n^{(M)}(h)(x) - h(x) \le \frac{(a_2+2)a_2}{n}$ which implies $B_n^{(M)}(f)(x) - f(x) \le \frac{(a_2+2)a_2}{n}$. If h is as in the second case, that is linear on [0, 1], then h is a concave function and by Corollary 2.1.10 it follows that $B_n^{(M)}(h)(x) - h(x) \le 2\omega_1(f, \frac{1}{n}) \le \frac{2a_2}{n} \le \frac{(a_2+2)a_2}{n}$ and again we get $B_n^{(M)}(f)(x) - f(x) \le \frac{(a_2+2)a_2}{n}$.

Case (ii) Let $j \in \{0, 1, \ldots, n-1\}$ be such that $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$ and let $k_0 \in \{j, j+1, \ldots, n\}$ be such that $B_n^{(M)}(f)(x) = f_{k_0,n,j}(x)$.
If $k_0 = j$, then it is immediate that

$$B_n^{(M)}(f)(x) - f(x) \le \omega_1(f, 1/n) \le \frac{\max\{a_1, a_2\}}{n}.$$

If $\frac{k_0}{n} \in [c_1, c_2]$, then we get $f_{k_0,n,j}(x) = g_{k_0,n,j}(x)$, where

$$g_{k,n,j}(x) = m_{k,n,j}(x)g(k/n),$$

and this implies $B_n^{(M)}(f)(x) = f_{k_0,n,j}(x) = g_{k_0,n,j}(x) \le B_n^{(M)}(g)(x)$.
Since $f(x) = g(x)$, it follows that

$$B_n^{(M)}(f)(x) - f(x) = B_n^{(M)}(f)(x) - g(x) \le B_n^{(M)}(g)(x) - g(x).$$

Clearly, g satisfies the hypothesis of Proposition 2.2.5, which combined with the above inequality implies $B_n^{(M)}(f)(x) - f(x) \le \frac{(a_1+2)a_1}{n}$.

The last possibility is when $\frac{k_0}{n} \in [c_2, 1]$. (Indeed, if we would have $\frac{k_0}{n} < c_1$ that would imply $c_1 > \frac{k_0}{n} \geq \frac{j+1}{n} > \frac{j+1}{n+1} \geq x$, a contradiction with $x \in [c_1, c_2]$). Therefore, this implies $f_{k_0,n,j}(x) = h_{k_0,n,j}(x)$, where does not matter which h we choose. We have here two subcases: (ii)$_a$ $a_1 \geq a_2$ and (ii)$_b$ $a_1 < a_2$.

Subcase (ii)$_a$. By simple geometrical reasonings, it is immediate that $f(\frac{k_0}{n}) = h(\frac{k_0}{n}) \leq g(\frac{k_0}{n})$, which immediately implies $f_{k_0,n,j}(x) = h_{k_0,n,j}(x) \leq g_{k_0,n,j}(x)$ and further on, $B_n^{(M)}(f)(x) \leq B_n^{(M)}(g)(x)$. This leads to the same conclusion as above, that is

$$B_n^{(M)}(f)(x) - f(x) \leq B_n^{(M)}(g)(x) - g(x) \leq \frac{(a_1+2)a_1}{n}.$$

Subcase (ii)$_b$. In this case, by simple geometrical reasonings we have $f(x) \geq h(x)$ for all $x \in [0, 1]$ (does not matter here which definition for h we choose) and we get

$$\begin{aligned} &B_n^{(M)}(f)(x) - f(x) \\ &= h_{k_0,n,j}(x) - f(x) \leq h_{k_0,n,j}(x) - h(x) \leq B_n^{(M)}(h)(x) - h(x). \end{aligned}$$

Clearly, in both definitions h satisfies the hypothesis of Proposition 2.2.5, which combined with the above inequality implies $B_n^{(M)}(f)(x) - f(x) \leq \frac{(a_2+2)a_2}{n}$.

Case (iii). As in the proof of Proposition 2.2.5, we get $B_n^{(M)}(f)(x) - f(x) \leq B_n^{(M)}(f)(c_1) - f(c_1)$ and since for c_1 the case (ii) is applicable we immediately obtain $B_n^{(M)}(f)(x) - f(x) \leq \frac{\left(\max_{i\in\{1,2\}} (a_i+2)a_i\right)}{n}$.

Collecting all the estimates in the above cases and subcases we get the desired conclusion. □

Proposition 2.2.7 (Coroianu–Gal [52]). *Let us consider the nondecreasing continuous function* $f : [0, 1] \to [0, \infty)$,

$$f(x) = \begin{cases} \alpha; & x \in [0, c], \\ ax + b; & x \in [c, 1], \end{cases}$$

where $\alpha > 0$. *Then, we have the estimate*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \frac{(2 + \frac{a}{\alpha})a}{n}.$$

Proof. Let us consider the function

$$g(x) = \begin{cases} 1; & x \in [0, c], \\ \frac{1}{\alpha}(ax + b); & x \in [c, 1]. \end{cases}$$

By Proposition 2.2.5, we get $\left|B_n^{(M)}(g)(x) - g(x)\right| \leq \frac{(a/\alpha+2)a/\alpha}{n}$. By the homogeneity of $B_n^{(M)}$, we get

$$\begin{aligned}&\left|B_n^{(M)}(f)(x) - f(x)\right| \\ &= \left|B_n^{(M)}(\alpha g)(x) - \alpha g(x)\right| = \alpha \left|B_n^{(M)}(g)(x) - g(x)\right|.\end{aligned}$$

This implies the desired conclusion. □

Proposition 2.2.8 (Coroianu–Gal [52]). *Let us consider the nondecreasing continuous function* $f : [0, 1] \to [0, \infty)$,

$$f(x) = \begin{cases} \alpha; & x \in [0, c_1], \\ a_1 x + b_1; & x \in [c_1, c_2], \\ a_2 x + b_2; & x \in [c_2, 1], \end{cases}$$

where $\alpha > 0$. *Then, we have the estimate*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \frac{\left(\max_{i \in \{1,2\}} (2 + \frac{a_i}{\alpha}) a_i\right)}{n}.$$

Proof. The proof is analogous with the proof of Proposition 2.2.7, so we omit it. □

Remark. By Propositions 2.2.7 and 2.2.8 it follows that if f is a strictly positive function on $[0, 1]$ and satisfies the hypothesis in Proposition 2.2.7 or Proposition 2.2.8, then we have

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \frac{(2 + \frac{\overline{a_0}}{f(0)})\overline{a_0}}{n}, x \in [0, 1],$$

where in the first case we have $\overline{a_0} = a$ and in the second case we have $\overline{a_0} = \max\{a_1, a_2\}$.

In what follows, we extend the above results to any monotone, continuous, and strictly positive polygonal line on $[0, 1]$.

Definition 2.2.9. Let $a, b \in \mathbb{R}$, $a < b$ and let $a = x_0 < x_1 < \cdots < x_l = b$ be a division of the interval $[a, b]$. A function $f : [a, b] \to \mathbb{R}$ will be called a continuous polygonal line if f is continuous on $[a, b]$ and for any $i \in \{0, 1, \ldots, l-1\}$, there exists a polynomial function of degree less than or equal to 1, $f_i : \mathbb{R} \to \mathbb{R}$, such that $f(x) = f_i(x) = a_i x + b_i$ for all $x \in [x_i, x_{i+1}]$. We denote $f = (f_{0[x_0,x_1]}, f_{1[x_1,x_2]}, \ldots, f_{l-1[x_{l-1},x_l]})$.

Theorem 2.2.10 (Coroianu–Gal [52]). *For* $f_i(x) = a_i x + b_i$, $i = 0, \ldots, l-1$, *let* $f = (f_{0[x_0,x_1]}, f_{1[x_1,x_2]}, \ldots, f_{l-1[x_{l-1},x_l]})$ *be a continuous, nondecreasing, and strictly positive on* $[0, 1]$ *polygonal line. Then for all* $x \in [0, 1]$ *and* $n \in \mathbb{N}$ *we have the estimate*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \frac{(2 + \frac{a_{i_0}}{f(0)})a_{i_0}}{n},$$

where $a_{i_0} = \max\{a_0, a_1, \ldots, a_{l-1}\}$.

Proof. We prove the theorem by mathematical induction on the variable $l \in \{1, 2 \ldots, \}$, representing the number of intervals given by the division of the interval $[0, 1]$.

If $l = 1$, then it is immediate that f is linear of the form $f(x) = ax + b$, $x \in [0, 1]$. Then, by Corollary 2.1.10 it follows that

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1(f, \frac{1}{n}) \leq \frac{2a}{n} \leq \frac{(2 + \frac{a}{f(0)})a}{n}.$$

Suppose now that the assertion of the theorem holds for $l - 1$. We denote by $\overline{a} = \max\{a_0, a_1, \ldots, a_{l-2}\}$. Also we need the functions

$$g = (f_{0[x_0,x_1]}, f_{1[x_1,x_2]}, \ldots, f_{l-2[x_{l-2},x_l]})$$

and

$$h(x) = \begin{cases} f(0); & x \in [0, c], \\ a_{i_0}x + f(x_{l-1}) - a_{i_0}x_{l-1}; & x \in [c, x_{l-1}], \\ f(x) = f_{l-1}(x); & x \in [x_{l-1}, 1], \end{cases}$$

where $c \in [0, 1]$ is such that $a_{i_0}c + f(x_{l-1}) - a_{i_0}x_{l-1} = f(0)$. Since $a_{i_0} = \max\{a_0, a_1, \ldots, a_{l-1}\}$, by simple geometrical reasonings we get $f(x) \geq h(x)$ for all $x \in [0, 1]$. In addition, it is easy to check that h is continuous on $[0, 1]$.

For arbitrary $x \in [0, 1]$, we distinguish two cases: (i) $x \in [0, x_{l-1}]$ and (ii) $x \in [x_{l-1}, 1]$.

Case (i). Let $j \in \{0, 1, \ldots, n - 1\}$ be such that $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$ and let $k_0 \in \{j, \ldots, n\}$ be such that $B_n^{(M)}(f)(x) = f_{k_0,n,j}(x)$. If $\frac{k_0}{n} \leq x_{l-1}$, then it is immediate that $f_{k_0,n,j}(x) = g_{k_0,n,j}(x)$ which immediately implies $B_n^{(M)}(f)(x) \leq B_n^{(M)}(g)(x)$. Recall here that everywhere in the proof we denoted $f_{k,n,j}(x) = m_{k,n,j}(x)f(k/n)$, $g_{k,n,j}(x) = m_{k,n,j}(x)g(k/n)$, and $h_{k,n,j}(x) = m_{k,n,j}(x)h(k/n)$.

Since g is split in $l - 1$ intervals, from our assumption we get

$$\left|B_n^{(M)}(g)(x) - g(x)\right| \leq \frac{(2 + \frac{\overline{a}}{g(0)})\overline{a}}{n}.$$

Since $g(0) = f(0)$ and $\overline{a} \leq a_{i_0}$, we get

$$\begin{aligned} &B_n^{(M)}(f)(x) - f(x) \\ &= B_n^{(M)}(f)(x) - g(x) \leq B_n^{(M)}(g)(x) - g(x) \\ &\leq \frac{(2 + \frac{\overline{a}}{g(0)})\overline{a}}{n} \leq \frac{(2 + \frac{a_{i_0}}{f(0)})a_{i_0}}{n}. \end{aligned}$$

If $\frac{k_0}{n} > x_{l-1}$, then clearly $f_{k_0,n,j}(x) = h_{k_0,n,j}(x)$ which implies $B_n^{(M)}(f)(x) \leq B_n^{(M)}(h)(x)$. By the Remark after the proof of Proposition 2.2.8 we get

$$\left|B_n^{(M)}(h)(x) - h(x)\right| \leq \frac{(2 + \frac{a_{i_0}}{h(0)})a_{i_0}}{n}.$$

Since $f(0) = h(0)$ and $f(x) \geq h(x)$, we obtain

$$\begin{aligned} &B_n^{(M)}(f)(x) - f(x) \\ &\leq B_n^{(M)}(f)(x) - h(x) \leq B_n^{(M)}(h)(x) - h(x) \\ &\leq \frac{(2 + \frac{a_{i_0}}{f(0)})a_{i_0}}{n}. \end{aligned}$$

Case (ii). Let $j \in \{0, 1, \ldots, n-1\}$ be such that $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$ and let $k_0 \in \{j, \ldots, n\}$ be such that $B_n^{(M)}(f)(x) = f_{k_0,n,j}(x)$. If $k_0 = j$, then we get

$$\begin{aligned} &B_n^{(M)}(f)(x) - f(x) \\ &\leq \omega_1(f, \frac{1}{n}) \leq \frac{a_{i_0}}{n} \leq \frac{(2 + \frac{a_{i_0}}{f(0)})a_{i_0}}{n}. \end{aligned}$$

If $k_0 > j$, then we have $\frac{k_0}{n} \geq x_{l-1}$ which immediately implies $f_{k_0,n,j}(x) = h_{k_0,n,j}(x)$. Noting that $f(x) = h(x)$, and reasoning as in the above Case (i), we easily get

$$B_n^{(M)}(f)(x) - f(x) \leq \frac{(2 + \frac{a_{i_0}}{f(0)})a_{i_0}}{n}$$

and the proof is complete. □

In order to obtain a similar result in the case of nonincreasing polygonal lines, we need first the following simple result.

Lemma 2.2.11 (Coroianu–Gal [52]). *For any function $f : [0,1] \to [0,\infty)$, we have*

$$B_n^{(M)}(f)(x) = B_n^{(M)}(g)(1-x), x \in [0,1],$$

where $g(x) = f(1-x)$ for all $x \in [0,1]$.

Proof. By direct calculation we get

$$\begin{aligned} &B_n^{(M)}(g)(1-x) \\ &= \frac{\bigvee_{k=0}^{n} p_{n,k}(1-x) g\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} p_{n,k}(1-x)} = \frac{\bigvee_{k=0}^{n} p_{n,n-k}(x) f\left(\frac{n-k}{n}\right)}{\bigvee_{k=0}^{n} p_{n,n-k}(x)} = B_n^{(M)}(f)(x). \end{aligned}$$

□

Theorem 2.2.12 (Coroianu–Gal [52]). *For $f_i(x) = a_i x + b_i$, $i = 0, \ldots, l-1$, let $f = (f_{0[x_0,x_1]}, f_{1[x_1,x_2]}, \ldots, f_{l-1[x_{l-1},x_l]})$ be a continuous, nonincreasing, and strictly positive on $[0,1]$ polygonal line. Then we have the estimate*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \frac{(2 + \frac{\overline{a_{i_0}}}{f(1)})\overline{a_{i_0}}}{n}, x \in [0,1],$$

where $\overline{a_{i_0}} = \max\{|a_0|, |a_1|, \ldots, |a_{l-1}|\}$.

Proof. Consider the function $g : [0,1] \to [0,\infty)$, $g(x) = f(1-x)$. Then evidently g is nondecreasing and g has the form

$$g = (g_{0[y_0,y_1]}, g_{1[y_1,y_2]}, \ldots, g_{l-1[y_{l-1},y_l]}),$$

where $y_i = 1 - x_{l-i}$, $i \in \{0,1,\ldots,l\}$ and $g_i(x) = f_{l-i-1}(1-x) = c_i x + d_i$, $i \in \{0,1,\ldots,l-1\}$. Moreover, it is easy to check that

$$\max\{c_0, c_1, \ldots, c_{l-1}\} = \max\{|a_0|, |a_1|, \ldots, |a_{l-1}|\} =: \overline{a_{i_0}}.$$

By Theorem 2.2.10 it follows that

$$\left|B_n^{(M)}(g)(x) - g(x)\right| \leq \frac{(2 + \frac{\overline{a_{i_0}}}{g(0)})\overline{a_{i_0}}}{n}.$$

Taking into account the above Lemma 2.2.11, we obtain

$$\begin{aligned}
&\left|B_n^{(M)}(f)(x) - f(x)\right| \\
&= \left|B_n^{(M)}(g)(1-x) - g(1-x)\right| \leq \frac{(2 + \frac{\overline{a_{i_0}}}{g(0)})\overline{a_{i_0}}}{n} \\
&= \frac{(2 + \frac{\overline{a_{i_0}}}{f(1)})\overline{a_{i_0}}}{n}
\end{aligned}$$

and the theorem is proved. □

As consequences of the results on the approximation of polygonal lines, we will get the main results of this section. Note that in all the proofs of the approximation results, according to Lemma 2.2.4 we may always assume that $B_n^{(M)}(f)(x) > f(x)$.

Theorem 2.2.13 (Coroianu–Gal [52]). *If $f : [0,1] \to [0,\infty)$ is a continuous, nondecreasing, and strictly positive function on $[0,1]$, then we have the estimate*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{n\omega_1(f, \frac{1}{n})}{f(0)} + 3\right)\omega_1(f, \frac{1}{n}), x \in [0,1], n \in \mathbb{N}.$$

Proof. For $n \in \mathbb{N}$, we consider the function

$$g = (g_{0[x_0,x_1]}, g_{1[x_1,x_2]}, \ldots, g_{n-1[x_{n-1},x_n]}),$$

where $x_i = \frac{i}{n}$, $i \in \{0, 1, \ldots, n\}$ and $g_{i-1}(x) = \frac{(x-x_i)[f(x_i)-f(x_{i-1})]}{x_i-x_{i-1}} + f(x_i)$ for all $x \in [x_{i-1}, x_i]$, $i \in \{1, \ldots, n\}$. Since $f(\frac{k}{n}) = g(\frac{k}{n})$ for all $k \in \{0, 1, \ldots, n\}$, by the definition of $B_n^{(M)}(f)$ too, it follows that $B_n^{(M)}(f)(x) = B_n^{(M)}(g)(x)$ for all $x \in [0, 1]$. Also, it is immediate that $f(0) = g(0)$ and that $|f(x) - g(x)| \le \omega_1(f, \frac{1}{n})$ for all $x \in [0, 1]$. Indeed, for $x \in [0, 1]$ let $i \in \{0, 1, \ldots, n-1\}$ be such that $x \in [x_i, x_{i+1}]$. Since g is nondecreasing on the interval $[0, 1]$ we get $|f(x) - g(x)| \le \max\{|f(x) - f(x_i)|, |f(x) - f(x_{i+1})|\} \le \omega_1(f, \frac{1}{n})$. Since g is nondecreasing, by Theorem 2.2.10 we get

$$\left|B_n^{(M)}(g)(x) - g(x)\right| \le \frac{(2 + \frac{a_{i_0}}{g(0)})a_{i_0}}{n}, x \in [0, 1],$$

where

$$a_{i_0} = \max_{i\in\{1,\ldots,n\}} \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right).$$

We have $\frac{a_{i_0}}{n} = f(x_{i_0+1}) - f(x_{i_0}) \le \omega_1(f, \frac{1}{n})$. On the other hand, it is immediate that $\frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}} \le n\omega_1(f, \frac{1}{n})$ for all $i \in \{1, \ldots, n\}$. Therefore, we obtain

$$\left|B_n^{(M)}(g)(x) - g(x)\right| \le \left(\frac{n\omega_1(f, \frac{1}{n})}{f(0)} + 2\right)\omega_1(f, \frac{1}{n}), x \in [0, 1].$$

For $x \in [0, 1]$, we get

$$\begin{aligned} &\left|B_n^{(M)}(f)(x) - f(x)\right| \\ &= \left|B_n^{(M)}(g)(x) - f(x)\right| \le \left|B_n^{(M)}(g)(x) - g(x)\right| + |f(x) - g(x)| \\ &\le \left(\frac{n\omega_1(f, \frac{1}{n})}{f(0)} + 2\right)\omega_1(f, \frac{1}{n}) + \omega_1(f, \frac{1}{n}) = \left(\frac{n\omega_1(f, \frac{1}{n})}{f(0)} + 3\right)\omega_1(f, \frac{1}{n}) \end{aligned}$$

and the theorem is proved. □

Corollary 2.2.14 (Coroianu–Gal [52]). *If $f : [0, 1] \to [0, \infty)$ is a continuous, nonincreasing, and strictly positive function on $[0, 1]$, then we have the estimate*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le \left(\frac{n\omega_1(f, \frac{1}{n})}{f(1)} + 3\right)\omega_1(f, \frac{1}{n}), x \in [0, 1], n \in \mathbb{N}.$$

Proof. Take $g(x) = f(1-x)$, $x \in [0,1]$. Clearly, g satisfies the hypothesis in Theorem 2.2.13, which means that

$$\begin{aligned}&\left|B_n^{(M)}(g)(x) - g(x)\right| \\ &\leq \left(\frac{n\omega_1(g, \frac{1}{n})}{g(0)} + 3\right)\omega_1(g, \frac{1}{n}), x \in [0,1].\end{aligned}$$

Since $f(1) = g(0)$ and $\omega_1(f, \frac{1}{n}) = \omega_1(g, \frac{1}{n})$, by Lemma 2.2.11 too, for $x \in [0,1]$ we get

$$\begin{aligned}&\left|B_n^{(M)}(f)(x) - f(x)\right| \\ &= \left|B_n^{(M)}(g)(1-x) - g(1-x)\right| \leq \left(\frac{n\omega_1(g, \frac{1}{n})}{g(0)} + 3\right)\omega_1(g, \frac{1}{n}) \\ &= \left(\frac{n\omega_1(f, \frac{1}{n})}{f(1)} + 3\right)\omega_1(f, \frac{1}{n}),\end{aligned}$$

which proves the corollary. □

In all what follows, for a continuous function $f : [0,1] \to \mathbb{R}$, we denote $m_f = \min\{f(x) : x \in [0,1]\}$.

Theorem 2.2.15 (Coroianu–Gal [52]). *If $f : [0,1] \to [0,\infty)$ is a continuous, quasiconvex, and strictly positive function on $[0,1]$, then we have the estimate*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{n\omega_1(f, \frac{1}{n})}{m_f} + 3\right)\omega_1(f, \frac{1}{n}), x \in [0,1], n \in \mathbb{N}.$$

Proof. Since f is quasiconvex, it follows that there exists $c \in [0,1]$ such that f is nonincreasing on $[0,c]$ and nondecreasing on $[c,1]$. In addition, it is immediate that $f(c) = m_f$. Let us introduce the functions

$$g(x) = \begin{cases} m_f; & x \in [0,c], \\ f(x); & x \in [c,1] \end{cases}$$

and

$$h(x) = \begin{cases} f(x); & x \in [0,c], \\ m_f; & x \in [c,1]. \end{cases}$$

It is easy to verify that $\max\{\omega_1(g, \frac{1}{n}), \omega_1(h, \frac{1}{n})\} \leq \omega_1(f, \frac{1}{n})$. Since $f = g \bigvee h$, by the property satisfied by $B_n^{(M)}$, we can write

$$B_n^{(M)}(f)(x) = B_n^{(M)}(g)(x) \bigvee B_n^{(M)}(h)(x), x \in [0,1].$$

In addition, we observe that g satisfies the hypothesis in Theorem 2.2.13 and h satisfies the hypothesis in Corollary 2.2.14. Therefore, we have

$$\left|B_n^{(M)}(g)(x) - g(x)\right| \le \left(\frac{n\omega_1(g, \frac{1}{n})}{g(0)} + 3\right)\omega_1(g, \frac{1}{n}), x \in [0, 1]$$

and

$$\left|B_n^{(M)}(h)(x) - h(x)\right| \le \left(\frac{n\omega_1(h, \frac{1}{n})}{h(1)} + 3\right)\omega_1(h, \frac{1}{n}), x \in [0, 1].$$

Let us choose arbitrary $x \in [0, 1]$. If $B_n^{(M)}(f)(x) = B_n^{(M)}(g)(x)$, then we have

$$\begin{aligned} & B_n^{(M)}(f)(x) - f(x) \\ &= B_n^{(M)}(g)(x) - g(x) \bigvee h(x) \le B_n^{(M)}(g)(x) - g(x) \\ &\le \left(\frac{n\omega_1(g, \frac{1}{n})}{m_f} + 3\right)\omega_1(g, \frac{1}{n}) \le \left(\frac{n\omega_1(f, \frac{1}{n})}{m_f} + 3\right)\omega_1(f, \frac{1}{n}). \end{aligned}$$

If $B_n^{(M)}(f)(x) = B_n^{(M)}(h)(x)$, then we have

$$\begin{aligned} & B_n^{(M)}(f)(x) - f(x) \\ &= B_n^{(M)}(h)(x) - g(x) \bigvee h(x) \le B_n^{(M)}(h)(x) - h(x) \\ &\le \left(\frac{n\omega_1(h, \frac{1}{n})}{m_f} + 3\right)\omega_1(h, \frac{1}{n}) \le \left(\frac{n\omega_1(f, \frac{1}{n})}{m_f} + 3\right)\omega_1(f, \frac{1}{n}). \end{aligned}$$

This proves the theorem. □

Theorem 2.2.16 (Coroianu–Gal [52]). *If $f : [0, 1] \to [0, \infty)$ is a continuous, quasiconcave, and strictly positive function on $[0, 1]$, then we have the estimate*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le \left(\frac{n\omega_1(f, \frac{1}{n})}{m_f} + 3\right)\omega_1(f, \frac{1}{n}), x \in [0, 1], n \in \mathbb{N}.$$

Proof. Since f is quasiconcave, it follows that there exists $c \in [0, 1]$ such that f is nondecreasing on $[0, c]$ and nonincreasing on $[c, 1]$. Let us introduce the functions

$$g(x) = \begin{cases} f(x); \ x \in [0, c], \\ f(c); \ x \in [c, 1] \end{cases}$$

and

$$h(x) = \begin{cases} f(c); \ x \in [0, c], \\ f(x); \ x \in [c, 1]. \end{cases}$$

It is immediate that $f(0) = g(0), f(1) = h(1)$ and that $\max\{\omega_1(g, \frac{1}{n}), \omega_1(h, \frac{1}{n})\} \leq \omega_1(f, \frac{1}{n})$. In addition, since $f \leq g$ and $f \leq h$, by the monotonicity of $B_n^{(M)}$, we get

$$B_n^{(M)}(f)(x) \leq \min\{B_n^{(M)}(g)(x), B_n^{(M)}(h)(x)\}, x \in [0, 1].$$

In order to prove our assertion, we distinguish two cases: (i) $x \in [0, c]$ and (ii) $x \in [c, 1]$.

Case (i). Noting that $f(x) = g(x)$ and that g satisfies the hypothesis in Theorem 2.2.13, we get

$$\begin{aligned}
&B_n^{(M)}(f)(x) - f(x) \\
&= B_n^{(M)}(f)(x) - g(x) \leq B_n^{(M)}(g)(x) - g(x) \leq \left(\frac{n\omega_1(g, \frac{1}{n})}{g(0)} + 3\right)\omega_1(g, \frac{1}{n}) \\
&\leq \left(\frac{n\omega_1(f, \frac{1}{n})}{f(0)} + 3\right)\omega_1(f, \frac{1}{n}).
\end{aligned}$$

Case (ii). Noting that $f(x) = h(x)$ and that h satisfies the hypothesis in Corollary 2.2.14, we get

$$\begin{aligned}
&B_n^{(M)}(f)(x) - f(x) \\
&= B_n^{(M)}(f)(x) - h(x) \leq B_n^{(M)}(h)(x) - h(x) \leq \left(\frac{n\omega_1(h, \frac{1}{n})}{h(1)} + 3\right)\omega_1(h, \frac{1}{n}) \\
&\leq \left(\frac{n\omega_1(f, \frac{1}{n})}{f(1)} + 3\right)\omega_1(f, \frac{1}{n}).
\end{aligned}$$

Collecting all the estimates in the above cases (i) and (ii) and since $m_f = \min\{f(0), f(1)\}$, we easily get the estimate in the statement. □

Theorem 2.2.17 (Coroianu–Gal [52]). *Let* $f : [0, 1] \to [0, \infty)$ *be a continuous and strictly positive function and suppose that there exists a division of the interval* $[0, 1]$, $0 = x_0 < x_1 < \cdots < x_l = 1$ *such that* f *is monotone on each interval* $[x_i, x_{i+1}]$, $i \in \{0, 1, .., l-1\}$ *and of opposite monotonicity on each two consecutive intervals. Then*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{n\omega_1(f, \frac{1}{n})}{m_f} + 3\right)\omega_1(f, \frac{1}{n}), x \in [0, 1], n \in \mathbb{N},$$

where $m_f = \min\{f(x); x \in [0, 1]\}$.

Proof. We prove the theorem by mathematical induction on the variable $l \in \{1, 2 \ldots, \}$ representing the number of intervals given by the division of the interval $[0, 1]$. If $l = 1$, then it is immediate that f is monotone and the conclusion follows from Theorem 2.2.13 or Corollary 2.2.14, respectively. If $l = 2$, then the conclusion follows from Theorem 2.2.15 or Theorem 2.2.16, respectively. Suppose now that the conclusion of the lemma holds for any p, $1 \leq p \leq l-1$. We have two cases: (i) f is nonincreasing on $[x_{l-1}, 1]$ and (ii) f is nondecreasing on $[x_{l-1}, 1]$.

Case (i). First we define the function

$$g(x) = \begin{cases} f(x); & x \in [0, x_{l-1}], \\ f(x_{l-1}); & x \in [x_{l-1}, 1]. \end{cases}$$

Then, we introduce the function h depending on the value $f(x_{l-1})$. If x_{l-1} is the global maximum point of f, then we consider

$$h(x) = \begin{cases} f(x_{l-1}); & x \in [0, x_{l-1}], \\ f(x); & x \in [x_{l-1}, 1]. \end{cases}$$

Otherwise, let $c \in [0, x_{l-2}]$ be the point of maximum value where the graph of f intersects the line $y = f(x_{l-1})$. We define

$$h(x) = \begin{cases} f(x); & x \in [0, c], \\ f(x_{l-1}); & x \in [c, x_{l-1}], \\ f(x); & x \in [x_{l-1}, 1]. \end{cases}$$

Since on the interval $[x_{l-2}, 1]$, g is monotone, it follows that the interval $[0, 1]$ can be split in p intervals, $p < l$, satisfying the hypothesis in the present theorem. This statement holds for h too. From our assumption it follows that

$$\left|B_n^{(M)}(g)(x) - g(x)\right| \leq \left(\frac{n\omega_1(g, \frac{1}{n})}{m_g} + 3\right) \omega_1(g, \frac{1}{n}), x \in [0, 1].$$

and

$$\left|B_n^{(M)}(h)(x) - h(x)\right| \leq \left(\frac{n\omega_1(h, \frac{1}{n})}{m_h} + 3\right) \omega_1(h, \frac{1}{n}), x \in [0, 1].$$

Now, let us choose arbitrary $x \in [0, 1]$. If $x \in [0, x_{l-1}]$, then $f(x) = g(x)$ and $B_n^{(M)}(f)(x) \leq B_n^{(M)}(g)(x)$. This implies

$$B_n^{(M)}(f)(x) - f(x) \leq \left(\frac{n\omega_1(g, \frac{1}{n})}{m_g} + 3\right) \omega_1(g, \frac{1}{n}).$$

It is easy to check that $\omega_1(g, \frac{1}{n}) \leq \omega_1(f, \frac{1}{n})$ and that $m_f \leq m_g$. Therefore, we obtain the desired conclusion in this case.

If $x \in [x_{l-1}, 1]$, then $f(x) = h(x)$ and $B_n^{(M)}(f)(x) \le B_n^{(M)}(h)(x)$. This implies

$$B_n^{(M)}(f)(x) - f(x) \le \left(\frac{n\omega_1(h, \frac{1}{n})}{m_h} + 3 \right) \omega_1(h, \frac{1}{n}).$$

Again, it is easy to prove that $\omega_1(h, \frac{1}{n}) \le \omega_1(f, \frac{1}{n})$ and that $m_f \le m_h$. Hence, we get the conclusion of the theorem in this case too.

Case (ii). We construct the function g exactly as in the above case (i). If x_{l-1} is a global minimum point for f, then we take

$$h(x) = \begin{cases} f(x_{l-1}); \ x \in [0, x_{l-1}], \\ f(x); \quad x \in [x_{l-1}, 1]. \end{cases}$$

Otherwise, let $c \in [0, x_{l-2}]$ be the point of maximum value where the graph of f intersects the line $y = f(x_{l-1})$. We take

$$h(x) = \begin{cases} f(x); \qquad x \in [0, c], \\ f(x_{l-1}); \ x \in [c, x_{l-1}], \\ f(x); \quad x \in [x_{l-1}, 1]. \end{cases} .$$

Clearly, we may suppose that for g and h we have the same estimations as in the above case (i). Since $f = g \bigvee h$ it follows that

$$B_n^{(M)}(f)(x) = B_n^{(M)}(g)(x) \bigvee B_n^{(M)}(h)(x).$$

From now on the proof goes on the same pattern as in the proof of Theorem 2.2.15 and noting that $\max\{\omega_1(g, \frac{1}{n}), \omega_1(h, \frac{1}{n})\} \le \omega_1(f, \frac{1}{n})$ and that $m_f = \min\{m_g, m_h\}$ we easily get the desired conclusion in this case too and the proof is complete. □

We present now the following most general approximation result for continuous strictly positive functions.

Theorem 2.2.18 (Coroianu–Gal [52]). *Let $f : [0, 1] \to [0, \infty)$ be a continuous and strictly positive function. Then*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le \left(\frac{n\omega_1(f, \frac{1}{n})}{m_f} + 4 \right) \omega_1(f, \frac{1}{n}), x \in [0, 1], n \in \mathbb{N}, \tag{2.6}$$

where $m_f = \min\{f(x); x \in [0, 1]\}$.

Proof. As in the proof of Theorem 2.2.13, for $n \in \mathbb{N}$, we consider the function $g = (g_{0[x_0,x_1]}, g_{1[x_1,x_2]}, \ldots, g_{n-1[x_{n-1},x_n]})$. It is immediate that g satisfies the hypothesis in Theorem 2.2.17. In addition, we have $\omega_1(g, \frac{1}{n}) \le \omega_1(f, \frac{1}{n})$ and $m_f \le m_g$.

Furthermore, since g is monotone on any interval of the form $[x_i, x_{i+1}]$, $i \in \{0, 1, \ldots, n-1\}$, we get $|f(x) - g(x)| \leq \omega_1(f, \frac{1}{n})$ for all $x \in [0, 1]$ as in the proof of Theorem 2.2.13. Taking into account the proof of Theorem 2.2.13, we get

$$\begin{aligned}
&\left|B_n^{(M)}(f)(x) - f(x)\right| \\
&= \left|B_n^{(M)}(g)(x) - f(x)\right| \leq \left|B_n^{(M)}(g)(x) - g(x)\right| + |f(x) - g(x)| \\
&\leq \left(\frac{n\omega_1(f, \frac{1}{n})}{m_f} + 3\right)\omega_1(f, \frac{1}{n}) + \omega_1(f, \frac{1}{n}) = \left(\frac{n\omega_1(f, \frac{1}{n})}{m_f} + 4\right)\omega_1(f, \frac{1}{n}).
\end{aligned}$$

which proves the theorem. □

Corollary 2.2.19 (Coroianu–Gal [52]). *If $f : [0,1] \to [0, \infty)$ is a strictly positive function satisfying the Lipschitz condition, then there exists a constant C independent of n and x but depending on f, such that*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \frac{C}{n}, x \in [0, 1], n \in \mathbb{N}.$$

Proof. Since f satisfies the Lipschitz condition, it follows that there exists $C_0 > 0$ such that $\omega_1(f, \frac{1}{n}) \leq \frac{C_0}{n}$. Substituting in (2.6) we obtain

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{C_0}{m_f} + 4\right)\frac{C_0}{n}, x \in [0, 1].$$

For $C = \left(\frac{C_0}{m_f} + 4\right) C_0$ we get the desired conclusion. □

Remarks. 1) Theorem 2.2.18 gives the order of uniform approximation (with the constant in O depending on f)

$$O\left\{n\left[\omega_1\left(f; \frac{1}{n}\right)\right]^2 + \omega_1\left(f; \frac{1}{n}\right)\right\},$$

which for the classes of Lipschitz functions Lip_α gives the approximation order $1/n^{2\alpha-1}$, that for $\alpha \in (2/3, 1]$ is essentially better than the general approximation order $O[\omega_1(f; 1/\sqrt{n})] = O[1/n^{\alpha/2}]$.

2) Comparing with the approximation error given by the linear Bernstein polynomials $B_n(f)(x)$ in Subsection 1.1.1, point (i), formula (1.2), case when in order to get, for example, the order of approximation $O\left(\frac{1}{n}\right)$, we have to suppose that f' is a Lipschitz 1-function, we see that in the case of approximation by $B_n^{(M)}(f)$, this order can be achieved under the less restrictive condition that f is a Lipschitz 1-function. This shows that the saturation class for the max-product Bernstein operator $B_n^{(M)}$ differs (in fact it is much larger) from the saturation class for the linear Bernstein polynomials.

Now, since $B_n^{(M)}(f)(0) - f(0) = B_n^{(M)}(f)(1) - f(1) = 0$, it is natural to look for a better pointwise estimate near to the endpoints 0 and 1. In this sense, we present the following two results.

Theorem 2.2.20 (Coroianu–Gal [52]). *Let $f : [0,1] \to [0,\infty)$ be a continuous function. Then*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le 24\omega_1\left(f, \sqrt{\frac{x(1-x)}{n}}\right),$$

for all $x \in [0, 1/(n+1)] \cup [n/(n+1), 1]$ and $n \in \mathbb{N}$, $n \ge 2$.

Proof. First, let us choose arbitrary $x \in [0, 1/(n+1)]$. By relation (2.1) in the proof of Theorem 2.1.5, we have

$$|B_n^{(M)}(f)(x) - f(x)| \le \left(1 + \frac{1}{\delta} B_n^{(M)}(\varphi_x)(x)\right) \omega_1(f;\delta), \tag{2.7}$$

where $\varphi_x(t) = |t - x|$, $t \in [0,1]$ and $\delta > 0$ is chosen arbitrary. So, it is enough to estimate

$$E_n(x) := B_n^{(M)}(\varphi_x)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x)\left|\frac{k}{n} - x\right|}{\bigvee_{k=0}^{n} p_{n,k}(x)}.$$

Since $x \in [0, 1/(n+1)]$, by Lemma 2.1.4 we get $\bigvee_{k=0}^{n} p_{n,k}(x) = p_{n,0}(x)$, which immediately implies $E_n(x) = \bigvee_{k=0}^{n} \binom{n}{k}\left(\frac{x}{1-x}\right)^k \left|\frac{k}{n} - x\right|$. Let $k_0 \in \{0, 1, \ldots, n\}$ be such that $E_n(x) = \binom{n}{k_0}\left(\frac{x}{1-x}\right)^{k_0}\left|\frac{k_0}{n} - x\right|$. If $k_0 = 0$, then $E_n(x) = x$. If $k_0 > 0$, then we get

$$\begin{aligned}
E_n(x) &= \binom{n}{k_0}\left(\frac{x}{1-x}\right)^{k_0}\left(\frac{k_0}{n} - x\right) \\
&\le \binom{n}{k_0}\left(\frac{x}{1-x}\right)^{k_0} \cdot \frac{k_0}{n} \\
&= \binom{n-1}{k_0-1}\left(\frac{x}{1-x}\right)^{k_0} = \binom{n-1}{k_0-1}\left(\frac{x}{1-x}\right)^{k_0-1} \cdot \frac{x}{1-x} \\
&\le \left(1 + \frac{x}{1-x}\right)^{n-1} \cdot \frac{x}{1-x} \le \left(1 + \frac{x}{1-x}\right)^{n-1} \cdot \frac{x}{1 - 1/(n+1)} \\
&= \frac{n+1}{n}\left(1 + \frac{x}{1-x}\right)^{n-1} \cdot x.
\end{aligned}$$

Since the function $g(x) = \frac{x}{1-x}$ is nondecreasing on $(0, 1/(n+1)]$, we get

$$E_n(x) \leq 2x\left(1 + \frac{1/(n+1)}{1 - 1/(n+1)}\right)^{n-1} = 2x\left(1 + \frac{1}{n}\right)^{n-1} \cdot \frac{n+1}{n}$$
$$= 2x\left(1 + \frac{1}{n}\right)^{n} \leq 2ex.$$

From the above estimates we get $E_n(x) \leq 2ex$ for all $x \in [0, 1/(n+1)]$. Now, taking $\delta = 2ex$ in relation (2.7), we get

$$|B_n^{(M)}(f)(x) - f(x)| \leq 2\omega_1(f, 2ex) \leq 12\omega_1(f, x),$$

where we used the well-known property $\omega_1(f, \lambda x) \leq ([\lambda] + 1)\omega_1(f, x)$. Because $x \in [0, 1/(n+1)] \subset [0, 1/2]$ implies $1 - x \geq 1/2$, we get

$$|B_n^{(M)}(f)(x) - f(x)|$$
$$\leq 12\omega_1(f, x) \leq 24\omega_1\left(f, \frac{1}{2} \cdot \sqrt{x} \cdot \sqrt{x}\right) \leq 24\omega_1\left(f, \sqrt{x(1-x)} \cdot \sqrt{x}\right)$$
$$\leq 24\omega_1\left(f, \sqrt{\frac{x(1-x)}{n}}\right).$$

Now, let us chose arbitrary $x \in [n/(n+1), 1]$. Take $g : [0, 1] \to \mathbb{R}$, $g(x) = f(1-x)$. Because $1 - x \in [0, 1/(n+1)]$ and $\omega_1\left(f, \sqrt{\frac{x(1-x)}{n}}\right) = \omega_1\left(g, \sqrt{\frac{x(1-x)}{n}}\right)$ and since $\left|B_n^{(M)}(f)(x) - f(x)\right| = \left|B_n^{(M)}(g)(1-x) - g(1-x)\right|$ we immediately obtain the same estimate as in the previous case and the theorem is proved. □

Combining Theorem 2.2.18 with Theorem 2.2.20, we obtain the following mixed pointwise-uniform estimate, essentially better near to 0 and 1.

Corollary 2.2.21 (Coroianu–Gal [52]). *Let $f : [0, 1] \to [0, \infty)$ be a continuous and strictly positive function. Then, for all $n \in \mathbb{N}$, $n \geq 2$, we have the estimates:*

$$|B_n^{(M)}(f)(x) - f(x)| \leq 24\omega_1\left(f, \sqrt{\frac{x(1-x)}{n}}\right),$$

for all $x \in [0, 1/(n+1)] \cup [n/(n+1), 1]$, and

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \leq \left(\frac{n\omega_1(f, \frac{1}{n})}{m_f} + 4\right)\omega_1(f, \frac{1}{n}),$$

for all $x \in [1/(n+1), n/(n+1)]$.

Remark. Since for $x \in [0, 1/(n+1)] \cup [n/(n+1), 1]$ we easily have $\sqrt{\frac{x(1-x)}{n}} \leq \frac{1}{n}$, even the uniform estimate generated in this way by Corollary 2.2.21 is obviously better than the uniform estimate in Theorem 2.2.18.

In the previous section, by Corollary 2.1.18, it was proved that the Bernstein max-prod operator preserves the quasiconvexity. In this section we will prove that the discussed operator preserves the quasiconcavity too. In this sense, we present the following shape preserving results.

Theorem 2.2.22 (Coroianu–Gal [52]). *Let us consider the function* $f : [0, 1] \to \mathbb{R}_+$ *and let us fix* $n \in \mathbb{N}$, $n \geq 1$. *Suppose in addition that there exists* $c \in [0, 1]$ *such that* f *is nondecreasing on* $[0, c]$ *and nonincreasing on* $[c, 1]$. *Then, there exists* $c' \in [0, 1]$ *such that* $B_n^{(M)}(f)$ *is nondecreasing on* $[0, c']$ *and nonincreasing on* $[c', 1]$. *In addition we have* $|c - c'| \leq \frac{1}{n+1}$ *and* $\left|B_n^{(M)}(f)(c) - f(c)\right| \leq \omega_1(f, \frac{1}{n+1})$.

Proof. Let $j_c \in \{0, 1, \ldots, n\}$ be such that $c \in [\frac{j_c}{n+1}, \frac{j_c+1}{n+1}]$. We will study the monotonicity on each interval of the form $[\frac{j}{n+1}, \frac{j+1}{n+1}]$, $j \in \{0, 1, \ldots, n\}$, then by the continuity of $B_n^{(M)}(f)$ we will be able to determine the monotonicity of $B_n^{(M)}(f)$ on $[0, 1]$.

Let us choose arbitrary $j \in \{0, 1, \ldots, j_c - 1\}$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. By the monotonicity of f, it follows that $f(\frac{j}{n}) \geq f(\frac{j-1}{n}) \geq \cdots \geq f(0)$. By Lemma 2.2.1, (ii), it easily follows that $f_{j,n,j}(x) \geq f_{j-1,n,j}(x) \geq \cdots \geq f_{0,n,j}(x)$. Now, by Lemma 2.2.2 it follows that $B_n^{(M)}(f)(x) = \bigvee_{k=j}^{n} f_{k,n,j}(x)$. Since $B_n^{(M)}(f)$ is defined as the maximum of nondecreasing functions, it follows that it is nondecreasing on $[\frac{j}{n+1}, \frac{j+1}{n+1}]$. Taking into account the continuity of $B_n^{(M)}(f)$, it is immediate that f is nondecreasing on $[0, \frac{j_c}{n+1}]$.

Now, let us chose arbitrary $j \in \{j_c + 1, j_c + 2, \ldots, n\}$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$. By the monotonicity of f, it follows that $f(\frac{j}{n}) \geq f(\frac{j+1}{n}) \geq \cdots \geq f(1)$. By Lemma 2.2.1, (i), it easily follows that $f_{j,n,j}(x) \geq f_{j+1,n,j}(x) \geq \cdots \geq f_{n,n,j}(x)$. Now, by Lemma 2.2.2 it follows that $B_n^{(M)}(f)(x) = \bigvee_{k=0}^{j} f_{k,n,j}(x)$. Since $B_n^{(M)}(f)$ is defined as the maximum of nonincreasing functions, it follows that it is nonincreasing on $[\frac{j}{n+1}, \frac{j+1}{n+1}]$. Taking into account the continuity of $B_n^{(M)}(f)$, it is immediate that f is nonincreasing on $[\frac{j_c+1}{n+1}, 1]$.

Finally, let us discuss the case when $j = j_c$. If $\frac{j}{n} \leq c$, then by the monotonicity of f it follows that $f(\frac{j_c}{n}) \geq f(\frac{j_c-1}{n}) \geq \cdots \geq f(0)$. Therefore, in this case we obtain that f is nondecreasing on $[\frac{j_c}{n+1}, \frac{j_c+1}{n+1}]$. It follows that f is nondecreasing on $[0, \frac{j_c+1}{n+1}]$ and nonincreasing on $[\frac{j_c+1}{n+1}, 1]$. In addition, $c' = \frac{j_c+1}{n+1}$ is the maximum point of $B_n^{(M)}(f)$ and it is easy to check that $|c - c'| \leq \frac{1}{n+1}$. If $\frac{j_c}{n} \geq c$, then by the monotonicity of f it follows that $f(\frac{j_c}{n}) \geq f(\frac{j_c+1}{n}) \geq \cdots \geq f(1)$. Therefore, in this case we obtain that

f is nonincreasing on $[\frac{j_c}{n+1}, \frac{j_c+1}{n+1}]$. It follows that f is nondecreasing on $[0, \frac{j_c}{n+1}]$ and nonincreasing on $[\frac{j_c}{n+1}, 1]$. In addition, $c' = \frac{j_c}{n+1}$ is the maximum point of $B_n^{(M)}(f)$ and again, it is easy to check that $|c - c'| \leq \frac{1}{n+1}$.

We prove now the last part of the theorem. First, let us notice that $B_n^{(M)}(f)(x) \leq f(c)$ for all $x \in [0, 1]$. Indeed, this is immediate by the definition of $B_n^{(M)}(f)$ and by the fact that c is the global maximum point of f. This implies

$$\left|B_n^{(M)}(f)(c) - f(c)\right|$$

$$= f(c) - B_n^{(M)}(f)(c) = f(c) - \bigvee_{k=0}^{n} f_{k,n,j_c}(c) \leq f(c) - f_{j_c,n,j_c}(c)$$

$$= f(c) - f(\frac{j_c}{n}).$$

Since $c, \frac{j_c}{n} \in [\frac{j_c}{n+1}, \frac{j_c+1}{n+1}]$, we easily get $f(c) - f(\frac{j_c}{n}) \leq \omega_1(f, \frac{1}{n+1})$ and the theorem is proved completely. □

Corollary 2.2.23 (Coroianu–Gal [52]). *If $f : [0, 1] \to \mathbb{R}_+$ is continuous and quasiconcave on $[0, 1]$, then for all $n \in \mathbb{N}$, $n \geq 1$, $B_n^{(M)}(f)$ is quasiconcave on $[0, 1]$.*

Proof. It is immediate by the Remark after the Definition 2.2.3 and by the Theorem 2.2.22. □

2.3 Saturation Results

All the results in the previous sections put in evidence the potential of the max-product Bernsten operator.

The goal of this section is to determine the saturation order together with its corresponding special class and to obtain a local inverse result for the max-product Bernstein operator.

Firstly, we need the following auxiliary result.

Lemma 2.3.1 (Coroianu–Gal [53]). *If $f : [0, 1] \to \mathbb{R}_+$, then for all $n \in \mathbb{N}$, $n \geq 1$ and for all $j \in \{0, 1, \ldots, n\}$, we have*

$$B_n^{(M)}(f)(j/(n+1)) \geq f(j/n).$$

Proof. Let us choose arbitrary $j \in \{0, 1, \ldots, n\}$. By relation just before the Lemma 2.1.6, one has

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x), x \in [j/(n+1), (j+1)/(n+1)], \tag{2.8}$$

where

$$f_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \cdot \left(\frac{x}{1-x}\right)^{k-j} \cdot f(k/n)$$

for all $k \in \{0, 1, \ldots, n\}$. Relation (2.8) implies $B_n^{(M)}(f)(x) \geq f_{k,n,j}(x)$ for all $x \in [j/(n+1), (j+1)/(n+1)]$ and $k \in \{0, 1, \ldots, n\}$. In particular, for $x = j/(n+1)$ and $k = j$, we get $B_n^{(M)}(f)(j/(n+1)) \geq f_{j,n,j}(j/(n+1))$. But since $f_{j,n,j}(j/(n+1)) = f(j/n)$, we immediately obtain the desired conclusion. □

The next result establishes the saturation order for the Bernstein max-product operator.

Theorem 2.3.2 (Coroianu–Gal [53]). *Denote* $C_+[0, 1] = \{f : [0, 1] \to \mathbb{R}_+; f$ *is continuous on* $[0, 1]\}$ *and* $\|f\| = \sup\{|f(x)|; x \in [0, 1]\}$. *The saturation order is* $\frac{1}{n}$ *and* $\|B_n^{(M)}(f) - f\| = o(1/n)$ *if and only if* $f \in C_+[0, 1]$ *is a constant function on* $[0, 1]$.

Proof. Firstly, it is immediate that for $f \in C_+[0, 1]$ constant function on $[0, 1]$ we have $B_n^{(M)}(f)(x) - f(x) = 0$ for all $x \in [0, 1]$.

Now, let us suppose that $B_n^{(M)}(f)$ approximates $f \in C_+[0, 1]$ with an order of approximation better than $\frac{1}{n}$. In this case, there exists $a_n \in \mathbb{R}$, $n \in \mathbb{N}$ with the property $a_n \searrow 0$ as $n \to +\infty$, such that

$$\left|f(x) - B_n^{(M)}(f)(x)\right| \leq \frac{a_n}{n}, \text{ for all } x \in [0, 1] \text{ and } n \in \mathbb{N}.$$

Let us choose arbitrary $\varepsilon > 0$. Since $a_n \searrow 0$ as $n \to +\infty$, it follows that there exists $n_0 \in \mathbb{N}$ such that $a_n < \varepsilon$ for all $n \in \mathbb{N}$, $n \geq n_0$. From the above relation we get

$$\left|f(x) - B_n^{(M)}(f)(x)\right| \leq \frac{\varepsilon}{n}, \text{ for all } x \in [0, 1] \text{ and } n \in \mathbb{N}, n \geq n_0.$$

Clearly, this implies

$$B_n^{(M)}(f)(x) - f(x) \leq \frac{\varepsilon}{n}, \text{ for all } x \in [0, 1] \text{ and } n \in \mathbb{N}, n \geq n_0.$$

In particular, for $x = j/(n + 1)$, we obtain

$$B_n^{(M)}(f)(j/(n + 1)) - f(j/(n + 1)) \leq \frac{\varepsilon}{n}, \text{ for all } j,n \in \mathbb{N}, n \geq n_0, j \leq n.$$

Now, since by Lemma 2.3.1 we have $B_n^{(M)}(f)(j/(n + 1)) \geq f(j/n)$, it follows that

$$f(j/n) - f(j/(n + 1)) \leq \frac{\varepsilon}{n}, \text{ for all } j,n \in \mathbb{N}, n \geq n_0, j \leq n. \tag{2.9}$$

Then, from the uniform continuity of f it results the existence of $n_1 \in \mathbb{N}$ such that

$$|f(x) - f(y)| \leq \varepsilon \text{ for all } x, y \in [0, 1] \text{ and } n \in \mathbb{N},\ |x - y| \leq 1/n, n \geq n_1. \tag{2.10}$$

We will obtain the desired conclusion by proving that f is constant on any arbitrary interval $[a, b]$ with $0 < a < b < 1$. Indeed, if this property holds, then owing to the continuity of f on $[0, 1]$ we immediately get the desired conclusion, that is the function f is a constant function. So, let us choose arbitrary $a, b \in [0, 1]$ such that $0 < a < b < 1$. Then, let $x_0 \in [a, b]$ and $y_0 \in [a, b]$ be the points where f attains its minimum and, respectively, the maximum on the interval $[a, b]$. If $x_0 = y_0$, then it is immediate that f is constant on the interval $[a, b]$. Therefore, without any loss of generality we may suppose that $x_0 \neq y_0$. We have two cases: 1) $x_0 < y_0$ and 2) $x_0 > y_0$.

Case 1) Let us choose arbitrary $n \in \mathbb{N}$, $n \geq n_2$, where $n_2 = \max\{n_0, n_1\}$. Since $\lim\limits_{l \to \infty} \frac{n}{n+l} = 0$ and since $x_0 > 0$, it follows that there exists sufficiently large l_0, $k_0 \in \mathbb{N}$, $l_0 > k_0 \geq 1$, such that

$$\frac{n}{n + l_0 + 1} \leq x_0 \leq \frac{n}{n + l_0} \leq \frac{n}{n + l_0 - 1} \leq \cdots \leq \frac{n}{n + k_0} \leq y_0 \leq \frac{n}{n + k_0 - 1}.$$

Also, we obtain

$$\left| x_0 - \frac{n}{n + l_0} \right| \leq \frac{n}{n + l_0} - \frac{n}{n + l_0 + 1} = \frac{n}{(n + l_0)(n + l_0 + 1)} < \frac{1}{n} \leq \frac{1}{n_1},$$

which by relation (2.10) implies that

$$|f(x_0) - f(n/(n + l_0))| \leq \varepsilon. \tag{2.11}$$

By similar reasonings we get that

$$|f(y_0) - f(n/(n + k_0))| \leq \varepsilon. \tag{2.12}$$

Now, applying successively relation (2.9), we obtain

$$f(n/(n + k_0)) - f(n/(n + k_0 + 1)) \leq \frac{\varepsilon}{n + k_0},$$

$$f(n/(n + k_0 + 1)) - f(n/(n + k_0 + 2)) \leq \frac{\varepsilon}{n + k_0 + 1},$$

$$f(n/(n + k_0 + 2)) - f(n/(n + k_0 + 3)) \leq \frac{\varepsilon}{n + k_0 + 2},$$

$$\vdots$$

$$f(n/(n + l_0 - 1)) - f(n/(n + l_0)) \leq \frac{\varepsilon}{n + l_0 - 1}.$$

Taking the sum of all these inequalities, after some simple calculations we get

$$\begin{aligned}&f(n/(n+k_0))-f(n/(n+l_0))\\ &\le \varepsilon\left(\frac{1}{n+k_0}+\frac{1}{n+k_0+1}+\ldots+\frac{1}{n+l_0-1}\right)\le\frac{(l_0-k_0)\varepsilon}{n}.\end{aligned}$$

Then, from relations (2.11)–(2.12) combined with the above inequality, we get

$$\begin{aligned}0\le f(y_0)-f(x_0)&\le |f(y_0)-f(n/(n+k_0))|+f(n/(n+k_0))-f(n/(n+l_0))\\ &\quad+|f(n/(n+l_0))-f(x_0)|\le\frac{(l_0-k_0)\varepsilon}{n}+2\varepsilon.\end{aligned}\tag{2.13}$$

Since $x_0\le\frac{n}{n+l_0}$ (see the first line of inequalities of this case 1)) we immediately get $l_0\le\frac{n(1-x_0)}{x_0}$ and then $l_0-k_0\le\frac{n(1-x_0)}{x_0}$. Using this inequality in relation (2.13) we obtain

$$0\le f(y_0)-f(x_0)\le\frac{\varepsilon(1-x_0)}{x_0}+2\varepsilon=\frac{\varepsilon(1+x_0)}{x_0},$$

where $\varepsilon>0$ was chosen arbitrary. Therefore, passing in the previous inequalities with $\varepsilon\searrow 0$, we obtain $f(x_0)=f(y_0)$ (here, it is important that $x_0>0$). This clearly implies that f is a constant function on the interval $[a,b]$.

Case 2) Take $g:[0,1]\to\mathbb{R}_+$, $g(x)=f(1-x)$. Since we obviously have $B_n^{(M)}(f)(x)=B_n^{(M)}(g)(1-x)$ for all $x\in[0,1]$ we get

$$\|B_n^{(M)}(f)-f\|=\|B_n^{(M)}(g)-g\|,\ \text{ for all } n\in\mathbb{N}.$$

Clearly, this means

$$\left|g(x)-B_n^{(M)}(g)(x)\right|\le\frac{a_n}{n},\ \text{ for all } x\in[0,1] \text{ and } n\in\mathbb{N}.$$

Then, it is elementary to prove that $f([a,b])=g([1-b,1-a])$ and since $f(x_0)=g(1-x_0)$ and $f(y_0)=g(1-y_0)$, it is immediate that $1-x_0$ and $1-y_0$ are the minimum point, respectively, the maximum point of the function g on the interval $[1-b,1-a]$. By $1-x_0<1-y_0$, we can apply the conclusion of case 1) for the function g on the interval $[1-b,1-a]$. Therefore, it follows that g is constant on the interval $[1-b,1-a]$, which easily implies that f is constant on the interval $[a,b]$. This finishes the proof of the theorem. □

Remark. The positivity of f in Theorem 2.3.2 can be dropped. Indeed, suppose that $f:[0,1]\to\mathbb{R}$ is bounded and of variable sign on $[0,1]$. Choose a constant $c^*>0$ such that $f(x)+c^*>0$, for all $x\in[0,1]$ and define the new max-product kind operator

$$A_n^{(M)}(f)(x)=B_n^{(M)}(f+c^*)(x)-c^*,\ \text{ for all } x\in[0,1].$$

Then, Theorem 2.3.2 holds for $A_n^{(M)}(f)$ with f not necessarily positive on $[0, 1]$ (see Theorem 2.9.1, (ix) in the last section of this chapter).

According to Corollary 2.2.19, the saturation order $\frac{1}{n}$ in the above Theorem 2.3.2 is attained for strictly positive Lipschitz functions, on $[0, 1]$. Conversely, more general if we replace the strict positivity by the positivity, then we can present the following local inverse result.

Theorem 2.3.3 (Coroianu–Gal [53]). *Let $f : [0, 1] \to [0, +\infty)$ and $0 < \alpha < \beta < 1$ be such that f is continuous on $[\alpha, \beta]$. If there exists a constant $M > 0$(independent of n but depending on f, α and β) such that*

$$\|B_n^{(M)}(f) - f\|_{[\alpha,\beta]} \leq M/n, \text{ for all } n \in \mathbb{N},$$

then $f|_{[\alpha,\beta]} \in Lip_L 1\,([\alpha, \beta])$ with $L = M + \max\left\{\frac{1}{\alpha}, \frac{1}{1-\beta}\right\} \cdot \|f\|_{[\alpha,\beta]}$, that is f is a Lipschitz 1 function on $[\alpha, \beta]$. Here $\|f\|_{[\alpha,\beta]} = \sup\{|f(x)|; x \in [\alpha, \beta]\}$ and

$$Lip_L 1\,([\alpha, \beta]) = \{g : [\alpha, \beta] \to \mathbb{R}; |g(x) - g(y)| \leq L|x - y|, \text{ for all } x, y \in [\alpha, \beta]\}.$$

The proof of Theorem 2.3.3 requires the next lemma.

Lemma 2.3.4. *Let $f : [0, 1] \to \mathbb{R}$ and $0 \leq \alpha < \beta \leq 1$ be fixed, such that f is continuous on $[\alpha, \beta]$. For $n \in \mathbb{N}$ satisfying $n \geq 2/(\beta - \alpha)$, denote*

$$\begin{aligned} &M_n(\alpha, \beta) \\ &= \max\left\{\left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right| : k \in \{1, \ldots, n-2\}, \alpha \leq \frac{k}{n}, \frac{k+1}{n} \leq \beta\right\}. \end{aligned}$$

Then

$$\limsup\,\{M_n(\alpha, \beta) : n \in \mathbb{N},\, n \geq 2/(\beta - \alpha)\} \leq C/n$$

(with $C > 0$ independent of n), if and only if f is Lipschitz 1 on $[\alpha, \beta]$ with the Lipschitz constant $C > 0$.

Proof. Since the proof of the converse implication is trivial, it is omitted. In order to prove the direct implication, let us choose arbitrary $x, y \in [\alpha, \beta]$ with $x < y$ and $n \in \mathbb{N}$ with $n \geq 2/(\beta - \alpha)$. Since f is uniformly continuous on $[\alpha, \beta]$, there exists $m_0 \in \mathbb{N}$ such that for any $u, v \in [\alpha, \beta]$, $|u - v| \leq 1/m_0$, we have

$$|f(u) - f(v)| \leq \frac{1}{n}.$$

For this m_0, let $k_0, l_0 \in \mathbb{N}$ be such that

$$\frac{k_0 - 1}{m_0} \leq x \leq \frac{k_0}{m_0} \leq \cdots \leq \frac{k_0 + l_0}{m_0} \leq y \leq \frac{k_0 + l_0 + 1}{m_0}. \tag{2.14}$$

We have

$$
\begin{aligned}
&|f(x)-f(y)| \\
&\le \left|f(x)-f\left(\frac{k_0}{m}\right)\right| + \sum_{p=0}^{l_0-1}\left[f\left(\frac{k_0+p+1}{m}\right)-f\left(\frac{k_0+p}{m}\right)\right] \\
&\quad + \left|f(y)-f\left(\frac{k_0+l_0}{m}\right)\right|.
\end{aligned}
$$

The way m_0 was chosen implies that

$$
\left|f(x)-f\left(\frac{k_0}{m}\right)\right| + \left|f(y)-f\left(\frac{k_0+l_0}{m}\right)\right| \le \frac{2}{n}.
$$

On the other hand, the hypothesis implies that

$$
\sum_{p=0}^{l_0-1}\left[f\left(\frac{k_0+p+1}{m}\right)-f\left(\frac{k_0+p}{m}\right)\right] \le \frac{l_0 C}{m}.
$$

We observe that relation (2.14) implies that $l_0/m \le y-x$ and hence we get

$$
\sum_{p=0}^{l_0-1}\left[f\left(\frac{k_0+p+1}{m}\right)-f\left(\frac{k_0+p}{m}\right)\right] \le C(y-x).
$$

Summarizing, we obtain $|f(x)-f(y)| \le C(y-x)+\frac{2}{n}$ and letting $n\to\infty$, it follows $|f(x)-f(y)| \le C(y-x)$. Since x, y are arbitrary in $[\alpha,\beta]$ it is immediate that f is Lipschitz 1 on $[\alpha,\beta]$ with the constant C. The proof is complete. □

Now we are in position to prove Theorem 2.3.3.

Proof of Theorem 2.3.3. For $n\in\mathbb{N}$ with $n\ge 2/(\beta-\alpha)$, let us choose $k(n)\in\{1,\ldots,n-2)$ such that $\alpha\le\frac{k(n)}{n}<\frac{k(n)+1}{n}\le\beta$ and (keeping the notations for $M_n(\alpha,\beta)$ in Lemma 2.3.4)

$$
M_n(\alpha,\beta)=\left|f\left(\frac{k(n)+1}{n}\right)-f\left(\frac{k(n)}{n}\right)\right|.
$$

Since $\frac{k(n)}{n}\in\left[\frac{k(n)}{n+1},\frac{k(n)+1}{n+1}\right]$ and $\frac{k(n)+1}{n}\in\left[\frac{k(n)+1}{n+1},\frac{k(n)+2}{n+1}\right]$, by Lemma 3.4 in [21] (see also Lemma 2.1.4) we get $\bigvee_{k=0}^{n} p_{n,k}(k(n)/n) = p_{n,k(n)}(k(n)/n)$ and $\bigvee_{k=0}^{n} p_{n,k}((k(n)+1)/n) = p_{n,k(n)+1}((k(n)+1)/n)$. We have two cases: (i) $f\left(\frac{k(n)+1}{n}\right)\ge f\left(\frac{k(n)}{n}\right)$ and (ii) $f\left(\frac{k(n)+1}{n}\right)<f\left(\frac{k(n)}{n}\right)$.

Case (i) We have

$$B_n^{(M)}(f)\left(\frac{k(n)}{n}\right) - f\left(\frac{k(n)}{n}\right) = \frac{\bigvee_{k=0}^{n} p_{n,k}(k(n)/n)\cdot f(k/n)}{\bigvee_{k=0}^{n} p_{n,k}(k(n)/n)} - f\left(\frac{k(n)}{n}\right)$$

$$= \frac{\bigvee_{k=0}^{n} p_{n,k}(k(n)/n)\cdot f(k/n)}{p_{n,k(n)}(k(n)/n)} - f\left(\frac{k(n)}{n}\right)$$

$$\geq \frac{p_{n,k(n)+1}(k(n)/n)\cdot f((k(n)+1)/n)}{p_{n,k(n)}(k(n)/n)} - f\left(\frac{k(n)}{n}\right)$$

$$= \frac{k(n)}{k(n)+1}\cdot f\left(\frac{k(n)+1}{n}\right) - f\left(\frac{k(n)}{n}\right).$$

This implies

$$B_n^{(M)}(f)\left(\frac{k(n)}{n}\right) - f\left(\frac{k(n)}{n}\right) \geq \left| f\left(\frac{k(n)+1}{n}\right) - f\left(\frac{k(n)}{n}\right)\right| - \frac{1}{k(n)+1}\cdot f\left(\frac{k(n)+1}{n}\right),$$

that is

$$M_n(\alpha,\beta) \leq B_n^{(M)}(f)\left(\frac{k(n)}{n}\right) - f\left(\frac{k(n)}{n}\right) + \frac{1}{k(n)+1}\cdot f\left(\frac{k(n)+1}{n}\right).$$

Obviously, this implies

$$M_n(\alpha,\beta) \leq \left\|B_n^{(M)}(f) - f\right\|_{[\alpha,\beta]} + \frac{1}{k(n)+1}\cdot \|f\|_{\alpha,\beta]}$$

and taking into account the hypothesis we get

$$M_n(\alpha,\beta) \leq \frac{C}{n} + \frac{1}{k(n)+1}\cdot \|f\|_{\alpha,\beta]}\,. \tag{2.15}$$

Case (ii). By similar reasonings with those in the Case (i), we get

$$M_n(\alpha,\beta) \leq \frac{C}{n} + \frac{1}{n-k(n)}\cdot \|f\|_{\alpha,\beta]}\,. \tag{2.16}$$

We may suppose without any loss of generality that the sequence $(\frac{k(n)}{n})_{n\in N}$ is convergent and let L be its limit. From $0 < \alpha \leq \frac{k(n)}{n} \leq \beta < 1$, clearly we have $L \in (0,1)$. Then it is immediate that $O\left(\frac{n}{k(n)+1}\right) = O(1)$ and $O\left(\frac{n}{n-k(n)}\right) = O(1)$. More exactly, we easily get

$$\frac{1}{\beta} \leq \frac{n}{k(n)+1} \leq \frac{1}{\alpha} \text{ and } \frac{1}{1-\alpha} \leq \frac{n}{n-k(n)} \leq \frac{1}{1-\beta}.$$

Combining the above inequalities with relations (2.15)–(2.16), we obtain

$$M_n(\alpha,\beta) \leq \frac{1}{n} \cdot \left[C + \max\left\{\frac{1}{\alpha}, \frac{1}{1-\beta}\right\} \cdot \|f\|_{[\alpha,\beta]}\right].$$

Now, by Lemma 2.3.4 we easily obtain the desired conclusion. □.

Remarks. 1) Applying the Remark after the proof of Theorem 2.3.2 to Theorem 2.3.3 too, it is immediate that for properly chosen c^*, Theorem 2.3.3 still holds for $A_n^{(M)}(f)$ with f not necessarily positive on $[0,1]$ (see Theorem 2.9.1, (x) in the last section of this chapter).

2) From the statement of Theorem 2.3.3, it is clear that if $\alpha \searrow 0$ or/and $\beta \nearrow 1$ then the Lipschitz constant $L \nearrow +\infty$, which shows that Theorem 2.3.3 could not be stated for the whole interval $[0,1]$.

3) It is a natural question to ask if for $\gamma \in (0,1)$, from an inequality of the form $\|B_n^{(M)}(f) - f\|_{[\alpha,\beta]} \leq M/n^\gamma$, $n \in \mathbb{N}$, it could be deduced that f is a Lipschitz γ function on $[\alpha,\beta]$. Analyzing the proof of Theorem 2.3.3, the answer seems to be, in general, negative, at least for the method of proof used, see the last part of the proof (the analogues of relations (2.15)–(2.16) and the next lines).

2.4 Localization Results

In this section, for the class of strictly positive functions strong localization results are obtained in approximation by the max-product Bernstein operators. The results allow to approximate locally bounded strictly positive functions with very good accuracy, with potential applications in, e.g., image processing and in the approximation of fuzzy numbers, which are useful concepts in statistics, computer programming, engineering (especially communications), and experimental science.

It is very important to note that the strict positivity of f in all the results of this section could be dropped. Indeed, suppose that $f : [0,1] \to \mathbb{R}$ is bounded and of variable sign on $[0,1]$. Choose a constant $c^* > 0$ such that $f(x) + c^* > 0$, for all $x \in [0,1]$ and define the new max-product kind operator

$$A_n^{(M)}(f)(x) = B_n^{(M)}(f + c^*)(x) - c^*, \text{ for all } x \in [0,1].$$

Then, all the results in this section hold for $A_n^{(M)}(f)$ with f not necessarily positive on $[0,1]$ (see Theorem 2.9.1, (xi), (xii), (xiii), (xiv), and (xv) in the last section of this chapter).

The plan of the present section goes as follows. Firstly, a strong localization result is obtained and as consequences, a local direct result and some interesting local shape preserving properties are proved.

It is worth noting the strong localization result expressed by the next Theorem 2.4.1 that shows that if the bounded functions f and g with strictly positive lower bounds coincide on a subinterval $[\alpha, \beta] \subset [0,1]$, then for sufficiently large values of n, $B_n^{(M)}(f)$ and $B_n^{(M)}(g)$ coincide on subintervals sufficiently close to $[\alpha, \beta]$. Clearly, the next Corollary 2.4.3 shows that $B_n^{(M)}(f)$ is very suitable to approximate strictly positive functions which are constant on some subintervals, namely if f is a strictly positive function which is constant on some subintervals $[\alpha_i, \beta_i]$, $i = 1, \ldots, p$, of $[0,1]$, then for sufficiently large n, $B_n^{(M)}(f)$ takes the same constant values on subintervals sufficiently close to each $[\alpha_i, \beta_i]$, $i = 1, \ldots, p$. This fact is illustrated by a simple graph inserted at the end of the section, on which, in addition, the approximation by the Bernstein max-product operator is compared with the approximation by the Bernstein polynomials.

The main result of this section is the localization result Theorem 2.4.1, from which, as consequences, a local direct saturation result and local shape preserving properties for the Bernstein max-product operator will directly be obtained.

Theorem 2.4.1 (Coroianu–Gal [54]). *Let $f, g : [0,1] \to [0,\infty)$ be both bounded on $[0,1]$ with strictly positive lower bounds and suppose that there exists $a, b \in [0,1]$, $0 < a < b < 1$ such that $f(x) = g(x)$ for all $x \in [a,b]$. Then for all $c, d \in [a,b]$ satisfying $a < c < d < b$ there exists $\tilde{n} \in \mathbb{N}$ depending only on f, g, a, b, c, d such that $B_n^{(M)}(f)(x) = B_n^{(M)}(g)(x)$ for all $x \in [c,d]$ and $n \in \mathbb{N}$ with $n \geq \tilde{n}$.*

Proof. Let us choose arbitrary $x \in [c,d]$ and for each $n \in \mathbb{N}$ let $j_x \in \{0, 1, \ldots, n\}$ (j_x depends on n too, but there is no need at all to complicate on the notations) be such that $x \in [j_x/(n+1), (j_x+1)/(n+1)]$. Then, by the relation just before the Lemma 2.1.6 we have

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j_x}(x),$$

where for $k \in \{0, 1, \ldots, n\}$ we have

$$f_{k,n,j_x}(x) = \frac{\binom{n}{k}}{\binom{n}{j_x}} \left(\frac{x}{1-x}\right)^{k-j_x} f\left(\frac{k}{n}\right).$$

Since $x \in [c,d] \cap [j_x/(n+1), (j_x+1)/(n+1)]$ and since $a < c < d < b$ it is immediate that for $n \geq n_0$ where n_0 is chosen such that $1/n_0 < \min\{c-a, d-b\}$, we obtain $a < j_x/(n+1) < j_x/n < b$. Indeed, if would exist $n \geq n_0$ with $\frac{j_x}{n+1} \leq a$, since $\frac{j_x+1}{n+1} \geq x \geq c > a$ would follow that $\frac{1}{n} \leq \frac{1}{n_0} < c - a \leq \frac{1}{n+1}$, a contradiction. Analogously, if would exist $n \geq n_0$ with $b \leq \frac{j_x}{n}$, then by $\frac{j_x+1}{n+1} \geq \frac{j_x}{n} \geq b > d \geq x \geq \frac{j_x}{n+1}$, we would get the contradiction $\frac{1}{n+1} \geq b - d > \frac{1}{n_0} > \frac{1}{n}$.

Therefore we obtain $na < j_x < nb$ for all $n \geq n_0$. It is important to notice here that n_0 does not depend on x. From the previous inequality it follows that if $n \geq n_0$ then for any $x \in [c, d]$ there exists $\alpha_x \in [a, b]$ such that $j_x = n\alpha_x$.

In what follows, it will serve to our purpose to use the sequence $(a_n)_{n\geq 1}$, $a_n = \left[\sqrt[3]{n^2}\right]$ (here $[a]$ denotes the integer part of a). For this sequence there exists $n_1 \in \mathbb{N}$ such that $na - a_n > 0$ for all $n \geq n_1$.

The first main step is to prove that there exists a constant $N_0 \in \mathbb{N}$ which does not depend on $x \in [c, d]$, such that for any $n \geq N_0$ and $x \in [c, d]$ we have $B_n^{(M)}(f)(x) = \bigvee_{k\in I_{n,x}} f_{k,n,j_x}(x)$, where $I_{n,x} = \{k \in \{0, 1, \ldots, n\} : j_x - a_n \leq k \leq j_x + a_n\}$ does not depend on f. In order to obtain this conclusion, for $n \geq \max\{n_0, n_1\}$ let us choose $k \in \{0, 1, \ldots, n\} \setminus I_{n,x}$. We have two cases: i) $k + a_n < j_x$, and ii) $j_x + a_n < k$.

Case i) Since $x \in [j_x/(n+1), (j_x+1)/(n+1)]$, we observe that $\frac{x}{1-x} \geq \frac{j_x/(n+1)}{1-j_x/(n+1)} = \frac{j_x}{n+1-j_x}$ and noting that $j_x = n\alpha_x$, after some simple calculations we obtain

$$\begin{aligned}&\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)}\\ &= \frac{(n - n\alpha_x + 1) \cdot (n - n\alpha_x + 2) \ldots \cdot (n - k)}{(k + 1) \cdot (k + 2) \ldots \cdot n\alpha_x} \cdot \left(\frac{x}{1 - x}\right)^{n\alpha_x - k} \cdot \frac{f(j_x/n)}{f(k/n)}\end{aligned}$$

$$\geq \frac{(n - n\alpha_x + 1) \cdot (n - n\alpha_x + 2) \ldots \cdot (n - k)}{(k + 1) \cdot (k + 2) \ldots \cdot n\alpha_x} \cdot \left(\frac{n\alpha_x}{n + 1 - n\alpha_x}\right)^{n\alpha_x - k} \cdot \frac{f(j_x/n)}{f(k/n)}.$$

We have two subcases: i$_a$) $n - n\alpha_x + 1 \leq k + 1$ and i$_b$) $n - n\alpha_x + 1 > k + 1$.

Case i$_a$) It is clear that $0 < \frac{n-n\alpha_x+1}{k+1} \leq \frac{n-n\alpha_x+2}{k+2} \leq \cdots \leq \frac{n-k}{n\alpha_x}$, which implies

$$\begin{aligned}\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} &\geq \left(\frac{n - n\alpha_x + 1}{k + 1}\right)^{n\alpha_x - k} \cdot \left(\frac{n\alpha_x}{n + 1 - n\alpha_x}\right)^{n\alpha_x - k} \cdot \frac{f(j_x/n)}{f(k/n)}\\ &= \left(\frac{n\alpha_x}{k + 1}\right)^{n\alpha_x - k} \cdot \frac{f(j_x/n)}{f(k/n)}.\end{aligned}$$

Since $k < n\alpha_x - a_n$ it is immediate that $k + 1 \leq n\alpha_x$ and since $n\alpha_x - a_n \geq k + 1$, these all together imply that

$$\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} \geq \left(\frac{n\alpha_x}{k + 1}\right)^{a_n} \cdot \frac{f(j_x/n)}{f(k/n)} \geq \left(\frac{n\alpha_x}{n\alpha_x - a_n}\right)^{a_n} \cdot \frac{f(j_x/n)}{f(k/n)}.$$

Then, noting that $\frac{n\alpha_x}{n\alpha_x - a_n} \geq \frac{nb}{nb-a_n} > 0$ and denoting $0 < m_f$ and $0 < M_f$ the lower and upper bounds of f on $[0, 1]$, we get that

$$\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} \geq \left(\frac{nb}{nb-a_n}\right)^{a_n} \cdot \frac{m_f}{M_f} = \left(1 + \frac{a_n}{nb-a_n}\right)^{a_n} \cdot \frac{m_f}{M_f}.$$

We observe that $\lim_{n\to\infty}\left(1 + \frac{a_n}{nb-a_n}\right)^{a_n} = e^{\lim_{n\to\infty} \frac{a_n^2}{nb-a_n}} = +\infty$. It follows that there exists $n_2 \in \mathbb{N}$, $n_2 \geq \max\{n_0, n_1\}$ such that $\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} > 1$ for all $x \in [c, d]$, $n \geq n_2$ and $k \in \{0, 1, \ldots, n\}$, $k < j_x - a_n$. In addition, it is important to notice that n_2 does not depend on $x \in [c, d]$ but of course it depends on f.

Case i$_b$) It is clear that $\frac{n-n\alpha_x+1}{k+1} \geq \frac{n-n\alpha_x+2}{k+2} \geq \cdots \geq \frac{n-k}{n\alpha_x}$, which implies

$$\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} \geq \left(\frac{n-k}{n\alpha_x}\right)^{j_x-k} \cdot \left(\frac{n\alpha_x}{n+1-n\alpha_x}\right)^{j_x-k} \cdot \frac{f(j_x/n)}{f(k/n)}$$
$$\geq \left(\frac{n-k}{n+1-n\alpha_x}\right)^{j_x-k} \cdot \frac{m_f}{M_f}.$$

Since $n - k > n - n\alpha_x + a_n \geq n + 1 - n\alpha_x$ and since $n\alpha_x - k > a_n$, we get

$$\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} \geq \left(\frac{n-n\alpha_x+a_n}{n+1-n\alpha_x}\right)^{a_n} \cdot \frac{m_f}{M_f}. \tag{2.17}$$

Reasoning as in the previous case we will obtain that there exists an absolute constant $n_3 \in \mathbb{N}$, $n_3 \geq \max\{n_0, n_1\}$ such that $\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} > 1$ for all $x \in [c, d]$, $n \geq n_3$ and $k \in \{0, 1, \ldots, n\}$, $k < j_x - a_n$.

Summarizing the case (i), we conclude that there exists a constant $N_1 = \max\{n_2, n_3\}$ (depending only on f, a, b, c, d), such that $\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} > 1$ for all $x \in [c, d]$, $n \geq N_1$ and $k \in \{0, 1, \ldots, n\}$, $k < j_x - a_n$.

Case ii) We observe that we can provide the same type of reasonings as in the previous case, if instead of the intervals $[a, b]$ and $[c, d]$, respectively, we work with the intervals $[1 - b, 1 - a]$ and $[1 - d, 1 - c]$, if instead of f we work with $h(u) = f(1 - u)$ and if instead of the final constant N_1 we use a final constant denoted with N_2.

Since N_1 depends only on f, a, b, c, d, it is clear that N_2 will also depend only on h, a , b, c, d, that is N_2 will depend only on f, a, b, c, d.

Indeed, suppose that $j_x + a_n < k$. Then, for $x \in [c, d]$ and $n \geq N_2$, denoting $y = 1 - x \in [1 - d, 1 - c]$, it is immediate that $j_y = n - j_x$, which implies that $n - k < j_y - a_n$.

Therefore, denoting $h_{k,n,j}(y) = \frac{\binom{n}{k}}{\binom{n}{j}} \left(\frac{y}{1-y}\right)^{k-j} h(k/n)$, similar to (2.18) we get

$$\frac{h_{j_y,n,j_y}(y)}{h_{n-k,n,j_y}(y)} > 1, y \in [1-d, 1-c], n \geq N_2, \ n-k < j_y - a_n. \tag{2.18}$$

Since it is immediate that for any $k \in \{0, 1, \ldots, n\}$ we have $h_{n-k,n,j_y}(y) = f_{k,n,j_x}(x)$ and $h_{j_y,n,j_y}(y) = f_{j_x,n,j_x}(x)$, by the relation (2.18) we get

$$\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} > 1, x \in [c,d], n \geq N_2, \ j_x + a_n < k,$$

that is the Case ii) is proved too.

Analyzing the results obtained in the Cases i)-ii), it results that for all $x \in [c,d]$, $n \geq N_0$, $N_0 = \max\{N_1, N_2\}$ and $k \in \{0, 1, \ldots, n\}$, with $k < j_x - a_n$ or $k > j_x + a_n$, we have $\frac{f_{j_x,n,j_x}(x)}{f_{k,n,j_x}(x)} > 1$. In conclusion, we obtain our preliminary result, that is

$$B_n^{(M)}(f)(x) = \bigvee_{k \in I_{n,x}} f_{k,n,j_x}(x), x \in [c,d], \ n \geq N_0,$$

where $I_{n,x} = \{k \in \{0, 1, \ldots, n\} : j_x - a_n \leq k \leq j_x + a_n\}$.

Next, let us choose arbitrary $x \in [c,d]$ and $n \in \mathbb{N}$ so that $n \geq N_0$. If there exists $k \in I_{n,x}$ such that $k/n \notin [c,d]$, then we distinguish two cases. Either $k/n < c$ or $k/n > d$. In the first case we observe that

$$0 < c - \frac{k}{n} \leq x - \frac{k}{n} \leq \frac{j_x + 1}{n+1} - \frac{k}{n} \leq \frac{j_x + 1}{n} - \frac{k}{n} \leq \frac{a_n + 1}{n}.$$

Since $\lim_{n \to \infty} \frac{a_n+1}{n} = 0$, for sufficiently large n we necessarily have $\frac{a_n+1}{n} < c - a$, which clearly implies that $k/n \in [a,c]$. In the same manner, when $k/n > d$, for sufficiently large n we necessarily have $k/n \in [d,b]$.

Summarizing, there exists $\widetilde{N}_1 \in \mathbb{N}$ independent of any $x \in [c,d]$, such that

$$B_n^{(M)}(f)(x) = \bigvee_{k \in I_{n,x}} f_{k,n,j_x}(x), x \in [c,d], \ n \geq \widetilde{N}_1$$

and for any $x \in [c,d]$, $n \geq \widetilde{N}_1$ and $k \in I_{n,x}$, we have $k/n \in [a,b]$. Also, it is easy to check that $\widetilde{N}_1$ depends only on a, b, c, d, and f, while j_x and $I_{n,x}$ are independent of f.

Reasoning for the function g exactly as for the function f, there exists $\widetilde{N}_2 \in \mathbb{N}$ which depends only on a, b, c, d and g, such that

$$B_n^{(M)}(g)(x) = \bigvee_{k \in I_{n,x}} g_{k,n,j_x}(x), x \in [c,d], \ n \geq \widetilde{N}_2$$

and in addition for any $x \in [c,d]$, $n \geq \widetilde{N}_2$ and $k \in I_{n,x}$, we have $k/n \in [a,b]$. Taking $\tilde{n} = \max\{\widetilde{N}_1, \widetilde{N}_2\}$ we easily obtain the desired conclusion. $\square$

Remark. The localization result for the Bernstein max-prod operator $B_n^{(M)}$ in Theorem 2.4.1 is the best possible and it is much stronger than the corresponding localization for the classical Bernstein polynomial B_n, given by the following (see DeVore–Lorentz [78], p. 308, relationship (3.3)): if $f = g$ on $[a, b]$, then for any $[c, d]$ included in the open interval (a, b), we have

$$B_n(f)(x) - B_n(g)(x) = o(1/n),\ x \in [c, d].$$

Recall here that $a_n = o(b_n)$ means that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

A local direct approximation result is now an immediate consequence of the localization result in Theorem 2.4.1, as follows.

Corollary 2.4.2 (Coroianu–Gal [54]). *Let $f : [0, 1] \to [0, \infty)$ be bounded on $[0, 1]$ with the lower bound strictly positive and $0 < a < b < 1$ be such that $f|_{[a,b]} \in Lip\, [a, b]$. Then, for any $c, d \in [0, 1]$ satisfying $a < c < d < b$, we have*

$$\left|B_n^{(M)}(f)(x) - f(x)\right| \le \frac{C}{n} \text{ for all } n \in \mathbb{N} \text{ and } x \in [c, d],$$

where the constant C depends only on f and a, b, c, d.

Proof. Let us define the function $F : [0, 1] \to \mathbb{R}$,

$$F(x) = \begin{cases} f(a) \text{ if } x \in [0, a], \\ f(x) \text{ if } x \in [a, b], \\ f(b) \text{ if } x \in [b, 1]. \end{cases}$$

The hypothesis immediately implies that F is a strictly positive Lipschitz function on $[0, 1]$. Then, according to Theorem 2.2.18 it results that

$$\left|B_n^{(M)}(F)(x) - F(x)\right| \le \left(\frac{n\omega_1(F, \frac{1}{n})}{m_F} + 4\right)\omega_1(F, \frac{1}{n})_{[0,1]}, \text{for } x \in [0, 1], n \in \mathbb{N},$$

where $m_F = \min\{F(x); x \in [0, 1]\} > 0$. Since by the definition of F we have $\omega_1(F, \frac{1}{n})_{[0,1]} = \omega_1(f, \frac{1}{n})_{[a,b]}$ and by the hypothesis on f we get $\omega_1(f, \frac{1}{n})_{[a,b]} \le C_0/n$ for all $n \in \mathbb{N}$, taking into account that $m_F \ge m_f$ (here we denoted $m_f = \inf\{f(x);\ x \in [0, 1]\} > 0$), it follows

$$\left|B_n^{(M)}(F)(x) - F(x)\right| \le \left(\frac{C_0}{m_f} + 4\right)\frac{C_0}{n}, x \in [0, 1], n \in \mathbb{N},$$

that is

$$\left|B_n^{(M)}(F)(x) - F(x)\right| \le \frac{C_1}{n}, x \in [0, 1], n \in \mathbb{N},$$

where $C_1 = C_0\left(C_0/m_f + 4\right)$ depends only on f, a, b.

Now, let us choose arbitrary $c, d \in [a, b]$ such that $a < c < d < b$. Then, by Theorem 2.4.1 it results the existence of $\tilde{n} \in \mathbb{N}$ which depends only on a, b, c, d, f, F such that $B_n^{(M)}(F)(x) = B_n^{(M)}(f)(x)$ for all $x \in [c, d]$. But since actually the function F depends on the function f, by simple reasonings we get that in fact $\tilde{n}$ depends only on a, b, c, d, and f.

Therefore, for arbitrary $x \in [c, d]$ and $n \in \mathbb{N}$ with $n \geq \tilde{n}$ we obtain

$$\left|B_n^{(M)}(f)(x) - f(x)\right| = \left|B_n^{(M)}(F)(x) - F(x)\right| \leq \frac{C_1}{n},$$

where C_1 and $\tilde{n}$ depend only on a, b, c, d, and f.

Now, denoting $C_2 = \max_{1 \leq n < \tilde{n}}\{n \cdot \|B_n^{(M)}(f) - f\|_{[c,d]}\}$, we finally obtain

$$|B_n^{(M)}(f)(x) - f(x)| \leq \frac{C}{n}, \text{ for all } n \in \mathbb{N},\ x \in [c, d],$$

with $C = \max\{C_1, C_2\}$ depending only on a, b, c, d, and f. □

It is known (see Theorem 2.1.15 and Corollary 2.1.16) that if f is monotone on $[0, 1]$ then so is $B_n^{(M)}(f)$ on $[0, 1]$ and (see Corollary 2.1.18 and Corollary 2.2.23) if f is quasiconvex (or quasiconcave) on $[0, 1]$, then so is $B_n^{(M)}(f)$.

As consequences of the localization result in Theorem 2.4.1, we present a series of local shape preserving properties for the Bernstein max-product operator attached to strictly positive functions. Thus, we will prove that if a strictly positive function f is monotonous (or quasiconvex or quasiconcave) on a subinterval $[a, b] \subset [0, 1]$, then for sufficiently large n, $B_n^{(M)}(f)$ preserves the monotonicity (or the quasiconvexity or quasiconcavity, respectively) on any strict subinterval of $[a, b]$.

Firstly we present:

Corollary 2.4.3 (Coroianu–Gal [54]). *Let $f : [0, 1] \to [0, \infty)$ be bounded on $[0, 1]$ with strictly positive lower bound and suppose that there exists $a, b \in [0, 1]$, $0 < a < b < 1$, such that f is constant on $[a, b]$ with the constant value α. Then for any $c, d \in [a, b]$ with $a < c < d < b$, there exists $\tilde{n} \in \mathbb{N}$ depending only on a, b, c, d, and f, such that $B_n^{(M)}(f)(x) = \alpha$ for all $x \in [c, d]$ and $n \in \mathbb{N}$ with $n \geq \tilde{n}$.*

Proof. Let $g : [0, 1] \to \mathbb{R}_+$ be given by $g(x) = \alpha > 0$ for all $x \in [0, 1]$. Since $f(x) = g(x)$ for all $x \in [a, b]$ and since obviously $B_n^{(M)}(g)(x) = \alpha$ for all $x \in [0, 1]$, by Theorem 2.4.1 we easily obtain the desired conclusion. □

Corollary 2.4.4 (Coroianu–Gal [54]). *Let $f : [0, 1] \to [0, \infty)$ be bounded on $[0, 1]$ with strictly positive lower bound and suppose that there exists $a, b \in [0, 1]$, $0 < a < b < 1$, such that f is nondecreasing (nonincreasing) on $[a, b]$. Then for any $c, d \in [a, b]$ with $a < c < d < b$, there exists $\tilde{n} \in \mathbb{N}$ depending only on a, b, c, d, and f, such that $B_n^{(M)}(f)$ is nondecreasing (nonincreasing) on $[c, d]$ for all $n \in \mathbb{N}$ with $n \geq \tilde{n}$.*

Proof. Suppose, for example, that f is nondecreasing on $[a, b]$. Defining the function $F : [0, 1] \to \mathbb{R}$ exactly as in the proof of Corollary 2.4.2, clearly that F is continuous, nondecreasing, and strictly positive on $[0, 1]$. Then, by Theorem 2.1.15, it follows that $B_n^{(M)}(F)$ is nondecreasing on $[0, 1]$ for all $n \in \mathbb{N}$. Let $a < c < d < b < 1$. By Theorem 2.4.1 (applicable to F and f), there exists $\tilde{n} \in \mathbb{N}$ depending only on a, b, c, d, and f, such that $B_n^{(M)}(f)(x) = B_n^{(M)}(F)(x)$ for all $x \in [c, d]$ and $n \geq \tilde{n}$, which proves the required assertion.

The proof in the case when f is nonincreasing on $[a, b]$ is similar. $\square$

Finally we present:

Corollary 2.4.5 (Coroianu–Gal [54]). *Let $f : [0, 1] \to [0, \infty)$ be a continuous and strictly positive function on $[0, 1]$ and suppose that there exists $a, b \in [0, 1]$, $0 < a < b < 1$, such that f is quasiconvex (quasiconcave) on $[a, b]$. Then for any $c, d \in [a, b]$ with $a < c < d < b$,there exists $\tilde{n} \in \mathbb{N}$ depending only on a, b, c, d, and f, such that $B_n^{(M)}(f)$ is quasiconvex (quasiconcave) on $[c, d]$ for all $n \in \mathbb{N}$ with $n \geq \tilde{n}$.*

Proof. Let us suppose, for example, that f is quasiconvex on $[a, b]$. By Popoviciu [128], the continuous function f is quasiconvex on $[a, b]$ equivalently means that there exists a point $\xi \in [a, b]$ such that f is nonincreasing on $[0, \xi]$ and nondecreasing on $[\xi, 1]$. Then, clearly that the function $F : [0, 1] \to \mathbb{R}$ defined as in the proof of Corollary 2.4.2, is strictly positive, continuous, and quasiconvex on $[0, 1]$, which by Corollary 2.1.18 implies that $B_n^{(M)}(F)$ is quasiconvex on $[0, 1]$ for all $n \in \mathbb{N}$. Let $a < c < d < b < 1$. By Theorem 2.4.1 (applicable to f and F), there exists $\tilde{n} \in \mathbb{N}$ depending only on a, b, c, d, and f, such that $B_n^{(M)}(f)(x) = B_n^{(M)}(F)(x)$ for all $x \in [c, d]$ and $n \geq \tilde{n}$.

Now, since by the above characterization of quasiconvex functions in Popoviciu [128] it is clear that any restriction to a subinterval of a quasiconvex function remains quasiconvex on that subinterval, we get the required assertion for quasiconvexity.

Suppose now that f is quasiconcave on $[a, b]$. By the Remark after Definition 2.1.17, it follows that there exists $\xi \in [a, b]$ such that f is nondecreasing on $[a, \xi]$ and nonincreasing on $[\xi, b]$. Then the function F mentioned above clearly remains quasiconcave on $[0, 1]$, which by Corollary 2.2.23 implies that $B_n^{(M)}(F)$ is quasiconcave on $[0, 1]$ for all $n \in \mathbb{N}$. Continuing the reasonings as in the case of quasiconvexity, we get the required assertion. $\square$

At this end, we illustrate graphically the property of $B_n^{(M)}(f)$ in Corollary 2.4.3 by a very simple example. Thus, let us consider the function $f : [0, 1] \to \mathbb{R}_+$,

$$f(x) = \begin{cases} 3x^2 + 0.25 & \text{if} \quad 0 \leq x \leq 0.5 \\ 1 & \text{if} \quad 0.5 < x \leq 0.75 \\ -3.6x + 3.7 & \text{if} \quad 0.75 < x \leq 1 \end{cases}$$

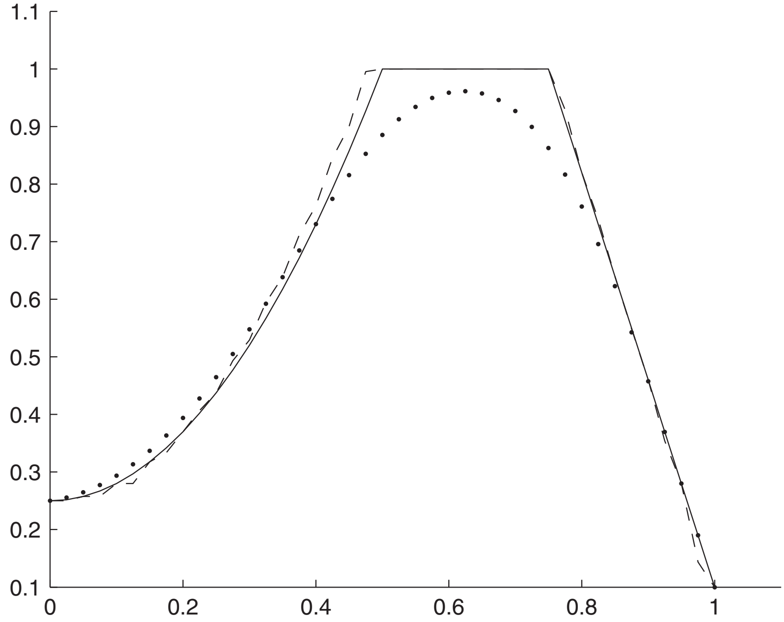

Fig. 2.1 Solid line: f; dotted line: Bernstein polynomial; dashed line: Bernstein max-product operator.

In Figure 2.1 we compare the approximation property of $B_n^{(M)}(f)(x)$, with the approximation property of the Bernstein polynomial $B_n(f)(x)$, for $n = 20$.

2.5 Iterations and Fixed Points

In this section we study the sequence of successive approximations, the fixed points, and the Ishikawa iterates for the max-product Bernstein operator.

For the classical Bernstein polynomials $B_n(f)(x)$, in the paper of Rus [134] the well-known Kelisky–Rivlin's result in [104] stating that for all $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$ it holds $\lim_{m\to\infty} B_n^m(f)(x) = f(0) + [f(1) - f(0)]x = B_1(f)(x)$ (here $B_n^m(f)$ denotes the mth iterate of the sequence of successive approximations), is proved in a very simple and elegant manner, by using the Banach fixed point theorem. Note here that $B_1(f)(x) = f(0) + [f(1) - f(0)]x$ is a fixed point for the operator B_n.

Also, if $m = m_n$ depends on n and if $\lim_{n\to\infty} \frac{m_n}{n} = 0$, then it is known that (see, e.g., [104]) $\lim_{n\to\infty} B_n^{m_n}(f)(x) = f(x)$ uniformly in $[0, 1]$.

Similar studies for the iterates of other kinds of Bernstein-type operators were obtained via fixed point theory in, e.g., Agratini [5], Rus [135], and Agratini–Rus [6].

The main aim of this section is to make a similar study for the iterates of the Bernstein max-product operator $B_n^{(M)}$. It is worth noting that due to the fact that $B_n^{(M)}$ is not a contraction (is only a non-expansive operator), the methods used in the case of Bernstein polynomials cannot be used for the Bernstein max-product operators, so that new methods are required.

The plan of the section goes as follows.

Although the Bernstein max-product operator is not a contraction, as an analogue of the above-mentioned Kelisky–Rivlin's results for the Bernstein polynomial, firstly we prove by a direct method that for any fixed $n \in \mathbb{N}$ and $f : [0, 1] \to [0, +\infty)$, the sequence of successive approximations of the nonlinear operator $B_n^{(M)}$, denoted by $a_n^m(f)(x) = [B_n^{(M)}]^m(f)(x)$, still uniformly converges for $m \to \infty$ to a fixed point of $B_n^{(M)}$. Also, the limits of the double sequence $(a_n^m(f))_{m,n\in\mathbb{N}}$ for other interdependences between m and n are calculated and important subsets of the set of fixed points of the operator $B_n^{(M)}$ are concretely determined.

Finally, we study the convergence of the so-called Ishikawa iterates for the operator $B_n^{(M)}$.

For the proof of the convergence of the sequence of successive approximations of $B_n^{(M)}$, we need the following three auxiliary results.

The first result obtained one refers to the fact that unlike the classical Bernstein (linear) operator $B_n(f)$ which is a contraction, the max-product Bernstein (nonlinear) operator $B_n^{(M)}(f)$ is only a nonexpansive operator. This means that the Banach fixed point theorem cannot be applied in this case.

Theorem 2.5.1 (Balaj–Coroianu–Gal–Muresan [10]). *For any $n \in \mathbb{N}$, the max-product Bernstein operator $B_n^{(M)} : C_+[0, 1] \to C_+[0, 1]$ is nonexpansive, that is*

$$\|B_n^{(M)}(f) - B_n^{(M)}(g)\| \leq \|f - g\|, \text{ for all } f, g \in C_+[0, 1],$$

where $C_+[0, 1] = \{f : [0, 1] \to \mathbb{R}_+; f$ is continuous on $[0, 1]\}$, $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$ and $\|\cdot\|$ denote the uniform norm in $C_+[0, 1]$.

Proof. We easily get

$$|B_n^{(M)}(f)(x) - B_n^{(M)}(g)(x)| \leq \frac{\bigvee_{k=0}^{n} |p_{n,k}(x)f(k/n) - p_{n,k}(x)g(k/n)|}{\bigvee_{k=0}^{n} p_{n,k}(x)}$$
$$\leq \|f - g\|,$$

which proves the theorem. □

Remarks. 1) In general, the inequality in Theorem 2.5.1 is not strict, that is there exists $f, g \in C_+[0, 1]$, such that $\|B_n^{(M)}(f) - B_n^{(M)}(g)\| = \|f - g\|$. Indeed, let us choose, for example, f nonincreasing on $[0, 1]$ and $g = 0$ on $[0, 1]$. By Corollary 2.1.16, it follows that $B_n^{(M)}(f)$ is also nonincreasing on $[0, 1]$, which implies that $\|f\| = f(0)$, $\|B_n^{(M)}(f)\| = B_n^{(M)}(f)(0)$ and by the obvious relationship $B_n^{(M)}(f)(0) = f(0)$, it implies $\|B_n^{(M)}(f) - B_n^{(M)}(g)\| = \|B_n^{(M)}(f)\| = f(0) = \|f\| = \|f - g\|$.

2) Note that Lemma 2.5 in [64] (see also Lemma 9.1.5 in this book) shows that for any bounded $f : [0,1] \to \mathbb{R}_+$ and $n \in \mathbb{N}$, $B_n^{(M)}(f) \in \mathrm{Lip}_L 1$, with $L = Cn^2\|f\|$, $C > 0$ being a constant independent of f and n, where

$$\mathrm{Lip}_L 1 = \{f : [0,1] \to \mathbb{R}; |f(x) - f(y)| \le L|x-y|, \text{ for all } x, y \in [0,1]\}.$$

In the next result we obtain an explicit value for C in the above Remark 2.

Theorem 2.5.2 (Balaj–Coroianu–Gal–Muresan [10]). *For all $f \in C_+[0,1]$ and $h \ge 0$ we have*

$$\omega_1(B_n^{(M)}(f); h) \le 6\pi e^2 n^2 \|f\| h.$$

Proof. Analyzing the proof of Lemma 2.5 in [64] (see also Lemma 9.1.5 in this book), we get $\omega_1(B_n^{(M)}(f); h) \le \frac{1}{c_1^2} n^2 \|f\| h$, where it is easy to observe that the constant $c_1 > 0$ (independent of x and n) comes from Lemma 2.4 in [64] (see also Lemma 9.1.4) as satisfying the inequality $\bigvee_{k=0}^{n} p_{n,k}(x) \ge \frac{c_1}{\sqrt{n}}$, for all $x \in [0,1]$ and $n \in \mathbb{N}$.

Analyzing now the proof of Lemma 2.4 in [64] (see also the proof of Lemma 9.1.4), it easily follows that $c_1 = c_2 \cdot \frac{1}{e}$, where $c_2 > 0$ is now the constant that appears in the statement of Lemma 2.3 in [64] (see also Lemma 9.1.3) as satisfying

$$\min\left\{p_{n,j}(\frac{j}{n+1}), p_{n,j}(\frac{j+1}{n+1})\right\} \ge \frac{c_2}{\sqrt{n}},$$

for all $n \in \mathbb{N}$, and $j \in \{0, 1, \ldots, n\}$, where $c_2 > 0$ is an absolute constant independent of n and j.

In continuation, analyzing the proof of Lemma 2.3 in [64] (see also the proof of Lemma 9.1.3) and denoting $A_n = \frac{(2^n n!)^2}{(2n)!} \cdot \frac{1}{\sqrt{2n+1}}$, since $\lim_{n\to\infty} A_n = \sqrt{\frac{\pi}{2}}$ and because it is easy to prove that $(A_n)_n$ is increasing, we get

$$\frac{2}{\sqrt{3}} < A_n < \sqrt{\frac{\pi}{2}}, \text{ for all } n \in \mathbb{N}.$$

This immediately implies

$$\frac{(2n)!}{4^n (n!)^2} > \sqrt{\frac{2}{3\pi}} \cdot \frac{1}{\sqrt{n}}, \text{ for all } n \in \mathbb{N}.$$

Therefore, following the lines in the proof of Lemma 2.3 in [64] (see also the proof of Lemma 9.1.3), case (i), we immediately obtain

$$p_n\left(\frac{j}{n+1}\right) > \frac{1}{\sqrt{e}} \cdot \sqrt{\frac{2}{3\pi}} \cdot \frac{1}{\sqrt{n}} = \frac{\sqrt{2}}{\sqrt{3\pi e}} \cdot \frac{1}{\sqrt{n}}.$$

Similarly, following the lines in the proof of Lemma 2.3 in [64] (see also the proof of Lemma 9.1.3), case (ii), we get

$$p_{n,n_1}\left(\frac{n_1+1}{n+1}\right) = \frac{(2n_1)!}{4^{n_1}(n_1)^2}\cdot\frac{2n_1+1}{2n_1+2} > \sqrt{\frac{2}{3\pi}}\cdot\frac{1}{\sqrt{n}}\cdot\frac{1}{2} = \frac{1}{\sqrt{6\pi}}\cdot\frac{1}{\sqrt{n}}.$$

Combining the cases (i) and (ii) in the proof of Lemma 2.3 in [64] (see also the proof of Lemma 9.1.3), since $\frac{\sqrt{2}}{\sqrt{3\pi e}} > \frac{1}{\sqrt{6\pi}}$, it follows that the constant c_2 in the statement of Lemma 2.3 in [64] (see also Lemma 9.1.3) can be chosen as $c_2 = \frac{1}{\sqrt{6\pi}}$.

In conclusion, going back with the values of the constants, we obtain $c_1 = \frac{1}{\sqrt{6\pi}}\cdot\frac{1}{e}$ and $\frac{1}{c_1^2} = 6\pi e^2$, which finish the proof. □

Also, we present:

Lemma 2.5.3 (Balaj–Coroianu–Gal–Muresan [10]). *For any $f \in C_+[0,1]$ and $n \in \mathbb{N}$ we have*

$$B_n^{(M)}[B_n^{(M)}(f)](x) \geq B_n^{(M)}(f)(x), \textit{ for all } x \in [0,1].$$

Proof. Let us choose arbitrary $j \in \{0,1,\ldots,n\}$. By the relation just before the Lemma 2.1.6, one has

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x),\ x \in [j/(n+1),(j+1)/(n+1)], \tag{2.19}$$

where

$$f_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}}\cdot\left(\frac{x}{1-x}\right)^{k-j}\cdot f(k/n)$$

for all $k \in \{0,1,\ldots,n\}$. Relation (2.19) implies $B_n^{(M)}(f)(x) \geq f_{k,n,j}(x)$ for all $x \in [j/(n+1),(j+1)/(n+1)]$ and $k \in \{0,1,\ldots,n\}$. In particular, for $x = j/n \in [j/(n+1),(j+1)/(n+1)]$ and $k = j$, we get $B_n^{(M)}(f)(j/n) \geq f_{j,n,j}(j/n) = f(j/n)$, $j \in \{0,1,\ldots,n\}$. Therefore, taking into account the relationship of definition for $B_n^{(M)}(f)(x)$, we immediately get the statement of the lemma. □

We are now in position to prove the first main result of this section.

Theorem 2.5.4 (Balaj–Coroianu–Gal–Muresan [10]). *For a fixed $f \in C_+[0,1]$, let us consider the iterative sequence of successive approximations $a_m^{(n)}(f)(x) = [B_n^{(M)}]^m(f)(x)$, $m,n \in \mathbb{N}$, $x \in [0,1]$. Here $[B_n^{(M)}]^2(f)(x) = B_n^{(M)}[B_n^{(M)}(f)](x)$ and so on.*

(i) For any fixed $n \in \mathbb{N}$*, there exists* $f_n : [0, 1] \to \mathbb{R}_+$*, such that* $f_n \in C_+[0, 1]$*,* $f_n \in \mathrm{Lip}_L\, 1$ *with* $L = 6\pi e^2 n^2 \|f\|$*,* $f_n(0) = f(0)$*,* $f_n(1) = f(1)$*,*

$$\lim_{m\to+\infty} a_m^{(n)}(f) = f_n, \ \ \textit{uniformly in } [0, 1],$$

$B_n^{(M)}(f_n)(x) = f_n(x)$ *for all* $x \in [0, 1]$ *(that is,* f_n *is a fixed point for the operator* $B_n^{(M)}$*) and*

$$B_n^{(M)}(f)(x) = a_1^{(n)}(f)(x) \leq a_m^{(n)}(f)(x) \leq a_{m+1}^{(n)}(f)(x) \leq f_n(x) \leq \|f\|,$$

for all $x \in [0, 1]$*,* $m \in \mathbb{N}$ *;*

(ii) For all $m, n \in \mathbb{N}$ *and* $x \in [0, 1]$*, we have the estimate*

$$|[B_n^{(M)}]^m(f)(x) - f(x)| \leq 12 \cdot \omega_1 \left(f; \frac{m}{\sqrt{n+1}}\right);$$

(iii) For any fixed $m \in \mathbb{N}$ *we have* $\lim_{n\to\infty} a_m^{(n)}(f)(x) = f(x)$*, uniformly in* $[0, 1]$ *;*

(iv) Let $m = m_n$ *depending on n such that* $\lim_{n\to\infty} \frac{m_n}{\sqrt{n}} = 0$*. Then we have* $\lim_{n\to\infty} a_{m_n}^{(n)}(f)(x) = f(x)$*, uniformly in* $[0, 1]$ *;*

(v) Suppose, in addition, that $f \in \mathrm{Lip}_L\, 1$ *and that it is strictly positive on* $[0, 1]$*. Then, for all* $m, n \in \mathbb{N}$ *we have the estimate*

$$\|[B_n^{(M)}]^m(f) - f\| \leq \frac{m}{n} \cdot L \left(\frac{L}{m_f} + 4\right),$$

where $m_f = \inf\{f(x); x \in [0, 1]\} > 0$ *;*

(vi) Suppose that $f \in \mathrm{Lip}_L\, 1$ *and that it is strictly positive on* $[0, 1]$*. Let* $m = m_n$ *depending on n such that* $\lim_{n\to\infty} \frac{m_n}{n} = 0$*. Then uniformly on* $[0, 1]$ *we have* $\lim_{n\to\infty} a_{m_n}^{(n)}(f)(x) = f(x)$*.*

(vii) Suppose that $f \in C_+[0, 1]$ *is such that for any* $n \in \mathbb{N}$*, the function* $B_n^{(M)}(f)$ *is a fixed point for the operator* $B_n^{(M)}$*. Then, for any sequence of natural numbers,* $(m_n)_{n\in\mathbb{N}}$*, the sequence of iterates* $a_{m_n}^{(n)}(f) = [B_n^{(M)}]^{m_n}(f)$ *converges uniformly on* $[0, 1]$ *to f, as* $n \to \infty$*.*

Proof. (i) From the above Lemma 2.5.3, easily follow the inequalities

$$0 \leq B_n^{(M)}(f)(x) = a_1^{(n)}(f)(x) \leq \cdots \leq a_m^{(n)}(f)(x) \leq a_{m+1}^{(n)}(f)(x) \leq \cdots \leq \|f\|,$$

for all $m, n \in \mathbb{N}$. The last inequality follows from the obvious inequality $0 \leq B_n^{(M)}(f)(x) \leq \|f\|$.

Fixing $n \in \mathbb{N}$ and $x \in [0, 1]$, the sequence of positive numbers $(a_m^{(n)}(f)(x))_{m\in\mathbb{N}}$ is bounded and monotonically nondecreasing, which implies, for $m \to +\infty$, its convergence to a limit, denote it by $f_n(x)$. Since $B_n^{(M)}(f)(x) \leq \|f\|$, we easily obtain

$a_m^{(n)}(f)(x) \leq \|f\|$, for all m, that is, the sequence $(a_m^{(n)}(f))_{m\in\mathbb{N}}$ is uniformly bounded. Passing to limit with $m \to +\infty$ we get $f_n(x) \leq \|f\|$ for all $x \in [0, 1]$, $n \in \mathbb{N}$.

Also, since it is easy to check that $B_n^{(M)}(f)(0) = f(0)$ and $B_n^{(M)}(f)(1) = f(1)$, it is immediate that $a_m^{(n)}(f)(0) = f(0)$ and $a_m^{(n)}(f)(1) = f(1)$ for all $m \in \mathbb{N}$, which therefore implies that $f_n(0) = f(0), f_n(1) = f(1)$.

Now, from $\|B_n(f)\| \leq \|f\|$ and applying successively Theorem 2.5.2, we easily obtain that $a_m^{(n)}(f) = [B_n^{(M)}]^m(f) \in \mathrm{Lip}_L 1$, for all $m \in \mathbb{N}$. Therefore, the sequence (of functions of successive approximation) $(a_m^{(n)}(f))_{m\in\mathbb{N}}$ clearly is equicontinuous, which combined with the fact that the sequence is uniformly bounded, by the Arzela-Ascoli theorem implies that it contains a subsequence $(a_{m_k}^{(n)}(f))_{k\in\mathbb{N}}$, uniformly convergent. Because the whole sequence is pointwise convergent to $f_n(x)$, we get that $\lim_{k\to\infty} a_{m_k}^{(n)}(f) = f_n$ uniformly in $[0, 1]$ and as a consequence, it immediately follows that $f_n \in C_+[0, 1]$, in fact moreover, that $f_n \in \mathrm{Lip}_L 1$ with $L = 6\pi e^2 n^2\|f\|$.

Applying now the well-known Dini's theorem to the pointwise convergent monotone sequence of continuous functions $(a_m^{(n)}(f))_{m\in\mathbb{N}}$, it follows that in fact we have $\lim_{m\to\infty} a_m^{(n)}(f) = f_n$ uniformly in $[0, 1]$.

Also, the monotonicity of the sequence $(a_m^{(n)})_{m\in\mathbb{N}}$ implies $a_m^{(n)}(f)(x) \leq f_n(x)$ for all $x \in [0, 1]$, $m, n \in \mathbb{N}$.

Finally, since $a_{m+1}^{(n)}(f) = B_n^{(M)}[a_m^{(n)}(f)]$ and $\lim_{m\to\infty} a_{m+1}^{(n)}(f) = f_n$ uniformly in $[0, 1]$, taking also into account that by Theorem 2.5.1, $B_n^{(M)}$ is nonexpansive, for any fixed n it follows that for all $m \in \mathbb{N}$ we have

$$\begin{aligned}\|B_n^{(M)}(f_n) - f_n\| &\leq \|B_n^{(M)}(f_n) - a_{m+1}^{(n)}(f)\| + \|a_{m+1}^{(n)}(f) - f_n\| \\ &\leq \|f_n - a_m^{(n)}(f)\| + \|a_{m+1}^{(n)}(f) - f_n\|.\end{aligned}$$

Passing here with $m \to \infty$, we get $\|B_n^{(M)}(f_n) - f_n\| = 0$, that is $B_n^{(M)}(f_n)(x) - f_n(x) = 0$, for all $x \in [0, 1]$.

(ii) For any fixed $m \in \mathbb{N}$ and $n \in \mathbb{N}$ variable, it is easy to see that the sequence $([B_n^{(M)}]^m(f))_{n\in\mathbb{N}}$ satisfies the Theorem 1.1.2, that is for all $\delta > 0$ we get

$$|[B_n^{(M)}]^m(f)(x) - f(x)| \leq \left[1 + \frac{1}{\delta}[B_n^{(M)}]^m(\varphi_x)(x)\right]\omega_1(f; \delta), x \in [0, 1],$$

where $\varphi_x(t) = |t - x|$, for all $t \in [0, 1]$.

In what follows we prove by mathematical induction that

$$[B_n^{(M)}]^m(\varphi_x)(x) \leq 6 \cdot \frac{m}{\sqrt{n+1}}, \text{ for all } m, n \in \mathbb{N},\ x \in [0, 1],$$

which replaced in the above estimate and by choosing then $\delta = 6 \cdot \frac{m}{\sqrt{n+1}}$, will immediately imply

$$|[B_n^{(M)}]^m(f)(x) - f(x)| \leq 12 \cdot \omega_1\left(f; \frac{m}{\sqrt{n+1}}\right).$$

Indeed, denoting

$$m_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}}\left(\frac{x}{1-x}\right)^{k-j},$$

we can write

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} m_{k,n,j}(x) f\left(\frac{k}{n}\right), \text{ for all } x \in [j/(n+1), (j+1)/(n+1)].$$

This immediately implies

$$[B_n^{(M)}]^2(f)(x) = \bigvee_{k=0}^{n} m_{k,n,j}(x) B_n^{(M)}(f)(k/n)$$
$$= \bigvee_{k=0}^{n} m_{k,n,j}(x)\left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n) f(i/n)\right].$$

Replacing here $f(t) = |t - x| = \varphi_x(t)$ with x fixed, and taking into account the inequality

$$\left|\frac{i}{n} - x\right| \leq \left|\frac{i}{n} - \frac{k}{n}\right| + \left|\frac{k}{n} - x\right|,$$

for all $x \in [j/(n+1), (j+1)/(n+1)]$ we get

$$[B_n^{(M)}]^2(\varphi_x)(x) = \bigvee_{k=0}^{n} m_{k,n,j}(x)\left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n)\left|\frac{i}{n} - x\right|\right]$$
$$\leq \bigvee_{k=0}^{n} m_{k,n,j}(x)\left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n)\left|\frac{k}{n} - \frac{i}{n}\right|\right]$$
$$+ \bigvee_{k=0}^{n} m_{k,n,j}(x)\left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n)\left|\frac{k}{n} - x\right|\right]$$

$$= \bigvee_{k=0}^{n} m_{k,n,j}(x)\left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n)\left|\frac{k}{n}-\frac{i}{n}\right|\right]$$
$$+ \bigvee_{k=0}^{n} m_{k,n,j}(x)\left|\frac{k}{n}-x\right|\left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n)\right]$$
$$\leq 6\cdot\frac{1}{\sqrt{n+1}} + 6\cdot\frac{1}{\sqrt{n+1}} = 6\cdot\frac{2}{\sqrt{n+1}}.$$

For the last estimate we used the inequalities which follow from the relationship (2.2) in the proof of Theorem 2.1.5

$$m_{k,n,j}(x)\left|\frac{k}{n}-x\right| \leq \frac{6}{\sqrt{n+1}},\quad m_{i,n,k}(k/n)\left|\frac{k}{n}-\frac{i}{n}\right| \leq \frac{6}{\sqrt{n+1}}$$

and the inequalities obtained from Lemma 2.1.2

$$m_{k,n,j}(x) \leq 1,\quad m_{i,n,k}(k/n) \leq 1.$$

Similarly, taking into account that for all $x \in [j/(n+1),(j+1)/(n+1)]$ we can write

$$[B_n^{(M)}]^3(f)(x)$$
$$= \bigvee_{k=0}^{n} m_{k,n,j}(x)\left[\bigvee_{i=0}^{n} m_{i,n,k}(k/n)\left[\bigvee_{l=0}^{n} m_{l,n,i}(i/n)f(l/n)\right]\right],$$

replacing here $f(t) = |t-x| = \varphi_x(t)$, taking into account the inequality

$$\left|\frac{l}{n}-x\right| \leq \left|\frac{l}{n}-\frac{i}{n}\right| + \left|\frac{i}{n}-\frac{k}{n}\right| + \left|\frac{k}{n}-x\right|,$$

and reasoning exactly as in the case of $[B_n^{(M)}]^2$, we easily obtain

$$[B_n^{(M)}]^3(\varphi_x)(x) \leq 6\cdot\frac{3}{\sqrt{n+1}},\ x \in [j/(n+1),(j+1)/(n+1)],$$

valid for all $j = 0, 1, \ldots, n$. Therefore, the above inequality is in fact valid for all $x \in [0, 1]$.

Reasoning now by mathematical induction, we get the desired estimate in the statement for arbitrary $m \in \mathbb{N}$.

(iii) It is immediate by passing to limit with $n \to \infty$ in the inequality from the above point (ii).

(iv) It is immediate by replacing m with m_n in the estimate in (ii), by passing to limit with $n \to \infty$ and taking into account that $\lim_{n\to\infty} \frac{m_n}{\sqrt{n+1}} = 0$.

(v) We obviously can write

$$\|[B_n^{(M)}]^m(f) - f\| \leq \sum_{j=1}^{m} \|[B_n^{(M)}]^j(f) - [B_n^{(M)}]^{j-1}(f)\|,$$

where by convention $[B_n^{(M)}]^0(f)(x) = f(x)$.

But by applying successively Theorem 2.5.1, we easily get that

$$\|[B_n^{(M)}]^j(f) - [B_n^{(M)}]^{j-1}(f)\| \leq \|[B_n^{(M)}]^{j-1}(f) - [B_n^{(M)}]^{j-2}(f)\|$$
$$\leq \cdots \leq \|[B_n^{(M)}](f) - (f)\| \leq \omega_1\left(f; \frac{1}{n}\right) \cdot \left[\frac{n \cdot \omega_1(f; 1/n)}{m_f} + 4\right],$$

where for the last estimate above we used Theorem 2.2.18, valid for strictly positive functions only.

Now, taking into account that $f \in \text{Lip}_L\, 1$, from the above estimate we get

$$\|[B_n^{(M)}]^j(f) - [B_n^{(M)}]^{j-1}(f)\| \leq \frac{1}{n}\left[L\left(\frac{L}{m_f} + 4\right)\right],$$

for all $j = 1, \ldots, m$, which finally implies

$$\|[B_n^{(M)}]^m(f) - f\| \leq \frac{m}{n}\left[L\left(\frac{L}{m_f} + 4\right)\right].$$

(vi) It is immediate by taking $m = m_n$ and passing to limit in the estimate from the above point (v).

(vii) By hypothesis, we have $B_n^{(M)}[B_n^{(M)}(f)] = B_n^{(M)}(f)$, for all $n \in \mathbb{N}$, and therefore it easily follows that $[B_n^{(M)}]^{m_n}(f) = B_n^{(M)}(f)$, for all $n \in \mathbb{N}$. Consequently, by Theorem 2.1.5, we obtain

$$\|[B_n^{(M)}]^{m_n}(f)(x) - f(x)| = |B_n^{(M)}(f)(x) - f(x)| \leq 12 \cdot \omega_1(f; 1/\sqrt{n+1}),$$

and passing to limit with $n \to \infty$, we immediately get the desired conclusion. □

Remarks. 1) In the class of Lipschitz, strictly positive functions, Theorem 2.5.4, (vi), is more general than Theorem 2.5.4, (iv). Indeed, while $\lim_{n\to\infty} \frac{m_n}{\sqrt{n}} = 0$ implies $\lim_{n\to\infty} \frac{m_n}{n} = 0$, the converse is not true. Note that the case of Theorem 2.5.4, (vi), is similar to what happens in the case of the iterates of Bernstein polynomials.

2) As a consequence of the well-known Trotter's approximation result in the theory of the semigroups of linear operators (see, e.g., [103]), it is known that in the case of Bernstein polynomials $B_n(f)(x)$, if f is twice differentiable and $\lim_{n\to\infty} \frac{m_n}{n} = t > 0$, then $\lim_{n\to\infty} B_n^{m_n}(f)(x) = e^{tA(x)}$, where $A(x) = \frac{x(1-x)f''(x)}{2}$, for all $x \in [0, 1]$.

   It remains as an interesting open question what happens with the iterates $[B_n^{(M)}]^{m_n}(f)$, when $\lim_{n\to\infty} \frac{m_n}{n} = t > 0$. Let us first observe that by Theorem 2.5.4, (vii), if f satisfies the hypothesis there, then $[B_n^{(M)}]^{m_n}(f)$ uniformly converges to f on $[0, 1]$. It is worth mentioning that by the next Theorems 2.5.5 and 2.5.6, we put in evidence large classes of functions f satisfying the hypothesis in Theorem 2.5.4, (vii). Therefore, the above-mentioned open problem for the Bernstein max-product operator gets a sense only if f does not satisfy the hypothesis in Theorem 2.5.4, (vii). Also, notice here that the Bernstein max-product operator $[B_n^{(M)}]^{m_n}$ is not linear.
3) If f is a fixed point of $B_n^{(M)}$, i.e. $f(x) = B_n^{(M)}(f)(x)$ for all $x \in [0, 1]$, we easily get $a_m^{(n)}(f)(x) = B_n^{(M)}(f)(x)$, for all $m \in \mathbb{N}$, $x \in [0, 1]$, therefore in this case it is trivial in Theorem 2.5.4, (i), that $f_n(x) = B_n^{(M)}(f)(x)$, for all $x \in [0, 1]$.
4) According to Theorem 2.5.4, (i), for each fixed $n \in \mathbb{N}$ it is important to determine the set of the fixed points for $B_n^{(M)}$. In this sense, we present the following results.

Theorem 2.5.5 (Balaj–Coroianu–Gal–Muresan [10]).

(i) If $f : [0, 1] \to [0, \infty)$ is nondecreasing and such that the function $g : (0, 1] \to [0, \infty)$, $g(x) = \frac{f(x)}{x}$ is nonincreasing, then for any $n \in \mathbb{N}$, $B_n^{(M)}(f)$ is a fixed point for the operator $B_n^{(M)}$, that is $B_n^{(M)}[B_n^{(M)}(f)](x) = B_n^{(M)}(f)(x)$, for all $x \in [0, 1]$;
(ii) If $f : [0, 1] \to [0, \infty)$ is nonincreasing and such that the function $h : [0, 1) \to [0, \infty)$, $h(x) = \frac{f(x)}{1-x}$ is nondecreasing, then for any $n \in \mathbb{N}$, $B_n^{(M)}(f)$ is a fixed point for the operator $B_n^{(M)}$, that is $B_n^{(M)}[B_n^{(M)}(f)](x) = B_n^{(M)}(f)(x)$, for all $x \in [0, 1]$.

Proof. (i) From the proof of Corollary 2.1.11, (i), for all $x \in [j/(n+1), (j+1)/(n+1)]$ and $j \in \{0, 1, \ldots, n-1\}$ we can write

$$B_n^{(M)}(f)(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}$$

and

$$B_n^{(M)}(f)(x) = f(1), \text{ for } x \in [n/(n+1), 1],$$

where

$$f_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \cdot \left(\frac{x}{1-x}\right)^{k-j} \cdot f(k/n).$$

Taking above $x = j/n$, by simple calculation we obtain

$$B_n^{(M)}(f)(j/n) = \max\{f(j/n), f[(j+1)/n] \cdot j/(j+1)\},$$

which by the property of the auxiliary function g in hypothesis, implies $f(j/n) \geq \frac{j}{j+1} f[(j+1)/n]$, which replaced in the above equality gives $B_n^{(M)}(f)(j/n) = f(j/n)$.

But it is clear that if for $f \in C_+[0,1]$ we have $B_n^{(M)}(f)(j/n) = f(j/n)$ for all $j \in \{0, 1, \ldots, n\}$, then $g = B_n^{(M)}(f)$ is a fixed point for $B_n^{(M)}$, which implies the desired conclusion.

(ii) From the proof of Corollary 2.1.11, (ii), for all $x \in [j/(n+1), (j+1)/(n+1)]$ and $j \in \{1, \ldots, n\}$ we can write

$$B_n^{(M)}(f)(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x)\},$$

and

$$B_n^{(M)}(f)(x) = f(0), \text{ for } x \in [0, 1/(n+1)].$$

Taking above $x = j/n$, by simple calculation we obtain

$$B_n^{(M)}(f)(j/n) = \max\{f[(j-1)/n] \cdot (n-j)/(n-j+1), f(j/n)\},$$

which by the property of the auxiliary function g in hypothesis, implies $f(j/n) \geq \frac{n-j}{n-j+1} f[(j-1)/n]$, which replaced in the above equality gives $B_n^{(M)}(f)(j/n) = f(j/n)$.

Therefore, we again get the desired conclusion. □

Remarks. 1) According to the Remark after the proof of Corollary 2.1.11, if $f : [0,1] \to [0,\infty)$ is a convex, nondecreasing function satisfying $\frac{f(x)}{x} \geq f(1)$ for all $x \in [0,1]$, or if $f : [0,1] \to [0,\infty)$ is a convex, nonincreasing function satisfying $\frac{f(x)}{1-x} \geq f(0)$, then again f satisfies the hypothesis in Theorem 2.5.5, (i) and (ii), respectively, and consequently we get $B_n^{(M)}[B_n^{(M)}(f)](x) = B_n^{(M)}(f)(x)$, for all $x \in [0,1]$.

2) Denote by $\mathcal{S}[0,1]$ the class of all functions f which satisfy the hypothesis in the statement of Theorem 2.5.5 (i), or of Theorem 2.5.5 (ii), or in the above Remark 1. Also, for any fixed arbitrary $n \in \mathbb{N}$, let us denote

$$\mathcal{G}_n^{(M)}[0,1] = B_n^{(M)}(\mathcal{S}[0,1])$$
$$= \{F \in C_+[0,1]; \exists f \in \mathcal{S}[0,1] \text{ such that } F(x) = B_n^{(M)}(f)(x), \ \forall x \in [0,1]\}.$$

Then, if we denote by

$$\mathcal{F}_n^{(M)}[0,1] = \{F : [0,1] \to [0,+\infty); B_n^{(M)}(F)(x) = F(x), \text{ for all } x \in [0,1]\},$$

the set of all fixed points of the operator $B_n^{(M)} : C_+[0,1] \to C_+[0,1]$, the statement of Theorem 2.5.5 together with the above Remark 1 means that we have $\mathcal{G}_n^{(M)}[0,1] \subset \mathcal{F}_n^{(M)}[0,1]$.

3) By Lemma 2.1.9, any nondecreasing concave function satisfies the hypothesis of Theorem 2.5.5, (i), and any nonincreasing concave function satisfies the hypothesis of Theorem 2.5.5, (ii). Therefore, the class of all positive, monotone, and concave functions on $[0,1]$ denoted by $MK_+[0,1]$ has the property $MK_+[0,1] \subset S[0,1]$, that is the function $H = B_n^{(M)}(f)$ satisfies $B_n^{(M)}(H)(x) = H(x)$, for all $x \in [0,1]$.
4) It is easy to consider concrete examples of functions in $S[0,1]$ (other than the constant functions which obviously are fixed points for $B_n^{(M)}$), like

$$x,\ e^x, 1+x^2, \sin(x), \cos(x), \ln(1+x), e^{-x}, 1+x^3.$$

Indeed, it is easy to check that x, e^x, and $1+x^2$ satisfy the first type of hypothesis in the above Remark 1, $\sin(x)$, $\cos(x)$ and $\ln(1+x)$ belong to the class $MK_+[0,1]$ defined in the above Remark 3, while e^{-x} satisfy the second type of hypothesis in the above Remark 1. Therefore, for any f in this remark we have $B_n^{(M)}[B_n^{(M)}(f)](x) = B_n^{(M)}(f)(x)$, for all $x \in [0,1]$ and $n \in \mathbb{N}$.

The results expressed by the above Remark 3 can be generalized to the whole class of concave functions, as follows.

Theorem 2.5.6 (Balaj–Coroianu–Gal–Muresan [10]). *If $f : [0,1] \to [0,\infty)$ is a continuous concave function then we have $B_n^{(M)}\left[B_n^{(M)}(f)\right] = B_n^{(M)}(f)$ for all $n \in \mathbb{N}$.*

Proof. By the proof of Corollary 2.1.10 we get

$$B_n^{(M)}(f)(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x), f_{j+1,n,j}(x)\},$$

for all $x \in [j/(n+1), (j+1)/(n+1)]$ and $j \in \{1, \ldots, n-1\}$,

$$B_n^{(M)}(f)(x) = \max\{f_{0,n,0}(x), f_{0,n,1}(x)\} \text{ for all } x \in [0, 1/(n+1)]$$

and

$$B_n^{(M)}(f)(x) = \max\{f_{n,n,n-1}(x), f_{n,n,n}(x)\}, \text{ for all } x \in [n/(n+1), 1].$$

Here recall that

$$f_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \cdot \left(\frac{x}{1-x}\right)^{k-j} \cdot f(k/n).$$

Since $j/n \in [j/(n+1), (j+1)/(n+1)]$, replacing $x = j/n$ in the above formulas for $B_n^{(M)}(f)(x)$, we easily obtain (see the reasonings in the proof of Theorem 2.5.5, (i) and (ii)) that $B_n^{(M)}(f)(j/n) = f(j/n)$ for all $j \in \{0, 1, \ldots, n\}$, which form the formula of definition of $B_n^{(M)}(f)(x)$ easily implies the desired conclusion. $\square$

Remarks. 1) Theorems 2.5.5 and 2.5.6 put in evidence large classes of functions $f \in C_+[0, 1]$, with the property that $B_n^{(M)}(f)$ is a fixed point for the operator $B_n^{(M)}$, for all $n \in \mathbb{N}$.

The following example of f is that of a function for which there exists $n \in \mathbb{N}$ (in fact an infinity of such of n) such that $B_n^{(M)}(f)$ is not anymore fixed point for the operator $B_n^{(M)}$. Indeed, let $f : [0, 1] \to [0, \infty)$ be defined by $f(x) = 1/2 - x$ if $x \in [0, 1/2]$ and $f(x) = x - 1/2$ if $x \in (1/2, 1]$. For $n = 5$, by the formula of definition of $B_n^{(M)}(f)(x)$, we easily get

$$B_5^{(M)}(f)(0) = B_5^{(M)}(f)(1) = 1/2,$$
$$B_5^{(M)}(f)(1/5) = B_5^{(M)}(f)(4/5) = 2/5,$$
$$B_5^{(M)}(f)(2/5) = B_5^{(M)}(f)(3/5) = 9/40,$$

and

$$B_5^{(M)}(B_5^{(M)}(f))(2/5) = 3/10.$$

Therefore, it follows $B_5^{(M)}(B_5^{(M)}(f))(2/5) \neq B_5^{(M)}(f)(2/5)$, which clearly implies that $B_5^{(M)}(f)$ is not a fixed point for the operator $B_5^{(M)}$.

In fact, by using, for example, MATLAB, one can easily show that for many other values of n (sufficiently large), again we get the same conclusion.

2) Theorem 2.5.6 is also useful to show that the method in the case of Bernstein polynomials in [134] cannot be used here, because for any $a, b \in \mathbb{R}_+$, the operator $B_n^{(M)}$ cannot be a contraction on the subspace $U_{a,b} = \{f \in C_+[0, 1]; f(0) = a\,, f(1) = b\}$.

In this sense, we can prove that for any natural number n, there exist two continuous functions $f, g : [0, 1] \to [0, \infty)$ satisfying $f(0) = g(0) = a, f(1) = g(1) = b$ and such that $\left\| B_n^{(M)}(f) - B_n^{(M)}(g) \right\| = \|f - g\|$.

Indeed, let us define as $y = f(x)$ the equation of the straight line passing through the points $(0, a)$ and $(1, b)$ and let g be the function whose graph is the polygonal line passing through the points $(0, a)$, $(1/2, c)$ and $(1, b)$, where the value c can be any real number which satisfies $c > f(1/2)$. (Note that the graphs of both functions f and g form a triangle.)

By simple geometrical reasonings we get that $\|f - g\| = g(1/2) - f(1/2)$.

Firstly, we suppose that n is even. Since f and g are concave functions, by the proof of the above Theorem 2.5.6, we get $B_n^{(M)}(f)(j/n) = f(j/n)$ and similarly, $B_n^{(M)}(g)(j/n) = g(j/n)$ for all $j \in \{0, 1, \ldots, n\}$. Therefore, taking $j(n) = n/2$, we obtain that $B_n^{(M)}(f)(1/2) = f(1/2)$ and $B_n^{(M)}(g)(1/2) = g(1/2)$. In conclusion, we have

$$g(1/2) - f(1/2) = \|f - g\| \geq \left\| B_n^{(M)}(f) - B_n^{(M)}(g) \right\|$$
$$\geq \left| B_n^{(M)}(f)(1/2) - B_n^{(M)}(g)(1/2) \right| = g(1/2) - f(1/2),$$

which implies $\left\| B_n^{(M)}(f) - B_n^{(M)}(g) \right\| = \|f - g\|$, for any even natural number n.

The reasoning is similar in the case when n is an odd natural number, because it suffices to replace the pair $(1/2, c)$ in the definition of g with $(n_0/(2n_0 + 1), c)$ where $n = 2n_0 + 1$.

The next results in this section are based on the following two well-known results.

Theorem 2.5.7 (Ishikawa [101]). *Let C be a compact convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be nonexpansive. For $(\lambda_m)_{m\in\mathbb{N}}$ a sequence in $[0, b]$ with $b < 1$ and such that $\sum_{m=0}^{\infty} \lambda_m = +\infty$, let us define the iterates in X by*

$$x_{m+1} := (1 - \lambda_m)x_m + \lambda_m T(x_m).$$

Then for any starting point $x_0 \in C$, the sequence $(x_m)_{m\in\mathbb{N}}$ converges to a fixed point of T.

Theorem 2.5.8 (Ishikawa [101]). *Let C be a closed bounded convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ be nonexpansive. Let $(\lambda_m)_m$ be as in Theorem 2.5.7. Then for any starting point $x_0 \in C$, the following sequence, $(\|x_m - T(x_m)\|)_{m\in\mathbb{N}}$, converges to 0 (i.e., $(x_m)_n$ is a so-called approximate fixed-point sequence).*

Now, in order to apply to our case the above Theorems 2.5.7 and 2.5.8, firstly we need to identify bounded closed convex and compact convex subsets in $C_+[0, 1]$. For example, it is easy to check that the subset

$$C_K^+[0, 1] = \{f \in C_+[0, 1]; \|f\| \leq K\},$$

is bounded, closed, and convex. Also, it is easy to check that the subset $C_{L,K} = C_K^+[0, 1] \bigcap \mathrm{Lip}_L 1$ is bounded, closed, convex, and equicontinuous, which by the Arzela-Ascoli theorem implies that $C_{L,K}$ is a convex compact subset in $C_+[0, 1]$ endowed with the uniform norm.

Another important hypothesis in the Theorems 2.5.7 and 2.5.8 is the invariance property of T. In our case, we need this invariance property for the Bernstein max-product operator. For this purpose, we will make use of the Theorem 2.5.2.

We have

Theorem 2.5.9 (Balaj–Coroianu–Gal–Muresan [10]).

(i) If $f \in C_K^+[0, 1]$, then for all $n \in \mathbb{N}$ we have $B_n^{(M)}(f) \in C_K^+[0, 1]$;
(ii) Let $K > 0$ and $L \geq 6\pi e^2 K$ be fixed constants and denote $C_{L,K} = C_K^+[0, 1] \bigcap Lip_L 1$. Then, for all $n \in \mathbb{N}$ satisfying the inequality $n^2 \leq \frac{L}{6\pi e^2 K}$, the invariance property $B_n^{(M)}(C_{L,K}) \subset C_{L,K}$ holds.

Proof. (i) Since $0 \leq f(k/n) \leq \|f\|$ for all $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$, it is immediate by the formula of definition of $B_n^{(M)}(f)(x)$, because we easily get $|B_n^{(M)}(f)(x)| \leq \|f\|$, for all $x \in [0,1]$, which implies $\|B_n^{(M)}\| \leq \|f\| \leq K$, for all $n \in \mathbb{N}$.

(ii) Let $f \in C_{L,K}$. By (i) it follows that $\|B_n^{(M)}(f)\| \leq K$ for all $n \in \mathbb{N}$ and by (i) it follows that $B_n^{(M)}(f) \in \mathrm{Lip}_{6\pi e^2 n^2 \|f\|} 1 \subset \mathrm{Lip}_{6\pi e^2 n^2 K} 1$, for all $n \in \mathbb{N}$. Then, by $n^2 \leq \frac{L}{6\pi e^2 K}$ we get $B_n^{(M)}(f) \in \mathrm{Lip}_{6\pi e^2 n^2 K} 1 \subset \mathrm{Lip}_L 1$, which leads to the conclusion that $B_n^{(M)}(f) \in C_{L,K}$ for n satisfying $n^2 \leq \frac{L}{6\pi e^2 K}$. □

As immediate consequences of the above considerations, we get the following two results.

Corollary 2.5.10 (Balaj–Coroianu–Gal–Muresan [10]). *Let $K > 0$ and $L \geq 6\pi e^2 K$ be fixed constants and $C_{L,K} = C_K^+[0,1] \bigcap \mathrm{Lip}_L 1$. Also, let $(\lambda_m)_{m\in\mathbb{N}}$ be sequence in $[0,b]$ with $b < 1$ and such that $\sum_{m=0}^{\infty} \lambda_m = +\infty$. For any $n \in \mathbb{N}$ and $f_{n,1} \in C_{L,K}$ fixed, let us define the iterated sequence of functions*

$$f_{n,m+1}(x) = (1-\lambda_m) f_{n,m}(x) + \lambda_m \cdot B_n^{(M)}(f_{n,m})(x),\ m \in \mathbb{N},\ x \in [0,1].$$

Then, for any fixed $n \in \mathbb{N}$ satisfying the inequality $n^2 \leq \frac{L}{6\pi e^2 K}$, the sequence of functions $(f_{n,m}(x))_{m\in\mathbb{N}}$ converges as $m \to \infty$ in the uniform norm, to a fixed point of the operator $B_n^{(M)}$.

Proof. Firstly, it is clear that $C_+[0,1]$ endowed with the uniform norm is a Banach space. By Theorem 2.5.1, by the comments between the statements of the Theorems 2.5.8 and 2.5.9 and by Theorem 2.5.9, (ii), the operator $B_n^{(M)} : C_{L,K} \to C_{L,K}$ is nonexpansive on the compact convex set $C_{L,K}$. Then the corollary is an immediate consequence of Theorem 2.5.7. □

Corollary 2.5.11 (Balaj–Coroianu–Gal–Muresan [10]). *Let $K > 0$ and $C_K^+[0,1] = \{f \in C_+[0,1]; \|f\| \leq K\}$. Also, let $(\lambda_m)_m$ and the iterated sequence $(f_{n,m+1}(x))_{m\in\mathbb{N}}$ be defined as in the statement of Corollary 2.5.10. Then, for any $n \in \mathbb{N}$ and $f_{n,1} \in C_K^+[0,1]$ fixed, we have*

$$\lim_{m\to\infty} \|f_{n,m} - B_n^{(M)}(f_{n,m})\| = 0,$$

where $\|\cdot\|$ denotes the uniform norm.

Proof. By Theorem 2.5.1, by the comments between the statements of the Theorems 2.5.8 and 2.5.9 and by Theorem 2.5.9, (i), the operator $B_n^{(M)} : C_K^+[0,1] \to C_K^+[0,1]$ is nonexpansive on the bounded, closed, and convex subset $C_K^+[0,1]$. Then the corollary is an immediate consequence of Theorem 2.5.8. □

Remark. The methods in this section can be extended to other max-product operators of Bernstein-type.

2.6 Applications to Approximation of Fuzzy Numbers

In this section, firstly we extend from $[0, 1]$ to an arbitrary compact interval $[a, b]$, the definition of the nonlinear Bernstein operators of max-product kind, denoted by $B_n^{(M)}(f; [a, b])$, $n \in \mathbb{N}$, by proving that their order of uniform approximation to f on $[a, b]$ is $\omega_1(f, 1/\sqrt{n})_{[a,b]}$ and that they preserve the quasiconcavity of f. Since if f is a fuzzy number, then $B_n^{(M)}(f; [a, b])$ generates in a simple way a fuzzy number $\widetilde{B}_n^{(M)}(f; [a, b])$ of the same support $[a, b]$ with f, it turns out that these results are very suitable in the approximation of the fuzzy numbers, approximating the (non-degenerate) segment core with the order $1/n$. In addition, in the case when the fuzzy numbers are given in the form of a pair $u = (u^-, u^+)$, the max-product operator $B_n^{(M)}$ generates a fuzzy number $\overline{B}_n^{(M)}(u) = (B_n^{(M)}(u^-; [0, 1]), B_n^{(M)}(u^+; [0, 1]))$, whose widths, expected intervals, ambiguities, and expected values approximate (for $n \to \infty$) the width, the expected interval, the ambiguity, and the expected value of u. Finally, the order $1/n$ in approximation of some subclasses in the L^1-metric is obtained, with applications to the approximation of some subclasses of fuzzy numbers.

Recently, many papers made investigations on the approximation of fuzzy numbers by trapezoidal or triangular fuzzy numbers (see [1–11, 13–15, 22, 51, 91–94, 155–158, 161]) and by nonlinear side functions (see [12, 16, 124, 137, 159]). The main aim of this section is to use the max-product Bernstein operator for approximating fuzzy numbers with continuous membership functions.

Since the restriction of a continuous fuzzy number to its compact support is a quasiconcave function, naturally it is suggested that the generalization of $B_n^{(M)}$ to $[a, b]$, denoted by $B_n^{(M)}(f; [a, b])$, could be used to approximate the restriction of a fuzzy number f to its support $[a, b]$. On the other hand, since $B_n^{(M)}(f; [a, b])$ preserves the monotonicity of f on $[a, b]$, we can use $B_n^{(M)}(f; [a, b])$ to approximate fuzzy numbers f given in the parametric LU-form too.

In our considerations, in this section we use the preliminaries on fuzzy numbers in Subsection 1.2.1 of Chapter 1.

The plan of this section goes as follows. In Subsection 2.6.1 we define the max-product Bernstein operator on a compact interval $[a, b]$ and obtain quantitative approximation and shape preserving properties on $[a, b]$. Applications to the uniform approximation of fuzzy numbers, preserving the support, approximating the core with the order $1/n$ and approximating the expected interval, the width, the ambiguity, and the expected value of the fuzzy number are obtained. Also, on a concrete example, a graphic which clearly illustrates the advantage of the approximation of fuzzy numbers by the max-product Bernstein operators with respect to the approximation by the associated linear Bernstein polynomials is presented. Then, in Subsection 2.6.2 we obtain quantitative estimates of order $1/n$ in approximation by the max-product Bernstein operators in the L^1-metric, which then are applied to the approximation of fuzzy numbers.

2.6.1 Uniform Approximation and Preservation of Characteristics

Given a fuzzy number u, it is a natural problem to construct approximating sequences of simple fuzzy numbers, $(u_n)_n$, converging to u in some given metric and, in addition, providing good approximations to the support, core, expected interval, width, ambiguity and expected value of u. The already known results in approximation theory could be a good source of inspiration. Thus, due to the fact that the Bernstein polynomials have interpolation properties at the endpoints and that they preserve the quasiconcavity, we could use them to approximate the fuzzy numbers, as follows.

Let u be a continuous fuzzy number with $\text{supp}(u) = [a, b]$, $a < b$ and $\text{core}(u) = [c, d]$, $c < d$. For any $n \in \mathbb{N}$, we can define

$$\widetilde{B}_n(u)(x) = 0 \text{ for } x \text{ outside } [a, b]$$

and

$$\widetilde{B}_n(u)(x) = B_n(u, [a, b]) = \sum_{k=0}^{n} p_{n,k}(x) \cdot u\left(a + (b - a)k/n\right),\ x \in [a, b],$$

where $p_{n,k}(x) = \left(\frac{x-a}{b-a}\right)^k \cdot \left(\frac{b-x}{b-a}\right)^{n-k}$, $k \in \{0, 1, \ldots, n\}$ are the fundamental Bernstein polynomials.

Since u is continuous and since $\|u\| = 1$ it results that $\left\|\widetilde{B}_n(u)\right\| < 1$ for every $n \in \mathbb{N}$.

For this reason, in order to produce proper fuzzy numbers we need to normalize $\widetilde{B}_n(u)$. Thus, we get the sequence of fuzzy numbers $\left(\frac{1}{\|\widetilde{B}_n(u)\|} \cdot \widetilde{B}_n(u)\right)_{n\geq 1}$. Now, it is well known that $\widetilde{B}_n(u)$ converges uniformly to u on $[a, b]$, since there exists an absolute constant C such that

$$|B_n(u, [a, b]) - u(x)| \leq C\omega_1(u; 1/\sqrt{n})_{[a,b]},$$

which easily implies that $\frac{1}{\|\widetilde{B}_n(u)\|} \cdot \widetilde{B}_n(u)$ converges to u with respect to the metric D_C.

Here $\omega_1(u; \delta)_{[a,b]}$ denotes the classical modulus of continuity of u on $[a, b]$, defined by $\omega_1(u; \delta)_{[a,b]} = \sup\{|u(x) - u(y)|; x, y \in [a, b], |x - y| \leq \delta\}$.

But, on the other hand, it is easy to prove that the core of $\frac{1}{\|\widetilde{B}_n(u)\|} \cdot \widetilde{B}_n(u)$ is reduced to a single element, which means that it does not hold the convergence of the core of $\frac{1}{\|\widetilde{B}_n(u)\|} \cdot \widetilde{B}_n(u)$ to the core of u (which is nondegenerated).

For this reason, we propose the max-product Bernstein operators, which in the case of approximation of fuzzy numbers, not only will fix the above-mentioned shortcoming, but also will preserve the other characteristics of a fuzzy number too.

From now on, throughout this section, we denote by $C(I)$ and $C_+(I)$, respectively, the space of continuous functions defined on an interval I and the space of positive continuous functions defined on I, respectively.

For a function $f \in C_+([a,b])$, we recall the corresponding max-product Bernstein operator on $[a,b]$ by

$$B_n^{(M)}(f;[a,b])(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f(a + k \cdot \frac{b-a}{n})}{\bigvee_{k=0}^{n} p_{n,k}(x)}, x \in [a,b],$$

where $p_{n,k}(x) = \binom{n}{k}\left(\frac{x-a}{b-a}\right)^k \cdot \left(\frac{b-x}{b-a}\right)^{n-k}$.

In this subsection we will prove that $B_n^{(M)} : C_+([a,b]) \to C_+([a,b])$ has the same order of uniform approximation as the linear Bernstein operator and that it preserves the quasiconcavity too.

We can now present the first main results of this subsection.

Theorem 2.6.1 (Bede–Coroianu–Gal [23]).

(i) If $a, b \in \mathbb{R}$, $a < b$ and $f : [a,b] \to \mathbb{R}_+$ is continuous, then we have the estimate

$$|B_n^{(M)}(f;[a,b])(x) - f(x)| \leq 12([b-a]+1)\omega_1\left(f; \frac{1}{\sqrt{n+1}}\right)_{[a,b]},$$

for all $n \in \mathbb{N}$, $x \in [a,b]$.

(ii) If $f : [a,b] \to \mathbb{R}_+$ is concave on $[a,b]$, then we have the estimate

$$|B_n^{(M)}(f;[a,b])(x) - f(x)| \leq 2([b-a]+1)\omega_1\left(f; \frac{1}{n}\right)_{[a,b]},$$

for all $n \in \mathbb{N}$, $x \in [a,b]$.

Proof. (i) Let us consider the function $g : [0,1] \to \mathbb{R}$, $g(y) = f(a + (b-a)y)$. It is easy to check that $g(k/n) = f(a + k \cdot \frac{b-a}{n})$ for all $k \in \{0, 1, \ldots, n\}$. Now, let us choose arbitrary $x \in [a,b]$ and let $y \in [0,1]$ be such that $x = a + (b-a)y$. This implies $y = (x-a)/(b-a)$ and $1 - y = (b-x)/(b-a)$. From these equalities and noting the expressions for $g(k/n)$, we obtain $B_n^{(M)}(f;[a,b])(x) = B_n^{(M)}(g;[0,1])(y)$. By Theorem 2.1.15, we get

$$\begin{aligned}\left|B_n^{(M)}(f;[a,b])(x) - f(x)\right| &= \left|B_n^{(M)}(g;[0,1])(y) - g(y)\right| \\ &\leq 12\omega_1\left(g; \frac{1}{\sqrt{n+1}}\right)_{[0,1]}.\end{aligned}$$

Since $\omega_1\left(g; \frac{1}{\sqrt{n+1}}\right)_{[0,1]} \leq \omega_1\left(f; \frac{b-a}{\sqrt{n+1}}\right)_{[a,b]}$ and by the property $\omega_1(f;\lambda\delta)_{[a,b]} \leq ([\lambda]+1)\omega_1(f;\delta)_{[a,b]}$ we obtain $\omega_1\left(g; \frac{1}{\sqrt{n+1}}\right)_{[0,1]} \leq ([b-a]+1)\omega_1\left(f; \frac{1}{\sqrt{n+1}}\right)_{[a,b]}$, which proves (i).

(ii) Keeping the notation from the above point (i), we get $B_n^{(M)}(f;[a,b])(x) = B_n^{(M)}(g;[0,1])(y)$, where $g(y) = f(a+(b-a)y)$ for all $y \in [0,1]$. The last equality is equivalent to $f(u) = g\left(\frac{u-a}{b-a}\right)$ for al $u \in [a,b]$. Writing now the property of concavity for f,

$$f(\lambda u_1 + (1-\lambda)u_2) \geq \lambda f(u_1) + (1-\lambda)f(u_2), \text{ for all } \lambda \in [0,1], u_1, u_2 \in [a,b],$$

in terms of g can be written as

$$g\left(\lambda \frac{u_1 - a}{b-a} + (1-\lambda)\frac{u_2 - a}{b-a}\right) \geq \lambda g\left(\frac{u_1 - a}{b-a}\right) + (1-\lambda) g\left(\frac{u_2 - a}{b-a}\right).$$

Denoting $y_1 = \frac{u_1 - a}{b-a} \in [0,1]$ and $y_2 = \frac{u_2-a}{b-a} \in [0,1]$, this immediately implies the concavity of g on $[0,1]$. Then, by Corollary 2.1.10, we get

$$\begin{aligned} \left|B_n^{(M)}(f;[a,b])(x) - f(x)\right| &= \left|B_n^{(M)}(g;[0,1])(y) - g(y)\right| \\ &\leq 2([b-a]+1)\omega_1\left(g; \frac{1}{n}\right)_{[0,1]}. \end{aligned}$$

Reasoning now exactly as in the above point (i), we get the desired conclusion. □

Theorem 2.6.2 (Bede–Coroianu–Gal [23]). *Let us consider the function* $f : [a,b] \to \mathbb{R}_+$ *and let us fix* $n \in \mathbb{N}$, $n \geq 1$. *Suppose in addition that there exists* $c \in [a,b]$ *such that* f *is nondecreasing on* $[a,c]$ *and nonincreasing on* $[c,b]$. *Then, there exists* $c' \in [a,b]$ *such that* $B_n^{(M)}(f;[a,b])$ *is nondecreasing on* $[a,c']$ *and nonincreasing on* $[c',b]$. *In addition we have* $|c - c'| \leq \frac{b-a}{n+1}$ *and* $\left|B_n^{(M)}(f;[a,b])(c) - f(c)\right| \leq ([b-a]+1)\,\omega_1(f, \frac{1}{n+1})_{[a,b]}$.

Proof. We construct the function g as in the previous theorem. Let $c_1 \in [0,1]$ be such that $g(c_1) = c$. Since g is the composition between f and the linear nondecreasing function $t \to a + (b-a)t$, we get that g is nondecreasing on $[0, c_1]$ and nonincreasing on $[c_1, 1]$. By Theorem 2.2.22 it results that there exists $c_1' \in [0,1]$ such that $B_n^{(M)}(g;[0,1])$ is nondecreasing on $[0, c_1']$, nonincreasing on $[c_1', 1]$ and in addition we have $\left|B_n^{(M)}(g;[0,1])(c_1) - g(c_1)\right| \leq \omega_1(g, \frac{1}{n+1})$ and $\left|c_1 - c_1'\right| \leq \frac{1}{n+1}$. Let $c' = a + (b-a)c_1'$. If $x_1, x_2 \in [a,c']$ with $x_1 \leq x_2$, then let $y_1, y_2 \in [0, c_1']$ be such that $x_1 = a + (b-a)y_1$ and $x_2 = a + (b-a)y_2$. Then, it follows that $B_n^{(M)}(f;[a,b])(x_1) = B_n^{(M)}(g;[0,1])(y_1)$ and $B_n^{(M)}(f;[a,b])(x_2) = B_n^{(M)}(g;[0,1])(y_2)$. The monotonicity of $B_n^{(M)}(g;[0,1])$ implies $B_n^{(M)}(g;[0,1])(y_1) \leq B_n^{(M)}(g;[0,1])(y_2)$, that is $B_n^{(M)}(f;[a,b])(x_1) \leq B_n^{(M)}(f;[a,b])(x_2)$. We thus obtain that $B_n^{(M)}(f;[a,b])$ is nondecreasing on $[a,c']$. Using the same type of reasoning, we obtain that $B_n^{(M)}(f;[a,b])$ is nonincreasing on $[c',b]$. For the rest of the proof, noting that $\left|c_1 - c_1'\right| \leq \frac{1}{n+1}$ we get $|c - c'| = \left|(b-a)(c_1 - c_1')\right| \leq \frac{b-a}{n+1}$. Finally, noting that

$$\left|B_n^{(M)}(g;[0,1])(c_1)-g(c_1)\right| \le \omega_1(g,\frac{1}{n+1})_{[0,1]}$$

and taking into account that $\omega_1(g,\frac{1}{n+1})_{[0,1]} \le ([b-a]+1)\,\omega_1(f,\frac{1}{n+1})_{[a,b]}$, we obtain

$$\begin{aligned}\left|B_n^{(M)}(f;[a,b])(c)-f(c)\right| &= \left|B_n^{(M)}(g;[0,1])(c_1)-g(c_1)\right| \\ &\le \omega_1(g,\frac{1}{n+1})_{[0,1]} \\ &\le ([b-a]+1)\,\omega_1(f,\frac{1}{n+1})_{[a,b]}\end{aligned}$$

and the proof is complete. □

Remarks. 1) From the above theorem it results that if $f:[a,b]\to\mathbb{R}_+$ is continuous and quasiconcave then $B_n^{(M)}(f)$ is quasiconcave too.

2) It is known from Section 2.1 that for functions in the space $C_+([0,1])$, $B_n^{(M)}$ preserves the monotonicity and the quasiconvexity. Reasoning similarly, it can be proved that these preservation properties hold in the general case of the space $C_+([a,b])$.

3) It is worth noting that all the previous approximation results in this Section 2.6 can easily be extended to bounded functions f which are not necessarily positive (i.e., $f:[a,b]\to\mathbb{R}$ are of arbitrary sign on $[a,b]$) as follows. If $c>0$ is a positive constant such that $f(x)+c\ge 0$ for all $x\in[a,b]$, then defining the new max-product kind operator

$$P_n^{(M)}(f;[a,b])(x)=B_n^{(M)}(f+c;[a,b])(x)-c,$$

and taking into account that $\omega_1(f;\delta)_{[a,b]}=\omega_1(f+c;\delta)_{[a,b]}$, we get the same estimates for $\left|P_n^{(M)}(f;[a,b])(x)-f(x)\right|$ as in the previous theorems in this section.

In what follows we present approximation results with respect to the metrics D_C and $\widetilde{D}_C$.

Firstly, we need some auxiliary results.

Lemma 2.6.3 (Bede–Coroianu–Gal [23]). *Let $a,b\in\mathbb{R}$, $a<b$. For $n\in\mathbb{N}$, k, $j\in\{0,1,\dots,n\}$ and $x\in(a+j\cdot\frac{b-a}{n+1},a+(j+1)\cdot\frac{b-a}{n+1})$, let*

$$m_{k,n,j}(x)=\frac{p_{n,k}(x)}{p_{n,j}(x)},$$

where recall that $p_{n,k}(x)=\binom{n}{k}\left(\frac{x-a}{b-a}\right)^k\cdot\left(\frac{b-x}{b-a}\right)^{n-k}$.

Then $m_{k,n,j}(x)<1$ for all $j\in\{0,1,\dots,n\}$ and $k\in\{0,1,\dots,n\}\setminus\{j\}$.

Proof. Without any loss of generality we may suppose that $a = 0$ and $b = 1$, because using the same reasoning as in the proof of Theorems 2.6.1 and 2.6.2, we easily obtain the conclusion of the lemma in the general case. So, let us fix $x \in (j/(n+1), (j+1)/(n+1))$. According to the proof of Lemma 2.1.2, we have

$$m_{0,n,j}(x) \le m_{1,n,j}(x) \le \cdots \le m_{j,n,j}(x),$$
$$m_{j,n,j}(x) \ge m_{j+1,n,j}(x) \ge \cdots \ge m_{n,n,j}(x).$$

Since $m_{j,n,j}(x) = 1$, it suffices to prove that $m_{j+1,n,j}(x) < 1$ and $m_{j-1,n,j}(x) < 1$. By direct calculations we get

$$\frac{m_{j,n,j}(x)}{m_{j+1,n,j}(x)} = \frac{j+1}{n-j} \cdot \frac{1-x}{x}.$$

Since the function $g(y) = (1-y)/y$ is strictly decreasing on the interval $[j/(n+1), (j+1)/(n+1)]$, it results that

$$\frac{1-x}{x} > \frac{1-(j+1)/(n+1)}{(j+1)/(n+1)} = \frac{n-j}{j+1}.$$

Clearly, this implies $m_{j,n,j}(x)/m_{j+1,n,j}(x) > 1$, that is $m_{j+1,n,j}(x) < 1$. By similar reasonings we get that $m_{j-1,n,j}(x) < 1$ and the proof is complete. □

Lemma 2.6.4 (Bede–Coroianu–Gal [23]). *If $a, b \in \mathbb{R}$, $a < b$ and $f : [a, b] \to \mathbb{R}_+$ is bounded, then for all $j \in \{0, 1, \ldots, n\}$, we have $B_n^{(M)}(f; [a, b])(a + j(b-a)/n) \ge f(a + j(b-a)/n)$.*

Proof. From Lemma 2.6.3, since $a + j(b-a)/n \in (a + j(b-a)/(n+1), a + (j+1)(b-a)/(n+1))$ and since $m_{k,n,j}(a + j(b-a)/n) = \frac{p_{n,k}(a+j(b-a)/n)}{p_{n,j}(a+j(b-a)/n)}$ for all $k \in \{0, 1, \ldots, n\}$, it follows that $\bigvee_{k=0}^{n} p_{n,k}(a + j(b-a)/n) = p_{n,j}(a + j(b-a)/n)$. Then, we have

$$\begin{aligned}
&B_n^{(M)}(f; [a, b])(a + j(b-a)/n)\\
&\qquad = \frac{\bigvee_{k=0}^{n} p_{n,k}(a + j(b-a)/n) f(a + k(b-a)/n)}{p_{n,j}(a + j(b-a)/n)}\\
&\qquad \ge \frac{p_{n,j}(a + j(b-a)/n) f(a + j(b-a)/n)}{p_{n,j}(a + j(b-a)/n)}\\
&\qquad = f(a + j(b-a)/n)
\end{aligned}$$

and the lemma is proved. □

Let us consider now a function $f \in C_+([a, b])$. Combining formula for $B_n^{(M)}(f; [a, b])(x)$ just before the statement of Theorem 2.6.1 with the conclusion of Lemma 2.6.4, we can simplify the method to compute $B_n^{(M)}(f;[a,b])(x)$ for some $x \in [a,b]$. Let us choose $j \in \{0,1,\dots,n\}$ and $x \in [a+(b-a)j/(n+1), a+(b-a)(j+1)/(n+1)]$. By properties of continuous functions, an immediate consequence of Lemma 2.6.3 is that $m_{k,n,j}(x) \le 1$ for all $k \in \{0,1,\dots,n\}$. This implies that

$$\bigvee_{k=0}^{n} p_{n,k}(x) = p_{n,j}(x), x \in \left[a+\frac{(b-a)j}{n+1}, a+\frac{(b-a)(j+1)}{n+1}\right]. \tag{2.20}$$

Therefore, denoting for each $k \in \{0,1,\dots,n\}$ and $x \in [a+(b-a)j/(n+1), a+(b-a)(j+1)/(n+1)]$,

$$f_{k,n,j}(x) = m_{k,n,j}(x) \cdot f(a+(b-a)k/n), \tag{2.21}$$

by the formula for $B_n^{(M)}(f;[a,b])(x)$ just before the statement of Theorem 2.6.1 and by (2.20) we obtain

$$B_n^{(M)}(f;[a,b])(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x), x \in \left[a+\frac{(b-a)j}{n+1}, a+\frac{(b-a)(j+1)}{n+1}\right]. \tag{2.22}$$

In what follows, suppose that u is a fuzzy number such that $\text{supp}(u) = [a,b]$ and $\text{core}(u) = [c,d]$. For $n \in \mathbb{N}$ we introduce the function $\widetilde{B}_n^{(M)}(u;[a,b]) : \mathbb{R} \to [0,1]$, $\widetilde{B}_n^{(M)}(u;[a,b])(x) = 0$ for all x outside $[a,b]$ and we have $\widetilde{B}_n^{(M)}(u;[a,b])(x) = B_n^{(M)}(u;[a,b])(x)$ for all $x \in [a,b]$. From Theorem 2.6.1, it results that the order of uniform approximation of the fuzzy number u by $\widetilde{B}_n^{(M)}(u;[a,b])$ is $\omega_1(u, 1/\sqrt{n})_{[a,b]}$. Then, since the restriction of u on the interval $[a,b]$ is a function like those considered in Theorem 2.6.2, it results that $\widetilde{B}_n^{(M)}(u;[a,b])$ is a quasiconcave function on $[a,b]$. Moreover, we have the following.

Theorem 2.6.5 (Bede–Coroianu–Gal [23]). *Let u be a fuzzy number with $\text{supp}(u) = [a,b]$ and $\text{core}(u) = [c,d]$ such that $a \le c < d \le b$. Then for sufficiently large n, it results that $\widetilde{B}_n^{(M)}(u;[a,b])$ is a fuzzy number such that:*

(i) $\text{supp}(u) = \text{supp}\left(\widetilde{B}_n^{(M)}(u;[a,b])\right)$;

(ii) If $\text{core}(\widetilde{B}_n^{(M)}(u;[a,b])) = [c_n, d_n]$, *then* $|c-c_n| \le \frac{b-a}{n}$ *and* $|d-d_n| \le \frac{b-a}{n}$;

(iii) If, in addition, u is continuous on $[a,b]$, then

$$\left|\widetilde{B}_n^{(M)}(u;[a,b])(x) - u(x)\right| \le 12([b-a]+1)\omega_1\left(u; \frac{1}{\sqrt{n+1}}\right)_{[a,b]},$$

for all $x \in \mathbb{R}$.

Proof. Let $n \in \mathbb{N}$, such that $\frac{b-a}{n} < d-c$. By Theorem 2.6.2 it follows that there exists $c' \in [a,b]$ such that $\widetilde{B}_n^{(M)}(u;[a,b])$ is nondecreasing on $[a,c']$ and nonincreasing on $[c', b]$. On the other hand, from the definition of $\widetilde{B}_n^{(M)}(u; [a, b])$, it results that $\left\|\widetilde{B}_n^{(M)}(u; [a, b])\right\| \leq \|u\|$ and since $\|u\| = 1$, it follows that $\left\|\widetilde{B}_n^{(M)}(u)\right\| \leq 1$. (Here $\|\cdot\|$ denotes the uniform norm on $B([a, b])$-the space of bounded functions on $[a, b]$.) Therefore, to prove that $\widetilde{B}_n^{(M)}(u)$ is a fuzzy number, it suffices to prove the existence of $\alpha \in [a, b]$ such that $\widetilde{B}_n^{(M)}(u) = 1$. Let $\alpha = a + j(b-a)/n$ where j is chosen such that $c < \alpha < d$. Such j exists since $\frac{b-a}{n} < d - c$. Since $\alpha \in \mathrm{core}(u)$, it results $u(\alpha) = 1$. On the other hand, by Lemma 2.6.4 it follows that $\widetilde{B}_n^{(M)}(u; [a, b])(\alpha) \geq u(\alpha)$ and clearly this implies that $\widetilde{B}_n^{(M)}(u; [a, b])$ is a fuzzy number. In what follows we prove punctually the rest of the theorem.

(i) Firstly we have $B_n^{(M)}(u; [a, b])(a) = u(a)$ and $B_n^{(M)}(u; [a, b])(b) = u(b)$. Noting the definitions of u and $\widetilde{B}_n^{(M)}(u; [a, b])$, it follows that $\widetilde{B}_n^{(M)}(u; [a, b])(x) = 0$ is outside of $[a, b]$. Now, by $u(x) > 0$ and $\widetilde{B}_n^{(M)}(u; [a, b])(x) = B_n^{(M)}(u; [a, b])(x)$ for all $x \in (a, b)$, we easily get that $\widetilde{B}_n^{(M)}(u; [a, b])(x) > 0$ for all $x \in (a, b)$, which proves (i).

(ii) Let us choose $n \in \mathbb{N}$ such that $(b-a)/n \leq d - c$. Then let $k(n, c), k(n, d) \in \{1, \ldots, n-1\}$ be such that

$$a + (b-a)\left(k(n, c) - 1\right)/n < c \leq a + (b-a)k(n, c)/n$$

and

$$a + (b-a)k(n, d)/n \leq d < a + (b-a)\left(k(n, d) + 1\right)/n.$$

Since $(b-a)/n \leq d-c$ it is immediate that $k(n, c) \leq k(n, d)$. In addition, by the way $k(n, c)$ and $k(n, d)$ were chosen, we observe that $u\left(a + (b-a)k/n\right) = 1$ for any $k \in \{k(n, c), \ldots, k(n, d)\}$ and $u\left(a + (b-a)k/n\right) < 1$ for any $k \in \{0, \ldots, n\} \setminus \{k(n, c), \ldots, k(n, d)\}$. In what follows, we will often make use of formulas (2.21)–(2.22) by adapting the notations to our case. Thus, for some $x \in [a + k(n, c)(b-a)/(n+1), a + (k(n, c) + 1)(b-a)/(n+1)]$, we have

$$\widetilde{B}_n^{(M)}(u; [a, b])(x) = \bigvee_{k=0}^{n} u_{k,n,k(n,c)}(x).$$

We observe that

$$u_{k(n,c),n,k(n,c)}(x) =$$
$$= m_{k(n,c),n,k(n,c)}(x) u\left(a + (b-a)k(n, c)/n\right) = u\left(a + (b-a)k(n, c)/n\right) = 1$$

and by the definition of $k(n, c)$ and by Lemma 2.6.3 (see also formula (2.20)) it is immediate that for any $k \in \{0, 1, \ldots, n\}$, we have

$$u\left(a+(b-a)k(n,c)/n\right)$$
$$\geq u\left(a+(b-a)k/n\right) \geq m_{k,n,k(n,c)}(x) \cdot u\left(a+(b-a)k/n\right) = u_{k,n,k(n,c)}(x)$$

and hence

$$\widetilde{B}_n^{(M)}(u;[a,b])(x) = u\left(a+(b-a)k(n,c)/n\right) = 1,$$
$$(\forall)\, x \in [a+k(n,c)(b-a)/(n+1), a+(k(n,c)+1)(b-a)/(n+1)]\,.$$

Performing similar reasonings we get that

$$\widetilde{B}_n^{(M)}(u;[a,b])(x) = u\left(a+(b-a)k(n,d)/n\right) = 1,$$
$$(\forall)\, x \in [a+k(n,d)(b-a)/(n+1), a+(k(n,d)+1)(b-a)/(n+1)]\,.$$

Now let us choose arbitrarily $x \in (a+(b-a)(k(n,c)-1)/(n+1), a+(b-a)k(n,c)/(n+1))$. We have

$$\widetilde{B}_n^{(M)}(u;[a,b])(x) = \bigvee_{k=0}^{n} u_{k,n,k(n,c)-1}(x).$$

If $k \in \{k(n,c), \ldots, k(n,d)\}$, then

$$u_{k,n,k(n,c)-1}(x) = m_{k,n,k(n,c)-1}(x)u\left(a+(b-a)k/n\right) < u\left(a+(b-a)k/n\right)$$
$$= u\left(a+(b-a)k(n,c)/n\right) = \widetilde{B}_n^{(M)}(u;[a,b])(a+(b-a)k(n,c)/(n+1)),$$

where we have used that $m_{k,n,k(n,c)-1} < 1$ since $k \neq k(n,c)-1$ (see Lemma 2.6.3). If $k \notin \{k(n,c), \ldots, k(n,d)\}$, then

$$u_{k,n,k(n,c)-1}(x) = m_{k,n,k(n,c)-1}(x)u\left(a+(b-a)k/n\right)$$
$$\leq u\left(a+(b-a)k/n\right) < u\left(a+(b-a)k(n,c)/n\right)$$
$$= \widetilde{B}_n^{(M)}(u;[a,b])(a+(b-a)k(n,c)/(n+1)).$$

Summarizing, we get that $\bigvee_{k=0}^{n} u_{k,n,k(n,c)-1}(x) < \widetilde{B}_n^{(M)}(u;[a,b])(a+(b-a)k(n,c)/(n+1))$ and this implies that

$$\widetilde{B}_n^{(M)}(u;[a,b])(x) < \widetilde{B}_n^{(M)}(u;[a,b])(a+(b-a)k(n,c)/(n+1)),$$
$$(\forall) x \in (a+(b-a)(k(n,c)-1)/(n+1), a+(b-a)k(n,c)/(n+1)).$$

By the quasiconcavity of $\widetilde{B}_n^{(M)}(u;[a,b])$ on $[a,b]$ it easily results that

$$\widetilde{B}_n^{(M)}(u;[a,b])(x) < \widetilde{B}_n^{(M)}(u;[a,b])(a+(b-a)k(n,c)/(n+1)),$$
$$(\forall) x \in [a, a+(b-a)k(n,c)/(n+1)).$$

By similar reasonings we get that

$$\widetilde{B}_n^{(M)}(u;[a,b])(x) < \widetilde{B}_n^{(M)}(u;[a,b])(a+(b-a)(k(n,d)+1)/(n+1)),$$
$$(\forall)x \in (a+(b-a)(k(n,d)+1)/(n+1), b].$$

From the above inequalities, noting that

$$\widetilde{B}_n^{(M)}(u;[a,b])(a+(b-a)k(n,c)/(n+1)) =$$
$$= u\,(a+(b-a)k(n,c)/n) = u(a+(b-a)k(n,d)/n)$$
$$= \widetilde{B}_n^{(M)}(u;[a,b])(a+(b-a)(k(n,d)+1)/(n+1)) = 1,$$

we obtain that $\widetilde{B}_n^{(M)}(u;[a,b])$ attains its maximum exclusively on the interval $[a+(b-a)k(n,c)/(n+1), a+(b-a)(k(n,d)+1)/(n+1)]$ which by the definition of $\widetilde{B}_n^{(M)}(u;[a,b])$ implies that $\text{core}\Big(\widetilde{B}_n^{(M)}(u;[a,b])\Big) = [a+(b-a)k(n,c)/(n+1), a+(b-a)(k(n,d)+1)/(n+1)]$. Now, denoting $c_n = a+(b-a)k(n,c)/(n+1)$ we observe that both c_n and c belong to the interval $[a+(b-a)\,(k(n,c)-1)\,/n, a+(b-a)k(n,c)/n]$ of length $(b-a)/n$ and hence $|c-c_n| \le (b-a)/n$. Similarly, denoting $d_n = a+(b-a)(k(n,d)+1)/(n+1)$ we get that $|d-d_n| \le (b-a)/n$ and the proof of statement (ii) is complete.

(iii) The proof is immediate by Theorem 2.6.1, taking into account the continuity of u. □

The following corollary is an immediate consequence of the previous theorem.

Corollary 2.6.6 (Bede–Coroianu–Gal [23]). *Let u be a continuous fuzzy number with* $\text{supp}(u) = [a,b]$ *and* $\text{core}(u) = [c,d]$ *such that $a \le c < d \le b$. Then we have*

(i) $\text{core}(\widetilde{B}_n^{(M)}(u;[a,b])) \to \text{core}(u)$.
(ii) $\lim\limits_{n\to\infty} D_C(\widetilde{B}_n^{(M)}(u;[a,b]), u) = 0$.

Proof. Relation (i) immediately follows from assertion (ii) of Theorem 2.6.5. Then since for large enough n we have supp $(\widetilde{B}_n^{(M)}(u;[a,b]))$ =supp(u) and by assertion (iii) of Theorem 2.6.5, it results that

$$D_C\left(\widetilde{B}_n^{(M)}(u;[a,b]), u\right) = \sup_{x\in[a,b]} \left|\widetilde{B}_n^{(M)}(u;[a,b])(x) - u(x)\right|$$
$$\le 12([b-a]+1)\omega_1\left(u; \frac{1}{\sqrt{n+1}}\right)_{[a,b]}$$

and since by the continuity of u we have $\omega_1\left(u; \frac{1}{\sqrt{n+1}}\right)_{[a,b]} \to 0$, we immediately obtain that (ii) holds too. □

From Theorem 2.6.5 and Corollary 2.6.6, it follows that the sequence of Bernstein max-product operators attached to a continuous fuzzy number fulfil the approximation and shape preserving properties mentioned in Sections 2.1–2.2 and hence they are a good example of an efficient convergent sequence of fuzzy numbers.

Remarks. 1) If the fuzzy number u is unimodal, that is $c = d$, then $\widetilde{B}_n^{(M)}(u; [a, b])$ is not necessarily a fuzzy number. But normalizing $\widetilde{B}_n^{(M)}(u; [a, b])$, we obtain the fuzzy number $\frac{1}{\left\|\widetilde{B}_n^{(M)}(u;[a,b])\right\|}\widetilde{B}_n^{(M)}(u; [a, b])$. (Recall that $\|\cdot\|$ denotes the uniform norm). Since $\widetilde{B}_n^{(M)}(u; [a, b]) \to u$ uniformly, we easily get that

$$\frac{1}{\left\|\widetilde{B}_n^{(M)}(u; [a, b])\right\|}\widetilde{B}_n^{(M)}(u; [a, b]) \to u,$$

uniformly. Interestingly, similarly to the case of fuzzy numbers with non-degenerated core, we can determine precisely the core of $\frac{1}{\left\|\widetilde{B}_n^{(M)}(u;[a,b])\right\|}\widetilde{B}_n^{(M)}$ $(u; [a, b])$. For simplicity, lets us denote

$$\frac{1}{\left\|\widetilde{B}_n^{(M)}(u; [a, b])\right\|}\widetilde{B}_n^{(M)}(u; [a, b]) = \widehat{B}_n^{(M)}(u; [a, b]), n \geq 1.$$

Firstly, let us notice that for sufficiently large n, since u is quasiconcave on $[a, b]$ it results the existence of $k_0 \in \{0, \ldots, n\}$ and $l_0 \in \mathbb{N}$, $k_0 + l_0 \leq n$, such that

$$u(a+k_0(b-a)/n) = u(a+(k_0+1)(b-a)/n)) = \ldots = u(a+(k_0+l_0)(b-a)/n))$$

and such that $u(a + k(b - a)/n) < u(a + k_0(b - a)/n)$, for any $k \in \mathbb{N}$ satisfying $k < k_0$ or $k_0 + l_0 < k \leq n$. Note that for sufficiently large n we have $1 \leq k_0 \leq k_0 + l_0 \leq n - 1$. Now reasoning as in the proof of Theorem 2.6.5, (ii), we easily obtain that

$$\text{core}\left(\widehat{B}_n^{(M)}(u; [a, b])\right) = [a+(b-a)k_0/(n+1), a+(b-a)(k_0+l_0+1)/(n+1)].$$

We must notice that in most situations there exists a neighborhood of c, $[c_1, c_2]$ such that u strictly increases on $[c_1, c]$ and u strictly decreases on $[c, c_2]$. For sufficiently large n, let $k(c) \in \{0, \ldots, n\}$ be such that

$$c \in [a + (b - a)k(c)/n, a + (b - a)(k(c) + 1)/n]$$

and

$$c_1 < a + (b - a)k(c)/n < a + (b - a)(k(c) + 1)/n < c_2.$$

By the monotonicity of u it results that

$$u\,(a+(b-a)k/n) < \max\{u(a+(b-a)k(c)/n), u(a+(b-a)(k(c)+1)/n)\},$$

for any $k \in \{0,1,\ldots,n\} \setminus \{k(c),k(c)+1\}$. From here it is immediate that $\mathrm{core}\left(\widehat{B}_n^{(M)}(u;[a,b])\right) \to \mathrm{core}(u)$.

Or, for $n \in \mathbb{N}$ we introduce the fuzzy number u_n as follows. First, we choose $k(c,n)$ such that

$$a+(b-a)\cdot\frac{k(c,n)}{(n+1)} \le c \le a+(b-a)\cdot\frac{(k(c,n)+1}{(n+1)}.$$

For x outside the interval $(a+(b-a)\cdot(k(c,n)-1)/(n+1), a+(b-a)\cdot(k(c,n)+2)/(n+1))$, we take $u_n(x)=u(x)$. For $x \in [a+(b-a)\cdot k(c,n)/(n+1), a+(b-a)\cdot(k(c,n)+1)/(n+1)]$ we take $u_n(x)=1$. Finally, in the missing intervals we take linear functions so that the continuity of u_n is ensured.

In addition, it follows that there exists a constant C independent of n, such that

$$\omega_1\left(u_n;\frac{1}{\sqrt{n+1}}\right)_{[a,b]} \le C\omega_1\left(u;\frac{1}{\sqrt{n+1}}\right)_{[a,b]}.$$

Indeed, it is clear that it suffices to compare the two moduli only on one of the two subintervals (each of them of length $(b-a)/(n+1)$) where $u_n(x)$ is a linear function. If $\omega_1(u_n;1/\sqrt{n+1})$ is attained on the left-hand side interval, it easily follows that it is less than

$$\begin{aligned}&|u(c)-u[c-2(b-a)/(n+1)]|\\ &\le [2(b-a)+1]\omega_1(u;1/(n+1))_{[a,b]}\\ &\le [2(b-a)+1]\omega_1(u;1/\sqrt{n+1})_{[a,b]}.\end{aligned}$$

If $\omega_1(u_n;1/\sqrt{n+1})_{[a,b]}$ is attained in an interval where $u_n(x)$ is not entirely linear, by decomposing that interval into two consecutive subintervals, such that on one $u_n(x)$ is linear and on the other one coincides with $u(x)$ (by construction), by the triangle inequality it easily follows that

$$\begin{aligned}\omega_1(u_n;1/\sqrt{n+1})_{[a,b]} &\le \omega_1(u;1/\sqrt{n+1})_{[a,b]}\\ &+[2(b-a)+1]\omega_1(u;1/\sqrt{n+1})_{[a,b]}.\end{aligned}$$

Now, since $a+(b-a)\cdot k(c,n)/n \in \mathrm{core}(u_n)$, it follows that $u_n(a+(b-a)\cdot k(c,n)/n)=1$, which by Lemma 2.6.4 implies

$$\widetilde{B}_n^{(M)}(u_n;[a,b])(a+(b-a)\cdot k(c,n)/n)=1.$$

Consequently, we get that $\widetilde{B}_n^{(M)}(u_n;[a,b])$ is a proper fuzzy number and since $\lim\limits_{n\to\infty}\text{core}(u_n)=c$, by Theorem 2.6.5, (ii), we get that $\lim\limits_{n\to\infty}(\text{core}\widetilde{B}_n^{(M)}(u_n;[a,b]))=c$. We prove now that $\widetilde{B}_n^{(M)}(u_n;[a,b]) \to u$, uniformly on $[a,b]$. We have

$$\left|\widetilde{B}_n^{(M)}(u_n;[a,b])(x)-u(x)\right| \leq \left|\widetilde{B}_n^{(M)}(u_n;[a,b])(x)-u_n(x)\right| + |u_n(x)-u(x)|$$
$$\leq 12([b-a]+1)\omega_1\left(u_n;\frac{1}{\sqrt{n+1}}\right)_{[a,b]} + |u_n(x)-u(x)|$$
$$\leq 12C([b-a]+1)\omega_1\left(u;\frac{1}{\sqrt{n+1}}\right)_{[a,b]} + |u_n(x)-u(x)|.$$

Since $|u_n(x)-u(x)| \leq 2([b-a]+1)\omega_1\left(u;\frac{1}{n+1}\right)_{[a,b]}$, we obtain

$$\left|\widetilde{B}_n^{(M)}(u_n;[a,b])(x)-u(x)\right|$$
$$\leq 12C([b-a]+1)\omega_1\left(u;\frac{1}{\sqrt{n+1}}\right)_{[a,b]} + 2([b-a]+1)\omega_1\left(u;\frac{1}{n+1}\right)_{[a,b]}$$

and this proves that $\widetilde{B}_n^{(M)}(u_n;[a,b]) \to u$, uniformly on $[a,b]$.

2) From Theorem 2.6.5 it follows that the max-product Bernstein operator, $B_n^{(M)}$, is more convenient for approximating fuzzy numbers than the classical linear Bernstein operator, B_n. While the order of uniform approximation is the same, the max-product Bernstein operator preserves better the shape of the approximated fuzzy number. In fact, it is easy to prove that if the fuzzy number u has a continuous membership function, then as n increases to ∞ we have $\|B_n(u;[a,b])\| < 1$. Of course, if we normalize $B_n(u;[a,b])$, then we obtain a fuzzy number (it is known that the linear Bernstein operator preserves the quasiconcavity, see, e.g., Section 2.2), but the core of the normalized linear Bernstein operator one reduces to a point which is inconvenient in the case when the core of u is a proper interval.
3) For practical considerations it is useful to study the problem of approximating fuzzy numbers that are of Lipschitz-type. For example, let us suppose that the fuzzy number u is α-Lipschitz on $[a,b]$, of order $\alpha\in(0,1]$, i.e.,

$$|u(x)-u(y)| \leq M|x-y|^{\alpha}, \forall x,y\in[a,b],$$

with some absolute constant M. By Theorem 2.6.5, (iii), we have

$$\left|\widetilde{B}_n^{(M)}(u;[a,b])(x)-u(x)\right| \leq 12([b-a]+1)M(n+1)^{-\alpha/2},$$

Now let $\varepsilon>0$ be arbitrary. Then we have

$$(12(b-a)+1)M(n+1)^{-\alpha/2} < \varepsilon,$$

for any $n \geq n_0 = [(\frac{C}{\varepsilon})^{(2/\alpha)}]+1$, with $C=(12(b-a)+1)M$, where $[\cdot]$ stands for the integer part of x.

4) It can be easily proved that if u is a unimodal continuous fuzzy number then the sequence $\left(\widetilde{B}_n^{(M)}(u_n;[a,b])\right)$ (no matter which construction is used from those presented in Remark 1) then all the requirements of Corollary 2.6.6 hold.

Example. Let us consider the fuzzy number given $\tilde{f}(x)=f(x)$ if $x \in [0,1]$, $\tilde{f}(x)=0$, otherwise, where f is the example of the function from the end of Section 2.4.

In Figure 2.1, we can compare the classical and the nonlinear max-product Bernstein operators in approximating this fuzzy number. We can easily see that on the support of $\tilde{f}$, the classical linear operator marked with dotted line is outperformed by the max-product operator marked with dashed line, this being almost coincident with the target fuzzy number at its core. The theoretical conclusions of the present section are well illustrated by this particular example too.

In what follows, for a fuzzy number written in the parametric form $u = (u^-, u^+)$, we can attach the max-product Bernstein operators $B_n^{(M)}(u^-;[0,1])$ and $B_n^{(M)}(u^+;[0,1])$. Since $B_n^{(M)}$ preserves the monotonicity, it follows that $B_n^{(M)}(u^-;[0,1])$ is nondecreasing and $B_n^{(M)}(u^+;[0,1])$ is nonincreasing. In addition we have

$$B_n^{(M)}(u^-;[0,1])(0) = u^-(0),\ B_n^{(M)}(u^-;[0,1])(1) = u^-(1),$$

$B_n^{(M)}(v^-;[0,1])(0) = v^-(0)$ and $B_n^{(M)}(v^-;[0,1])(1) = v^-(1)$. In conclusion we obtain that $\overline{B}_n^{(M)}(u) = (B_n^{(M)}(u^-;[0,1]), B_n^{(M)}(u^+;[0,1]))$ is a proper fuzzy number which in addition preserves the core and the support of u.

The following result holds.

Theorem 2.6.7 (Bede–Coroianu–Gal [23]). *Let $u = (u^-, u^+)$ be a positive fuzzy number with the level functions u^- and u^+ continuous. Then, denoting $u_n := (u_n^-, u_n^+) = \overline{B}_n^{(M)}(u)$, we have*

(i)

$$\widetilde{D}_C\left(\overline{B}_n^{(M)}(u), u\right)$$
$$\leq 12 \max\left\{\omega_1\left(u^-;\frac{1}{\sqrt{n+1}}\right)_{[0,1]}, \left(u^+;\frac{1}{\sqrt{n+1}}\right)_{[0,1]}\right\},\ \textit{for all } n \in \mathbb{N};$$

(ii)

$$EI(u_n) \to EI(u),$$

$\mathrm{width}(u_n) \to \mathrm{width}(u)$ *and*

$$Amb_s(u_n) \to Amb_s(u),$$
$$Amb_s(u_n) \to Amb_s(u), k \in \mathbb{N}.$$

for any reduction function $s : [0,1] \to [0,1]$.

Proof. (i) The proof is immediate by the continuity of u and the definition of $\overline{B}_n^{(M)}(u)$, taking into account Theorem 2.1.5 too.

(ii) We will use relations (1.41), (1.42), and (1.43) which define the expected interval, ambiguity, and value, respectively, and the formula for the width. Analyzing these formulas, from $u_n = \overline{B}_n^{(M)}(u) := (u_n^-, u_n^+)$, we conclude that in order to obtain the required convergence of the expected interval, width, ambiguity and of the expected value of u_n, it suffices to prove that

$$\lim_{n\to\infty}\int_0^1 s(\alpha)u_n^-(\alpha)d\alpha = \int_0^1 s(\alpha)u^-(\alpha)d\alpha$$

and

$$\lim_{n\to\infty}\int_0^1 s(\alpha)u_n^+(\alpha)d\alpha = \int_0^1 s(\alpha)u^+(\alpha)d\alpha,$$

for any $s(\alpha) = \alpha^k$, with fixed $k \in \mathbb{N}\bigcup\{0\}$.

Indeed, taking $s = \alpha^0 = 1$, we easily get the convergence of the expected interval and of the width. Therefore, let now $k \geq 1$ be fixed. For every $n \in \mathbb{N}$, we easily get

$$\begin{aligned}\left|\int_0^1 s(\alpha)u_n^-(\alpha)d\alpha - \int_0^1 s(\alpha)u^-(\alpha)d\alpha\right| &\leq s(1)\int_0^1 \left|u_n^-(\alpha) - u^-(\alpha)\right| d\alpha \\ &\leq s(1)\widetilde{D}_C(u_n, u),\end{aligned}$$

which easily implies that $\lim_{n\to\infty}\int_0^1 s(\alpha)u_n^-(\alpha)d\alpha = \int_0^1 s(\alpha)u^-(\alpha)d\alpha$. The proof of the second equality (for u^+) is similar, so that we omit the details. □

If the fuzzy number u is not positive, *i.e.* $u^-(0) < 0$, then there are many possibilities to attach a modified max-product Bernstein operator. For example, we can define

$$\overline{P}_n^{(M)}(u) = (B_n^{(M)}(u^- - u^-(0); [0,1]) + u^-(0), B_n^{(M)}(u^+ - u^-(0); [0,1])) + u^-(0)).$$

It is easily seen that $\text{supp}\left(\overline{P}_n^{(M)}(u)\right)$ =supp(u) and $\text{core}\left(\overline{P}_n^{(M)}(u)\right)$ =core(u). If u^- and u^+ both are continuous, then it is immediate that we have the same kind of estimates in Theorem 2.6.7, when we replace there $\overline{B}_n^{(M)}(u)$ with $\overline{P}_n^{(M)}(u)$.

2.6.2 L^1-Approximation

In this subsection, approximations results with respect to the metrics D_1 and d_1 are presented. In this sense, we will prove that using the max-product Bernstein operator, for some particular classes of fuzzy numbers we obtain a better approximation with respect to the metric D_1 than that with respect to the metric D_C.

Recall first that a function $f : [a,b] \to \mathbb{R}$ is called of bounded variation if there exists a positive constant $C > 0$, such that for any $m \in \mathbb{N}$ and any partition of $[a,b]$, $a = x_0 < x_1 < \cdots < x_m = b$, we have

$$\sum_{j=0}^{m} |f(x_{j+1}) - f(x_j)| \le C.$$

The supremum of the above sum after all the possible partitions of $[a,b]$ is called the total variation of f on $[a,b]$ and it is denoted by $V_a^b(f)$.

It is known that a function of bounded variation is not necessarily continuous on $[a,b]$. For example, any monotonous function f is of bounded variation and $V_a^b(f) = |f(b) - f(a)|$. Another important fact is that according to the Jordan's theorem, a function $f : [a,b] \to \mathbb{R}$ is with bounded variation on $[a,b]$ if and only if there exist two nondecreasing functions $f_1, f_2 : [a,b] \to \mathbb{R}$ such that $f = f_1 - f_2$ on $[a,b]$.

It is worth noting that any fuzzy number is of bounded variation on its support.

Indeed, let u be an arbitrary fuzzy number so that $\mathrm{supp}(u) = [a,b]$ and $\mathrm{core}(u) = [c,d]$, with $a \le c \le d \le b$. By Definition 1.2.1, there exist $l_s : [a,c] \to \mathbb{R}$ nondecreasing and $l_d : [d,b]$ nonincreasing, such that $u(x) = l_s(x)$ for $x \in [a,c]$, $u(x) = l_d(x)$ for $x \in [d,b]$ and $u(x) = 1$ for $x \in [c,d]$.

We have

Lemma 2.6.8 (Bede–Coroianu–Gal [23]). *If u is a fuzzy number defined as above, then $V_a^b(u) \le 2$ and we can write*

$$u(x) = u_1(x) - u_2(x), \text{ for all } x \in [a,b],$$

where u_1 and u_2 are nondecreasing and are given by

$$u_1(x) = l_s(x), \text{ if } x \in [a,c],\ u_1(x) = 1 \text{ if } x \in [c,b],$$

and

$$u_2(x) = 0, \text{ if } x \in [a,d],\ u_2(x) = 1 - l_d(x) \text{ if } x \in [d,b].$$

Proof. Let us consider an arbitrary partition of $[a,b]$, $a = x_0 < x_1 < \cdots < x_m = b$. Without any loss of generality we may suppose that there exists $k_0 \in \{1,2,\ldots,m\}$ such that $x_{k_0} = d$. Indeed, if not, then we can consider a finer partition of $[a,b]$ which contains d and for which the corresponding sum is larger than $\sum_{j=0}^{m} |u(x_{j+1}) - u(x_j)|$. Therefore, it suffices to find an upper bound for sums corresponding to partitions which contain d. Then it is immediate that

$$\sum_{j=0}^{m}|u(x_{j+1})-u(x_j)|=\sum_{j=0}^{k_0-1}|u(x_{j+1})-u(x_j)|+\sum_{j=k_0}^{m}|u(x_{j+1})-u(x_j)|$$

and by the monotonicity properties of u on $[a,b]$ it easily results that

$$\sum_{j=0}^{m}|u(x_{j+1})-u(x_j)|=u(d)-u(a)+u(d)-u(b)=2.$$

The decomposition is immediate, which proves the lemma. □

Because of the Lemma 2.6.8, we will deal only with the approximation of functions with bounded variation, for simplicity firstly considered defined on $[0,1]$.

For any $k,j\in\{0,1,..n\}$, let us define the functions $f_{k,n,j}:[\frac{j}{n+1},\frac{j+1}{n+1}]\to\mathbb{R}$,

$$f_{k,n,j}(x)=\frac{\binom{n}{k}}{\binom{n}{j}}\left(\frac{x}{1-x}\right)^{k-j}f\left(\frac{k}{n}\right).$$

By the notations from Lemma 2.6.3 (for $a=0$ and $b=1$) it follows that $f_{k,n,j}(x)=m_{k,n,j}(x)f\left(\frac{k}{n}\right)$ for every $k,j\in\{0,1,..n\}$ and, by the conclusion of the same lemma, it results that

$$f_{k,n,j}(x)\le f\left(\frac{k}{n}\right),$$

for all $k,j\in\{0,1,..n\}$ and $x\in\left[\frac{j}{n+1},\frac{j+1}{n+1}\right]$. Then it is known that for any $j\in\{0,1,\ldots,n\}$ and $x\in[\frac{j}{n+1},\frac{j+1}{n+1}]$ we can write

$$B_n^{(M)}(f;[0,1])(x)=\bigvee_{k=0}^{n}f_{k,n,j}(x).$$

The first main result is the following.

Theorem 2.6.9 (Bede–Coroianu–Gal [23]). *Let $f:[0,1]\to[0,\infty)$ be with bounded variation on $[0,1]$, such that $g(t)=f(t)/t$ is nonincreasing on $(0,1]$ and $h(t)=f(t)/(1-t)$ is nondecreasing on $[0,1)$.*

Then for all $n\in\mathbb{N}$ we have

$$\int_0^1|B_n^{(M)}(f;[0,1])(x)-f(x)|dx\le\frac{C}{n+1},$$

where $f=f_1-f_2$, with f_1,f_2 nondecreasing on $[a,b]$ and $C=2[V_0^1(f_1)+V_0^1(f_2)+\|f\|]$ ($\|\cdot\|$ denotes here the uniform norm).

Proof. By the hypothesis on g and h and by the proof of Corollary 2.1.10, we can write

$$B_n^{(M)}(f;[0,1])(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x), f_{j+1,n,j}(x)\},$$

for all $x \in [j/(n+1), (j+1)/(n+1)]$, $j = 0, 1, \ldots, n-1$ (with $f_{-1,n,0}(x) = 0$ by convention).

For $x \in [j/(n+1), (j+1)/(n+1)]$, we have two cases:

Case A. $B_n^{(M)}(f;[0,1])(x) = f_{j,n,j}(x)$ or $B_n^{(M)}(f;[0,1])(x) = f_{j+1,n,j}(x)$, which will imply $B_n^{(M)}(f;[0,1])(x) = \max\{f_{j,n,j}(x), f_{j+1,n,j}(x)\}$;

Case B. $B_n^{(M)}(f;[0,1])(x) = f_{j-1,n,j}(x)$, which will imply

$$B_n^{(M)}(f;[0,1])(x) = \max\{f_{j-1,n,j}(x), f_{j,n,j}(x)\}.$$

Case A. Since f is with bounded variation let $f_1, f_2 : [0,1] \to \mathbb{R}$ be nondecreasing functions satisfying $f = f_1 - f_2$.

Firstly, we easily notice that

$$\int_{n/(n+1)}^{1} |B_n^{(M)}(f;[0,1])(x) - f(x)|dx \le \frac{2\,\|f\|}{n+1}.$$

Indeed, this is immediate since by the definition of $B_n^{(M)}(f;[0,1])$ it easily follows that $\left\|B_n^{(M)}(f;[0,1])\right\| \le \|f\|$. It will be useful later to write the above inequality as

$$\begin{aligned}
&\int_{n/(n+1)}^{1} |B_n^{(M)}(f;[0,1])(x) - f(x)|dx \\
&\le \frac{1}{n+1} \cdot |f_1(n/(n+1)) - f_1(1)| \\
&\quad + \frac{1}{n+1} \cdot |f_2(n/(n+1)) - f_2(1)| + \frac{2\,\|f\|}{n+1}.
\end{aligned} \tag{2.23}$$

Let us suppose now that $j \in \{0, 1, \ldots, n-1\}$. We distinguish two cases:

Case (i). Let $x \in [j/(n+1), (j+1)/(n+1)]$ be fixed such that $B_n^{(M)}(f;[0,1])(x) = f_{j,n,j}(x)$. Since $f_{j,n,j}(x) = f(\frac{j}{n}) = f_1(\frac{j}{n}) - f_2(\frac{j}{n})$, it follows that

$$\begin{aligned}
&\left|B_n^{(M)}(f;[0,1])(x) - f(x)\right| \\
&= \left|f(\frac{j}{n}) - f(x)\right| = \left|\left(f_1(\frac{j}{n}) - f_2(\frac{j}{n})\right) - (f_1(x) - f_2(x))\right| \\
&\le \left|f_1(\frac{j}{n}) - f(x)\right| + \left|f_2(\frac{j}{n}) - f(x)\right|
\end{aligned}$$

$$\leq |f_1(j/(n+1)) - f_1((j+1)/(n+1))|$$
$$+ |f_2(j/(n+1)) - f_2((j+1)/(n+1))|,$$

by the monotonicity of f_1 and f_2 and by $j/n \in [j/(n+1), (j+1)/(n+1)]$.

Case (ii). Let $x \in [j/(n+1), (j+1)/(n+1)]$ be such that $B_n^{(M)}(f;[0,1])(x) = f_{j+1,n,j}(x)$. We have two subcases:

(ii_a) $B_n^{(M)}(f;[0,1])(x) \leq f(x)$, when evidently $f_{j,n,j}(x) \leq f_{j+1,n,j}(x) \leq f(x)$ and we immediately get

$$\begin{aligned}
&\left|B_n^{(M)}(f;[0,1])(x) - f(x)\right| \\
&= \left|f_{j+1,n,j}(x) - f(x)\right| \\
&= f(x) - f_{j+1,n,j}(x) \leq f(x) - f(j/n) = |f(j/n) - f(x)| \\
&\leq |f_1(j/(n+1)) - f_1((j+1)/(n+1))| \\
&\quad + |f_2(j/(n+1)) - f_2((j+1)/(n+1))|.
\end{aligned}$$

(ii_b) $B_n^{(M)}(f;[0,1])(x) > f(x)$, when

$$\begin{aligned}
\left|B_n^{(M)}(f;[0,1])(x) - f(x)\right| &= f_{j+1,n,j}(x) - f(x) \leq f((j+1)/n) - f(x) \\
&\leq |f_1((j+1)/n) - f_1(x)| + |f_2((j+1)/n) - f_2(x)| \\
&\leq |f_1((j+1)/(n+1)) - f_1(j/(n+1))| \\
&\quad + |f_1((j+2)/(n+1)) - f_1((j+1)/(n+1))| \\
&\quad + |f_2((j+1)/(n+1)) - f_2(j/(n+1))| \\
&\quad + |f_2((j+2)/(n+1)) - f_2((j+1)/(n+1))|.
\end{aligned}$$

Therefore, for all $x \in [j/(n+1), (j+1)/(n+1)]$ and $j \in \{0, 1, \ldots, n-1\}$ we get

$$\begin{aligned}
&|B_n^{(M)}(f;[0,1])(x) - f(x)| \\
&\leq |f_1((j+1)/(n+1)) - f_1(j/(n+1))| \\
&+ |f_1((j+2)/(n+1)) - f_1((j+1)/(n+1))| \\
&+ |f_2((j+1)/(n+1)) - f_2(j/(n+1))| \\
&+ |f_2((j+2)/(n+1)) - f_2((j+1)/(n+1))|
\end{aligned}$$

and integrating this inequality on $[j/(n+1), (j+1)/(n+1)]$, it follows

$$\int_{j/(n+1)}^{(j+1)/(n+1)} |B_n^{(M)}(f;[0,1])(x) - f(x)| dx$$

$$\leq \frac{1}{n+1}[|f_1((j+1)/(n+1)) - f_1(j/(n+1))|]$$
$$+ \frac{1}{n+1}[|f_1((j+2)/(n+1)) - f_1((j+1)/(n+1))|]$$
$$+ \frac{1}{n+1}[|f_2((j+1)/(n+1)) - f_2(j/(n+1))|]$$
$$+ \frac{1}{n+1}[|f_2((j+2)/(n+1)) - f_2((j+1)/(n+1))|].$$

Summing for j from 0 to $n-1$, we immediately get

$$\int_0^{n/(n+1)} |B_n^{(M)}(f;[0,1])(x) - f(x)|dx$$
$$\leq \frac{1}{n+1} \cdot \sum_{j=0}^{n-1} |f_1((j+1)/(n+1)) - f_1(j/(n+1))|$$
$$+ \frac{1}{n+1} \cdot \sum_{j=0}^{n-1} |f_2((j+1)/(n+1)) - f_2(j/(n+1))|$$

$$+ \frac{1}{n+1} \sum_{j=0}^{n-1} |f_1((j+2)/(n+1)) - f_1((j+1)/(n+1))|$$
$$+ \frac{1}{n+1} \sum_{j=0}^{n-1} |f_2((j+2)/(n+1)) - f_2((j+1)/(n+1))|$$

$$= \frac{1}{n+1} \cdot \sum_{j=0}^{n-1} |f_1((j+1)/(n+1)) - f_1(j/(n+1))|$$
$$+ \frac{1}{n+1} \cdot \sum_{j=0}^{n-1} |f_2((j+1)/(n+1)) - f_2(j/(n+1))|$$

$$+ \frac{1}{n+1} \cdot \sum_{j=1}^{n} |f_1((j+1)/(n+1)) - f_1(j/(n+1))|$$
$$+ \frac{1}{n+1} \cdot \sum_{j=1}^{n} |f_2((j+1)/(n+1)) - f_2(j/(n+1))|$$

Taking now into account the inequality (2.23) from the beginning of the proof too, it follows

$$\int_0^1 |B_n^{(M)}(f;[0,1])(x) - f(x)|dx$$
$$\leq \frac{1}{n+1} \cdot \sum_{j=0}^{n} |f_1((j+1)/(n+1)) - f_1(j/(n+1))|$$
$$+ \frac{1}{n+1} \cdot \sum_{j=0}^{n} |f_2((j+1)/(n+1)) - f_2(j/(n+1))|$$

$$+ \frac{1}{n+1} \cdot \sum_{j=1}^{n} |f_1((j+1)/(n+1)) - f_1(j/(n+1))|$$
$$+ \frac{1}{n+1} \cdot \sum_{j=1}^{n} |f_2((j+1)/(n+1)) - f_2(j/(n+1))| + \frac{2\,\|f\|}{n+1}$$

$$\leq \frac{2}{n+1} \sum_{j=0}^{n} |f_1((j+1)/(n+1)) - f_1(j/(n+1))|$$
$$+ \frac{2}{n+1} \sum_{j=0}^{n} |f_2((j+1)/(n+1)) - f_2(j/(n+1))| + \frac{2\,\|f\|}{n+1}$$
$$= \frac{2}{n+1} \left(V_0^1(f_1) + V_0^1(f_2) + \|f\|\right)$$

and taking $C = 2\left(V_0^1(f_1) + V_0^1(f_2) + \|f\|\right)$ we easily obtain the desired conclusion for the Case A.

Case B. The reasonings are absolutely similar to the Case A, so that we omit the proof. □

Corollary 2.6.10 (Bede–Coroianu–Gal [23]).

(i) Let $f : [0,1] \to [0,\infty)$ be nondecreasing on $[0,1]$, such that $g(t) = f(t)/t$ is nonincreasing on $(0,1]$.

Then for all $n \in \mathbb{N}$ we have

$$\int_0^1 |B_n^{(M)}(f;[0,1])(x) - f(x)|dx \leq \frac{2[f(1) - f(0)]}{n+1}.$$

(ii) Let $f : [0,1] \to [0,\infty)$ be nonincreasing on $[0,1]$, such that $h(t) = f(t)/(1-t)$ is nondecreasing on $(0,1]$.

Then for all $n \in \mathbb{N}$ *we have*

$$\int_0^1 |B_n^{(M)}(f;[0,1])(x) - f(x)|dx \leq \frac{2[f(0) - f(1)]}{n+1}.$$

Proof. (i) By the hypothesis it is clear that $h(t) = f(t)/(1-t)$ is nondecreasing on $[0, 1)$.

Now, since f is nondecreasing, for all $x \in [j/(n+1), (j+1)/(n+1)]$, $j = 0, 1, .., n$, we have

$$B_n^{(M)}(f;[0,1])(x) = \bigvee_{k=j}^{n} f_{k,n,j}(x)$$

and for $j = n$ we get $B_n^{(M)}(f;[0,1])(x) = f(1)$ for all $x \in [n/(n+1), 1]$.

Therefore, it is clear that for all $x \in [n/(n+1), 1]$ we have

$$|B_n^{(M)}(f;[0,1])(x) - f(x)| \leq |f(n/(n+1)) - f(1)|,$$

which immediately implies

$$\int_{n/(n+1)}^1 |B_n^{(M)}(f;[0,1])(x) - f(x)|dx \leq \frac{1}{n+1} \cdot |f(n/(n+1)) - f(1)|.$$

Reasoning as in the above Theorem 2.6.9 for $f_1 = f$ and $f_2 = 0$ we easily obtain that we can take $C = 2[f(1) - f(0)]$ in the proof of the previous theorem.

(ii) By the hypothesis it is clear that $g(t) = f(t)/t$ is nonincreasing on $(0, 1]$. Reasoning as in the above case (i), we easily get the desired estimate. □

Corollary 2.6.11 (Bede–Coroianu–Gal [23]). *If* $f : [0,1] \to [0,+\infty)$ *is a concave and monotonous function on* $[0, 1]$*, then*

$$\int_0^1 \left|B_n^{(M)}(f;[0,1])(x) - f(x)\right| dx \leq \frac{2|f(1) - f(0)|}{n+1}, \text{ for all } n \in \mathbb{N}.$$

Proof. By Lemma 2.1.9, since f is concave it follows that $g(t) = f(t)/t$ is nonincreasing on $(0, 1]$ and $h(t) = f(t)/(1-t)$ is nondecreasing on $[0, 1)$. Then, the desired estimate is a direct consequence of Corollary 2.6.10. □

Remarks. 1) By Corollary 2.1.11, for concave monotonous functions, the approximation in the uniform norm $\|\cdot\|$ is given by

$$\|B_n^{(M)}(f;[0,1]) - f\| \leq 2\omega_1(f;1/n)_{[0,1]}, \text{ for all } n \in \mathbb{N}.$$

Comparing with the estimate in the above Corollary 2.6.11, it is clear that the estimate in the uniform norm by the Bernstein max-product operator is weaker than that in the L^1-norm. Indeed, it suffices to take $f(x) = \sqrt{x}$, $x \in [0, 1]$, which is a concave nondecreasing function on $[0, 1]$. Therefore, Corollary 2.6.11 gives the approximation order $\frac{1}{n}$, while in the uniform norm we get the approximation order $\omega_1(f; 1/n)_{[0,1]} = f(1/n) - f(0) = \frac{1}{\sqrt{n}}$.

2) By simple reasonings, we can deduce other classes of functions satisfying the hypothesis in Corollary 2.6.10. For example, it follows that if $f : [0, 1] \to [0, \infty)$ is a convex, nondecreasing function satisfying $\frac{f(x)}{x} \geq f(1)$ for all $x \in [0, 1]$, then the function $g : (0, 1] \to [0, \infty), g(x) = \frac{f(x)}{x}$ is nonincreasing and as a consequence for f is valid the conclusion of Corollary 2.6.10, (i). Indeed, for simplicity let us suppose that $f \in C^1[0, 1]$ and denote $F(x) = xf'(x) - f(x)$, $x \in [0, 1]$. Then $g'(x) = \frac{F(x)}{x^2}$, for all $x \in (0, 1]$. Since the inequality $\frac{f(x)}{x} \geq f(1)$ can be written as $\frac{f(1)-f(x)}{1-x} \leq f(1)$, for all $x \in [0, 1)$, passing to limit with $x \to 1$ it follows $f'(1) \leq f(1)$, which implies (since f' is nondecreasing)

$$F(x) \leq xf'(1) - f(x) \leq xf'(1) - xf(1) = x[f'(1) - f(1)] \leq 0, \text{ for all } x \in (0, 1].$$

This means that $g(x)$ is nonincreasing.

Analogously, if $f : [0, 1] \to [0, \infty)$ is a convex, nonincreasing function satisfying $\frac{f(x)}{1-x} \geq f(0)$, then for f is valid the conclusion of Corollary 2.6.10, (ii).

It is worth noting that according to the Remark after the proof of Corollary 2.1.11, for these classes of functions too, the error estimate in the uniform norm by Bernstein max-product operator is weaker than that in the L^1-norm.

Now, in order to apply the above results to the approximation of fuzzy numbers, reasoning as in the previous section we will extend them to an arbitrary compact interval $[a, b]$, $a < b$, as follows.

For any $k, j \in \{0, 1, \ldots, n\}$ we define (similarly as for the case $a = 0, b = 1$) the functions $f_{k,n,j} : \left[a + \frac{(b-a)j}{n+1}, a + \frac{(b-a)(j+1)}{n+1}\right] \to \mathbb{R}$,

$$f_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \left(\frac{x-a}{b-x}\right)^{k-j} f\left(a + \frac{k(b-a)}{n}\right).$$

Thus, we can prove the following.

Theorem 2.6.12 (Bede–Coroianu–Gal [23]). *Let $f : [a, b] \to [0, \infty)$ be with bounded variation on $[a, b]$, such that $f(y)/(y - a)$ is nonincreasing on $(a, b]$ and $f(y)/(b - y)$ is nondecreasing on $[a, b)$. Then there exists $C > 0$ which depends only on f such that*

$$\int_a^b |B_n^{(M)}(f; [a, b])(x) - f(x)|dx \leq \frac{C}{n+1}, \text{ for all } n \in \mathbb{N}.$$

Proof. We define the function $G : [0,1] \to \mathbb{R}$, $G(y) = f(a + (b-a)y)$. It is immediate that (see the beginning in the proof of Theorem 2.6.1, (i)),

$$\int_a^b |B_n^{(M)}(f;[a,b])(x) - f(x)|dx = (b-a)\int_0^1 |B_n^{(M)}(G;[0,1])(x) - G(x)|dx.$$

Then, it is easy to see that G satisfies the hypothesis of Theorem 2.6.9, namely that G is of bounded variation on $[0,1]$, $g(t) = G(t)/t$ is nonincreasing on $(0,1]$ and $h(t) = G(t)/(1-t)$ is nondecreasing on $[0,1)$. Indeed, this follows immediately denoting $y = a+(b-a)t \in (a,b]$ for all $t \in (0,1]$, which implies $t = (y-a)/(b-a)$ and

$$g(t) = G(t)/t = f(a + (b-a)t)/t = (b-a)f(y)/(y-a)$$

and

$$h(t) = G(t)/(1-t) = f(a + (b-a)t)/(1-t) = (b-a)f(y)/(b-y).$$

Therefore, it results the existence of a constant C_G which depends only on G, such that

$$\int_0^1 |B_n^{(M)}(G;[0,1])(x) - G(x)|dx \leq \frac{C_G}{n+1}, \text{ for all } n \in \mathbb{N}.$$

But since the function G depends on f we easily obtain that actually C_G depends only on f. Now taking $C = (b-a)C_G$, we get the desired conclusion. □

The following application to the approximation of fuzzy numbers holds.

Theorem 2.6.13 (Bede–Coroianu–Gal [23]). *If u denotes a fuzzy number with $supp(u) = [a,b]$ and $core(u) = [c,d]$ and the restriction of u to the interval $[a,b]$ satisfies the hypotheses of Theorem 2.6.12 (the condition to be of bounded variation is implicitly satisfied by Lemma 2.6.8), then for all $n \in \mathbb{N}$ we have*

$$D_1(\widetilde{B}_n^{(M)}(u;[a,b]),u) = \int_{\mathbb{R}} \left|\widetilde{B}_n^{(M)}(u;[a,b])(x) - u(x)\right| dx \leq \frac{6(b-a)}{n+1}.$$

Proof. Reasoning as in the proof of the previous theorem we define the fuzzy number v, where $v(x) = 0$ outside $[0,1]$ and $v(x) = u(a+(b-a)x)$ for all $x \in [0,1]$. It is immediate that $supp(v) = [0,1]$. Reasoning for v as we did for G in the previous theorem, for some $n \in \mathbb{N}$ we get

$$\int_{\mathbb{R}} \left|\widetilde{B}_n^{(M)}(u;[a,b])(x) - u(x)\right| dx \leq \frac{(b-a)C_v}{n+1},$$

where, since the restriction of v to the interval $[0,1]$ satisfies the hypothesis of Theorem 2.6.9, we can take

$$C_v = 2[V_0^1(v_1) + V_0^1(v_2) + \|v\|].$$

Here v_1 and v_2 are defined on the same pattern as u_1 and u_2 in Lemma 2.6.8. Therefore, it is immediate that $V_0^1(v_1) = V_0^1(v_2) = 1$ and since, on the other hand, $\|v\| = 1$, we obtain that $C_v = 6$. This finishes the proof. □

Remark. In the above theorem we can assume that for sufficiently large n, $\widetilde{B}_n^{(M)}(u;[a,b])$ is a proper fuzzy number. Indeed, if u is unimodal, then by Remark 1 after the proof of Corollary 2.6.6 we can construct a fuzzy number which coincides with $\widetilde{B}_n^{(M)}(u;[a,b])$, excepting an interval of length $3(b-a)/n$ and this easily implies that if in the integral, we replace $\widetilde{B}_n^{(M)}(u;[a,b])$ with its corresponding fuzzy number, then we obtain the same type of estimation.

Finally, since u^- and u^+ are monotonous functions, taking into account the notations for the fuzzy number $\overline{B}_n^{(M)}(u)$ just before the statement of Theorem 2.6.7, with respect to the metric d_1 we immediately obtain the following result.

Corollary 2.6.14 (Bede–Coroianu–Gal [23]). *Suppose that $u = (u^-, u^+)$ is a fuzzy number such that u^- satisfies the hypothesis of Corollary 2.6.10, (i), and u^+ satisfies the hypothesis of Corollary 2.6.10, (ii). Then, for all $n \in \mathbb{N}$ we have*

$$d_1(u, \overline{B}_n^{(M)}(u)) \le \frac{2[u^-(1) - u^-(0) + u^+(0) - u^+(1)]}{n+1}$$

and

$$|width(u) - width(\overline{B}_n^{(M)}(u))| \le d_1(u, \overline{B}_n^{(M)}(u)).$$

2.7 Bivariate Max-Product Bernstein Operators

In this section, starting from the two kinds of Bernstein polynomials of two variables attached to a function of two variables, $f(x,y)$, defined by

$$B_{n,m}(f)(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)f(i/n, j/m)$$

$$= \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)f(i/n, j/m)}{\sum_{i=0}^{n}\sum_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)}, \ (x,y) \in [0,1]^2, \ n, m \in \mathbb{N},$$

where $f : [0,1] \times [0,1] \to \mathbb{R}$ (see Hildebrandt–Schoenberg [100], Butzer [40]) and

$$T_n(f)(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}x^i y^j (1-x-y)^{n-i-j} f(i/n, j/n)$$

$$= \frac{\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}x^i y^j (1-x-y)^{n-i-j} f(i/n, j/n)}{\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}x^i y^j (1-x-y)^{n-i-j}}, \; x \geq 0, y \geq 0, x+y \leq 1,$$

where $f : \Delta \to \mathbb{R}$, $\Delta = \{(x,y); x \geq 0, y \geq 0, x+y \leq 1\}$ (see the book Lorentz [113], p. 51), we make a similar study with that in the univariate case, for the following two bivariate max-product Bernstein operators, defined by

$$B_{n,m}^{(M)}(f)(x,y) = \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)f(i/n, j/m)}{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)}$$

$$= \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)f(i/n, j/m)}{\bigvee_{i=0}^{n} p_{n,i}(x) \cdot \bigvee_{j=0}^{m} p_{m,j}(y)}, \; (x,y) \in [0,1]^2, \; n, m \in \mathbb{N}, \tag{2.24}$$

and

$$T_n^{(M)}(f)(x,y) = \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}x^i y^j (1-x-y)^{n-i-j} f(i/n, j/n)}{\bigvee_{i=0}^{n}\bigvee_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}x^i y^j (1-x-y)^{n-i-j}}, \tag{2.25}$$

$$(x,y) \in \Delta, \; n \in \mathbb{N},$$

respectively.

Remarks. 1) Since we have $\bigvee_{i=0}^{n} p_{n,i}(x) \cdot \bigvee_{j=0}^{m} p_{m,j}(y) > 0$ for all $x, y \in [0,1]$ and by Lemma 2.1.4 in the univariate case, we explicitly can write

$$\bigvee_{i=0}^{n} p_{n,i}(x) \cdot \bigvee_{j=0}^{m} p_{m,j}(y) = p_{n,r}(x) \cdot p_{m,s}(y),$$

$$\text{for all } (x,y) \in \left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right], r = \overline{0,n}, \; s = \overline{0,m},$$

it follows that $B_{n,m}^{(M)}(f)(x,y)$ is well defined on $[0,1] \times [0,1]$.

In addition, $B_{n,m}^{(M)}(f)(x,y)$ is a continuous functions of (x,y) in $[0,1]^2$. Indeed, as function of (x,y), its denominator is a product of two univariate continuous functions of x and y variable, respectively, which immediately implies the continuity as function of the "global" variable (x,y). On the other hand, the numerator of $B_{n,m}^{(M)}(f)(x,y)$ can be written as a maximum of finite number of continuous bivariate functions, which by the general formula applied recurrently, $\max\{A,B\} = \frac{A+B+|A-B|}{2}$, immediately implies the continuity of the numerator too.

Also, if we denote

$$A_{i,n,r}(x) = \frac{p_{n,i}(x)}{p_{n,r}(x)} = \frac{\binom{n}{i}}{\binom{n}{r}}\left(\frac{x}{1-x}\right)^{i-r},$$

$$A_{j,m,s}(y) = \frac{p_{m,j}(y)}{p_{m,s}(y)} = \frac{\binom{m}{j}}{\binom{m}{s}}\left(\frac{y}{1-y}\right)^{j-s}$$

and

$$A_{i,n,r,j,m,s}(x,y) = A_{i,n,r}(x) \cdot A_{j,m,s}(y),$$

it follows that we can write the following formula useful in the proofs of the approximation results,

$$B_{n,m}^{(M)}(f)(x,y) = \bigvee_{i=0}^{n}\bigvee_{j=0}^{m} A_{i,n,r,j,m,s}(x,y)f(i/n, j/m), \tag{2.26}$$

for all $(x,y) \in \left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right]$, $r = \overline{0,n}$, $s = \overline{0,m}$.

Finally, we note that $B_{n,m}^{(M)}(f)(x,y)$ interpolates $f(x,y)$ on the peaks of the square $[0,1] \times [0,1]$, that is we have

$$B_{n,m}^{(M)}(f)(\alpha,\beta) = f(\alpha,\beta), \text{ for all } \alpha,\beta \in \{0,1\}.$$

2) It easily follows that we can write

$$B_{n,m}^{(M)}(f)(x,y) = B_{n,x}^{(M)}[B_{m,y}^{(M)}(f)](x,y),$$

where, if $G = G(x,y)$ then the notations $B_{n,x}^{(M)}(G)$ means that the univariate max-product Bernstein operator $B_n^{(M)}(G)$ is applied to G considered as function of x, while $B_{n,y}^{(M)}(G)$ means that the univariate max-product Bernstein operator $B_n^{(M)}(G)$ is applied to G considered as function of y. In other words, the bivariate max-product Bernstein operators are tensor products of the univariate max-product Bernstein operators.

3) Since $\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}x^i y^j(1-x-y)^{n-i-j} = 1$ for all $(x,y) \in \Delta$, it easily follows that $\bigvee_{i=0}^{n}\bigvee_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}x^i y^j(1-x-y)^{n-i-j} > 0$, for all $(x,y) \in \Delta$. Indeed, if contrariwise would exist $(x_0,y_0) \in \Delta$ with $\bigvee_{i=0}^{n}\bigvee_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}x_0^i y_0^j(1-x_0-y_0)^{n-i-j} \leq 0$, that would imply $\sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}x_0^i y_0^j(1-x_0-y_0)^{n-i-j} \leq 0$, which is a contradiction. Therefore, $T_n^{(M)}(f)(x,y)$ is well defined for all $(x,y) \in \Delta$ and, in addition, clearly that it is a continuous function on Δ.

Also, we note that $T_n^{(M)}(f)(x,y)$ interpolates $f(x,y)$ on the peaks of the triangle Δ, that is we have

$$T_n^{(M)}(f)(1,0)=f(1,0),\ T_n^{(M)}(f)(0,1)=f(0,1),\ T_n^{(M)}(f)(0,0)=f(0,0).$$

In order to obtain shape preserving properties, we need a few concepts of shapes in the bivariate case, which are natural extensions of the monotonicity and convexity in univariate case, and some of them are obtained by using the "tensor product" method.

Definition 2.7.1. Let $f:[0,1]\times[0,1]\to\mathbb{R}$.

(i) We say that $f(x,y)$ is *increasing (decreasing) with respect to* x on $[0,1]\times[0,1]$, if

$$f(x+h,y)-f(x,y)\geq 0\ (\leq 0),\ \ \forall\, y\in[0,1], \forall x, x+h\in[0,1],\ h\geq 0.$$

(ii) We say that $f(x,y)$ is *increasing (decreasing) with respect to* y on $[0,1]\times[0,1]$, if

$$f(x,y+k)-f(x,y)\geq 0\ (\leq 0),\ \ \forall\, x\in[0,1], \forall y, y+k\in[0,1], k\geq 0.$$

(iii) We say that $f(x,y)$ is *upper (lower) bidimensional monotone* on $[0,1]\times[0,1]$ (see, e.g., Marcus [118], p. 33) if

$$\Delta_2 f(x,y)=f(x+h,y+k)-f(x,y+k)-f(x+h,y)+f(x,y)\geq 0\ (\leq 0),$$

for all $x,x+h\in[0,1]$, $y,y+k\in[0,1]$, $h\geq 0$, $k\geq 0$.

(iv) We say that $f(x,y)$ is *totally upper (lower) monotone* on $[0,1]\times[0,1]$ (see Nicolescu [125] or R.C. Young [160]) if (i), (ii) and (iii) hold with all simultaneously ≥ 0 (or with all simultaneously ≤ 0).

(v) (Popoviciu [129], p. 78) The function f is called *convex of order* (n,m) *in the Popoviciu sense* (where $n,m\in\{0,1,\cdots,\}$) if for any $n+1$ distinct points $x_1<x_2<\cdots<x_{n+1}$ and any $m+1$ distinct points $y_1<y_2<\cdots<y_{m+1}$ in $[0,1]$, we have

$$\begin{bmatrix} x_1,x_2,\cdots,x_{n+1} & \\ & ;f \\ y_1,y_2,\cdots,y_{m+1} & \end{bmatrix}\geq 0,$$

where the symbol above represents the divided difference of a bivariate function and it is defined iteratively (by means of the divided difference of univariate functions) as (see Popoviciu [129], p. 64–65)

$$[x_1,\cdots,x_{n+1};[y_1,\cdots,y_{m+1};f(x,\cdot)]_y]_x=$$
$$[y_1,\cdots,y_{m+1};[x_1,\cdots,x_{m+1};f(\cdot,y)]_x]_y.$$

Here

$$[x_1, \cdots, x_p; g(\cdot)] = \sum_{i=1}^{p} \frac{g(x_i)}{(x_i - x_1)\cdots(x_i - x_{i-1})(x_i - x_{i+1})\cdots(x_i - x_p)},$$

represents the usual divided difference of univariate function g and $[g; x_1] = g(x_1)$.

(vi) The continuous function f is called *quasiconvex with respect to* x on $[0, 1] \times [0, 1]$, if for any fixed $y \in [0, 1]$, as function of x the function $f(x, y)$ is quasiconvex, that is $f(\lambda x_1 + (1 - \lambda)x_2, y) \leq \max\{f(x_1, y), f(x_2, y)\}$, for all $x_1, x_2, y \in [0, 1]$ and $\lambda \in [0, 1]$.

(vii) The continuous function on $[0, 1]\times[0, 1], f$, is *called quasiconvex with respect to* y on $[0, 1] \times [0, 1]$, if for any fixed $x \in [0, 1]$, as function of y the function $f(x, y)$ is quasiconvex, that is $f(x, \lambda y_1 + (1 - \lambda)y_2) \leq \max\{f(x, y_1), f(x, y_2)\}$, for all $y_1, y_2, x \in [0, 1]$ and $\lambda \in [0, 1]$.

(viii) The continuous function on $[0, 1]\times[0, 1], f$, is called *bidimensional quasiconvex* on $[0, 1] \times [0, 1]$, if

$$f(\lambda x_1 + (1 - \lambda)x_2, \mu y_1 + (1 - \mu)y_2)$$
$$\leq \max\{f(x_1, y_1), f(x_1, y_2), f(x_2, y_1), f(x_2, y_2)\},$$

for all $x_1, x_2, y_1, y_2 \in [0, 1]$ and all $\lambda, \mu \in [0, 1]$.

The continuous function f is called *totally quasiconvex* on $[0, 1] \times [0, 1]$, if f is bidimensional quasiconvex, and in addition satisfies (vi) and (vii).

(ix) The continuous function f is *(simply) quasiconvex* on $[0, 1] \times [0, 1]$, if for all $\lambda \in [0, 1]$ and all $x = (x_1, x_2), y = (y_1, y_2) \in [0, 1] \times [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$, that is written more explicit

$$f(\lambda x_1 + (1 - \lambda)y_1, \lambda x_2 + (1 - \lambda)y_2) \leq \max\{f(x_1, x_2), f(y_1, y_2)\}.$$

Remarks. 1) The most natural bivariate monotonicity seems to be that in Definition 2.7.1, (iv), because for such bivariate functions the set of discontinuity points is at most countable (see Nicolescu [125]).

2) In the case when f has partial derivatives, the conditions (i)–(iv) in Definition 2.7.1 can be expressed as follows:

(i) by $\frac{\partial f(x,y)}{\partial x} \geq 0\ (\leq 0),\ \forall\, x, y \in [0, 1]$,
(ii) by $\frac{\partial f(x,y)}{\partial y} \geq 0\ (\leq 0),\ \forall\, x, y \in [0, 1]$,
(iii) by $\frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0\ (\leq 0),\ \forall\, x, y \in [0, 1]$, (see Nicolescu [125]),

while (iv) is represented by all conditions (i)–(iii).

3) It is obvious that convexities of orders $(0, 1)$ and $(1, 0)$ in Popoviciu sense mean in fact $f(x, y)$ is increasing on $[0, 1]$ with respect to y (for any fixed $x \in [0, 1]$) and increasing with respect to x (for any fixed $y \in [0, 1]$), respectively.

Also, convexity of order $(1, 1)$ in Popoviciu sense one reduces to upper bidimensional monotonicity introduced in Marcus [118], p. 33, simultaneously convexities of order $(0, 1)$, $(1, 0)$ and $(1, 1)$ means the totally upper monotonicity in Nicolescu [125], convexity of order $(0, 2)$ means in fact that $f(x, y)$ is convex on $[0, 1]$ with respect to y (for any fixed x), and so on.

4) Suppose f is of C^{n+m} class on $[0, 1] \times [0, 1]$.

By the mean value theorem we get that if $\frac{\partial^{n+m} f(x,y)}{\partial x^n \partial y^m} \geq 0$, $\forall (x, y) \in [0, 1] \times [0, 1]$, then $f(x, y)$ is convex of order (n, m) in Popoviciu sense on $[0, 1] \times [0, 1]$.

5) If f is quasiconvex with respect to x and quasiconvex with respect to y, then it is easy to check by direct calculation that these imply that f is bidimensional quasiconvex.

In what follows, we deal with the approximation properties of the bivariate max-product Bernstein operator $B_{n,m}^{(M)}$. The first main result is the following.

Theorem 2.7.2. *(i) Let $f : [0, 1] \times [0, 1] \to \mathbb{R}_+$ be a continuous function. We have*

$$|B_{n,m}^{(M)}(f)(x, y) - f(x, y)| \leq 18\omega_1 \left(f; \frac{1}{\sqrt{n+1}}, \frac{1}{\sqrt{m+1}} \right),$$

valid for all $x, y \in [0, 1]$ and $n, m \in \mathbb{N}$. Here

$$\omega_1(f; \delta, \eta) = \sup\{|f(x, y) - f(u, v)|; x, y, u, v \in [0, 1], |x - u| \leq \delta, |y - v| \leq \eta\}.$$

(ii) Suppose that $f : [0, 1] \times [0, 1] \to \mathbb{R}_+$ is continuous on $[0, 1] \times [0, 1]$ and that is concave with respect to x (for all $y \in [0, 1]$ fixed) and concave with respect to y (for all $x \in [0, 1]$ fixed) (in other words, f is concave in Popoviciu sense of the orders $(0, 2)$ and $(2, 0)$). Then, for all $x, y \in [0, 1]$ and $m, n \in \mathbb{N}$, we have

$$|B_{n,m}^{(M)}(f)(x, y) - f(x, y)| \leq 4\omega_1 \left(f; \frac{1}{n}, \frac{1}{m} \right).$$

(iii) Suppose that $f : [0, 1] \times [0, 1] \to \mathbb{R}_+$ is continuous and strictly positive on $[0, 1] \times [0, 1]$. Then, for all $x, y \in [0, 1]$ and $m, n \in \mathbb{N}$, we have

$$|B_{n,m}^{(M)}(f)(x, y) - f(x, y)|$$
$$\leq 8\omega_1 \left(f; \frac{1}{n}, \frac{1}{m} \right) + \frac{1}{m_f} \left[n \cdot \left(\omega_{1,x} \left(f; \frac{1}{n} \right) \right)^2 + m \cdot \left(\omega_{1,y} \left(f; \frac{1}{m} \right) \right)^2 \right],$$

where $\omega_{1,x}(f; \delta) = \sup\{|f(x + h, y) - f(x, y)|; x, x + h, y \in [0, 1], 0 \leq h \leq \delta\}$, $\omega_{1,y}(f; \delta) = \sup\{|f(x, y + h) - f(x, y)|; x, y, y + h \in [0, 1], 0 \leq h \leq \delta\}$ and $m_f = \min\{f(x, y); x, y \in [0, 1]\}$.

In addition, if f is a Lipschitz function, that is $|f(x,y)-f(u,v)| \le L(|x-u|+|y-v|)$, for all $x, y, u, v, \in [0,1]$, then

$$|B_{n,m}^{(M)}(f)(x,y)-f(x,y)| \le \left(8L+\frac{L^2}{m_f}\right)\left(\frac{1}{n}+\frac{1}{m}\right).$$

Proof. (i) Taking into account the inequality valid for the positive numbers A_k, B_k, $k \in \{0,1,\ldots,p\}$,

$$|\max_{k\in\{0,1,\ldots,p\}}\{A_k\} - \max_{k\in\{0,1,\ldots,p\}}\{B_k\}| \le \max_{k\in\{0,1,\ldots,p\}}\{|A_k-B_k|\},$$

we obtain

$$\begin{aligned}
&|B_{n,m}^{(M)}(f)(x,y)-f(x,y)| \\
&= \left|\frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)f(i/n,j/m)}{\bigvee_{i=0}^{n} p_{n,i}(x)\cdot\bigvee_{j=0}^{m} p_{m,j}(y)} - \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)f(x,y)}{\bigvee_{i=0}^{n} p_{n,i}(x)\cdot\bigvee_{j=0}^{m} p_{m,j}(y)}\right| \\
&\le \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)|f(i/n,j/m)-f(x,y)|}{\bigvee_{i=0}^{n} p_{n,i}(x)\cdot\bigvee_{j=0}^{m} p_{m,j}(y)} \\
&\le \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)\omega_1(f;|i/n-x|,|j/m-y|)|}{\bigvee_{i=0}^{n} p_{n,i}(x)\cdot\bigvee_{j=0}^{m} p_{m,j}(y)} \\
&= \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)\omega_1\left(f;\delta\frac{|i/n-x|}{\delta},\eta\frac{|j/m-y|}{\eta}\right)}{\bigvee_{i=0}^{n} p_{n,i}(x)\cdot\bigvee_{j=0}^{m} p_{m,j}(y)} \\
&= \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)\left(1+\frac{|i/n-x|}{\delta}+\frac{|j/m-y|}{\eta}\right)\omega_1(f;\delta,\eta)}{\bigvee_{i=0}^{n} p_{n,i}(x)\cdot\bigvee_{j=0}^{m} p_{m,j}(y)} \\
&= \omega_1(f;\delta,\eta)\cdot\left(1+\frac{1}{\delta}\cdot\frac{\bigvee_{i=0}^{n} p_{n,i}(x)|i/n-x|}{\bigvee_{i=0}^{n} p_{n,i}(x)}+\frac{1}{\eta}\cdot\frac{\bigvee_{j=0}^{m} p_{m,j}(y)|j/m-y|}{\bigvee_{j=0}^{m} p_{m,j}(y)}\right).
\end{aligned}$$

Taking here into account the univariate case (see the formulas (2.1)–(2.2) in the proof of Theorem 2.1.5) and choosing $\delta = \frac{6}{\sqrt{n+1}}$, $\eta = \frac{6}{\sqrt{m+1}}$, we get

$$\begin{aligned}
|B_{n,m}^{(M)}(f)(x,y)-f(x,y)| &\le 3\omega_1\left(f;\frac{6}{\sqrt{n+1}},\frac{6}{\sqrt{m+1}}\right) \\
&\le 18\omega_1\left(f;\frac{1}{\sqrt{n+1}},\frac{1}{\sqrt{m+1}}\right).
\end{aligned}$$

In the proof of (i) we also used the following known properties: $\omega_1(f; \lambda\delta, \mu\eta) \leq (1 + \lambda + \mu)\omega_1(f; \delta, \eta)$ and $\omega_1(f; n\delta, n\mu) \leq n\omega_1(f; \delta, \eta)$, for all $n \in \mathbb{N}$, $\delta \geq 0$, $\eta \geq 0$, $\lambda \geq 0$ and $\mu \geq 0$ (see, e.g., Timan [146], p. 112, relation (3)).

(ii) Denoting $A_{i,n,r} = \frac{\binom{n}{i}}{\binom{n}{r}} \left(\frac{x}{1-x}\right)^{i-r}$, for the expression appearing in (2.26) we have $A_{i,n,r,j,m,s}(x, y) = A_{i,n,r}(x) \cdot A_{j,m,s}(y)$. Also, let us denote

$$F_{i,n,r,j,m,s}(x, y) = A_{i,n,r,j,m,s}(x, y) \cdot f\left(\frac{i}{n}, \frac{j}{m}\right),$$
$$\text{for all } (x, y) \in \left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right].$$

By (2.26) we can write

$$B_{n,m}^{(M)}(f)(x, y) = \bigvee_{i=0}^{n} \bigvee_{j=0}^{m} F_{i,n,r,j,m,s}(x, y),$$
$$(x, y) \in \left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right], r = \overline{0, n},\ s = \overline{0, m}.$$

Fixing r, j, m, s, and y and reasoning on $F_{i,n,r,j,m,s}(x, y)$ with respect to i (that is on $A_{i,n,r}(x)f\left(\frac{i}{n}, \frac{j}{m}\right)$) exactly as in the proof of Corollary 2.1.10 (see also the proof of Corollary 4.6 in Bede–Coroianu–Gal [21]), we get

$$\bigvee_{i=0}^{n} F_{i,n,r,j,m,s}(x, y)$$
$$= \max\{F_{r-1,n,r,j,m,s}(x, y), F_{r,n,r,j,m,s}(x, y), F_{r+1,n,r,j,m,s}(x, y)\},$$

because $f(x, y)$ is concave with respect to x. Note that above for $r = 0$ ($r = n$, respectively), the term $F_{r-1,n,r,j,m,s}(x, y)$ ($F_{r+1,n,r,j,m,s}(x, y)$, respectively) is not defined but in fact does not appear under the max operator.

On the other hand, fixing x and reasoning now on the terms of the form $A_{j,m,s}(y)f\left(\frac{i}{n}, \frac{j}{m}\right)$ in the expressions of each of the three terms in the above formula, from the proof of Corollary 2.1.10 (see also the proof of Corollary 4.6 in Bede–Coroianu–Gal [21]) and taking into account that $f(x, y)$ is concave with respect to y, we get

$$B_{n,m}^{(M)}(f)(x, y)$$
$$= \max\{F_{r-1,n,r,s-1,m,s}(x, y), F_{r-1,n,r,s,m,s}(x, y), F_{r-1,n,r,s+1,m,s}(x, y),$$
$$F_{r,n,r,s-1,m,s}(x, y), F_{r,n,r,s,m,s}(x, y)(x, y), F_{r,n,r,s+1,m,s}(x, y),$$
$$F_{r+1,n,r,s-1,m,s}(x, y), F_{r+1,n,r,s,m,s}(x, y), F_{r+1,n,r,s+1,m,s}(x, y)\},$$

for all $(x, y) \in \left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right]$.

Note again that for $r = 0$, or $r = n$, or $s = 0$, or $s = m$, the corresponding functions F in the above maximum operator which are not defined, in fact do not appear.

The expressions inside the maximum can be combined two by twos in $\binom{9}{2} = 36$ different ways and it is easy to see (by a finite mathematical induction) that if we can get

$$|\max\{F_{p,n,i,q,m,j}(x,y), F_{p',n,i,q',m,j}(x,y)\} - f(x,y)| \leq 4\omega_1\left(f; \frac{1}{n}, \frac{1}{m}\right),$$

for all $p, p' \in \{i-1, i, i+1\}$, $q, q' \in \{j-1, j, j+1\}$, then this implies

$$|B_{n,m}^{(M)}(f)(x,y) - f(x,y)| \leq 4\omega_1\left(f; \frac{1}{n}, \frac{1}{m}\right).$$

Reasoning exactly as in the proof of Lemma 2.1.6 (see also Lemma 4.2 in Bede–Coroianu–Gal [21]) and in the proof of Lemma 2.1.7 (see also Lemma 4.3 in Bede–Coroianu–Gal [21]), we easily get the above estimate.

(iii) We get

$$|B_{n,m}^{(M)}(f)(x,y) - f(x,y)|$$
$$= \left| \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)f(i/n, j/m)}{\bigvee_{i=0}^{n} p_{n,i}(x)\cdot\bigvee_{j=0}^{m} p_{m,j}(y)} - \frac{\bigvee_{i=0}^{n}\bigvee_{j=0}^{m} p_{n,i}(x)p_{m,j}(y)f(x,y)}{\bigvee_{i=0}^{n} p_{n,i}(x)\cdot\bigvee_{j=0}^{m} p_{m,j}(y)} \right|$$
$$\leq \left| \bigvee_{i=0}^{n}\left(\frac{p_{n,i}(x)}{\bigvee_{k=0}^{n} p_{n,k}(x)}\cdot\bigvee_{j=0}^{m}\frac{p_{m,j}(y)}{\bigvee_{s=0}^{m} p_{m,s}(y)} f(i/n, j/m)\right)\right.$$
$$\left. - \bigvee_{i=0}^{n}\left(\frac{p_{n,i}(x)}{\bigvee_{k=0}^{n} p_{n,k}(x)}\cdot\bigvee_{j=0}^{m}\frac{p_{m,j}(y)}{\bigvee_{s=0}^{m} p_{m,s}(y)} f(i/n, y)\right)\right|$$
$$+ \left| \bigvee_{i=0}^{n}\left(\frac{p_{n,i}(x)}{\bigvee_{k=0}^{n} p_{n,k}(x)}\cdot\bigvee_{j=0}^{m}\frac{p_{m,j}(y)}{\bigvee_{s=0}^{m} p_{m,s}(y)} f(i/n, y)\right)\right.$$
$$\left. - \bigvee_{i=0}^{n}\frac{p_{n,i}(x)}{\bigvee_{k=0}^{n} p_{n,k}(x)}\cdot\bigvee_{j=0}^{m}\frac{p_{m,j}(y)}{\bigvee_{s=0}^{m} p_{m,s}(y)} f(x,y)\right| \leq \bigvee_{i=0}^{n}\frac{p_{n,i}(x)}{\bigvee_{k=0}^{n} p_{n,k}(x)}\cdot$$
$$\cdot\left|\bigvee_{j=0}^{m}\frac{p_{m,j}(y)}{\bigvee_{s=0}^{m} p_{m,s}(y)} f(i/n, j/m) - \bigvee_{j=0}^{m}\frac{p_{m,j}(y)}{\bigvee_{s=0}^{m} p_{m,s}(y)} f(i/n, y)\right|$$
$$+ \bigvee_{j=0}^{m}\frac{p_{m,j}(y)}{\bigvee_{s=0}^{m} p_{m,s}(y)}\left|\bigvee_{i=0}^{n}\frac{p_{n,i}(x)}{\bigvee_{k=0}^{n} p_{n,k}(x)} f(i/n, y) - \bigvee_{i=0}^{n}\frac{p_{n,i}(x)}{\bigvee_{k=0}^{n} p_{n,k}(x)} f(x,y)\right|$$

(which by Theorem 2.2.18, see also Theorem 2.6 in Coroianu–Gal [52]

in the univariate case, implies)

$$\leq \left(\frac{m \cdot \omega_{1,y}\left(f; \frac{1}{m}\right)}{m_f} + 4\right) \cdot \omega_{1,y}\left(f; \frac{1}{m}\right) + \left(\frac{n \cdot \omega_{1,x}\left(f; \frac{1}{n}\right)}{m_f} + 4\right) \cdot \omega_{1,x}\left(f; \frac{1}{n}\right)$$

$$\leq 8\omega_1\left(f; \frac{1}{n}, \frac{1}{m}\right) + \frac{1}{m_f} \cdot \left[n\left(\omega_{1,x}\left(f; \frac{1}{n}\right)\right)^2 + m\left(\omega_{1,y}\left(f; \frac{1}{m}\right)\right)^2\right].$$

For the last inequality, we used the inequality $\omega_{1,x}\left(f; \frac{1}{n}\right) + \omega_{1,y}\left(f; \frac{1}{m}\right) \leq 2\omega_1\left(f; \frac{1}{n}, \frac{1}{m}\right)$ (see Anastassiou–Gal [8], p. 81).

For f Lipschitz function, the last estimate in the statement of (iii) is an immediate consequence of the above one. □

According to Theorem 2.2.2 in Gal [84], p. 116, the bivariate Bernstein polynomials $B_{n,m}(f)(x, y)$ preserve the Popoviciu convexities of any order (n, m) with $n, m \in \{0, 1, \ldots, \}$. On the other part, it is well known the fact that the usual convexity of $f(x, y)$ (in the geometric sense that $z = f(x, y)$ is a convex surface) is not preserved by the bivariate Bernstein polynomials $B_{n,m}(f)(x, y)$. Therefore, it is natural to see what shape preserving properties have the bivariate max-product Bernstein operators, $B_{n,m}^{(M)}(f)$.

Firstly, it is evident that $B_n^{(M)}(f)(x, y)$ does not preserve the convexity of $f(x, y)$ with respect to x (that is the $(2, 0)$ convexity in Popoviciu sense) and the convexity of $f(x, y)$ with respect to y. Indeed, by taking $f(x, y) = g(x)$ and $f(x, y) = h(y)$, from $B_{n,m}^{(M)}(g)(x, y) = B_n^{(M)}(g)(x)$ and $B_{n,m}^{(M)}(h)(x, y) = B_m(h)(y)$ would follow that the univariate max-product Bernstein operator preserve the convexity, which contradicts the fact that this does not hold in general, see Remark 1, after the proof of Lemma 2.1.19.

Taking into account that the univariate max-product Bernstein operators preserve only the monotonicity and the quasiconvexity, we expect that in the bivariate case, the max-product Bernstein operators to preserve exactly the kinds of bivariate monotonicity in Definition 2.7.1. In this sense, we present the following.

Theorem 2.7.3. *Let* $f : [0, 1] \times [0, 1] \to \mathbb{R}_+$ *be continuous on* $[0, 1] \times [0, 1]$. *Then for all* $n, m \in \mathbb{N}$*, we have*

(i) if $f(x, y)$ *is increasing (decreasing) with respect to* x *(meaning for all fixed, arbitrary* y*), then so is the operator* $B_{n,m}^{(M)}(f)(x, y)$ *;*

(ii) if $f(x, y)$ *is increasing (decreasing) with respect to* y*, then so is the operator* $B_{n,m}^{(M)}(f)(x, y)$ *;*

(iii) if $f(x, y)$ *is simultaneously monotonous with respect to* x *and* y *and of the same monotonicity (of opposite monotonicity, respectively), then the operator* $B_{n,m}^{(M)}(f)(x, y)$ *is upper (lower, respectively) bidimensional monotone on* $[0, 1] \times [0, 1]$ *;*

(iv) *if* $f(x,y)$ *is simultaneously monotonous with respect to* x *and* y *and of the same monotonicity (of opposite monotonicity, respectively), then the operator* $B_{n,m}^{(M)}(f)(x,y)$ *is totally upper (lower, respectively) monotone on* $[0,1]\times[0,1]$ *;*

(v) *if* $f(x,y)$ *is quasiconvex and continuous on* $[0,1]\times[0,1]$ *with respect to* x*, then so is the operator* $B_{n,m}^{(M)}(f)(x,y)$ *;*

(vi) *if* $f(x,y)$ *is quasiconvex and continuous on* $[0,1]\times[0,1]$ *with respect to* y*, then so is the operator* $B_{n,m}^{(M)}(f)(x,y)$ *;*

(vii) *if* $f(x,y)$ *is continuous on* $[0,1]\times[0,1]$ *and simultaneously quasiconvex with respect to* x *and quasiconvex with respect to* y*, then* $B_{n,m}^{(M)}(f)(x,y)$ *is bidimensional quasiconvex and continuous on* $[0,1]\times[0,1]$ *;*

(viii) *if* $f(x,y)$ *is continuous on* $[0,1]\times[0,1]$ *and simultaneously quasiconvex with respect to* x *and quasiconvex with respect to* y*, then the operator* $B_{n,m}^{(M)}(f)(x,y)$ *is totally quasiconvex and continuous on* $[0,1]\times[0,1]$.

Proof. (i), (ii). Are immediate from the preservation of the monotonicity of the univariate max-product Bernstein operators in Theorem 5.5, Corollary 5.6 in Bede–Coroianu–Gal [21] (see also Theorem 2.1.15, Corollary 2.1.16), because of the relationship (2.24) defining $B_{n,m}^{(M)}(f)(x,y)$, where we can also change the order of the two maximum operators from the numerator.

Indeed, from (2.24) we see that we can write

$$B_{n,m}^{(M)}(f)(x,y)$$
$$=\bigvee_{j=0}^{m}\nu_{m,j}(y)\cdot B_n^{(M)}(f(\cdot,j/m))(x)=\bigvee_{i=0}^{n}\lambda_{n,i}(x)\cdot B_m^{(M)}(f(i/n,\cdot))(y), \tag{2.27}$$

where

$$\nu_{m,j}(y)=\frac{p_{m,j}(y)}{\bigvee_{j=0}^{m}p_{m,j}(y)} \text{ and } \lambda_{n,i}(x)=\frac{p_{n,i}(x)}{\bigvee_{i=0}^{n}p_{n,i}(x)}.$$

As a consequence, the proof simply reduces to the obvious property that the maximum of a finite number of univariate monotone functions (of the same monotonicity) remains a monotone function.

(iii) Suppose, for example, that $f(x,y)$ is simultaneously nondecreasing with respect to x and y. From relation (2.26) and taking into account relation (5.6), p. 20 in [21] (see also the relation in the proof of Theorem 2.1.15), we easily get

$$B_{n,m}^{(M)}(f)(x,y)=\bigvee_{i=0}^{n}A_{i,n,r}(x)\bigvee_{j=s}^{m}A_{j,m,s}(y)f(i/n,j/m),$$

for all $(x,y)\in\left[\frac{r}{n+1},\frac{r+1}{n+1}\right]\times\left[\frac{s}{m+1},\frac{s+1}{m+1}\right]$, $r=\overline{0,n}$, $s=\overline{0,m}$.

Then, since as function of x, $G(x)=\bigvee_{j=s}^{m}A_{j,m,s}(y)f(x,j/m)$ is nondecreasing as maximum of nondecreasing functions of x, by the same relation (5.6), p. 20 in [21] (see also the relation in the proof of Theorem 2.1.15), we get

$$B_{n,m}^{(M)}(f)(x,y) = \bigvee_{i=r}^{n} A_{i,n,r}(x) \bigvee_{j=s}^{m} A_{j,m,s}(y) f(i/n, j/m)$$

$$= \bigvee_{i=r}^{n} \bigvee_{j=s}^{m} A_{i,n,r,j,m,s}(x,y) f(i/n, j/m),$$

for all $(x,y) \in \left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right]$, $r = \overline{0,n}$, $s = \overline{0,m}$.

It is easy to check that for all $(x,y) \in \left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right]$ and $i \geq r, j \geq s$, we have $\frac{\partial^2 A_{i,n,r,j,m,s}}{\partial x \partial y} \geq 0$. Taking into account the properties of the $\max\{F(x), G(x)\}$ function (that it has only a finite number of points of nondifferentiability), it is clear that $\left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right]$ can be decomposed into a finite grid of open bivariate subintervals, disjoint two by twos and covering (excepting their boundary) $\left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right]$, such that on each such of subinterval, $B_{n,m}^{(M)}(f)(x,y)$ coincides with one of the function $A_{i,n,r,j,m,s}(x,y)$, with $i \geq r$ and $j \geq s$. By the Remark 2, (iii), after the Definition 2.7.1, this means that on each kind of this open subinterval, $B_{n,m}^{(M)}(f)(x,y)$ is upper bidimensional monotone. Combining with the continuity of $B_{n,m}^{(M)}(f)$ on $[0,1] \times [0,1]$, by reduction to absurdum it can easily be proved that in fact $B_{n,m}^{(M)}(f)$ is upper bidimensional monotone on each $\left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right]$, which implies the upper bidimensional monotonicity on the whole $[0,1] \times [0,1]$.

If we suppose that $f(x,y)$ is simultaneously nonincreasing with respect to x and y, then applying now relation (5.8), p. 20 in [21] (see also the relation in the proof of Corollary 2.1.16), we get

$$B_{n,m}^{(M)}(f)(x,y) = \bigvee_{i=0}^{r} A_{i,n,r}(x) \bigvee_{j=0}^{s} A_{j,m,s}(y) f(i/n, j/m)$$

$$= \bigvee_{i=0}^{r} \bigvee_{j=0}^{s} A_{i,n,r,j,m,s}(x,y) f(i/n, j/m),$$

for all $(x,y) \in \left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right]$, $r = \overline{0,n}$, $s = \overline{0,m}$.

By similar reasonings as above, we arrive again at the conclusion that $B_{n,m}^{(M)}(f)$ is upper bidimensional monotone on $[0,1] \times [0,1]$.

Now, if we suppose, for example, that $f(x,y)$ is nondecreasing with respect to x and nonincreasing with respect to y, arrive at the formula

$$B_{n,m}^{(M)}(f)(x,y) = \bigvee_{i=r}^{n} A_{i,n,r}(x) \bigvee_{j=0}^{s} A_{j,m,s}(y) f(i/n, j/m)$$

$$= \bigvee_{i=r}^{n} \bigvee_{j=0}^{s} A_{i,n,r,j,m,s}(x,y) f(i/n, j/m),$$

for all $(x, y) \in \left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right], r = \overline{0, n},\ s = \overline{0, m}$. Taking now into account that for $i \geq r$ and $j \leq s$ we have $\frac{\partial^2 A_{i,n,r,j,m,s}}{\partial x \partial y} \leq 0$, by similar reasonings we get that $B_{n,m}^{(M)}(f)$ is lower bidimensional monotone on $[0, 1] \times [0, 1]$.

The case when $f(x, y)$ is nonincreasing with respect to x and nondecreasing with respect to y is similar.

(iv) It is an immediate consequence of (i), (ii), and (iii).

(v), (vi) Are immediate from the preservation of the quasiconvexity of the univariate max-product Bernstein operators in Corollary 5.9 in Bede–Coroianu–Gal [21] (see also Corollary 2.1.18), because of the relationship (2.24) defining $B_{n,m}^{(M)}(f)(x, y)$, where we can also change the order of the two maximum operators from the numerator.

Indeed, reasoning as in the case of the above points (i) and (ii), the proof simply reduces to the obvious property that the maximum of a finite number of univariate quasiconvex continuous functions remains a quasiconvex continuous function.

(vii) Since $f(x, y)$ is quasiconvex with respect to y (for any fixed x), by the last expression in (2.27) and by the above point (vi) it follows that $B_{n,m}^{(M)}(f)(x, y) = \bigvee_{i=0}^{n} \lambda_{n,i}(x) \cdot B_m^{(M)}(f(i/n, \cdot))(y)$ is quasiconvex as function of y, as finite maximum of quasiconvex functions of y (since $\lambda_{n,i}(x) \geq 0$). Similarly, from the first expression in (2.27) and by the above point (v), it follows that $B_{n,m}^{(M)}(f)(x, y) = \bigvee_{j=0}^{m} \nu_{m,j}(y) \cdot B_n^{(M)}(f(\cdot, j/m))(x)$ is quasiconvex as function of x, as finite maximum of quasiconvex functions of x (since $\nu_{m,j}(y) \geq 0$).

Finally, taking into account Remark 5 after Definition 2.7.1, it follows that $B_{n,m}^{(M)}(f)(x, y)$ is bidimensional quasiconvex on $[0, 1] \times [0, 1]$.

(viii) It is an immediate consequence of (v), (vi), (vii). □

Finally, note that by the above formula (2.26) and by the statement and proof of Theorem 2.1.20, we immediately get the following.

Theorem 2.7.4. *For any continuous function $f : [0, 1] \times [0, 1] \to \mathbb{R}_+$, the max-product operators $B_{n,m}^{(M)}(f)(x, y)$ is convex with respect to x, convex with respect to y and convex of order $(2, 2)$ in Popoviciu sense (see Definition 2.7.1, (v)) on any bidimensional subinterval of the form $\left[\frac{r}{n+1}, \frac{r+1}{n+1}\right] \times \left[\frac{s}{m+1}, \frac{s+1}{m+1}\right], r = \overline{0, n},\ s = \overline{0, m}$.*

Remarks. 1) It is an open question if $B_{n,m}^{(M)}(f)$ preserves the bidimensional monotonicity of f.

2) It is an open question if $B_{n,m}^{(M)}(f)$ preserves the bidimensional quasiconvexity of f.

3) It is an open question if $B_{n,m}^{(M)}(f)$ preserves the simply quasiconvexity of f in Definition 2.7.1, (ix).

4) There exist classes of bivariate functions f, for which $B_{n,m}^{(M)}(f)$ gives better approximation results than $B_{n,m}(f)$. For example, large subclasses of the class of bivariate Lipschitz functions are better approximated by $B_{n,m}^{(M)}(f)$ than by $B_{n,m}(f)$.

Indeed, by Theorem 2.2.2, p. 116 in [84], we have

$$\|B_{n,m}(f) - f\| \leq C\omega_2^{\varphi}(f; 1/\sqrt{n}, 1/\sqrt{m}),$$

where $\varphi^2(x) = x(1-x)$ and ω_2^{φ} is the Ditzian–Totik modulus of smoothness. Reasoning similar to the univariate case in Section 6, p. 25 in [21](see also Remark 2, from the end of Section 2.1), we easily can construct Lipschitz bivariate functions $f(x,y) = g(x)h(y)$ (where g and h are two univariate concave nondecreasing polygonal lines), such that $\omega_2^{\varphi}(f;\delta,\eta) \sim \delta + \eta$, which will immediately imply $\|B_{n,m}(f) - f\| \leq C(1/\sqrt{n} + 1/\sqrt{m})$.

On the other hand, for such (Lipschitz) bivariate functions, by Theorem 2.7.2, (iii), we have $\|B_{n,m}^{(M)}(f) - f\| \leq C(1/n + 1/m)$, which is essentially better.

Also, note that the shape preserving properties of $B_{n,m}^{(M)}(f)$ expressed by Theorem 2.7.3, (iii)–(iv) and (vii)–(viii), do not hold for the linear Bersntein operator $B_{n,m}(f)$.

A simple comparison of the properties of $B_{n,m}^{(M)}(f)$ and $B_{n,m}(f)$ can be seen in the next Figures 2.2, 2.3, and 2.4, for $f(x,y) = g(x)g(y)$, where $g : [0,1] \to \mathbb{R}_+$ is given by

$$g(x) = \begin{cases} 3x^2 + 0.25 & \mathit{if} \quad 0 \leq x \leq 0.5 \\ 1 & \mathit{if} \quad 0.5 < x \leq 0.75 \\ -3.6x + 3.7 & \mathit{if} \quad 0.75 < x \leq 1 \end{cases}$$

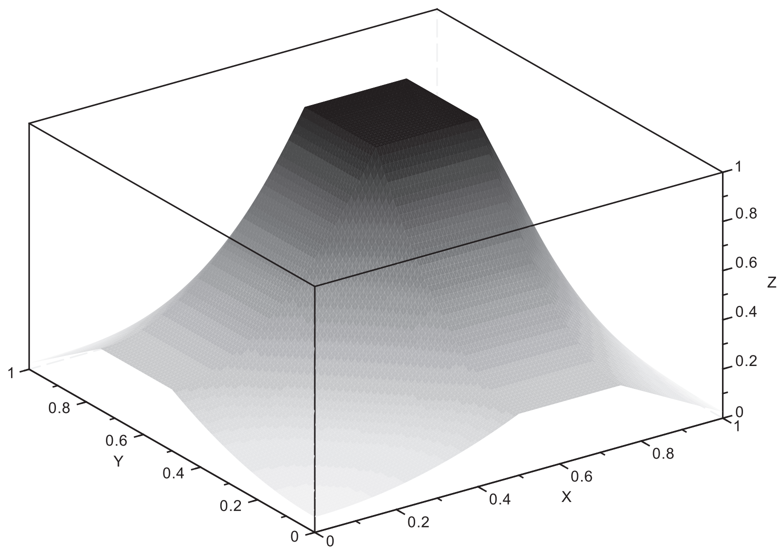

Fig. 2.2 Approximated bivariate function.

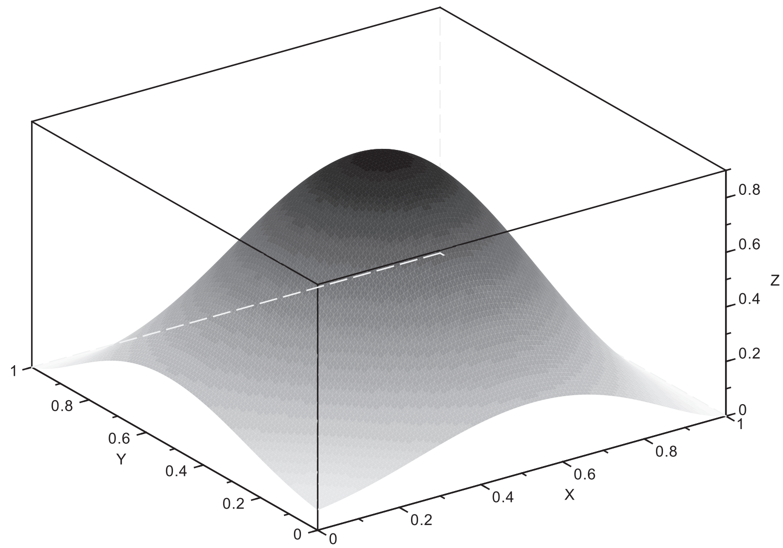

Fig. 2.3 Bivariate Bernstein polynomial.

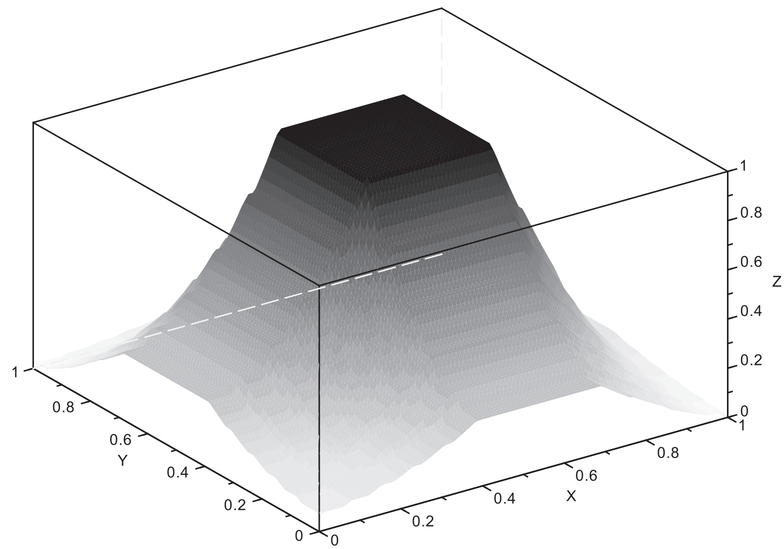

Fig. 2.4 Max-product Bernstein operator.

From these figures it is easily seen that for the given particular function, the bivariate max-product Bernstein operator has better approximation and shape preserving properties than the bivariate Bernstein polynomials.

In what follows we study the approximation properties of the bivariate max-product Bernstein operator $T_n^{(M)}$.

Firstly, we need some auxiliary results.

Theorem 2.7.5. *Denote*

$$C_+(\Delta) = \{f : \Delta \to [0, +\infty); f \text{ is continuous on } \Delta\},$$

where $\Delta = \{(x, y); x \geq 0, y \geq 0, \ x + y \leq 1\}$.

(i) *If* $f \in C_+(\Delta)$*, then* $T_n^{(M)}(f) \in C_+(\Delta)$*, for all* $n \in \mathbb{N}$ *;*
(ii) $T_n^{(M)}(\lambda f) = \lambda T_n^{(M)}(f)$*, for all* $\lambda \geq 0, f \in C_+(\Delta)$ *and* $n \in \mathbb{N}$ *;*
(iii) $T_n^{(M)}(f + g)(x, y) \leq T_n^{(M)}(f)(x, y) + T_n^{(M)}(g)(x, y)$*, for all* $f, g \in C_+(\Delta)$*,* $n \in \mathbb{N}$ *and* $(x, y) \in \Delta$ *;*
(iv) *If* $f, g \in C_+(\Delta)$ *satisfy* $f(x, y) \leq g(x, y)$ *for all* $(x, y) \in \Delta$*, then* $T_n^{(M)}(f)(x, y) \leq T_n^{(M)}(g)(x, y)$*, for all* $(x, y) \in \Delta$ *and* $n \in \mathbb{N}$.
(v) *Denoting* $e_0(x, y) = 1$ *for all* $(x, y) \in \Delta$*, we have* $T_n^{(M)}(e_0)(x, y) = e_0(x, y)$*, for all* $(x, y) \in \Delta$ *and* $n \in \mathbb{N}$.
(vi) *For all* $f, g \in C_+(\Delta)$*,* $n \in N$ *and* $(x, y) \in \Delta$ *we have*

$$|T_n^{(M)}(f)(x, y) - T_n^{(M)}(g)(x, y)| \leq T_n^{(M)}(|f - g|)(x, y).$$

Proof. The properties (i)–(v) are immediate from the definition of $T_n^{(M)}$ and from the properties of the maximum operator. To prove (vi), let $f, g \in C_+(\Delta)$. We have $f = f - g + g \leq |f - g| + g$, which by the above points (iii) and (iv) successively implies $T_n^{(M)}(f)(x, y \leq T_n^{(M)}(|f - g|)(x, y) + T_n^{(M)}(g)(x, y)$, that is $T_n^{(M)}(f)(x, y) - T_n^{(M)}(g)(x, y) \leq T_n^{(M)}(|f - g|)(x, y)$.

Writing now $g = g - f + f \leq |f - g| + f$ and applying the above reasonings, it follows $T_n^{(M)}(g)(x, y) - T_n^{(M)}(f)(x, y) \leq T_n^{(M)}(|f - g|)(x, y)$, which combined with the above inequality gives $|T_n^{(M)}(f)(x, y) - T_n^{(M)}(g)(x, y)| \leq T_n^{(M)}(|f - g|)(x, y)$. □

As a consequence, we get the following.

Corollary 2.7.6. *For all* $f \in C_+(\Delta)$*,* $n \in \mathbb{N}$ *and* $(x, y) \in \Delta$*, we have*

$$|f(x, y) - T_n^{(M)}(f)(x, y)|$$
$$\leq \left[1 + \frac{1}{\delta} \cdot T_n^{(M)}(\varphi_x)(x, y) + \frac{1}{\delta} \cdot T_n^{(M)}(\varphi_y)(x, y)\right] \omega_1(f; \delta)_\Delta,$$

where $\varphi_x(t) = |t - x|$*, where* t *is variable and* x *is supposed fixed,* $\varphi_y = |s - y|$*, where* s *is supposed variable and* y *is supposed fixed, and*

$$\omega_1(f; \delta)_\Delta = \sup\{|f(t, s) - f(x, y)|; (t, s), (x, y) \in \Delta, \sqrt{(t - x)^2 + (s - y)^2} \leq \delta\}.$$

Proof. By Theorem 2.7.5, (vi), it follows

$$|f(x, y) - T_n^{(M)}(f)(x, y)| = |T_n^{(M)}(f(x, y))(x, y) - T_n^{(M)}(f(t, s))(x, y)|$$
$$\leq T_n^{(M)}(|f(t, s) - f(x, y)|)(x, y).$$

Now, since for all $(t, s), (x, y) \in \Delta$ we have

$$|f(t,s) - f(x,y)| \leq \omega_1(f; \|(t,s) - (x,y)\|) \leq \left[\frac{1}{\delta}\|(t,s) - (x,y)\| + 1\right]\omega_1(f;\delta)_\Delta$$
$$= \left[\frac{1}{\delta} \cdot \sqrt{(t-x)^2 + (s-y)^2} + 1\right]\omega_1(f;\delta)_\Delta$$
$$\leq \omega_1(f;\delta)_\Delta \cdot \left[\frac{1}{\delta} \cdot |t-x| + \frac{1}{\delta} \cdot |s-y| + 1\right],$$

replacing in the above inequality we immediately obtain the estimate in the statement. □

Remarks. 1) From Corollary 2.7.6 it follows that the approximation properties of $T_n^{(M)}(f)(x,y)$ are controlled by the ratios

$$T_n^{(M)}(\varphi_x)(x,y) = \frac{\bigvee_{i=0}^n \bigvee_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j} x^i y^j (1-x-y)^{n-i-j} |i/n - x|}{\bigvee_{i=0}^n \bigvee_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j} x^i y^j (1-x-y)^{n-i-j}}$$

and

$$T_n^{(M)}(\varphi_y)(x,y) = \frac{\bigvee_{i=0}^n \bigvee_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j} x^i y^j (1-x-y)^{n-i-j} |j/n - y|}{\bigvee_{i=0}^n \bigvee_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j} x^i y^j (1-x-y)^{n-i-j}}.$$

2) Because in our considerations $(x, y) \in \Delta$, it is of interest to note that on the peaks of the triangle Δ, which are $(1, 0)$, $(0, 1)$ and $(0, 0)$, we have the values

$$T_n^{(M)}(\varphi_{x=1})(1,0) = 0,\ T_n^{(M)}(\varphi_{x=0})(0,1) = 0,\ T_n^{(M)}(\varphi_{x=0})(0,0) = 0,$$

and

$$T_n^{(M)}(\varphi_{y=1})(0,1) = 0,\ T_n^{(M)}(\varphi_{y=0})(1,0) = 0,\ T_n^{(M)}(\varphi_{y=0})(0,0) = 0.$$

The denominator of the $T_n(M)(f)$ operator can be exactly calculated, as follows.

Lemma 2.7.7. *Let $n \in \mathbb{N} \bigcup \{0\}$. We have*

$$\bigvee_{i=0}^{n} \bigvee_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j} x^i y^j (1-x-y)^{n-i-j} = \bigvee_{i=0}^{n} p_{n,i}(x) \cdot \left[\bigvee_{j=0}^{n-i} p_{n-i,j}\left(\frac{y}{1-x}\right)\right],$$

for all $(x, y) \in \Delta$ with $x < 1$. Recall here that $p_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$.

Proof. We can write (for $x \neq 1$)

$$\bigvee_{i=0}^{n}\bigvee_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}x^i y^j(1-x-y)^{n-i-j}$$

$$=\bigvee_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\bigvee_{j=0}^{n-i}\binom{n-i}{j}y^j\frac{(1-x-y)^{n-i-j}}{(1-x)^{n-i}}$$

$$=\bigvee_{i=0}^{n}p_{n,i}(x)\cdot\bigvee_{j=0}^{n-i}p_{n-i,j}\left(\frac{y}{1-x}\right).$$

□

Corollary 2.7.8. *For all $f \in C_+(\Delta)$, $n \in \mathbb{N}$ and $(x,y) \in \Delta$, we have*

$$|f(x,y) - T_n^{(M)}(f)(x,y)|$$

$$\leq 3\omega_1\left(f;\frac{6}{\sqrt{n+1}}\right)_\Delta \leq 18\omega_1\left(\frac{1}{\sqrt{n+1}},\frac{1}{\sqrt{n+1}}\right).$$

Proof. Reasoning exactly as in the proof of Lemma 2.7.7, we get

$$\bigvee_{i=0}^{n}\bigvee_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}x^i y^j(1-x-y)^{n-i-j}\left|\frac{i}{n}-x\right|$$

$$=\bigvee_{i=0}^{n}p_{n,i}(x)|i/n-x|\cdot\bigvee_{j=0}^{n-i}p_{n-i,j}\left(\frac{y}{1-x}\right),$$

which immediately implies

$$T_n^{(M)}(\varphi_x)(x,y) = B_n^{(M)}(\varphi_x)(x).$$

Also, due to the formula

$$\binom{n}{j}\cdot\binom{n-j}{k} = \binom{n}{k}\cdot\binom{n-k}{j},$$

we can write

$$\bigvee_{i=0}^{n}\bigvee_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}x^i y^j(1-x-y)^{n-i-j}$$

$$=\bigvee_{j=0}^{n}\bigvee_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}x^i y^j(1-x-y)^{n-i-j}$$

$$=\bigvee_{j=0}^{n}p_{n,j}(y)\cdot\bigvee_{i=0}^{n-j}p_{n-j,i}\left(\frac{x}{1-y}\right),$$

which immediately implies

$$T_n^{(M)}(\varphi_y)(x,y) = B_n^{(M)}(\varphi_y)(y).$$

By Corollary 2.7.6 combined with the inequality (2.2) in the proof of Theorem 2.1.5, we get

$$|f(x,y) - T_n^{(M)}(f)(x,y)| \le 3\omega_1\left(f; \frac{6}{\sqrt{n+1}}\right)_{\Delta},$$

which combined with the inequality $\omega_1(f;\delta)_{\Delta} \le \omega_1(f;\delta,\delta)$ (see, e.g., Anastassiou–Gal [8], p. 81) and with the property of $\omega_1(f;\delta,\delta)$ used in the proof of Theorem 2.7.2 too, complete the proof of the corollary. □

Remark. It remains an interesting open question which concept of shape in Definition 2.7.1 is preserved by $T_n^{(M)}$.

2.8 Applications to Image Processing

In this section we present a possible application where max-product approximation can be useful, namely in image processing (zooming). Our focus is to compare the classical bivariate Bernstein operator with the max-product Bernstein operator $B_{n,m}(f)(x,y)$, considered in the previous section. First we consider a grayscale image and we attach to it a function $f(x,y)$ representing the grayscale value of the pixel with coordinates (x,y). Then we rescale the image by downsampling, storing one pixel's grayscale value out of four neighbor pixels. Then we upscale the image to its original size by approximating the missing grayscale values by the value of the classical and the max-product Bernstein operators calculated at the coordinates of the missing pixels considered.

We compare the classical and the max-product approximation by using the Mean Square Error (MSE), which is the square of the Euclidean norm of the difference between the original and reconstructed image, divided by the dimension (no. of pixels) of the image. Another widespread image approximation measure is the Peak Signal to Noise Ratio defined as $PSNR = 10\log_{10}\frac{MAX^2}{MSE}$, where MAX represents the maximum grayscale value of the function. We also measure the blur of the image by using the mean magnitude of the discretized image gradient (we denote it by MGrad), that is the Euclidean norm of the discretized gradient of the grayscale function divided by the dimension of the image. This is a very simple way to measure the blur of an image and more sophisticated approaches can also be considered ([49]), however for a simple comparison or the sharpness of the original and reconstructed images, the average magnitude of the gradient of an image is sufficient.

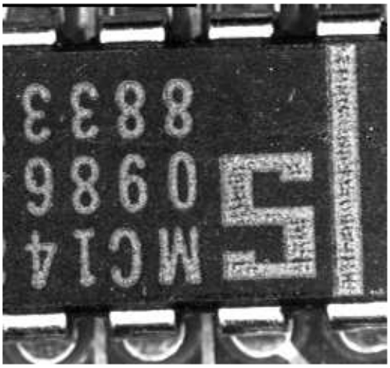

Fig. 2.5 Original Image Text.

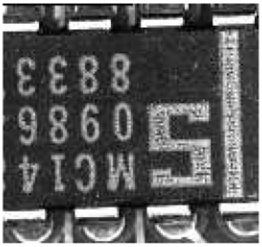

Fig. 2.6 Image Text of dimensions 64 × 64 pixels.

First, image Text having dimensions 256 × 256 pixels is considered. It is then downscaled to a 64×64 image (see Figures 2.5 and 2.6). Then we upscale the image to its original size by considering the classical bivariate Bernstein operator and the max-product bivariate Bernstein operator in both cases used on blocks of 8×8 pixels of the downscaled image. The results show that the better PSNR value is achieved by the classical Bernstein operator (approx. 21.4, see Figure 2.7) compared to the max-product operator (approx 19.9, see Figure 2.8). The MGrad for the original image was 21.5, for the classical Bernstein operator it is considerably less (18.6) than that obtained for the max-product operator (21.4). As a higher Mgrad value means a sharper (so less blurry) image we conclude that the max-product Bernstein operator outperforms in this sense the classical Bernstein operator by better preserving the sharpness of the image. This can be visually observed as well and we attribute it to the shape preserving properties of the bivariate Bernstein operator of max-product type.

We performed the same experiment on an image of larger dimensions and the conclusions are similar. Image Lena of size 512 × 512 is downsampled to the dimension 256 × 256 then approximated using the classical and max-product Bernstein operators. The size of the blocks for the approximation was 4 × 4 in this case. The PSNR value obtained by the classical Bernstein approximation is approx. 23.2, while for the max-product operator it is 21.8. This is in favor of the classical operator, but the MGrad values are respectively: 35.7 for the original image, 28 is obtained for the classical Bernstein operator and 31.5 for the operator of max-product type.

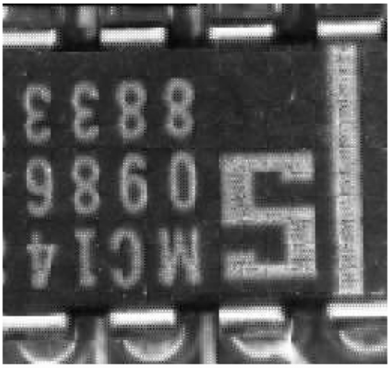

Fig. 2.7 Image text reconstructed using the classical Bernstein operator.

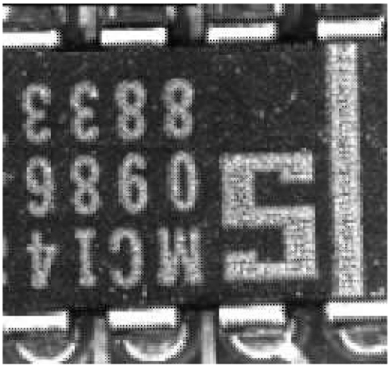

Fig. 2.8 Image text reconstructed using the max-product Bernstein operator.

Another experiment considered higher sampling rates. The Image Lena of dimension 504×504 was downsampled to 168×168. The reconstruction results using blocks of 9 pixels of the original image are shown in Figures 2.9 and 2.10. MGrad for the original image was 34.8. For the classical operator we have obtained PSNR=19.7 and MGrad=23.2 while for the max-product counterpart we have obtained PSNR=18.5 and MGrad=26.7

Surely, the proposed comparison has a limited scope but our goal was to compare classical and max-product versions of the same operators. The conclusion of these experiments is that using a max-product approximation rather than a classical one can be useful when the sharpness of an image is important. Future research in this direction is a study of max-product bicubic interpolation, using it in image approximation and comparing it to bicubic interpolation which is a widespread method for image rescaling. Another direction of interest is in medical image approximation [110], where an approximation method that better preserves contrast can be very useful.

Fig. 2.9 Image Lena reconstructed using the classical Bernstein operator(rate=9).

2.9 Notes

All the results in Sections 2.7, 2.8 and the below Theorem 1.9.1, Theorem 2.9.2, Corollary 2.9.3, Lemmas 2.9.4–2.9.7 and Theorem 2.9.8 are new and appear for the first time in this book.

Note 2.9.1. It is easy to observe that due to the definition formula, all the results in the previous sections concerning the max-product Bernstein operators $B_n^{(M)}(f)$, are necessarily proved only for positive (or strictly positive) functions on $[0, 1]$.

But all these results can be extended to functions of variable sign, as follows.

Theorem 2.9.1. *Suppose that* $f : [0, 1] \to \mathbb{R}$ *is bounded and of variable sign on* $[0, 1]$. *Choose a constant* $c^* > 0$ *such that* $f(x) + c^* > 0$, *for all* $x \in [0, 1]$ *and define the new max-product kind operator*

$$A_n^{(M)}(f)(x) = B_n^{(M)}(f + c^*)(x) - c^*, \text{ for all } x \in [0, 1].$$

Fig. 2.10 Image Lena reconstructed using the max-product Bernstein operator(rate=9).

We have

(i) *If* f *is continuous on* $[0,1]$, *then* $|A_n^{(M)}(f)(x) - f(x)| \le 12\omega_1(f; 1/\sqrt{n+1})$, *for all* $x \in [0,1]$, $n \in \mathbb{N}$ *;*

(ii) *If* f *is concave on* $[0,1]$, *then* $|A_n^{(M)}(f)(x) - f(x)| \le 2\omega_1(f; 1/n)$, *for all* $x \in [0,1]$, $n \in \mathbb{N}$ *;*

(iii) *If* f *is nondecreasing and* $g(x) = f(x)/x$ *is nonincreasing, then* $|A_n^{(M)}(f)(x) - f(x)| \le 2\omega_1(f; 1/n)$, *for all* $x \in [0,1]$, $n \in \mathbb{N}$ *;*

(iv) *If* f *is nonincreasing and* $g(x) = f(x)/(1-x)$ *is nondecreasing, then* $|A_n^{(M)}(f)(x) - f(x)| \le 2\omega_1(f; 1/n)$, *for all* $x \in [0,1]$, $n \in \mathbb{N}$ *;*

(v) *If* f *is monotone on* $[0,1]$, *then* $A_n^{(M)}(f)$ *is monotone and of the same monotonicity, for all* $n \in \mathbb{N}$ *;*

(vi) *If* f *is quasiconvex on* $[0,1]$, *then* $A_n^{(M)}(f)$ *is quasiconvex on* $[0,1]$, *for all* $n \in \mathbb{N}$ *;*

(vii) *If* f *is continuous on* $[0,1]$, *then*

$$|A_n^{(M)}(f)(x) - f(x)| \le C_f[n\omega_1^2(f; 1/n) + \omega_1(f; 1/n)], \text{ for all } x \in [0,1], n \in \mathbb{N};$$

(viii) *If f is a Lipschitz function on* $[0,1]$ *(of order* 1*), then*

$$|A_n^{(M)}(f)(x) - f(x)| \le \frac{C}{n},\ \textit{for all } x \in [0,1], n \in \mathbb{N};$$

(ix) *If f is continuous on* $[0,1]$*, then* $\|A_n^{(M)}(f) - f\| = o(1/n)$ *if and only if f is constant function (here* $\|\cdot\|$ *denotes the uniform norm in* $C[0,1]$*) ;*

(x) *Let* $0 < \alpha < \beta < 1$ *be such that f is continuous on* $[\alpha,\beta]$*. If there exists a constant* $M > 0$ *(independent of n but depending on f, α and β) such that*

$$\|A_n^{(M)}(f) - f\|_{[\alpha,\beta]} \le M/n,\ \textit{ for all } n \in \mathbb{N},$$

then f is a Lipschitz function (of order 1*) on* $[\alpha,\beta]$ *;*

(xi) *Let also* $g : [0,1] \to \mathbb{R}$ *be bounded on* $[0,1]$ *and suppose that there exists* $a, b \in [0,1]$*,* $0 < a < b < 1$ *such that* $f(x) = g(x)$ *for all* $x \in [a,b]$*. Then for all* $c, d \in [a,b]$ *satisfying* $a < c < d < b$*, there exists* $\tilde{n} \in \mathbb{N}$ *depending only on* f, g, a, b, c, d *such that* $A_n^{(M)}(f)(x) = A_n^{(M)}(g)(x)$ *for all* $x \in [c,d]$ *and* $n \in \mathbb{N}$ *with* $n \ge \tilde{n}$ *;*

(xii) *Let* $0 < a < b < 1$ *be such that* $f|_{[a,b]} \in \mathrm{Lip}[a,b]$*. Then, for any* $c, d \in [0,1]$ *satisfying* $a < c < d < b$*, we have*

$$\left|A_n^{(M)}(f)(x) - f(x)\right| \le \frac{C}{n}\ \textit{for all } n \in \mathbb{N}, x \in [c,d],$$

where the constant C depends only on f and a, b, c, d *;*

(xiii) *Suppose that there exist* $a, b \in [0,1]$*,* $0 < a < b < 1$*, such that f is constant on* $[a,b]$ *with the constant value* α*. Then for any* $c, d \in [a,b]$ *with* $a < c < d < b$*, there exists* $\tilde{n} \in \mathbb{N}$ *depending only on* a, b, c, d*, and f, such that* $A_n^{(M)}(f)(x) = \alpha$ *for all* $x \in [c,d]$ *and* $n \in \mathbb{N}$ *with* $n \ge \tilde{n}$ *;*

(xiv) *Suppose that there exist* $a, b \in [0,1]$*,* $0 < a < b < 1$*, such that f is nondecreasing (nonincreasing) on* $[a,b]$*. Then for any* $c, d \in [a,b]$ *with* $a < c < d < b$*, there exists* $\tilde{n} \in \mathbb{N}$ *depending only on* a, b, c, d*, and f, such that* $A_n^{(M)}(f)$ *is nondecreasing (nonincreasing) on* $[c,d]$ *for all* $n \in \mathbb{N}$ *with* $n \ge \tilde{n}$ *;*

(xv) *Suppose that there exist* $a, b \in [0,1]$*,* $0 < a < b < 1$*, such that f is quasiconvex (quasiconcave) on* $[a,b]$*. Then for any* $c, d \in [a,b]$ *with* $a < c < d < b$*, there exists* $\tilde{n} \in \mathbb{N}$ *depending only on* a, b, c, d*, and f, such that* $A_n^{(M)}(f)$*, is quasiconvex (quasiconcave) on* $[c,d]$ *for all* $n \in \mathbb{N}$ *with* $n \ge \tilde{n}$ *;*

(xvi) *If f is a concave and monotonous function on* $[0,1]$*, then*

$$\int_0^1 \left|A_n^{(M)}(f;[0,1])(x) - f(x)\right| dx \le \frac{2|f(1) - f(0)|}{n+1},\ \textit{for all } n \in \mathbb{N}.$$

Proof. The proofs are immediate from $A_n^{(M)}(f)(x) - f(x) = B_n^{(M)}(f + c^*)(x) - (f(x) + c^*)$ and from the fact that a constant added to a function does not modify its monotonicity, convexity (concavity), quasiconvexity (quasiconcavity), and the value of its modulus of continuity. As a consequence, (i) follows directly from Theorem 2.1.5, (ii) from Corollary 2.1.10, (iii)–(iv) from Corollary 2.1.11, (v) from Theorem 2.1.15, and from Corollary 2.1.16, (vi) from Corollary 2.1.18, (vii) from Theorem 2.2.18, (viii) from Corollary 2.2.19, (ix) from Theorem 2.3.2, (x) from Theorem 2.3.3, (xi) from Theorem 2.4.1, (xii) from Corollary 2.4.2, (xiii) from Corollary 2.4.3, (xiv) from Corollary 2.4.4, (xv) from Corollary 2.4.5, and (xvi) from Corollary 2.6.11. □

Note 2.9.2. It is easy to observe that due to the definition formulas, all the results in Section 2.7 concerning the bivariate max-product Bernstein operators $B_{n,m}^{(M)}(f)$ and $T_n^{(M)}(f)$ are necessarily proved only for positive (or strictly positive) bivariate functions on $[0, 1] \times [0, 1]$.

But all the results in Section 2.7 can easily be extended to bivariate bounded functions of variable sign, by defining the new operators of max-product kind $A_{n,m}^{(M)}(f)(x, y) = B_{n,m}^{(M)}(f + c^*)(x, y) - c^*$ and $S_n^{(M)}(f)(x, y) = T_n^{(M)}(f + c^*)(x, y) - c^*$, where $c^* > 0$ is a constant chosen such that $f(x, y) + c^* > 0$ for all $x, y \in [0, 1] \times [0, 1]$.

The proofs are immediate from the results in Section 2.7, from the relationships $A_{n,m}^{(M)}(f)(x, y) - f(x, y) = B_{n,m}^{(M)}(f + c^*)(x, y) - (f(x, y) + c^*)$, $S_n^{(M)}(f)(x, y) - f(x, y) = T_n^{(M)}(f + c^*)(x, y) - (f(x, y) + c^*)$ and from the fact that a constant added to a function does not modify any of its bivariate monotonicity and the value of its bivariate modulus of continuity.

Note 2.9.3. In this note, we present the approximation properties of two new positive nonlinear operators called Bernstein operators of sum-max kind attached to positive functions, defined by

$$B_n^{(SM)}(f)(x) = \frac{\sum_{k=0}^n \left[\bigvee_{j=0}^k p_{n,j}(x) f(j/n)\right]}{\sum_{k=0}^n \left[\bigvee_{j=0}^k p_{n,j}(x)\right]},$$

and

$$T_n^{(SM)}(f)(x) = \frac{\sum_{k=0}^n \left[\bigvee_{j=0}^k p_{n,j}(x) f(j/n)\right] + \sum_{k=0}^n \left[\bigvee_{j=n-k}^n p_{n,j}(x) f(j/n)\right]}{\sum_{k=0}^n \left[\bigvee_{j=0}^k p_{n,j}(x)\right] + \sum_{k=0}^n \left[\bigvee_{j=n-k}^n p_{n,j}(x)\right]}.$$

Remark. Since the denominators of $B_n^{(SM)}(f)(x)$ and $T_n^{(SM)}(f)(x)$ are sums of positive functions on $[0, 1]$, containing the strictly positive term on $[0, 1]$, $\bigvee_{j=0}^n p_{n,j}(x)$, it follows that $B_n^{(SM)}(f)(x)$ and $T_n^{(SM)}(f)(x)$ are well defined for all $x \in [0, 1]$.

The following properties are immediate:

Theorem 2.9.2. *Let $L_n^{(SM)}(f) = B_n^{(SM)}(f)$ for all $n \in \mathbb{N}$ or $L_n^{(SM)}(f) = T_n^{(SM)}(f)$ for all $n \in \mathbb{N}$ and denote*

$$C_+[0,1] = \{f : [0,1] \to [0,+\infty); f \text{ is continuous on } [0,1]\}.$$

(i) If $f \in C_+[0,1]$, then $L_n^{(SM)}(f) \in C_+[0,1]$, for all $n \in \mathbb{N}$;
(ii) $L_n^{(SM)}(\lambda f) = \lambda L_n^{(SM)}(f)$, for all $\lambda \geq 0$, $f \in C_+[0,1]$ and $n \in \mathbb{N}$;
(iii) $L_n^{(SM)}(f+g)(x) \leq L_n^{(SM)}(f)(x) + L_n^{(SM)}(g)(x)$, for all $f, g \in C_+[0,1]$, $n \in \mathbb{N}$ and $x \in [0,1]$;
(iv) Denoting $e_0(x) = 1$ for all $x \in [0,1]$, we have $L_n^{(SM)}(e_0)(x) = e_0(x)$, for all $x \in [0,1]$ and $n \in \mathbb{N}$;
(v) If $f, g \in C_+[0,1]$ satisfy $f \leq g$ on $[0,1]$, then $L_n^{(SM)}(f) \leq L_n^{(SM)}(g)$ on $[0,1]$.
(vi) For any $f \in C_+[0,1]$ we have $L_n^{(SM)}(f)(0) = f(0)$ and $L_n^{(SM)}(f)(1) = f(1)$, for all $n \in \mathbb{N}$.

Now, according to Theorem 1.1.2, from Theorem 2.9.2 we immediately get the following.

Corollary 2.9.3. *For all $f \in C_+[0,1]$, $n \in \mathbb{N}$ and $x \in [0,1]$, we have*

$$|f(x) - L_n^{(SM)}(f)(x)| \leq \left[1 + \frac{1}{\delta} L_n^{(SM)}(\varphi_x)(x)\right] \omega_1(f;\delta)_{[0,1]},$$

where $\varphi_x(t) = |t - x|$, for all $t \in [0,1]$(here x is supposed fixed.)

Remark. From Corollary 2.9.3, it follows that the approximation properties of $B_n^{(SM)}(f)(x)$ and $T_n^{(SM)}(f)(x)$ are controlled by the ratios

$$B_n^{(SM)}(\varphi_x)(x) = \frac{\sum_{k=0}^{n} \left[\bigvee_{j=0}^{k} p_{n,j}(x) |\frac{j}{n} - x|\right]}{\sum_{k=0}^{n} \left[\bigvee_{j=0}^{k} p_{n,j}(x)\right]}, \tag{2.28}$$

and

$$T_n^{(SM)}(\varphi_x)(x)$$
$$= \frac{\sum_{k=0}^{n} \left[\bigvee_{j=0}^{k} p_{n,j}(x) |\frac{j}{n} - x|\right] + \sum_{k=0}^{n} \left[\bigvee_{j=n-k}^{k} p_{n,j}(x) |\frac{j}{n} - x|\right]}{\sum_{k=0}^{n} \left[\bigvee_{j=0}^{k} p_{n,j}(x)\right] + \sum_{k=0}^{n} \left[\bigvee_{j=n-k}^{n} p_{n,j}(x)\right]}. \tag{2.29}$$

The denominator of the $B_n^{(SM)}(f)$ operator can be exactly calculated, as follows.

Lemma 2.9.4. *Let* $n \in \mathbb{N} \bigcup \{0\}$ *and* $x \in [0, 1]$. *Then we have*

(i)

$$\bigvee_{j=0}^{k} p_{n,j}(x) = p_{n,i}(x), \text{ if } x \in \left[\frac{i}{n+1}, \frac{i+1}{n+1}\right], i = \overline{0, k-1}, k \geq 1,$$

and

$$\bigvee_{j=0}^{k} p_{n,j}(x) = p_{n,k}(x), \text{ if } x \in \left[\frac{k}{n+1}, 1\right], k \geq 0;$$

(ii)

$$\bigvee_{j=k}^{n} p_{n,j}(x) = p_{n,k}(x), \text{ if } x \in \left[0, \frac{k+1}{n+1}\right], k \geq 0,$$

and

$$\bigvee_{j=k}^{n} p_{n,j}(x) = p_{n,i}(x), \text{ if } x \in \left[\frac{i}{n+1}, \frac{i+1}{n+1}\right], i = \overline{k, n}, k \geq 0;$$

(iii)

$$\sum_{k=0}^{n} \left[\bigvee_{j=0}^{k} p_{n,j}(x)\right] = p_{n,0}(x) + np_{n,0}(x) = (n+1)p_{n,0}(x),\ x \in \left[0, \frac{1}{n+1}\right],$$

and

$$\sum_{k=0}^{n} \left[\bigvee_{j=0}^{k} p_{n,j}(x)\right] = \sum_{j=0}^{i} p_{n,j}(x) + (n-i)p_{n,i+1}(x),\ x \in \left[\frac{i+1}{n+1}, \frac{i+2}{n+1}\right],$$

$i = \overline{0, n-1}$.

Proof. (i)–(ii) Reasoning exactly as in the proof of Lemma 2.1.4, the proof is immediate.

(iii) By taking successively in the formulas in (i), $k = 0, k = 1, \ldots, k = n$ and then summing after k from 0 to n, we immediately obtain the formulas in (iii). □

Also, from the above lemma we can explicit the denominator of the operator $T_n^{(SM)}(f)$ too.

Lemma 2.9.5. *Let $x \in [0,1]$, $n \in \mathbb{N}\bigcup\{0\}$ and let $l \in \{0,1,\dots,n\}$ be such that $x \in [\frac{l}{n+1}, \frac{l+1}{n+1}]$. Then we have*

$$\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\right] + \sum_{k=0}^{n}\left[\bigvee_{j=n-k}^{n} p_{n,j}(x)\right] = 1 + (n+1)p_{n,l}(x).$$

Proof. First we suppose that $l \in \{1,2,\dots,n-1\}$. Taking into account the equalities in Lemma 2.9.4, (i)–(ii) we get

$$\begin{aligned}
&\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\right] + \sum_{k=0}^{n}\left[\bigvee_{j=n-k}^{n} p_{n,j}(x)\right]\\
&=\sum_{k=0}^{l-1}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\right] + \sum_{k=l}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\right]\\
&+\sum_{k=0}^{n-l-1}\left[\bigvee_{j=n-k}^{n} p_{n,j}(x)\right] + \sum_{k=n-l}^{n}\left[\bigvee_{j=n-k}^{n} p_{n,j}(x)\right]\\
&= (n-l+1)p_{n,l}(x) + \sum_{k=0}^{l-1} p_{n,k}(x) + (l+1)p_{n,l}(x) + \sum_{k=0}^{n-l-1} p_{n,n-k}(x)\\
&=(n+1)p_{n,l}(x) + p_{n,l}(x) + \sum_{k=0}^{l-1} p_{n,k}(x) + \sum_{k=l+1}^{n} p_{n,k}(x)\\
&=(n+1)p_{n,l}(x) + \sum_{k=0}^{n} p_{n,k}(x) = 1 + (n+1)p_{n,l}(x).
\end{aligned}$$

In the case when $l \in \{0,n\}$ the reasoning is similar and therefore we omit the details. □

Taking into account that by Theorem 2.9.2, (iv) we have $B_n^{(SM)}(e_0)(x) = e_0(x)$ and $T_n^{(SM)}(e_0)(x) = e_0(x)$ would be of interest to exactly calculate $B_n^{(SM)}(f)(x)$ and $T_n^{(SM)}(f)(x)$ for $f(x) = e_1(x) = x$ too. Firstly, in the case of the $B_n^{(SM)}$ operator, we have the following.

Lemma 2.9.6. *For all $x \in [0,1]$ and $n \in \mathbb{N}$ we have*

$$B_n^{(SM)}(e_1)(x) = \frac{x[np_{n-1,0}(x)]}{(n+1)p_{n,0}(x)} = \frac{x}{1-x}\cdot\frac{n}{n+1}, \text{ if } x \in [0, 1/(n+1)],$$

$$B_n^{(SM)}(e_1)(x) == \frac{xnp_{n-1,0}(x)}{p_{n,0}(x) + np_{n,1}(x)} = \frac{nx}{1+(n^2-1)x}, \text{ if } x \in [1/(n+1), 1/n],$$

$$B_n^{(SM)}(e_1)(x) = \frac{x[p_{n-1,0}(x) + (n-1)p_{n-1,1}(x)]}{p_{n,0}(x) + np_{n,1}(x)} = \frac{x}{1-x} \cdot \frac{1-x+(n-1)^2x}{(1-x)+n^2}$$
$$\text{if } x \in [1/n, 2/(n+1)],$$
$$B_n^{(SM)}(e_1)(x) = \frac{x[p_{n-1,0}(x) + (n-1)p_{n-1,1}(x)]}{p_{n,0}(x) + p_{n,1}(x) + (n-1)p_{n,2}(x)}$$
$$= \frac{x[(1-x)+(n-1)^2x]}{(1-x)^2 + nx(1-x) + n(n-1)^2x^2/2}, \text{ if } x \in [2/(n+1), 2/n],$$
$$B_n^{(SM)}(e_1)(x) = \frac{x[p_{n-1,0}(x) + p_{n-1,1}(x) + (n-2)p_{n-1,2}(x)]}{p_{n,0}(x) + p_{n,1}(x) + (n-1)p_{n,2}(x)},$$
$$\text{if } x \in [2/n, 3/(n+1)],$$
$$B_n^{(M)}(e_1)(x) = \frac{x[p_{n-1,0}(x) + p_{n-1,1}(x) + (n-2)p_{n-1,2}(x)]}{p_{n,0} + p_{n,1}(x) + p_{n,2}(x) + (n-2)p_{n,3}(x)},$$
$$\text{if } x \in [3/(n+1), 3/n],$$

and so on, in general we have

$$B_n^{(SM)}(e_1)(x) = \frac{x[\sum_{i=0}^{j-1} p_{n-1,i}(x) + (n-j)p_{n-1,j}(x)]}{\sum_{i=0}^{j-1} p_{n,i}(x) + (n+1-j)p_{n,j}(x)},$$
$$\text{if } x \in [j/n, (j+1)/(n+1)],$$
$$B_n^{(SM)}(e_1)(x) = \frac{x[\sum_{i=0}^{j-1} p_{n-1,i}(x) + (n-j)p_{n-1,j}(x)]}{\sum_{i=0}^{j} p_{n,i}(x) + (n-j)p_{n,j+1}(x)},$$
$$\text{if } x \in [(j+1)/(n+1), (j+1)/n],$$

for $j \in \{0, 1, \ldots, n-1\}$. *Here by convention we take* $\sum_{i=0}^{-1} = 0$.

Proof. Firstly, note that we have

$$B_n^{(SM)}(e_1)(x) = \frac{\sum_{k=0}^{n} \left[\bigvee_{j=0}^{k} p_{n,j}(x)\frac{j}{n}\right]}{\sum_{k=0}^{n} \left[\bigvee_{j=0}^{k} p_{n,j}(x)\right]}.$$

Since

$$\bigvee_{j=0}^{k} p_{n,j}(x)\frac{j}{n} = \bigvee_{j=1}^{k} p_{n,j}(x)\frac{j}{n} = x \cdot \bigvee_{j=0}^{k-1} p_{n-1,j}(x),$$

to find explicitly $B_n^{(SM)}(e_1)(x)$, we will use for both the numerator and denominator the same Lemma 2.9.4.

Thus, for the numerator we get

$$\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k}p_{n,j}(x)\frac{j}{n}\right]=\sum_{k=1}^{n}\left[\bigvee_{j=0}^{k}p_{n,j}(x)\frac{j}{n}\right]=x\cdot\sum_{k=1}^{n}\left[\bigvee_{j=0}^{k-1}p_{n-1,j}(x)\right]$$
$$=x\cdot\sum_{i=0}^{n-1}\left[\bigvee_{j=0}^{i}p_{n-1,j}(x)\right]=x\cdot\sum_{k=0}^{n-1}\left[\bigvee_{j=0}^{k}p_{n-1,j}(x)\right].$$

Now, applying Lemma 2.9.4, (iii) to both expressions

$$A:=\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k}p_{n,j}(x)\right]\text{ and }B:=x\cdot\sum_{k=0}^{n-1}\left[\bigvee_{j=0}^{k}p_{n-1,j}(x)\right],$$

taking into account that we get the following division of the interval $[0, 1]$

$$0<\frac{1}{n+1}\leq\frac{1}{n}\leq\frac{2}{n+1}\leq\frac{2}{n}\leq\frac{3}{n+1}\leq\frac{3}{n}\leq\frac{4}{n+1}\leq\frac{4}{n}\cdots,$$

it follows:

$$\frac{B}{A}=\frac{xnp_{n-1,0}(x)}{(n+1)p_{n,0}(x)}=\frac{n}{n+1}\cdot\frac{x}{1-x},\ x\in\left[0,\frac{1}{n+1}\right],$$
$$\frac{B}{A}=\frac{xnp_{n-1,0}(x)}{p_{n,0}(x)+np_{n,1}(x)}=\frac{nx}{1+x(n^2-1)},\ x\in\left[\frac{1}{n+1},\frac{1}{n}\right],$$
$$\frac{B}{A}=\frac{x[p_{n-1,0}(x)+(n-1)p_{n-1,1}(x)]}{p_{n,0}(x)+np_{n,1}(x)}$$
$$=\frac{x}{1-x}\cdot\frac{1-x+(n-1)^2x}{(1-x)+n^2},\ x\in\left[\frac{1}{n},\frac{2}{n+1}\right],$$
$$\frac{B}{A}=\frac{x[p_{n-1,0}(x)+(n-1)p_{n-1,1}(x)]}{p_{n,0}(x)+np_{n,1}(x)}$$
$$=\frac{x}{1-x}\cdot\frac{1-x+(n-1)^2x}{(1-x)+n^2},\ x\in\left[\frac{1}{n},\frac{2}{n+1}\right],$$
$$\frac{B}{A}=\frac{x[p_{n-1,0}(x)+(n-1)p_{n-1,1}(x)]}{p_{n,0}(x)+p_{n,1}(x)+(n-1)p_{n,2}(x)}$$
$$=\frac{x[(1-x)+(n-1)^2x]}{(1-x)^2+nx(1-x)+n(n-1)^2x^2/2}\ x\in\left[\frac{2}{n+1},\frac{2}{n}\right],$$
$$\frac{B}{A}=\frac{x[p_{n-1,0}(x)+p_{n-1,1}(x)+(n-2)p_{n-1,2}(x)]}{p_{n,0}(x)+p_{n,1}(x)+(n-1)p_{n,2}(x)},$$

$$\text{if } x \in [2/n, 3/(n+1)],$$

$$\frac{B}{A} = \frac{x[p_{n-1,0}(x) + p_{n-1,1}(x) + (n-2)p_{n-1,2}(x)]}{p_{n,0} + p_{n,1}(x) + p_{n,2}(x) + (n-2)p_{n,3}(x)},$$

$$\text{if } x \in [3/(n+1), 3/n],$$

and so on, in general we have

$$\frac{B}{A} = \frac{x[\sum_{i=0}^{j-1} p_{n-1,i}(x) + (n-j)p_{n-1,j}(x)]}{\sum_{i=0}^{j-1} p_{n,i}(x) + (n+1-j)p_{n,j}(x)},$$

$$\text{if } x \in [j/n, (j+1)/(n+1)],$$

$$\frac{B}{A} = \frac{x[\sum_{i=0}^{j-1} p_{n-1,i}(x) + (n-j)p_{n-1,j}(x)]}{\sum_{i=0}^{j} p_{n,i}(x) + (n-j)p_{n,j+1}(x)},$$

$$\text{if } x \in [(j+1)/(n+1), (j+1)/n],$$

for $j \in \{0, 1, \ldots, n-1\}$. Here by convention we take $\sum_{i=0}^{-1} = 0$. □

Analogously, in the case of $T_n^{(SM)}$ operator, we can state:

Lemma 2.9.7. *For all $x \in [0, 1]$ and $n \in \mathbb{N}$ we have*

$$T_n^{(SM)}(e_1)(x) = \frac{1 + np_{n-1,0}(x)}{1 + (n+1)p_{n,0}(x)}, \text{ if } x \in [0, 1/(n+1)],$$

$$T_n^{(SM)}(e_1)(x) = \frac{1 + np_{n-1,0}(x)}{1 + (n+1)p_{n,1}(x)}, \text{ if } x \in [1/(n+1), 1/n],$$

$$T_n^{(SM)}(e_1)(x) = \frac{1 + np_{n-1,1}(x)}{1 + (n+1)p_{n,1}(x)}, \text{ if } x \in [1/n, 2/(n+1)],$$

$$T_n^{(SM)}(e_1)(x) = \frac{1 + np_{n-1,1}(x)}{1 + (n+1)p_{n,2}(x)}, \text{ if } x \in [2/(n+1), 2/n],$$

$$T_n^{(SM)}(e_1)(x) = \frac{1 + np_{n-1,2}(x)}{1 + (n+1)p_{n,2}(x)}, \text{ if } x \in [2/n, 3/(n+1)],$$

$$T_n^{(M)}(e_1)(x) = \frac{1 + np_{n-1,2}(x)}{1 + (n+1)p_{n,3}(x)}, \text{ if } x \in [3/(n+1), 3/n],$$

and so on, in general we have

$$T_n^{(SM)}(e_1)(x) = \frac{1 + np_{n-1,l}(x)}{1 + (n+1)p_{n,l}(x)}, \text{ if } x \in [l/n, (l+1)/(n+1)],$$

$$T_n^{(SM)}(e_1)(x) = \frac{1 + np_{n-1,l}(x)}{1 + (n+1)p_{n,l+1}(x)}, \text{ if } x \in [(l+1)/(n+1), (l+1)/n],$$

for $l \in \{0, 1, \ldots, n\}$.

Proof. Let us choose arbitrary $x \in [0,1]$ and let $l \in \{0,1,\ldots,n\}$ be such that $x \in [l/n,(l+1)/n]$. Firstly, we have

$$T_n^{(SM)}(e_1)(x) = \frac{\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\frac{j}{n}\right] + \sum_{k=0}^{n}\left[\bigvee_{j=n-k}^{n} p_{n,j}(x)\frac{j}{n}\right]}{\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\right] + \sum_{k=0}^{n}\left[\bigvee_{j=n-k}^{n} p_{n,j}(x)\right]}.$$

Then, we observe that

$$\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\frac{j}{n}\right] = \sum_{k=1}^{n}\left[\bigvee_{j=1}^{k} p_{n,j}(x)\frac{j}{n}\right] = x\sum_{k=1}^{n}\left[\bigvee_{j=1}^{k} p_{n-1,j-1}(x)\right]$$
$$= x\sum_{k=0}^{n-1}\left[\bigvee_{j=0}^{k} p_{n-1,j}(x)\right].$$

By similar reasonings we get that

$$\sum_{k=0}^{n}\left[\bigvee_{j=n-k}^{n} p_{n,j}(x)\frac{j}{n}\right] = x\sum_{k=0}^{n-1}\left[\bigvee_{j=n-1-k}^{n-1} p_{n-1,j}(x)\right].$$

The above two equalities give

$$\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\frac{j}{n}\right] + \sum_{k=0}^{n}\left[\bigvee_{j=n-k}^{n} p_{n,j}(x)\frac{j}{n}\right]$$
$$= x\left(\sum_{k=0}^{n-1}\left[\bigvee_{j=0}^{k} p_{n-1,j}(x)\right] + \sum_{k=0}^{n-1}\left[\bigvee_{j=n-1-k}^{n-1} p_{n-1,j}(x)\right]\right).$$

If $x \in [l/n,(l+1)/(n+1)]$ then using Lemma 2.9.5 for the numerator of $T_n^{(SM)}(f)(x)$ and for the denominator too, we obtain

$$T_n^{(SM)}(f)(x) = \frac{1 + np_{n-1,l}(x)}{1 + (n+1)p_{n,l}(x)}.$$

Now, if $x \in [(l+1)/(n+1),(l+1)/n]$, by the same Lemma 2.9.5 we obtain

$$T_n^{(SM)}(f)(x) = \frac{1 + np_{n-1,l}(x)}{1 + (n+1)p_{n,l+1}(x)}$$

and the proof is complete. □

In what follows, we study the approximation properties of $B_n^{(SM)}(f)$ and $T_n^{(SM)}(f)$, based on estimations of the quantities defined by (2.28) and (2.29).

Theorem 2.9.8. *Let $f : [0,1] \to \mathbb{R}_+$ be continuous on $[0,1]$.*

(i) We have

$$|T_n^{(SM)}(f)(x) - f(x)| \le 24\omega_1\left(f; \frac{1}{\sqrt{n+1}}\right), \text{for all } n \in \mathbb{N}, x \in [0,1].$$

(ii) For any $0 < a < 1$, there exist an index n_0 and a constant $C_a > 0$, both depending only on a (increasing with respect to a and satisfying $\lim_{a\nearrow 1} n_0(a) = \lim_{a\nearrow 1} C_a = +\infty$), such that

$$|B_n^{(SM)}(f)(x) - f(x)| \le C_a\omega_1\left(f; \frac{1}{\sqrt{n+1}}\right), \text{ for all } n \in \mathbb{N},\ n \ge n_0,\ x \in [0,a],$$

that is, $B_n^{(SM)}(f)$ converges uniformly to f on each compact subinterval of the form $[0,a]$, with $a < 1$.

Proof. (i) Since for $T_n^{(SM)}$ are valid the conclusions of Theorem 2.9.2 and Corollary 2.9.3, it follows that for any $x \in [0,1]$ and $\delta > 0$ we have

$$|T_n^{(SM)}(f)(x) - f(x)| \le \left[1 + \frac{1}{\delta}T_n^{(SM)}(\varphi_x)(x)\right]\omega_1(f;\delta). \tag{2.30}$$

Let us choose arbitrary $x \in [0,1]$. By the proof of Theorem 2.1.5 it results

$$\frac{\bigvee_{k=0}^n p_{n,k}(x)|\frac{k}{n} - x|}{\bigvee_{k=0}^n p_{n,k}(x)} \le \frac{6}{\sqrt{n+1}}.$$

Let $k_1, k_2 \in \{0, 1, \ldots, n\}$ be such that $\bigvee_{k=0}^n p_{n,k}(x)|\frac{k}{n} - x| = p_{n,k_1}(x)|\frac{k_1}{n} - x|$ and such that $x \in [k_2/(n+1), (k_2+1)/(n+1)]$. By Lemma 2.9.4, (i) we get that $\bigvee_{k=0}^n p_{n,k}(x) = p_{n,k_2}(x)$ and from the above inequality we obtain

$$\frac{p_{n,k_1}(x)|\frac{k_1}{n} - x|}{p_{n,k_2}(x)} \le \frac{6}{\sqrt{n+1}}. \tag{2.31}$$

In addition, it is immediate that

$$\sum_{k=0}^n \left[\bigvee_{j=0}^k p_{n,j}(x)|\frac{j}{n} - x|\right] + \sum_{k=0}^n \left[\bigvee_{j=n-k}^k p_{n,j}(x)|\frac{j}{n} - x|\right]$$
$$\le 2(n+1)p_{n,k_1}(x)|\frac{k_1}{n} - x|.$$

On the other hand, from Lemma 2.9.5 it follows that

$$\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\right]+\sum_{k=0}^{n}\left[\bigvee_{j=n-k}^{n} p_{n,j}(x)\right]=1+(n+1)p_{n,k_2}(x).$$

From the above two inequalities (taking into account relation (2.31)) we obtain

$$\begin{aligned}T_n^{(SM)}(\varphi_x)(x) &\leq \frac{2(n+1)p_{n,k_1}(x)|\frac{k_1}{n}-x|}{(n+1)p_{n,k_2}(x)}\\ &= \frac{2p_{n,k_1}(x)|\frac{k_1}{n}-x|}{p_{n,k_2}(x)} \leq \frac{12}{\sqrt{n+1}}.\end{aligned}$$

Now, taking $\delta=\frac{12}{\sqrt{n+1}}$ in relation (2.30) we obtain the desired conclusion. □

(ii) Again, we observe that for $B_n^{(SM)}$ are valid the conclusions of Theorem 2.9.2 and Corollary 2.9.3 and therefore it follows that for any $x\in[0,1]$ and $\delta>0$ we have

$$|B_n^{(SM)}(f)(x)-f(x)| \leq \left[1+\frac{1}{\delta}B_n^{(SM)}(\varphi_x)(x)\right]\omega_1(f;\delta). \tag{2.32}$$

Keeping the notations from the above point (i), the numerator of the expression $B_n^{(SM)}(\varphi_x)(x)$ can be upper estimated by the quantity $(n+1)p_{n,k_1}(x)\left|\frac{k_1}{n}-x\right|$.

On the other hand, we have to find a lower estimate for the denominator of the expression $B_n^{(SM)}(\varphi_x)(x)$ that is $\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\right]$. For that purpose, first suppose that $x\in[0,1/2]$. We have two cases: 1) n is odd. 2) n is even.

Case 1). We can write

$$\begin{aligned}\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\right] &= \left(p_{n,0}(x)+\bigvee_{j=0}^{1} p_{n,j}(x)+\ldots+\bigvee_{j=0}^{(n-1)/2} p_{n,j}(x)\right)\\ &+\left(\bigvee_{j=0}^{(n+1)/2} p_{n,j}(x)+\ldots+\bigvee_{j=0}^{n} p_{n,j}(x)\right):=S_1+S_2.\end{aligned}$$

Since $x\in[0,1/2]$, it follows that $k_2\in\{0,1,\ldots,(n-1)/2\}$. For this x, from Lemma 2.9.4, (i), it follows that all the terms in the sum denoted above by S_2 (including the term $\bigvee_{j=0}^{n} p_{n,j}(x)$), become equal to $p_{n,k_2}(x)$, which will imply that $S_1+S_2=S_1+\frac{n+1}{2}\bigvee_{j=0}^{n} p_{n,j}(x)\geq\frac{n+1}{2}\bigvee_{j=0}^{n} p_{n,j}(x)$.
From the above two inequalities and taking into account relation (2.31) we obtain

$$B_n^{(SM)}(\varphi_x)(x)\leq\frac{(n+1)p_{n,k_1}(x)|\frac{k_1}{n}-x|}{[(n+1)/2]p_{n,k_2}(x)}=\frac{2p_{n,k_1}(x)|\frac{k_1}{n}-x|}{p_{n,k_2}(x)}\leq\frac{12}{\sqrt{n+1}}.$$

Case 2). We can write

$$\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\right] = \left(p_{n,0}(x) + \bigvee_{j=0}^{1} p_{n,j}(x) + \ldots + \bigvee_{j=0}^{n/2} p_{n,j}(x)\right)$$
$$+\left(\bigvee_{j=0}^{n/2+1} p_{n,j}(x) + \ldots + \bigvee_{j=0}^{n} p_{n,j}(x)\right) := S_1 + S_2,$$

and reasoning as in the Case 1), we immediately obtain that $S_1 + S_2 = S_1 + \frac{n}{2}\bigvee_{j=0}^{n} p_{n,j}(x) \geq \frac{n}{2}\bigvee_{j=0}^{n} p_{n,j}(x)$. Therefore from (2.31) we get

$$B_n^{(SM)}(\varphi_x)(x) \leq \frac{(n+1)p_{n,k_1}(x)|\frac{k_1}{n} - x|}{[n/2]p_{n,k_2}(x)} \leq \frac{4p_{n,k_1}(x)|\frac{k_1}{n} - x|}{p_{n,k_2}(x)} \leq \frac{24}{\sqrt{n+1}}.$$

Summarizing, for all $x \in [0, 1/2]$ we have

$$B_n^{(SM)}(\varphi_x) \leq \frac{24}{\sqrt{n+1}}, \text{ for all } n \in \mathbb{N}$$

and choosing in (2.32), $\delta = \frac{24}{\sqrt{n+1}}$, we get

$$|B_n^{(SM)}(f)(x) - f(x)| \leq 48\omega_1\left(f; \frac{1}{\sqrt{n+1}}\right), \text{ for all } x \in [0, 1/2],\ n \in \mathbb{N}.$$

Now, let $\frac{1}{2} < a < 1$ be arbitrary, fixed and $n_0 = \frac{2}{1-a}$. For all $n > n_0$, it is immediate that we have $n > [an] + 2$ and $\frac{n+1}{n-[an]-2} \leq \frac{n+1}{n-an-2} = \frac{n+1}{n(1-a)-2}$. Since $\frac{n+1}{n(1-a)-2}$ is decreasing as function of n, it follows that the maximum value of $\frac{n+1}{n(1-a)-2}$ is attained for $n = \left[\frac{2}{1-a}\right] + 1$ and denote this value by M_a.

Let $x \in [0, a]$ and $n > \frac{2}{1-a}$. Now, writing

$$\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\right] = \left(p_{n,0}(x) + \bigvee_{j=0}^{1} p_{n,j}(x) + \ldots + \bigvee_{j=0}^{[an]+1} p_{n,j}(x)\right)$$
$$+\left(\bigvee_{j=0}^{[an]+2} p_{n,j}(x) + \ldots + \bigvee_{j=0}^{n} p_{n,j}(x)\right) := S_1 + S_2,$$

because $\frac{[an]+2}{n+1} > a$, by Lemma 2.9.4, (i) and reasoning exactly as in the case when $x \in [0, 1/2]$, it follows that

$$\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(x)\right] = S_1 + S_2 \geq S_2 = [n - ([an] + 2) + 1]\bigvee_{j=0}^{n} p_{n,j}(x)$$
$$= (n - [an] - 1)\bigvee_{j=0}^{n} p_{n,j}(x) > (n - [an] - 2)\bigvee_{j=0}^{n} p_{n,j}(x).$$

Therefore, taking into account (2.30) too, for all $x \in [0, a]$ and $n \geq \left[\frac{2}{1-a}\right] + 1$, we obtain

$$B_n^{(SM)}(\varphi_x)(x) \leq \frac{n+1}{n - [an] - 2} \cdot \frac{6}{\sqrt{n+1}} \leq \frac{6M_a}{\sqrt{n+1}}.$$

By choosing in (1.38), $\delta = \frac{6M_a}{\sqrt{n+1}}$, we immediately get the desired estimate in (ii), valid for all $x \in [0, a]$ and $n \geq \left[\frac{2}{1-a}\right] + 1$. It is clear that both n_0 and C_a depend increasingly with respect to $a \nearrow 1$ and satisfy $\lim_{a\nearrow 1} n_0(a) = \lim_{a\nearrow 1} C_a = +\infty$. □

Remarks. 1) It can be shown that in the whole space $C_+[0, 1]$, the order of uniform approximation by the operator $T_n^{(SM)}$ in Theorem 2.9.8, (i), is the best possible. Indeed, let us consider the function $f : [0, 1] \to [0, \infty)$, $f(x) = 0$ if $x \in [0, 1/2]$ and $f(x) = x - 1/2$ if $x \in [1/2, 1]$. In the paper [52] (see Example 3.1. there) we have proved that

$$B_n^{(M)}(f)(1/2) - f(\frac{1}{2}) \geq \frac{e^{-5}}{6}\omega_1(f, 1/\sqrt{n}), \tag{2.33}$$

where

$$B_n^{(M)}(f)(1/2) = \frac{\bigvee_{k=0}^{n} p_{n,k}(1/2)f(k/n)}{\bigvee_{k=0}^{n} p_{n,k}(1/2)}.$$

Let $k_0 \in \{0, 1, \ldots, n\}$ be such that $1/2 \in [k_0/(n+1), (k_0+1)/(n+1)]$ and let $k_1 \in \{0, 1, \ldots, n\}$ be such that $\bigvee_{k=0}^{n} p_{n,k}(1/2)f(k/n) = p_{n,k_1}(1/2)$. Noting that $\bigvee_{k=0}^{n} p_{n,k}(1/2) = p_{n,k_0}(1/2)$, it follows that

$$B_n^{(M)}(f)(1/2) = \frac{p_{n,k_1}(1/2)f(k_1/n)}{p_{n,k_0}(1/2)}.$$

On the other hand, since the relation $\sum_{k=0}^{n} p_{n,k}(1/2)=1$ implies $p_{n,k_0}(1/2) \geq 1/(n+1)$, by Lemma 2.9.5 it follows that

$$\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(1/2)\right]+\sum_{k=0}^{n}\left[\bigvee_{j=n-k}^{n} p_{n,j}(1/2)\right]$$
$$=1+(n+1)p_{n,k_0}(1/2)\leq(n+1)p_{n,k_0}(1/2)+(n+1)p_{n,k_0}(1/2)$$
$$=2(n+1)p_{n,k_0}(1/2).$$

Then one can easily prove that

$$\sum_{k=0}^{n}\left[\bigvee_{j=0}^{k} p_{n,j}(1/2)f(j/n)\right]+\sum_{k=0}^{n}\left[\bigvee_{j=n-k}^{n} p_{n,j}(x)f(j/n)\right]$$
$$\geq(n+2)p_{n,k_1}(1/2)f(k_1/n).$$

Indeed, this is immediate since for $k\in\{k_1,k_1+1,\ldots,n\}$ we have

$$\bigvee_{j=0}^{k} p_{n,j}(1/2)f(j/n)=p_{n,k_1}(1/2)f(k_1/n)$$

and for $k\in\{n-k_1,n-k_1+1,\ldots,n\}$ we have

$$\bigvee_{j=n-k}^{n} p_{n,j}(x)f(1/n)=p_{n,k_1}(1/2)f(k_1/n).$$

The above two inequalities imply

$$T_n^{(SM)}(f)(x)\geq\frac{(n+2)p_{n,k_1}(1/2)f(k_1/n)}{2(n+1)p_{n,k_0}(1/2)}\geq\frac{1}{2}\cdot B_n^{(M)}(f)(1/2).$$

Since $f(1/2)=0$ and taking into account relation (2.33), we get

$$T_n^{(SM)}(f)(1/2)-f(\frac{1}{2})\geq\frac{e^{-5}}{12}\omega_1(f,1/\sqrt{n}).$$

which proves the desired conclusion. □

2) It remains an interesting open question to see if the sum-max Bernstein operator possesses similar properties with the max-prod Bernstein operator.
3) The construction generated by the sum-max method applied above to the Bernstein polynomials, evidently that could be applied to any other Bernstein-type operator in the next chapters.
4) The positivity of the approximated function f in the approximation by the sum-max Bernstein operator could be dropped if we apply the same idea as in Theorem 2.9.1.

http://www.springer.com/978-3-319-34188-0

Approximation by Max-Product Type Operators
Bede, B.; Coroianu, L.; Gal, S.G.
2016, XV, 458 p. 12 illus., 1 illus. in color., Hardcover
ISBN: 978-3-319-34188-0