Preface

In this research monograph, we bring to light an interesting new direction in constructive approximation of functions by operators.

There are at least two natural justifications for these new operators, termed by us as max-product operators (for reasons we will see below). They are based on possibility theory, a mathematical theory dealing with certain types of uncertainties and which is considered as an alternative to probability theory.

The first justification is based on the interpretations of the max-product Bernstein operator as a possibilistic expectation of a particular fuzzy variable having a possibilistic Bernoulli distribution and on a Chebyshev-type inequality in the theory of possibility, facts which are in a perfect analogy with the probabilistic approach of Bernstein for the convergence of the classical Bernstein polynomials.

The second justification is based on the Feller scheme in terms of the possibilistic integral, which again is in perfect analogy with the classical Fellers's probabilistic scheme used for the construction of the convergent sequences of positive and linear operators.

The generality of these two approaches allows us to obtain convergence results for many discrete max-product-type operators, like the max-product Bernstein operators, max-product Meyer–König and Zeller operators, max-product Favard–Szász–Mirakjan operators, max-product Baskakov operators, max-product Picard operators, max-product Gauss–Weierstrass operators, and max-product Poisson–Cauchy operators.

For these reasons, the max-product-type operators could also be called possibilistic-type operators.

The method of directly obtaining the max-product operators can easily be formalized as follows (see Open Question 5.5.4, p. 324 in the book Gal [84]): for example, in the case of the classical Bernstein polynomials, we write them in the form

$$B_n(f)(x) = \frac{\sum_{k=0}^{n} f(k/n) \cdot p_{n,k}(x)}{\sum_{k=0}^{n} p_{n,k}(x)},$$

and then we replace the sum operation by the max operation (by keeping the product operation), obtaining

$$B_n^{(M)}(f)(x) = \frac{\max_{0 \le k \le n}\{f(k/n) \cdot p_{n,k}(x)\}}{\max_{0 \le k \le n}\{p_{n,k}(x)\}}.$$

This formalization can then easily be applied to the other classical Bernstein-type operators, like the Meyer–König and Zeller operators, Favard–Szász–Mirakjan operators, and Baskakov operators. More importantly, it can be applied to linear approximation operators which are not necessarily positive, like the interpolation-type operators.

All the max-product operators are nonlinear and piecewise rational, and they present, for many subclasses of functions, essentially better approximation properties than the classical linear operators.

It is worth mentioning that the starting point of this research is represented by the papers of Bede–Nobuhara–Fodor–Hirota [31] and Bede–Nobuhara–Dañkova–Di Nola [32], where instead of the classical linear and positive Shepard operator attached to a positive function $f : [0, 1] \to \mathbb{R}_+$ and to equidistant nodes,

$$S_{n,\lambda}(f)(x) = \frac{\sum_{k=0}^{n} f(k/n)|x - k/n|^{-\lambda}}{\sum_{k=0}^{n} |x - k/n|^{-\lambda}},$$

where $\lambda \ge 1$, $n \in \mathbb{N}$, the authors consider the following Shepard-type nonlinear operator

$$S_{n,\lambda}^{(M)}(f)(x) = \frac{\max_{0 \le k \le n}\{f(k/n) \cdot |x - k/n|^{-\lambda}\}}{\max_{0 \le k \le n}\{|x - k/n|^{-\lambda}\}}.$$

The new so-called max-product operator remains convergent to the continuous function f, with a Jackson-type rate, namely,

$$|S_{n,\lambda}^{(M)}(f)(x) - f(x)| \le \frac{3}{2}\omega_1(f; 1/n),$$

valid for all $x \in [0, 1]$, $n \in \mathbb{N}$ (see [31]).

Comparing with the estimates given by the classical Shepard operator in [141], we note that for $1 \le \lambda \le 2$, the operator $S_{n,\lambda}^{(M)}(f)$ gives essentially better estimates.

In a very long list of papers (see references) whose results are collected by this monograph, we study the nice approximation properties of many max-product Bernstein-type operators, interpolation-type operators, and sampling operators.

The book can be briefly described as follows.

In Chapter 1, we give a short account of all basic (classical) approximation operators with their most important properties. We introduce all of their corresponding max-product operators with their main characteristics and give other basic definitions and results which are important for the content of the book.

The structure of Chapter 2, which deals with the max-product Bernstein operators, is as follows:

In Section 2.1, we first apply, for the max-product Bernstein operator $B_n^{(M)}$, the general results for sublinear, monotone, and positive homogenous operators in Theorem 1.1.2 in Subsection 1.1.3. Also, for large subclasses of functions, like the concave functions, Jackson-type estimates are obtained. Concerning the shape-preserving properties, it is proved that $B_n^{(M)}(f)$ preserves the monotonicity and the quasiconvexity of f. Finally, a comparison with the approximation order given by the Bernstein polynomials is made.

In Section 2.2, improved error estimates in terms of $n[\omega_1(f; 1/n)]^2 + \omega_1(f; 1/n)$ for strictly positive functions f are obtained and the preservation of quasiconcavity of f is proved.

Section 2.3 deals with the saturation results for $B_n^{(M)}$, while Section 2.4 contains very strong localization results for $B_n^{(M)}$ (much stronger than those of the Bernstein polynomials). It is worth noting the strong localization result expressed by Theorem 2.4.1 that shows that if the bounded functions f and g with strictly positive lower bounds coincide on a subinterval $[\alpha, \beta] \subset [0, 1]$, then for sufficiently large values of n, $B_n^{(M)}(f)$ and $B_n^{(M)}(g)$ coincide on subintervals sufficiently close to $[\alpha, \beta]$. Then, Corollary 2.4.3 shows that $B_n^{(M)}(f)$ is very suitable to approximate strictly positive functions which are constant on some subintervals. Namely, if f is a strictly positive continuous function which is constant on some subintervals $[\alpha_i, \beta_i]$, $i = 1, \dots, p$, of $[0, 1]$, then for sufficiently large n, $B_n^{(M)}(f)$ takes the same constant values on subintervals sufficiently close to each $[\alpha_i, \beta_i]$, $i = 1, \dots, p$. This property is illustrated by a simple graphic.

In Section 2.5, we study the iterations and the fixed points for the operator $B_n^{(M)}$ and in Section 2.6 one applies the properties of $B_n(f)$ to the approximation of fuzzy numbers. It is also worth mentioning here some approximation results in the L^1-norm.

Section 2.7 deals with the approximation and shape-preserving properties for two kinds of bivariate max-product Bernstein operators.

Section 2.8 contains applications to image processing of the tensor product bivariate max-product Bernstein operator.

In Section 2.9, the max-product Bernstein operators are used to extend all the approximation results to the functions of variable sign, by introducing the new operator $A^{(M)}(f)(x) = B_n^{(M)}(f + c)(x) - c$, where $c > 0$ is a constant chosen such that $f(x) + c > 0$, for all $x \in [0, 1]$.

In Chapters 3, 4, 5, and 6, approximation and shape-preserving properties for the max-product Favard–Szász–Mirakjan operator (nontruncated and truncated cases), the max-product Baskakov operator (nontruncated and truncated cases), the max-product Bleimann–Butzer–Hahn operator, and the max-product Meyer–König and Zeller operator are obtained, respectively.

Chapter 7 studies in detail the approximation properties of various max-product Lagrange and Hermite–Fejér interpolation operators, on general knots, on equidistant knots, and on Chebyshev knots of the first and of the second kind. It is worth noting here that while in general, the classical interpolation operators even diverge, their associated max-product operators always give a Jackson-type error estimate.

In Chapter 8, we study in detail the max-product sampling operators, associated to the classical linear sampling operators used in signal theory. We mention here especially the max-product sampling operators based on the sinc/Wittaker kernel and on the Fejér kernel, since they present essentially better approximation properties.

Note that the method in Section 2.9, shortly mentioned above and used for the max-product Bernstein operator to extend all the approximation results to functions of variable sign, in fact works, for all kinds of max-product operators introduced in the Chapters 3, 4, 5, 6, 7, and 8, thus allowing the validity of all the approximation results for functions of variable sign.

Chapter 9 treats the global smoothness properties for the max-product Bernstein operator, for the max-product Hermite–Fejér operator based on Chebyshev knots of first kind, and for the max-product Lagrange operator based on the Chebyshev knots of second kind, plus ± 1.

Chapter 10 presents the two approaches (of Bernstein and of Feller mentioned above) of the max-product-type operators in the frame of possibility theory, which, besides representing natural motivations for them, open new directions of research.

In Chapter 11 we apply the max-product idea to some Weierstrass-type functions, by which we obtain interesting kinds of functions presenting fractal-type properties.

The fact that Chapters 2 up to 8 consider roughly the same properties for different types of max-product operators might give to the reader the impression of being too repetitive.

Unfortunately, the possibility to present the results in Chapters 2, 3, 4, 5, 6, 7, and 8 in a unified form is not possible, because each max-product operator in these chapters is expressed by different formulas and, in order to present credible proofs of their approximation properties (including quantitative estimates, saturation results, localization results, and shape-preserving properties), it is clear that we need to perform all the calculations for each operator, as they are different from an operator to another one.

However, the qualitative properties of convergence for these operators are presented in an unified form through the two alternative approaches in possibility theory described in detail by Chapter 10.

The book is mainly addressed to researchers in the fields of the approximation of functions, approximation of fuzzy numbers, mathematical analysis, numerical analysis, signal theory, and image processing. Also, it is suitable for graduate courses in the above domains.

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