Chapter 2
Fokker–Planck Equations

Abstract In this chapter, we review the Bakry–Emery approach from the PDE viewpoint (Sect. 2.1) and the original stochastic viewpoint (Sect. 2.3) and detail some known relations to convex Sobolev inequalities (Sect. 2.2). Our focus is the PDE viewpoint addressed by (Toscani, G, Entropy production and the rate of convergence to equilibrium for the Fokker-Planck equation, Quart. Appl. Math, **57**, 521–541, (1999) [43]), and we follow partially the presentation of Matthes, D, Entropy Methods and Related Functional Inequalities, Lecture Notes, Pavia, Italy, (2007) http://www-m8.ma.tum.de/personen/matthes/papers/lecpavia.pdf [34]. The original Bakry–Emery method in (Bakry, D, Emery, M, Diffusions hypercontractives. Séminaire de probabilités XIX, 1983/84, Lecture Notes in Mathmatics, vol. 1123, pp. 177–206, Springer, Berlin (1985) [7] has been elaborated by many authors, and we select some of its extensions, including intermediate asymptotics by Carrillo and Toscani, Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity, Indiana Univ. Math. J., **49**, 113–142, (2000) [13] (Sect. 2.4) and more general Fokker–Planck equations, investigated, e.g., by Carrillo et al., Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity, Indiana Univ. Math. J., **49**, 113–142, (2000); Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities, Monatsh. Math. **133**, 1–82, (2001) [13, 14] and Arnold et al., Large-time behavior of non-symmetric Fokker-Planck type equations, Commun. Stoch. Anal., **2**, 153–175, (2008); Sharp entropy decay for hypocoercive and non-symmetric Fokker–Planck equations with linear drift, Preprint (2014). arXiv:1409.5425 [2, 5] (Sects. 2.5–2.6). Because of limited space, we ignore many important developments and deep connections to, e.g., optimal transport and Riemannian geometry, and we just refer to Villani, C.: Optimal Transport Old and New. Springer, Berlin (2009) [46] and the numerous references therein for more information.

Keywords Bakry–Emery approach · Convex Sobolev inequality · Intermediate asymptotics · Nonlinear Fokker–Planck equations

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2.1 The PDE Viewpoint of the Bakry–Emery Approach

One idea of entropy methods is first to calculate the entropy production $\mathrm{P}[u] = -\mathrm{dH}/\mathrm{d}t$ associated to the solution u of an evolution equation and second to employ the inequality $\mathrm{P}[u] \geq \kappa \mathrm{H}[u]$. Then we infer from Gronwall's lemma the exponential decay of the solution to the steady state with rate κ. In this section, we detail how the exponential decay rate *and* the functional inequality can be proved *simultaneously*. The key idea, due to Bakry and Emery [7], is to compute the time derivative of the entropy production and to relate it to the entropy production. We restrict ourselves to formal arguments to highlight the ideas and refer to [6] for rigorous arguments.

In order to explain the idea, we focus on the linear Fokker–Planck equation

$$\partial_t u = \mathrm{div}(\nabla u + u\nabla V), \quad t > 0, \quad u(0) = u_0 \quad \text{in } \mathbb{R}^d. \tag{2.1}$$

Such equations arise in many applications like semiconductor theory [33], plasma physics [40], and stellar dynamics [22]. The function $V = V(x)$ models a potential. In probability theory, the Fokker–Planck equation is also called the Kolmogorov forward equation, describing the time evolution of the probability density function of a stochastic process [39]. Existence results are proved in [14].

We suppose that the initial datum $u_0 \in L^1(\mathbb{R}^d)$ is nonnegative and has unit mass, $\int_{\mathbb{R}^d} u_0 \mathrm{d}x = 1$. Then the solution $u(t)$ (which is assumed to exist) is nonnegative (by the maximum principle) and conserves mass, i.e. $\int_{\mathbb{R}^d} u(t)\mathrm{d}x = 1$ for all $t > 0$. The potential is assumed to be smooth, convex, and to satisfy $\mathrm{e}^{-V} \in L^1(\mathbb{R}^d)$. We may interpret $u(x,t)$ as the density of a particle system which is confined by the potential.

The Fokker–Planck equation possesses a unique steady state u_∞ which is computed from

$$0 = \nabla u_\infty + u_\infty \nabla V = u_\infty \nabla(\log u_\infty + V).$$

Hence, if $u_\infty > 0$ (see [14, Sect. 3.1] for the general case $u_\infty \geq 0$), $\log u_\infty + V$ is constant and consequently,

$$u_\infty(x) = Z\mathrm{e}^{-V(x)}, \quad Z = \left(\int_{\mathbb{R}^d} \mathrm{e}^{-V(y)}\mathrm{d}y\right)^{-1}. \tag{2.2}$$

In particular, u_∞ has unit mass.

Let $\phi : [0,\infty) \to \mathbb{R}$ be a smooth, strictly convex function such that $\phi(1) = 0$ and $1/\phi''$ is concave. Then $\widetilde{\phi}(s) = \phi(s) - \phi'(1)(s-1)$ is a linear perturbation of ϕ, it has the same properties as ϕ and satisfies $\widetilde{\phi}'(1) = 0$, so we may suppose without loss of generality that $\phi'(1) = 0$. The convexity implies that $\phi(s) \geq 0$ for $s \geq 0$ and in particular, $\phi''(1) > 0$. With such a function, we define the (relative) entropy

$$\mathrm{H}_\phi[u] = \int_{\mathbb{R}^d} \phi\left(\frac{u}{u_\infty}\right) u_\infty \mathrm{d}x.$$

Our main result is the following theorem; see [6, 7].

Theorem 2.1 (Exponential decay of the Fokker–Planck equation) *Let $\phi \in C^4([0,\infty))$ be convex, $\phi(1)=0$, and $1/\phi''$ is defined and concave. Furthermore, let $\mathrm{H}_\phi[u_0]<\infty$ and let the Bakry–Emery condition $\nabla^2 V \geq \lambda\mathbb{I}$ hold for some $\lambda>0$. Then any smooth solution to* (2.1) *converges exponentially fast to the steady state,*

$$\|u(t)-u_\infty\|_{L^1(\mathbb{R}^d)} \leq \mathrm{e}^{-\lambda t} K_\phi \mathrm{H}_\phi[u_0]^{1/2}, \quad t>0$$

where $K_\phi=\sqrt{2/\phi''(1)}$ is the constant in the general Csiszár–Kullback–Pinsker inequality (see Appendix A.1).

The condition $\nabla^2 V \geq \lambda\mathbb{I}$ means that $\nabla^2 V-\lambda\mathbb{I}$ is positive semi-definite, where $\nabla^2 V$ denotes the Hessian matrix of V. Admissible functions ϕ are, for instance,

$$\phi(s)=s(\log s-1)+1, \quad \phi(s)=s^\alpha-1 \quad (1<\alpha\leq 2).$$

Proof Our presentation is close to the proof of Theorem 2.3 in [34].

Step 1: first time derivative of the entropy. We compute the entropy production. For this, we set $\rho=u/u_\infty$ and observe that $\partial_t u=\operatorname{div}(u_\infty\nabla\rho)$. Then, by integrating by parts,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_\phi[u]=\int_{\mathbb{R}^d}\phi'(\rho)\partial_t u\,\mathrm{d}x=-\int_{\mathbb{R}^d}\phi''(\rho)|\nabla\rho|^2 u_\infty\,\mathrm{d}x\leq 0. \tag{2.3}$$

Since ϕ is convex, the entropy production $\mathrm{P}_\phi[u]=-\mathrm{d}\mathrm{H}_\phi/\mathrm{d}t$ is nonnegative.

Step 2: second time derivative of the entropy. The key idea of Bakry and Emery is to estimate the second time derivative of $\mathrm{H}_\phi[u]$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{H}_\phi[u]=-\int_{\mathbb{R}^d}\big(\phi'''(\rho)\partial_t u|\nabla\rho|^2+2\phi''(\rho)\nabla\rho\cdot\partial_t\nabla\rho u_\infty\big)\mathrm{d}x=:I_1+I_2.$$

For the first integral I_1, we insert (2.1) and integrate by parts:

$$\begin{aligned} I_1&=\int_{\mathbb{R}^d}\nabla(\phi'''(\rho)|\nabla\rho|^2)\cdot(u_\infty\nabla\rho)\mathrm{d}x\\ &=\int_{\mathbb{R}^d}\big(\phi''''(\rho)|\nabla\rho|^4+2\phi'''(\rho)\nabla\rho^\top\nabla^2\rho\nabla\rho\big)u_\infty\mathrm{d}x, \end{aligned} \tag{2.4}$$

where $\nabla\rho^\top\nabla^2\rho\nabla\rho=\sum_{i,j=1}^d\partial_i\rho\partial_{ij}^2\rho\partial_j\rho$ and $\partial_i=\partial/\partial x_i$. In order to determine the second integral I_2, we observe first that $\nabla\partial_t\rho=\nabla\Delta\rho-\nabla^2\rho\nabla V-\nabla^2 V\nabla\rho$. Inserting this expression into I_2 and writing $\nabla\rho\cdot\nabla\Delta\rho=\operatorname{div}(\nabla^2\rho\nabla\rho)-|\nabla^2\rho|^2$,

$$
\begin{aligned}
I_2 &= -2\int_{\mathbb{R}^d} \phi''(\rho)\big(\nabla\rho\cdot\nabla\Delta\rho - \nabla\rho^\top\nabla^2\rho\nabla V - \nabla\rho^\top\nabla^2 V\nabla\rho\big)u_\infty \mathrm{d}x \\
&= -2\int_{\mathbb{R}^d} \phi''(\rho)\big(\operatorname{div}(\nabla^2\rho\nabla\rho) - |\nabla^2\rho|^2 - \nabla\rho^\top\nabla^2\rho\nabla V - \nabla\rho^\top\nabla^2 V\nabla\rho\big)u_\infty \mathrm{d}x.
\end{aligned}
$$

Next, we integrate by parts in the first term and use the strict convexity of V in the last term:

$$
\begin{aligned}
I_2 \geq 2\int_{\mathbb{R}^d} &\big(\phi'''(\rho)\nabla\rho^\top\nabla^2\rho\nabla\rho u_\infty + \phi''(\rho)\nabla\rho^\top\nabla^2\rho(\nabla u_\infty + u_\infty\nabla V) \\
&+ \phi''(\rho)|\nabla^2\rho|^2 u_\infty\big)\mathrm{d}x + 2\lambda\int_{\mathbb{R}^d}\phi''(\rho)|\nabla\rho|^2 u_\infty \mathrm{d}x.
\end{aligned}
$$

By definition (2.2) of u_∞, the second term on the right-hand side vanishes. Moreover, by (2.3), the last integral equals the negative first time derivative of H_ϕ. Therefore, adding this expression and (2.4), it follows that

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{H}_\phi[u] &\geq \int_{\mathbb{R}^d}\big(\phi''''(\rho)|\nabla\rho|^4 + 4\phi'''(\rho)\nabla\rho^\top\nabla^2\rho\nabla\rho + 2\phi''(\rho)|\nabla^2\rho|^2\big)u_\infty \mathrm{d}x \\
&\quad - 2\lambda\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_\phi[u] \\
&= 2\int_{\mathbb{R}^d}\phi''(\rho)\left|\nabla^2\rho + \frac{\phi'''(\rho)}{\phi''(\rho)}\nabla\rho\otimes\nabla\rho\right|^2 u_\infty \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d}\left(\phi''''(\rho) - 2\frac{\phi'''(\rho)^2}{\phi''(\rho)}\right)|\nabla\rho|^4 u_\infty \mathrm{d}x - 2\lambda\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_\phi[u].
\end{aligned}
$$

Note that $\phi'''' - 2(\phi''')^2/\phi'' = -(\phi'')^2(1/\phi'')'' \geq 0$, since we assumed that $1/\phi''$ is concave. Together with the convexity of ϕ, we infer that

$$
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{H}_\phi[u] \geq -2\lambda\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_\phi[u], \quad t > 0. \tag{2.5}
$$

Step 3: exponential decay of the entropy. Integrating both sides of (2.5) over $t \in (s,\infty)$, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_\phi[u(s)] - \lim_{t\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_\phi[u(t)] \leq -2\lambda\left(\mathrm{H}_\phi[u(s)] - \lim_{t\to\infty}\mathrm{H}_\phi[u(t)]\right).
$$

We claim that the two limits vanish, leading to

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_\phi[u(s)] \leq -2\lambda\mathrm{H}_\phi[u(s)], \quad s \geq 0. \tag{2.6}
$$

Then Gronwall's lemma implies that the entropy decays exponentially fast,

$$\mathrm{H}_\phi[u(s)] \le \mathrm{H}_\phi[u_0]\mathrm{e}^{-2\lambda s}, \quad s \ge 0. \tag{2.7}$$

It remains to prove that the limits vanish. By definition of the entropy production, $\mathrm{P}_\phi[u] = -\mathrm{dH}_\phi/\mathrm{d}t$, so inequality (2.5) can be written as $\mathrm{dP}_\phi/\mathrm{d}t \le -2\lambda\mathrm{P}_\phi$. By Gronwall's lemma, we conclude that the entropy production decays exponentially,

$$\mathrm{P}_\phi[u(t)] \le \mathrm{P}[u_0]\mathrm{e}^{-2\lambda t}, \quad t > 0,$$

and consequently, $\lim_{t\to\infty} \mathrm{P}_\phi[u(t)] = 0$. For the remaining limit, we argue formally. Assuming that we can interchange the limit and the nonlinear functional, we have

$$0 = \lim_{t\to\infty} \mathrm{P}_\phi[u(t)] = \mathrm{P}_\phi\left[\lim_{t\to\infty} u(t)\right].$$

The functional P_ϕ vanishes exactly at $u = u_\infty$, which shows that $\lim_{t\to\infty} u(t) = u_\infty$. Therefore, interchanging the limits in H_ϕ and employing $\phi(1) = 0$,

$$\lim_{t\to\infty} \mathrm{H}_\phi[u(t)] = \mathrm{H}_\phi\left[\lim_{t\to\infty} u(t)\right] = \mathrm{H}_\phi[u_\infty] = 0,$$

which proves the claim. The rigorous proof is rather technical; see [6, Sect. 2].

Step 4: exponential decay in L^1. Finally, we apply the Csiszár–Kullback–Pinsker inequality (see Appendix A.1) and (2.7) to find that

$$\|u(t) - u_\infty\|^2_{L^1(\mathbb{R}^d)} \le \frac{2}{\phi''(1)}\mathrm{H}_\phi[u(t)] \le \frac{2}{\phi''(1)}\mathrm{H}_\phi[u_0]\mathrm{e}^{-2\lambda t}.$$

This finishes the proof. □

Remark 2.1 (Integration by parts) The proof of Theorem 2.1 is based on smart integrations by parts. One may ask the question whether it is possible to make this approach more systematic. The answer is affirmative and leads to the method of systematic integration by parts, which is presented in Chap. 3. □

Remark 2.2 (Comparison with spectral theory) We already mentioned that the inverse of the best constant in the Poincaré inequality is the smallest positive eigenvalue of the differential operator under consideration. We call the distance between the zero eigenvalue and the smallest positive eigenvalue the *spectral gap* (if it exists). The entropy method allows for the explicit computation of the spectral gap of, e.g., Schrödinger operators $-\Delta + V(x)$ in $\mathbb{R}^d$. Such a result can also be obtained by spectral theory [38], and it gives sharp results. Spectral theory, however, gives hardly explicit values for the spectral gap and it is difficult to generalize to nonlinear problems. Entropy methods often yield such explicit values, with conditions which are easy to check, and it is robust to many nonlinear generalizations (we give an example in Sect. 2.5). Yet, entropy methods often do not give sharp rates. □

Remark 2.3 (Nonlocal condition) A drawback of the Bakry–Emery method is that the decay rate is based on a pointwise estimate on the potential. It is shown in [20, Theorem 1.2] that the local estimate can be replaced by a weaker nonlocal condition related to the positivity of the first eigenvalue of a Schrödinger operator. □

2.2 Convex Sobolev Inequalities

The exponential convergence of solutions to the heat or DLLS equation in Sect. 1.3 was proved by using an entropy-entropy production inequality. It seems as if such an inequality was not used in the proof of Theorem 2.1. In fact, we did. The inequality is given by (2.6), and we have shown the exponential decay *and* the entropy-entropy production inequality *simultaneously*. We call the entropy-entropy production inequality also a convex Sobolev inequality (see Definition 1.3).

Corollary 2.1 (Convex Sobolev inequality) *Let ϕ and V satisfy the assumptions of Theorem 2.1, and let u_∞ be defined by (2.2). Then*

$$\int_{\mathbb{R}^d} \phi\left(\frac{u}{u_\infty}\right) u_\infty \mathrm{d}x \le \frac{1}{2\lambda} \int_{\mathbb{R}^d} \phi''\left(\frac{u}{u_\infty}\right) \left|\nabla \frac{u}{u_\infty}\right|^2 u_\infty \mathrm{d}x \tag{2.8}$$

for all nonnegative integrable functions u for which the integrals are defined.

Proof The left-hand side of the inequality equals $\mathrm{H}_\phi[u]$, the right-hand side equals $\mathrm{P}_\phi[u]/(2\lambda)$. Thus, the result follows from (2.6) by choosing $s = 0$, $u_0 = u$. □

Remark 2.4 The convexity requirement $\nabla^2 V \ge \lambda \mathbb{I}$ is called the *Bakry–Emery condition* . It can be significantly generalized, and we come back to this point in Sect. 2.3; see [8, 15]. When $V(x) = \lambda |x|^2/2$, the constant $1/(2\lambda)$ in (2.8) is optimal. □

Example 2.1 (Logarithmic Sobolev inequality) Let $\phi(s) = s(\log s - 1) + 1$ and let $\rho \in L^1(u_\infty \mathrm{d}x)$ be a function with unit mass with respect to the measure $u_\infty \mathrm{d}x$ and $\sqrt{\rho} \in H^1(u_\infty \mathrm{d}x)$. We still assume that $\int_{\mathbb{R}^d} u_\infty \mathrm{d}x = 1$. Then (2.8) can be written as a weighted logarithmic Sobolev inequality:

$$\int_{\mathbb{R}^d} \rho \log \rho u_\infty \mathrm{d}x \le \frac{2}{\lambda} \int_{\mathbb{R}^d} |\nabla \sqrt{\rho}|^2 u_\infty \mathrm{d}x. \tag{2.9}$$

If $\lambda = 1$ and we set $f = \sqrt{\rho}$, $\mathrm{d}\mu = u_\infty \mathrm{d}x$, it takes the equivalent form

$$\int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2 \mathrm{d}\mu}\right) \mathrm{d}\mu \le 2 \int_{\mathbb{R}^d} |\nabla f|^2 \mathrm{d}\mu. \tag{2.10}$$

We may allow for functions f^2 whose mass is possibly not equal to one. This form is the original one in the paper of Gross [24] (also see the earlier contributions by

Stam [41] and Federbush [21]). It is usually known as the *Gaussian form* of the logarithmic Sobolev inequality. A remarkable property is that the constant, which is optimal, does not depend on the dimension.

Inequality (2.8) can be formulated with respect to the Lebesgue measure only. For this, let $V(x) = \frac{1}{2}|x|^2$. Then $\lambda = 1$, $u_\infty(x) = (2\pi)^{-d/2}\mathrm{e}^{-|x|^2/2}$, and

$$\int_{\mathbb{R}^d} \phi\left(\frac{u}{u_\infty}\right) u_\infty \mathrm{d}x = \int_{\mathbb{R}^d} u \log u \mathrm{d}x + \frac{d}{2}\log(2\pi) + \frac{1}{2}\int_{\mathbb{R}^d} u|x|^2 \mathrm{d}x,$$
$$\int_{\mathbb{R}^d} \phi''\left(\frac{u}{u_\infty}\right)\left|\nabla \frac{u}{u_\infty}\right|^2 u_\infty \mathrm{d}x = 4\int_{\mathbb{R}^d} |\nabla\sqrt{u}|^2 \mathrm{d}x - 2d + \int_{\mathbb{R}^d} u|x|^2 \mathrm{d}x.$$

Inserting these expressions into (2.8), we find that

$$\int_{\mathbb{R}^d} u \log u \mathrm{d}x + \frac{d}{2}\log(2\pi) + d \le 2\int_{\mathbb{R}^d} |\nabla\sqrt{u}|^2 \mathrm{d}x,$$

which is the *Euclidean form* of the logarithmic Sobolev inequality (cf. (1.10)). □

Example 2.2 (Weighted Poincaré inquality) Choosing $\phi(s) = s^2 - 1$, (2.8) yields

$$\int_{\mathbb{R}^d} (\rho^2 - 1) u_\infty \mathrm{d}x \le \frac{1}{\lambda}\int_{\mathbb{R}^d} |\nabla\rho|^2 u_\infty \mathrm{d}x,$$

or, since $\int_{\mathbb{R}^d} \rho u_\infty \mathrm{d}x = \int_{\mathbb{R}^d} u_\infty \mathrm{d}x = 1$,

$$\int_{\mathbb{R}^d} \rho^2 u_\infty \mathrm{d}x - \left(\int_{\mathbb{R}^d} \rho u_\infty \mathrm{d}x\right)^2 \le \frac{1}{\lambda}\int_{\mathbb{R}^d} |\nabla\rho|^2 u_\infty \mathrm{d}x. \tag{2.11}$$

This is a *weighted Poincaré inequality*. Note that the standard Poincaré inequality $\|u\|^2_{L^2(\Omega)} \le C_P\|\nabla u\|^2_{L^2(\Omega)}$ for $u \in H^1_0(\Omega)$ is not valid when $\Omega = \mathbb{R}^d$; therefore, the presence of a weight in (2.11) is needed. The constant λ is linked to u_∞ by the convexity condition on the potential V appearing in the definition of u_∞ (see (2.2)).

In stochastics, the left-hand side of (2.11) is called the variance of ρ in $L^2(u_\infty \mathrm{d}x)$, $\mathrm{Var}(\rho)$, while the right-hand side is the Dirichlet integral, $\mathscr{E}(\rho) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\rho|^2 u_\infty \mathrm{d}x$. Thus, (2.11) can be written compactly as

$$\mathrm{Var}(\rho) \le \frac{2}{\lambda}\mathscr{E}(\rho).$$

We refer to [8, Sect. 4.2] for more comments on Poincaré inequalities on probability spaces. Also see Sect. 2.3. □

Example 2.3 (Beckner inequality) There is a family of inequalities which interpolate between the logarithmic Sobolev and the Poincaré inequality, the *Beckner inequalities* [10]. For this, let $\phi(s) = s^\alpha - 1$ with $1 < \alpha \le 2$. By (2.8) and with $\rho = u/u_\infty$ satisfying $\int_{\mathbb{R}^d} \rho u_\infty \mathrm{d}x = 1$,

$$\frac{1}{\alpha-1}\left(\int_{\mathbb{R}^d}\rho^\alpha u_\infty \mathrm{d}x-\left(\int_{\mathbb{R}^d}\rho u_\infty \mathrm{d}x\right)^\alpha\right)\le\frac{\alpha}{2\lambda}\int_{\mathbb{R}^d}\rho^{\alpha-2}|\nabla\rho|^2u_\infty \mathrm{d}x.$$

If $\alpha = 2$, we recover the Poincaré inequality (2.11). For $\alpha \to 1$, because of the pointwise convergence $(s^\alpha - s)/(\alpha-1) \to s\log s$, the Beckner inequality leads to the logarithmic Sobolev inequality. For more comments, we refer to [8, 30]. □

Remark 2.5 (Logarithmic Sobolev implies Poincaré) The logarithmic Sobolev inequality (2.9) implies the Poincaré inequality. Indeed, let $\rho = 1 + \varepsilon g$ for some smooth function g, where $\varepsilon > 0$ is sufficiently small such that $1 + \varepsilon g \ge 0$. Since $\int_{\mathbb{R}^d}\rho u_\infty \mathrm{d}x = \int_{\mathbb{R}^d} u_\infty \mathrm{d}x = 1$, we have $\int_{\mathbb{R}^d} g u_\infty \mathrm{d}x = 0$. Inserting $\rho = 1+\varepsilon g$ into (2.9) and expanding $\log(1+\varepsilon g) = \varepsilon g - \frac{1}{2}\varepsilon^2 g^2 + O(\varepsilon^3)$, some terms cancel and we end up with

$$\frac{\varepsilon^2}{2}\int_{\mathbb{R}^d} g^2u_\infty \mathrm{d}x + O(\varepsilon^3)\le\frac{1}{2\lambda}\int_{\mathbb{R}^d}\frac{\varepsilon^2|\nabla g|^2}{1+\varepsilon g}u_\infty \mathrm{d}x.$$

Dividing by ε^2 and passing to the limit $\varepsilon \to 0$ with dominated convergence, the Poincaré inequality

$$\int_{\mathbb{R}^d} g^2u_\infty \mathrm{d}x\le\frac{1}{\lambda}\int_{\mathbb{R}^d}|\nabla g|^2u_\infty \mathrm{d}x$$

follows for all functions $g \in H^1(u_\infty \mathrm{d}x)$ satisfying $\int_{\mathbb{R}^d} g u_\infty \mathrm{d}x = 0$.

The reverse implication (Poincaré implies logarithmic Sobolev) is generally false. An counterexample is given by the function $\exp(-|x|^\alpha)$ in $\mathbb{R}^d$ with $\alpha \in [1,2)$, which replaces $u_\infty(x) = \exp(-|x|^2)$ [9]. □

Remark 2.6 (Poincaré and Sobolev imply logarithmic Sobolev) In bounded domains, we do not need a confining potential. In this situation, the logarithmic Sobolev inequality follows from the Poincaré inequality *if additionally* the Sobolev inequality in $H^1(\Omega)$ holds. More precisely, let the bounded domain $\Omega \subset \mathbb{R}^d$ be such that the following inequalities hold for all $u \in H^1(\Omega)$:

$$\left\|u-\int_\Omega u\mathrm{d}x\right\|_{L^2(\Omega)}\le C_1\|\nabla u\|_{L^2(\Omega)},\quad \|u\|_{L^p(\Omega)}\le C_2\|u\|_{H^1(\Omega)},$$

where $2 < p \le 2d/(d-2)$ (and $2 < p < \infty$ if $d \le 2$). Then

$$\int_\Omega u^2\log\frac{u^2}{\|u\|^2_{L^2(\Omega)}}\mathrm{d}x\le C_3\int_\Omega|\nabla u|^2\mathrm{d}x,$$

where $C_3 > 0$ depends on C_1, C_2, and p (C_3 is decreasing in p). The proof is due to Stroock [42], and a short proof based on Jensen's inequality is given in [17]. □

Remark 2.7 (Optimality of the constant) The convex Sobolev inequality is optimal for the Boltzmann or quadratic entropy if the potential is quadratic in at least one coordinate direction (with convexity constant λ), since there exist extremal functions for which (2.8) becomes an equality [6, Sect. 3.5]. As pointed out in [4], the non-optimality for other entropies may have two reasons: either the constant λ is not the sharp convex Sobolev constant (e.g. for $V(x) = |x|^4$, $x \in \mathbb{R}$), or there is no extremal function, even for the sharp constant λ. This happens for the entropies $\phi(s) = s^\alpha$ with $1 < \alpha < 2$. The reason is that the linear dependence between the entropy and the entropy production is not optimal. Arnold and Dolbeault [4] have improved this relationship. They showed that there exists an increasing continuous function g satisfying $g(0) = 0$ and $g(s) > s$ for $s > 0$ such that

$$g\left(\int_{\mathbb{R}^d} \phi\left(\frac{u}{u_\infty}\right) u_\infty \mathrm{d}x\right) \leq \frac{1}{2\lambda} \int_{\mathbb{R}^d} \phi''\left(\frac{u}{u_\infty}\right) \left|\nabla \frac{u}{u_\infty}\right|^2 u_\infty \mathrm{d}x.$$

This inequality is referred to as a *refined convex Sobolev inequality* . □

2.3 The Stochastic Viewpoint of the Bakry–Emery Approach

The original method of Bakry and Emery [7] is of probabilistic nature. We review this approach briefly. As in the previous section, the computations are purely formal. Our presentation follows the lines of [8, Chap. 1] which also provides the functional framework.

Let $(X_t)_{t \geq 0}$ be a Markov process on $\Omega \subset \mathbb{R}^d$. Very loosely speaking, a Markov process is characterized by the (Markov) property that the future probabilities are determined by the present state only, i.e., the processes are "memoryless". An example is the Brownian motion. We associate to this Markov process a Markov semigroup $(P_t)_{t \geq 0}$ on the set of suitable measurable functions $f : \Omega \to \mathbb{R}$ by

$$P_t f(x) = \mathrm{E}(f(X_t)|X_0 = x), \quad t \geq 0, \ x \in \Omega,$$

where $\mathrm{E}(f(X_t)|X_0 = x)$ is the expectation of $f(X_t)$ conditional that the initial value X_0 equals x. Because of the Markov property, the semigroup satisfies the properties $P_0(f) = f$ and $P_s \circ P_t = P_{s+t}$ for all $s, t \geq 0$. We assume that the process $(X_t)_{t \geq 0}$ allows for a unique *invariant measure* μ_∞ which is characterized by

$$\int_\Omega P_t(f) \mathrm{d}\mu_\infty = \int_\Omega f \mathrm{d}\mu_\infty \tag{2.12}$$

for all $t \geq 0$ and all bounded positive measurable functions $f : \Omega \to \mathbb{R}$. We can associate to the semigroup P_t the *infinitesimal generator* (defined on some domain)

$$L(f) = \lim_{t \searrow 0} \frac{1}{t}(P_t(f) - f).$$

Then P_t is the solution to the evolution equation $\partial_t P_t = L P_t$, $t > 0$, and $P_0 = I$.

Example 2.4 (Ornstein–Uhlenbeck process) The generator L of the (generalized) Ornstein–Uhlenbeck process is given by

$$Lf(x) = \Delta f(x) - \nabla V(x) \cdot \nabla f(x), \quad x \in \mathbb{R}^d,$$

for suitable functions $f : \mathbb{R}^d \to \mathbb{R}$. Under regularity assumptions on the confinement potential $V(x)$ and the initial datum f_0, the associated Cauchy problem $\partial_t f = Lf$, $t > 0$, $f(x, 0) = f_0(x)$ possesses a unique solution which defines a semigroup P_t. Differentiating (2.12) at $t = 0$, we see that the invariant measure satisfies

$$0 = \int_{\mathbb{R}^d} Lf \mathrm{d}\mu_\infty.$$

If μ_∞ is absolutely continuous, the Radon–Nikodym derivative $u_\infty = \mathrm{d}\mu_\infty/\mathrm{d}x$ exists. Replacing $\mathrm{d}\mu_\infty$ in the above integral by $u_\infty \mathrm{d}x$ and integrating by parts, it follows that

$$0 = \int_{\mathbb{R}^d} (L^* u_\infty) f \mathrm{d}x \quad \text{for all admissible } f,$$

where $L^* u = \operatorname{div}(\nabla u + u \nabla V)$ is the formal adjoint operator of L. We recognize that L^* is exactly the Fokker–Planck operator considered in Sect. 2.1. If the set of admissible functions f is large enough (it should be dense in $L^1(\mathbb{R}^d)$) then $L^* u_\infty = 0$. Thus, u_∞ corresponds to the steady state of $\partial_t u = L^* u$ which is, according to Sect. 2.1, given by $u_\infty(x) = Z\mathrm{e}^{-V(x)}$, where $Z > 0$ ensures that u_∞ has unit mass. We conclude that the (unique) invariant measure of P_t is given by $\mathrm{d}\mu_\infty = u_\infty \mathrm{d}x$. □

Bakry and Emery formulated the convexity condition for the potential in a more abstract way. They defined the *carré du champ* operator and the Γ_2 (*gamma-deux*) operator by, respectively,

$$\Gamma(f, g) = \frac{1}{2}\big(L(fg) - gLf - fLg\big),$$
$$\Gamma_2(f, g) = \frac{1}{2}\big(L\Gamma(f, g) - \Gamma(Lf, g) - \Gamma(f, Lg)\big).$$

These operators are the first two members of a sequence of iterated-gradient operators which define a Lie algebra structure [31]. We abbreviate

$$\Gamma(f) = \Gamma(f, f), \quad \Gamma_2(f) = \Gamma_2(f, f).$$

Example 2.5 The carré du champ operator of the generator $Lf = \Delta f - \nabla V \cdot \nabla f$ of the Ornstein–Uhlenbeck process becomes

$$\Gamma(f, g) = \frac{1}{2}\big(\Delta(fg) - f\Delta g - g\Delta f\big) = \nabla f \cdot \nabla g.$$

The computation of the Γ_2 operator is more involved and we report only the result:

$$\Gamma_2(f) = |\nabla^2 f|^2 + \nabla f^\top \nabla^2 V \nabla f.$$

Observe that $\Gamma_2(f) \geq \lambda\Gamma(f)$ if V is strictly convex with constant λ. In fact, this condition generalizes the special convexity assumption on V. □

Bakry and Emery proved the following result [7] (also see [15]).

Theorem 2.2 (Convex Sobolev inequality) *Let $\phi \in C^2([0,\infty))$ be convex such that $1/\phi''$ is defined and concave. We assume that there exists $\lambda > 0$ such that for all suitable nonnegative functions f, the Bakry–Emery condition*

$$\Gamma_2(f) \geq \lambda\Gamma(f) \tag{2.13}$$

is satisfied. Then the following convex Sobolev inequality holds:

$$\int_\Omega \phi(f)\mathrm{d}\mu_\infty - \phi\left(\int_\Omega f\mathrm{d}\mu_\infty\right) \leq \frac{1}{2\lambda}\int_\Omega \phi''(f)\Gamma(f)\mathrm{d}\mu_\infty.$$

Proof As in Sect. 2.1, the idea of the proof is to differentiate the entropy $\mathrm{H}_\phi[f] = \int_\Omega \phi(f)\mathrm{d}\mu_\infty$ twice with respect to time. Lengthy computations similar to those in the proof of Theorem 2.1 give

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_\phi[P_t f]\Big|_{t=0} &= -\int_\Omega \phi''(f)\Gamma(f)\mathrm{d}\mu_\infty,\\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{H}_\phi[P_t f]\Big|_{t=0} &\geq 2\int_\Omega \phi''(f)^{-1}\Gamma_2(\phi'(f))\mathrm{d}\mu_\infty.\end{aligned}$$

Here, we have assumed that the functions are sufficiently regular and the boundary integrals vanish. The *carré du champ* operator satisfies a chain rule, $\Gamma(\phi'(f), g) = \phi''(f)\Gamma(f, g)$, which implies, together with condition (2.13), that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{H}_\phi[P_t f]\Big|_{t=0} &\geq 2\lambda\int_\Omega \phi''(f)^{-1}\Gamma(\phi'(f))\mathrm{d}\mu_\infty\\ &= 2\lambda\int_\Omega \phi''(f)\Gamma(f)\mathrm{d}\mu_\infty = -2\lambda\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_\phi[P_t f]\Big|_{t=0}.\end{aligned}$$

Now, we argue as in the proof of Theorem 2.1 to conclude the result. □

Remark 2.8 The Bakry–Emery condition (2.13) is also referred to as the curvature-dimension condition $CD(\lambda,\infty)$. When the diffusion operator is considered on a Riemannian manifold, this condition means that the Ricci curvature is bounded below by λ. For more details, we refer to [45, Chap. 14] and [8, Sect. 1.16]. □

Example 2.6 (Logarithmic Sobolev and Poincaré inequalities) With $\phi(s) = s\log s$ and $\phi(s) = s^2$ we obtain the logarithmic Sobolev and Poincaré inequality, respectively:

$$\int_\Omega f\log f\mathrm{d}\mu_\infty - \int_\Omega f\mathrm{d}\mu_\infty \log\left(\int_\Omega f\mathrm{d}\mu_\infty\right) \le \frac{1}{2\lambda}\int_\Omega \frac{\Gamma(f)}{f}\mathrm{d}\mu_\infty,$$
$$\int_\Omega (f - f_\infty)^2\mathrm{d}\mu_\infty \le \frac{1}{\lambda}\int_\Omega \Gamma(g)\mathrm{d}\mu_\infty, \quad f_\infty = \int_\Omega f\mathrm{d}\mu_\infty,$$

for admissible functions satisfying $\int_{\mathbb{R}^d} f\mathrm{d}\mu_\infty = \int_{\mathbb{R}^d} f_\infty\mathrm{d}\mu_\infty = 1$. Compared to the corresponding counterparts in Sect. 2.2, the above inequalities are formulated for rather general diffusion operators L satisfying the Bakry–Emery condition (2.13). If L is the generator of the Ornstein–Uhlenbeck process and $\mathrm{d}\mu_\infty = u_\infty\mathrm{d}x$, $\Gamma(f) = |\nabla f|^2$ follows (see Example 2.5) and we recover (2.9) and (2.11), respectively. □

2.4 Relaxation to Self-Similarity

We come back to the heat equation in the whole space,

$$\partial_t u = \Delta u, \quad t > 0, \quad u(0) = u_0 \quad \text{in } \mathbb{R}^d, \tag{2.14}$$

with nonnegative initial datum u_0 having unit mass. The following arguments are taken from [34, Sect. 1.5]. The solution to (2.14) can be written explicitly,

$$u(x,t) = \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} \mathrm{e}^{-|x-y|^2/(4t)}u_0(y)\mathrm{d}y.$$

We see immediately that

$$\|u(t)\|_{L^1(\mathbb{R}^d)} = 1, \quad \|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{(4\pi t)^{d/2}} \quad \text{for all } t > 0,$$

which allows us, by Hölder interpolation, to conclude decay estimates in any L^p norm with $p \in [1,\infty)$. Can we apply the entropy method to this equation? At first sight, this seems to be not possible since the logarithmic entropy $\mathrm{H}[u] = \int_{\mathbb{R}^d} u\log u\mathrm{d}x$ is a Lyapunov functional along solutions to (2.14) but

$$\mathrm{H}[u(t)] \le \int_{\mathbb{R}^d} u(t)\log\|u(t)\|_{L^\infty(\mathbb{R}^d)}\mathrm{d}x = \log\|u(t)\|_{L^\infty(\mathbb{R}^d)} \to -\infty$$

as $t \to \infty$. This is not surprising since the only integrable steady state to (2.14) is $u_\infty = 0$, and this function does not have unit mass.

However, the entropy is useful to study the intermediate asymptotics, namely the relaxation of $u(t)$ to the self-similar solution

$$U(x,t) = \frac{1}{(2\pi(2t+1))^{d/2}} \exp\left(-\frac{|x|^2}{2(2t+1)}\right), \quad x \in \mathbb{R}^d,\ t > 0. \tag{2.15}$$

The main result (with sharp decay rate $O(t^{-1/2})$) is as follows.

Theorem 2.3 (Relaxation to self-similarity) *Let $u_0 \in L^1(\mathbb{R}^d)$ be nonnegative with unit mass, $\int_{\mathbb{R}^d} |x|^2 u_0 \mathrm{d}x < \infty$, and $\int_{\mathbb{R}^d} u_0 \log u_0 \mathrm{d}x < \infty$. Let $u(t)$ be the solution to (2.14), $U(t)$ be defined by (2.15), and* H *be the Boltzmann entropy. Then*

$$\|u(t) - U(t)\|_{L^1(\mathbb{R}^d)} \le \frac{\sqrt{2\mathrm{H}[u_0]}}{\sqrt{2t+1}}, \quad t > 0.$$

Proof We suppose that the solution $u(t)$ is smooth which is the case when the initial data is smooth. For general initial datum, one may apply an approximation procedure. The idea of the proof of the theorem is a time-dependent rescaling [13, 18]. The advantage of such a scaling is that it preserves the initial datum and the rescaled equation usually has a steady state v_∞. Then the analysis of the intermediate asymptotics is equivalent to the study of large-time convergence to v_∞.

Set $y = x/\sqrt{2t+1}$, $s = \log\sqrt{2t+1}$, and $v(y,s) = \mathrm{e}^{ds} u(\mathrm{e}^s y, \frac{1}{2}(\mathrm{e}^{2s} - 1))$. An elementary computation shows that v solves

$$\partial_s v = \mathrm{div}(\nabla v + yv), \quad s > 0, \quad v(0) = u_0 \quad \text{in } \mathbb{R}^d. \tag{2.16}$$

This is a linear Fokker–Planck equation with quadratic potential $V(y) = \frac{1}{2}|y|^2$. The self-similar solution in the rescaled variables can be written as

$$(2t+1)^{d/2} U(x,t) = (2\pi)^{d/2} \mathrm{e}^{-|y|^2/2} =: v_\infty(y).$$

The Gaussian v_∞ is the unique steady state to (2.16). Theorem 2.1 shows that

$$\|v(s) - v_\infty\|_{L^1(\mathbb{R}^d)} \le \sqrt{2\mathrm{H}[u_0]}\mathrm{e}^{-s}, \quad s > 0. \tag{2.17}$$

It remains to transform back to the original variables. The substitutions $y = (2t+1)^{-1/2}x$ and $s = \log\sqrt{2t+1}$ lead to

$$\begin{aligned} \|v(s) - v_\infty(s)\|_{L^1(\mathbb{R}^d)} &= \|u(t) - U(t)\|_{L^1(\mathbb{R}^d)}, \\ \sqrt{2\mathrm{H}[u_0]}\mathrm{e}^{-s} &= \sqrt{2\mathrm{H}[u_0]}(2t+1)^{-1/2}. \end{aligned}$$

Inserting these expressions into (2.17) finishes the proof. $\square$

The method of time-dependent rescalings is a very powerful technique to analyze the intermediate asymptotics of diffusive equations, also for nonlinear equations; see, e.g., [3, 13, 18, 19, 29].

2.5 Nonlinear Fokker–Planck Equations

One of the strengths of the Bakry–Emery approach is its robustness against model variations. Otto [36], Carrillo and Toscani [13], and Del Pino and Dolbeault [16] found independently extensions of this method to porous-medium and fast-diffusion equations. In this section, we consider as in [14] nonlinear Fokker–Planck equations

$$\partial_t u = \operatorname{div}(\nabla f(u) + u\nabla V), \quad t > 0, \quad u(0) = u_0 \quad \text{in } \Omega, \tag{2.18}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded *convex* domain, and we impose the boundary conditions

$$(\nabla f(u) + u\nabla V) \cdot \mathrm{n} = 0 \quad \text{on } \partial\Omega,\ t > 0, \tag{2.19}$$

where n is the exterior unit normal vector on $\partial\Omega$. We may also choose $\Omega = \mathbb{R}^d$, assuming some decay properties of u at $|x| \to \infty$; see [13]. Nonlinear Fokker–Planck equations occur, for instance, in porous-medium flow [44], semiconductor modeling (where $f(u) = u^{5/3}$; see [27]), or population dynamics [11]. We also refer to the book [23]. The solution $u(x, t)$ can be interpreted in such applications as the particle density. The existence and uniqueness of weak solutions to (2.18) and (2.19) was proved, under appropriate assumptions, in [28] for bounded domains and in [14, Sect. 3.5] for the whole space.

The following presentation is close to that of Matthes [34, Chap. 3] and of Carrillo and Toscani [13]. It is possible to treat general nonlinearities f and general potentials V but the determination of the steady state is delicate [14, Sect. 3.1]. In order to avoid these technical issues, we consider the special functions

$$V(x) = \frac{\lambda}{2}|x|^2, \quad x \in \Omega, \quad \text{and} \quad f(u) = u^m, \quad u \geq 0,\ m \geq 1 - \frac{1}{d}. \tag{2.20}$$

Thus, we allow for slow diffusion ($m > 1$) and fast diffusion ($m < 1$) if the diffusion is not too fast. In this situation, the unique steady state to (2.18) and (2.19) is given by the compactly supported Barenblatt profile,

$$u_\infty(x) = \left(N - \frac{m-1}{2m}\lambda|x|^2\right)_+^{1/(m-1)},$$

where $z_+ = \max\{0, z\}$ denotes the positive part of $z \in \mathbb{R}$ and the constant N can be determined from mass conservation, i.e. $\int_\Omega u_\infty \mathrm{d}x = \int_{\mathbb{R}^d} u_0 \mathrm{d}x$. Since u_∞ vanishes outside of a bounded set, we cannot define the entropy exactly as in Sect. 2.1 using the quotient u/u_∞. Instead we consider the difference

$$\mathrm{H}^*[u] = \mathrm{H}[u] - \mathrm{H}[u_\infty],$$

where $\mathrm{H}[u]$ denotes the absolute entropy

$$\mathrm{H}[u] = \int_\Omega u\left(\frac{u^{m-1}}{m-1} + \frac{\lambda}{2}|x|^2\right)\mathrm{d}x.$$

This functional can be interpreted as the free energy of the system, consisting of the internal energy $u^m/(m-1)$ and the electric energy $uV = \lambda u|x|^2/2$. Newman [35] showed that the free energy is a Lyapunov functional for the porous-medium equation, and Ralston [37] used it to prove the L^1-convergence of the solution towards the self-similar profile by employing compactness arguments.

We prove now the analogue of Theorem 2.1 in Sect. 2.1.

Theorem 2.4 (Exponential decay for the nonlinear Fokker–Planck equation) *Let (2.20) hold, let $u_0 \in L^1(\Omega)$ be nonnegative with $\mathrm{H}[u_0] < \infty$, and let $u(t)$ be a solution to (2.18) and (2.19). Then*

$$\|u(t) - u_\infty\|_{L^1(\Omega)} \le C\mathrm{e}^{-\lambda t}, \quad t \ge 0,$$

where the constant $C > 0$ (also) depends on $\mathrm{H}[u_0]$.

We give only a sketch of the proof since it is highly technical; see [13, Sect. 1.3] and [14, Sect. 3.3] for a full proof. The first difficulty is that the porous-medium equation generally admits Hölder continuous solutions only and it is not as regularizing as the heat equation. Therefore, we need to regularize the Hölder continuous solution by smooth positive functions and then to pass to the deregularization limit. The second difficulty is to justify that the boundary terms vanish in the integrations by parts, as the integrals involve the quadratic potential.

Proof We split the proof in four steps. We assume that the solution to (2.18)–(2.19) is positive and smooth. This can be achieved by an approximation procedure [14].

Step 1: first time derivative of the entropy. We introduce the so-called entropy variable $\mu = mu^{m-1}/(m-1) + \lambda|x|^2/2$. Then u solves $\partial_t u = \mathrm{div}(u\nabla\mu)$ and, integrating by parts,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}^*[u] = \int_\Omega \mu\partial_t u\mathrm{d}x = -\int_\Omega u|\nabla\mu|^2\mathrm{d}x \le 0.$$

The boundary integral vanishes since $u\nabla\mu\cdot\mathrm{n} = 0$ on $\partial\Omega$, by (2.19).

Step 2: second time derivative of the entropy. We compute, integrating by parts,

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathrm{H}^*[u] &= -\int_\Omega \partial_t u|\nabla\mu|^2\mathrm{d}x - 2\int_\Omega u\nabla\partial_t\mu\cdot\nabla\mu\mathrm{d}x \\
&= 2\int_\Omega u\nabla\mu^\top\nabla^2\mu\nabla\mu\mathrm{d}x - 2\int_\Omega u\nabla\big(mu^{m-2}\operatorname{div}(u\nabla\mu)\big)\cdot\nabla\mu\mathrm{d}x \\
&= 2\int_\Omega u\nabla\mu^\top\left(\frac{m}{m-1}\nabla^2u^{m-1}+\lambda\mathbb{I}\right)\nabla\mu\mathrm{d}x + 2\int_\Omega mu^{m-2}\big(\operatorname{div}(u\nabla\mu)\big)^2\mathrm{d}x \\
&= 2\lambda\int_\Omega u|\nabla\mu|^2\mathrm{d}x + \frac{2m}{m-1}\int_\Omega u\nabla\mu^\top\nabla^2u^{m-1}\nabla\mu\mathrm{d}x \\
&\quad + 2m\int_\Omega u^{m-2}(\nabla u\cdot\nabla\mu + u\Delta\mu)^2\mathrm{d}x =: -2\lambda\frac{\mathrm{d}\mathrm{H}^*}{\mathrm{d}t}[u] + I_1 + I_2.
\end{aligned}$$

The boundary integrals vanish since $u\nabla\mu\cdot\mathrm{n} = 0$ on $\partial\Omega$. It remains to show that $I_1 + I_2$ is nonnegative since this implies that

$$\frac{\mathrm{d}^2\mathrm{H}^*}{\mathrm{d}t^2} + 2\lambda\frac{\mathrm{d}\mathrm{H}^*}{\mathrm{d}t} \geq 0, \quad t > 0. \tag{2.21}$$

In order to verify that $I_1 + I_2 \geq 0$, we integrate by parts in I_1 (again, the boundary integrals vanish):

$$\begin{aligned}
I_1 &= -\frac{2m}{m-1}\int_\Omega \operatorname{div}(u\nabla\mu)\nabla u^{m-1}\cdot\nabla\mu\mathrm{d}x - \frac{2m}{m-1}\int_\Omega u\nabla\mu^\top\nabla^2\mu\nabla u^{m-1}\mathrm{d}x \\
&= -\frac{2m}{m-1}\int_\Omega (\nabla u\cdot\nabla\mu)(\nabla u^{m-1}\cdot\nabla\mu)\mathrm{d}x - \frac{2m}{m-1}\int_\Omega u\Delta\mu(\nabla u^{m-1}\cdot\nabla\mu)\mathrm{d}x \\
&\quad - \frac{2m}{m-1}\int_\Omega u\nabla\mu^\top\nabla^2\mu\nabla u^{m-1}\mathrm{d}x.
\end{aligned} \tag{2.22}$$

Next, we expand the square $(\nabla u\cdot\nabla\mu + u\Delta\mu)^2$ in I_2:

$$I_2 = 2m\int_\Omega u^{m-2}(\nabla u\cdot\nabla\mu)^2\mathrm{d}x + 4m\int_\Omega u^{m-1}(\nabla u\cdot\nabla\mu)\Delta\mu\mathrm{d}x + 2m\int_\Omega u^m(\Delta\mu)^2\mathrm{d}x. \tag{2.23}$$

Observing that the sum of the first integrals in (2.22) and (2.23), respectively, cancel and combining the second integrals in both identities, we arrive at

$$I_1 + I_2 = 2\int_\Omega \nabla u^m\cdot\nabla\mu\Delta\mu\mathrm{d}x - 2\int_\Omega \nabla\mu^\top\nabla^2\mu\nabla u^m\mathrm{d}x + 2m\int_\Omega u^m(\Delta\mu)^2\mathrm{d}x.$$

Now, we integrate by parts in the first two integrals in order to remove the gradient from u^m. Here, we need to be careful with the boundary integrals. Indeed, the second integral yields the boundary term

$$-2\int_{\partial\Omega} u^m \nabla\mu^\top \nabla^2\mu \mathrm{n} d\sigma = -\int_{\partial\Omega} u^m \nabla(|\nabla\mu|^2)\cdot \mathrm{n} d\sigma.$$

Since Ω is convex and $\nabla\mu\cdot\mathrm{n} = 0$ on $\partial\Omega$, Lemma A.3 shows that $\nabla(|\nabla\mu|^2)\cdot\mathrm{n} \le 0$ on $\partial\Omega$. Hence, the boundary integral is nonnegative. The third-order derivatives, which appear after the integrations by parts, cancel, and we end up with

$$I_1 + I_2 \ge 2(m-1)\int_\Omega u^m (\Delta\mu)^2 dx + 2\int_\Omega u^m |\nabla^2\mu|^2 dx \ge 0,$$

which proves our claim (2.21).

Step 3: Exponential decay of the entropy. Integrating the differential inequality (2.21) in (t,∞), we arrive at

$$\frac{d\mathrm{H}^*}{dt} + 2\lambda\mathrm{H}^* \le 0, \quad t > 0. \tag{2.24}$$

For this result, we need to verify that $\lim_{t\to\infty}(d\mathrm{H}^*/dt)[u(t)] = 0$ and $\lim_{t\to\infty} \mathrm{H}^*[u(t)] = 0$. The proof of the first limit is a consequence of the differential inequality (2.21) which can be written as $d\mathrm{P}/dt \le -2\lambda\mathrm{P}$, where $\mathrm{P} = -d\mathrm{H}^*/dt$.

The proof of the second limit is more delicate. We proceed as in Step 1 of the proof of Theorem 11 in [14]; an alternative proof is given in [13, Theorem 3.1]. Step 1 shows that $t \mapsto \mathrm{H}^*[u(t)]$ is nonincreasing. Moreover, $\mathrm{H}^*[u(t)] \ge 0$, so $\mathrm{H}^*[u(t)]$ is bounded below. By the monotone convergence theorem, $\lim_{t\to\infty}\mathrm{H}^*[u(t)] =: \eta \ge 0$ exists. The goal is to prove that $\eta = 0$. To this end, we observe that the integral

$$0 \le \int_0^t \mathrm{P}[u(s)]ds = \mathrm{H}^*[u(0)] - \mathrm{H}^*[u(t)] \le \mathrm{H}^*[u_0]$$

is uniformly bounded in t and $\int_0^\infty \mathrm{P}[u(s)]ds < \infty$. Consequently, there exists a sequence $t_k \to \infty$ such that $\mathrm{P}[u(t_k)] \to 0$. It is possible to show that $t \mapsto u(\cdot + t_k)$ is equicontinuous and uniformly bounded in $L^\infty(0,T;L^\infty(\Omega))$ for any fixed $T > 0$ (use the maximum principle). By the theorem of Arzelà-Ascoli, there exists a subsequence of (t_k), which is not relabeled, such that $u(\cdot + t_k) \to u^*$ uniformly in $\overline{\Omega}\times[0,T]$ as $k\to\infty$. In particular, $u(t_k) \to u^*$ uniformly in $\overline{\Omega}\times[0,T]$. The uniform bound on the entropy production allows us to verify that $(\nabla f(u(t_k)))$ is bounded in $L^2(\Omega)$ and, in fact, $(f(u(t_k))$ is bounded in $H^1(\Omega)$. Hence, there exists a subsequence (not relabeled) such that $f(u(t_k))) \rightharpoonup f^*$ weakly in $H^1(\Omega)$ as $k\to\infty$. Since $(u(t_k))$ is bounded in $L^\infty(\Omega)$, we also have $f(u(t_k)) \to f(u^*)$ strongly in $L^2(\Omega)$. Thus, we can identify the limit, $f^* = f(u^*)$. The convergences $u(t_k) \to u^*$ uniformly in $\overline{\Omega}$ and $f(u(t_k)) \rightharpoonup f(u^*)$ weakly in $H^1(\Omega)$ allow us to deduce that $u^* = u_\infty$ (this step requires some effort). Then the convergence $u(t_k) \to u_\infty$ uniformly in $\overline{\Omega}$ is sufficient to conclude that $\lim_{k\to\infty}\mathrm{H}^*[u(t_k)] = \mathrm{H}^*[u_\infty] = 0$. But $t \mapsto \mathrm{H}^*[u(t)]$ is nonincreasing, so we can perform the limit for any sequence $t\to\infty$, i.e. $\lim_{t\to\infty}\mathrm{H}^*[u(t)] = 0$, which shows the claim.

Step 4: Exponential decay in L^1. As in the linear case, we need the Csiszár–Kullback–Pinsker inequality but because of the lack of positivity of u_∞, there is a technical difficulty. A proof is given in [34, Sect. 1.7] but we only present the main arguments. If the support of u is contained in some compact set, one shows by a Taylor expansion that $\|u-u_\infty\|_{L^1(\Omega)} \le C_1 \mathrm{H}^*[u]^{1/2}$, where $C_1 > 0$ depends on u_∞. The entropy also controls the mass of u outside the support of u_∞, i.e. $\mathrm{m}_R(u) := \int_{\{|x|>R\}} u\mathrm{d}x \le C_2\mathrm{H}^*[u]^{1/2}$, where $C_2 > 0$ depends on f, d, and λ. Next, we combine these two statements. Introduce $\widehat{u} := \alpha u \mathbf{1}_{\{|x|<R\}}$, where $\alpha = 1 - m_R(u)/M$, $M = \int_\Omega u\mathrm{d}x$, and $R = 2mN/(2(m-1)\lambda)$. This choice ensures that $\widehat{u}$ has the same mass and support as u_∞. By the triangle inequality and the definition of α,

$$\|u-u_\infty\|_{L^1(\Omega)} \le \|u-\widehat{u}\|_{L^1(\Omega)} + \|\widehat{u}-u_\infty\|_{L^1(\Omega)} \le 2\mathrm{m}_R(u) + \|\widehat{u}-u_\infty\|_{L^1(\Omega)}.$$

The first term is controlled by $\mathrm{H}^*[u]^{1/2}$, the second one by $\mathrm{H}^*[\widehat{u}]^{1/2}$. Finally, we verify that $\mathrm{H}^*[\widehat{u}] \le C_3\mathrm{H}^*[u]$ for some $C_3 > 0$ which depends on α and f. It follows that $\|u-u_\infty\|_{L^1(\Omega)} \le C_4\mathrm{H}^*[u]^{1/2}$, where $C_4 = 2C_2 + C_1C_3^{1/2}$. □

In Sect. 2.1, we have shown that the linear Fokker–Planck equation is related to a convex Sobolev inequality (see Corollary 2.1), including the Poincaré and logarithmic Sobolev inequalities. One may ask whether the nonlinear Fokker–Planck equation is related to a functional inequality too. This question was answered by Del Pino and Dolbeault [16] and leads to a Gagliardo–Nirenberg inequality.

Proposition 2.1 (Gagliardo–Nirenberg inequality) *Let $d \ge 2$, $p > 1$, and, if $d \ge 3$, $p \le d/(d-2)$. Then for all $v \in H^1(\mathbb{R}^d) \cap L^{p+1}(\mathbb{R}^d)$,*

$$\|v\|_{L^{2p}(\mathbb{R}^d)} \le C\|\nabla v\|^\theta_{L^2(\mathbb{R}^d)}\|v\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}, \quad \theta = \frac{d(p-1)}{p(d+2-(d-2)p)}, \tag{2.25}$$

where (Γ denotes the Gamma function and $q = (p+1)/(p-1)$)

$$C = \left(\frac{q(1-p)^2}{2\pi d}\right)^{\theta/2}\left(\frac{2q-d}{2q}\right)^{1/(2p)}\left(\frac{\Gamma(q)}{\Gamma(q-d/2)}\right)^{\theta/d}.$$

The constant C is optimal; equality is reached by the compactly supported function $(N+|x-x^*|^2)_+^{1/(1-p)}$ for any $N > 0$ and $x^* \in \mathbb{R}^d$.

Remark 2.9 Inequality (2.25) contains the optimal Sobolev inequality for $p = d/(d-2)$ (since $\theta = 1$) and the logarithmic Sobolev inequality with optimal constant as $p \to 1$. □

2.6 Extensions

The results of Sect. 2.1 can be extended in a number of ways, by generalizing the Fokker–Planck equation (2.1). In this section, we report some of these extensions, which are due to Arnold et al. [2, 5, 6]. We also refer to the recent review [1].

Fokker–Planck equation with variable diffusion. This generalization was analyzed in [6] and concerns the Fokker–Planck equation

$$\partial_t u = \operatorname{div}(D(x)(\nabla u + u\nabla V)), \quad t > 0, \quad u(0) = u_0 \quad \text{in } \mathbb{R}^d, \tag{2.26}$$

where $D(x)$ is assumed to be a symmetric, locally uniform positive definite $(d \times d)$-matrix on $\mathbb{R}^d$ with smooth coefficients. We suppose that u_0 is nonnegative and has unit mass and that the confinement potential $V(x)$ satisfies $\mathrm{e}^{-V(x)} \in L^1(\mathbb{R}^d)$. The unique steady state of (2.26) is given by $u_\infty(x) = Z\mathrm{e}^{-V(x)}$ and $Z > 0$ is such that $\int_{\mathbb{R}^d} u_\infty \mathrm{d}x = 1$. We define as in Sect. 2.1 the (relative) entropy

$$\mathrm{H}_\phi[u] = \int_{\mathbb{R}^d} \phi\left(\frac{u}{u_\infty}\right) u_\infty \mathrm{d}x, \tag{2.27}$$

where $\phi : [0, \infty) \to \mathbb{R}$ is a smooth function. The following result is proved in [6, Theorem 2.16].

Theorem 2.5 (Exponential decay for (2.26)) *We assume that $\phi \in C^4([0, \infty))$ is convex, $\phi(1) = 0$, and $1/\phi''$ is concave. Let $\mathrm{H}_\phi[u_0] < \infty$, and let either $D(x) = D$ be a constant matrix such that $\nabla^2 V \geq \lambda D^{-1}$ for some $\lambda > 0$, or $D(x) = a(x)\mathbb{I}$ with $a : \Omega \to \mathbb{R}$ such that*

$$\begin{aligned} &\left(\frac{1}{2} - \frac{d}{4}\right)\frac{1}{a}\nabla a \otimes \nabla a + \frac{1}{2}(\Delta a - \nabla a \cdot \nabla V)\mathbb{I} \\ &\quad + a\nabla^2 V + \frac{1}{2}(\nabla a \otimes \nabla V + \nabla V \otimes \nabla a) - \nabla^2 a \geq \lambda \mathbb{I} \quad \text{in } \Omega. \end{aligned} \tag{2.28}$$

Then, for a smooth solution $u(t)$ to (2.26),

$$\mathrm{H}_\phi[u(t)] \leq \mathrm{H}_\phi[u_0]\mathrm{e}^{-2\lambda t}, \quad t > 0,$$

and the following convex Sobolev inequality holds:

$$\mathrm{H}_\phi[u] \leq \frac{1}{2\lambda} \int_{\mathbb{R}^d} \phi\left(\frac{u}{u_\infty}\right) \nabla\left(\frac{u}{u_\infty}\right)^\top D(x) \nabla\left(\frac{u}{u_\infty}\right) \mathrm{d}x. \tag{2.29}$$

Proof The proof follows the lines of the proof of Theorem 2.1. We consider only the case $D(x) = a(x)\mathbb{I}$. Let $\rho = u/u_\infty$. Writing (2.26) in the symmetric form $\partial_t u = \operatorname{div}(D(x)u_\infty \nabla\rho)$, the first time derivative of the entropy becomes

$$\frac{\mathrm{dH}_\phi}{\mathrm{d}t} = -\int_{\mathbb{R}^d} \phi''(\rho)a(x)|\nabla\rho|^2 u_\infty \mathrm{d}x. \tag{2.30}$$

Employing condition (2.28), the second time derivative can be estimated as (see the proof of Lemma 2.13 in [6])

$$\frac{\mathrm{d}^2\mathrm{H}_\phi}{\mathrm{d}t^2} \geq \int_{\mathbb{R}^d} \mathrm{tr}(XY)u_\infty \mathrm{d}x + 2\lambda \int_{\mathbb{R}^d} \phi''(\rho)a(x)|\nabla\rho|^2 u_\infty \mathrm{d}x,$$

where “tr” denotes the trace of a matrix,

$$X = \begin{pmatrix} 2\phi''(\rho) & 2\phi'''(\rho) \\ 2\phi'''(\rho) & \phi''''(\rho) \end{pmatrix}, \tag{2.31}$$

and $Y = (Y_{ij}) \in \mathbb{R}^{2\times 2}$ is a symmetric matrix with elements

$$\begin{aligned} Y_{11} &= \left| a\nabla^2\rho + \frac{1}{2}(\nabla a \otimes \nabla\rho + \nabla\rho \otimes \nabla a) - \frac{1}{2}\nabla a \cdot \nabla\rho \mathbb{I} \right|^2, \\ Y_{12} &= Y_{21} = a^2 \nabla\rho^\top \nabla^2\rho\nabla\rho + \frac{1}{2}a|\nabla\rho|^2 \nabla\rho \cdot \nabla a, \\ Y_{22} &= a^2|\nabla\rho|^4. \end{aligned} \tag{2.32}$$

Since $1/\phi''$ is assumed to be concave, X is positive semidefinite. A computation shows that this is also true for Y. We infer that $\mathrm{tr}(XY) \geq 0$ and $\mathrm{d}^2\mathrm{H}_\phi/\mathrm{d}t^2 + 2\lambda \mathrm{dH}_\phi/\mathrm{d}t \geq 0$. Integrating this inequality and using $\lim_{t\to\infty}(\mathrm{dH}_\phi/\mathrm{d}t)[u(t)] = 0$ and $\lim_{t\to\infty} \mathrm{H}_\phi[u(t)] = 0$ (which needs to be proved), we obtain the convex Sobolev inequality $\mathrm{dH}_\phi/\mathrm{d}t + 2\lambda\mathrm{H}_\phi \leq 0$, from which we conclude the proof. $\square$

Non-symmetric Fokker–Planck equation. The Fokker–Planck equation (2.26) contains a conservative drift term involving V. We investigate now the situation in which the force is not a potential. More precisely, consider as in [2]

$$\partial_t u = \mathrm{div}\left(D(x)(\nabla u + u(\nabla V + F))\right), \quad t > 0, \quad u(0) = u_0 \quad \text{in } \mathbb{R}^d. \tag{2.33}$$

The assumptions on u_0, $D(x)$, and $V(x)$ are as above. In particular, $D(x)$ is a symmetric $\mathbb{R}^{d\times d}$ matrix. We suppose that the function $F(x,t)$ is smooth and satisfies

$$\mathrm{div}(DFu_\infty) = 0 \quad \text{for } x \in \Omega,\ t > 0. \tag{2.34}$$

Under this condition, u_∞ is still a steady state to (2.33). The idea is to decompose the Fokker–Planck operator into two parts: a symmetric and an anti-symmetric one,

$$L_{\mathrm{sym}}(u) = \mathrm{div}\left(u_\infty D\nabla\frac{u}{u_\infty}\right), \quad L_{\mathrm{as}}(u) = \mathrm{div}(DFu).$$

Clearly, $L_{\text{sym}}u_\infty = L_{\text{as}}u_\infty = 0$, using (2.34).

Lemma 2.1 *The operator L_{sym} is symmetric and L_{as} is anti-symmetric with respect to $L^2(u_\infty^{-1}\mathrm{d}x)$.*

Proof The symmetry of L_{sym} follows from the symmetry of $D(x)$ since

$$\int_{\mathbb{R}^d} L_{\text{sym}}(u)vu_\infty^{-1}\mathrm{d}x = -\int_{\mathbb{R}^d} u_\infty\nabla\left(\frac{u}{u_\infty}\right)^\top D\nabla\frac{v}{u_\infty}\mathrm{d}x = \int_{\mathbb{R}^d} uL_{\text{sym}}(v)u_\infty^{-1}\mathrm{d}x$$

for suitable functions u, v. To prove that L_{as} is anti-symmetric, we first integrate by parts in

$$\int_{\mathbb{R}^d} L_{\text{as}}(u)vu_\infty^{-1}\mathrm{d}x = -\int_{\mathbb{R}^d} uDF\cdot\left(u_\infty^{-1}\nabla v + v\nabla u_\infty^{-1}\right)\mathrm{d}x. \tag{2.35}$$

By (2.34), we have

$$0 = \operatorname{div}(DFu_\infty)uvu_\infty^{-2} = (\operatorname{div} D)\cdot Fuvu_\infty^{-1} - uv(DF)\cdot\nabla u_\infty^{-1} + D:\nabla Fuvu_\infty^{-1},$$

where $D:\nabla F = \sum_{i,j} D_{ij}\partial_{x_i}F_j$. Using this expression in (2.35), we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} L_{\text{as}}(u)vu_\infty^{-1}\mathrm{d}x &= -\int_{\mathbb{R}^d}\left(uDF\cdot\nabla v + (\operatorname{div} D)\cdot Fuv + D:\nabla Fuv\right)u_\infty^{-1}\mathrm{d}x\\ &= -\int_{\mathbb{R}^d} u\operatorname{div}(DFv)u_\infty^{-1}\mathrm{d}x = -\int_{\mathbb{R}^d} uL_{\text{as}}(v)u_\infty^{-1}\mathrm{d}x,\end{aligned}$$

which shows the lemma. □

The anti-symmetry of L_{as} indicates that the equilibration property is driven by the symmetric Fokker–Planck operator L_{sym} only. This is confirmed by the following theorem which is proved in [2]. For related results, we refer to [12].

Theorem 2.6 (Exponential decay for (2.33)) *Let $D(x) = a(x)\mathbb{I}$ and let (2.28), with ∇V replaced by $\nabla V - F$, and (2.34) hold. Let u be a (smooth) solution to (2.33). Then, with the entropy $\mathrm{H}_\phi[u]$ defined in (2.27),*

$$\mathrm{H}_\phi[u(t)] \le \mathrm{H}_\phi[u_0]\mathrm{e}^{-2\lambda t}, \quad t > 0.$$

Proof Set $\rho = u/u_\infty$. We compute the first time derivative of the entropy:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_\phi[u] &= \int_{\mathbb{R}^d}\phi'(\rho)\operatorname{div}(u_\infty D\nabla\rho + DFu)\mathrm{d}x\\ &= -\int_{\mathbb{R}^d}\phi''(\rho)\nabla\rho^\top D\nabla\rho u_\infty\mathrm{d}x + \int_{\mathbb{R}^d}\phi'(\rho)\operatorname{div}(DFu)\mathrm{d}x.\end{aligned}$$

The first integral on the right-hand side corresponds to the entropy production of the symmetric case. We claim that the second integral vanishes. Indeed, (2.34) implies that $\operatorname{div}(DF)u = \operatorname{div}(DFu_\infty)\rho - (DF)\cdot\nabla u_\infty\rho = -(DF)\cdot\nabla u_\infty\rho$, which yields

$$\begin{aligned}\int_{\mathbb{R}^d}\phi'(\rho)\operatorname{div}(DFu)\mathrm{d}x &= \int_{\mathbb{R}^d}\phi'(\rho)\big(-(DF)\cdot\nabla u_\infty\rho + (DF)\cdot\nabla u\big)\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\phi'(\rho)(DF)\cdot\nabla\rho u_\infty\mathrm{d}x = \int_{\mathbb{R}^d}\nabla\phi(\rho)\cdot(DF)u_\infty\mathrm{d}x \\ &= -\int_{\mathbb{R}^d}\phi(\rho)\operatorname{div}(DFu_\infty)\mathrm{d}x = 0.\end{aligned}$$

This shows that the entropy production is the same as for the symmetric Fokker–Planck equation. The second derivative $\mathrm{d}^2\mathrm{H}_\phi/\mathrm{d}t^2$, however, involves the nonpotential term. The computation is similar to, but more involved than the proof of Theorem 2.5; see [2, Lemma 2.3]. □

Remark 2.10 If D is a constant invertible matrix, the condition $\nabla^2 V \geq \lambda D^{-1}$ can be replaced by [2] $\nabla^2 V - \frac{1}{2}(\nabla F + \nabla F^\top) \geq \lambda D^{-1}$ in $\mathbb{R}^d$. A similar condition was derived by Bolley and Gentil [12]. They consider the Fokker–Planck equation without potential, $\partial_t u = \operatorname{div}(D(\nabla u + uF))$. Then the conclusion of Theorem 2.6 holds if $\frac{1}{2}(\nabla F + \nabla F^\top) \geq \lambda D^{-1}$ in $\mathbb{R}^d$. Thus, a division into gradient and divergence-free parts is not needed, but the steady state is no longer explicit. □

Remark 2.11 Theorem 2.6 also holds for matrix-valued coefficients $D(x)$ but the analogue of condition (2.28) is more involved; see [2, (2.13)]. □

Degenerate Fokker–Planck equation. The final result concerns Fokker–Planck equations whose diffusion matrix is only positive semi-definite, as analyzed by Arnold and Erb [5],

$$\partial_t u = \operatorname{div}(D\nabla u + Cxu), \quad t > 0, \quad u(0) = u_0 \quad \text{in } \mathbb{R}^d. \tag{2.36}$$

We assume that $D \in \mathbb{R}^{d\times d}$ is constant, symmetric, and positive semi-definite and that $C \in \mathbb{R}^{d\times d}$. The difficulty of the analysis is that the entropy production can vanish for functions other than the equilibrium state u_∞. Thus, the second time derivative of the entropy may change its sign along a trajectory. The idea of Arnold and Erb [5] is to employ a modified relative entropy; see (2.39) below.

We need two main assumptions: Let no nontrivial subspace of $\ker D$ be invariant under $C^\top$ and let C be positively stable. This means that

$$\text{for all eigenvectors } v \text{ of } C^\top,\ v \notin \ker D, \tag{2.37}$$

$$\text{for all eigenvalues } \lambda \text{ of } C^\top,\ \Re(\lambda) > 0, \tag{2.38}$$

where $\Re(\lambda)$ denotes the real part of λ. The first condition is equivalent to the hypoellipticity of $u \mapsto \partial_t u - \operatorname{div}(D\nabla u + Cxu)$ [26, §1]. This means that for positive L^1 initial data, the solution to (2.36) is smooth (even C^∞) and positive. Hypothesis (2.38) means that there is a confinement potential, and this allows us to obtain a steady state that is given by [5, Sect. 3]

$$u_\infty(x) = Z\exp\Big(-\frac{1}{2}x^\top K^{-1}x\Big), \quad x \in \mathbb{R}^d,$$

where $K \in \mathbb{R}^{d\times d}$ is the unique symmetric and positive definite solution to the continuous Lyapunov equation $2D = CK + KC^\top$ and $Z > 0$ is a normalization constant.

We can decompose the Fokker–Planck operator as above in the symmetric part $L_{\mathrm{sym}}u = \mathrm{div}(u_\infty D\nabla\rho)$ and the anti-symmetric part $L_{\mathrm{as}}u = \mathrm{div}(u_\infty R\nabla\rho)$, where $R = \frac{1}{2}(CK - KC^\top)$ is anti-symmetric, $\rho = u/u_\infty$, and the statements hold in the space $L^2(u_\infty^{-1}\mathrm{d}x)$. The symmetry of L_{sym} can be shown as in Lemma 2.1. The anti-symmetry of L_{as} is a consequence of the anti-symmetry of R [5, Theorem 3.5].

The idea of [5] is to introduce the modified entropy production

$$\mathrm{P}^*[u] = \int_{\mathbb{R}^d}\phi''(\rho)\nabla\rho^\top D_0\nabla\rho u_\infty \mathrm{d}x, \tag{2.39}$$

i.e., we replace the matrix D in (2.30) by a symmetric, positive definite matrix D_0. The goal is to choose D_0 such that an estimate between P^* and $\mathrm{dP}^*/\mathrm{d}t$ can be shown. Since D_0 is positive definite, there exists $\kappa > 0$ such that $D_0 \geq \kappa D$, which implies that $\mathrm{P}^* \geq \kappa\mathrm{P}$, where $\mathrm{P}[u] = \int_{\mathbb{R}^d}\phi''(\rho)\nabla\rho^\top D\nabla\rho u_\infty\mathrm{d}x$.

Theorem 2.7 (Exponential decay for (2.26)) *Let ϕ satisfy the assumptions stated in Theorem 2.5, let $\mathrm{H}_\phi[u_0] < \infty$, and let conditions (2.37) and (2.38) hold. Finally, let $u(t)$ be the smooth solution to (2.36) and let $\mu = \min\{\Re(\lambda) : \lambda$ eigenvector of $C\}$. If all eigenvalues λ of C with real part $\Re(\lambda) = \mu$ are non-defective (i.e. their geometric and algebraic multiplicities coincide), then there exists $c > 1$ such that*

$$\mathrm{H}_\phi[u(t)] \leq c\mathrm{H}_\phi[u_0]\mathrm{e}^{-2\mu t}, \quad t > 0.$$

If at least one eigenvalue λ of C with $\Re(\lambda) = \mu$ is defective, then for all $0 < \varepsilon < \mu$, there exists $c_\varepsilon > 1$ such that

$$\mathrm{H}_\phi[u(t)] \leq c_\varepsilon\mathrm{H}_\phi[u_0]\mathrm{e}^{-2(\mu-\varepsilon)t}, \quad t > 0,$$

Proof We consider the non-defective case and give a sketch of the proof only; see [5] for the full proof. It is possible to show that there exists a symmetric, positive definite matrix $D_0 \in \mathbb{R}^{d\times d}$ such that

$$SD_0 + D_0S^\top \geq 2\mu D_0, \quad \text{where } S = KC^\top K^{-1}.$$

Using this property, one shows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{P}^*[u] = &-\int_{\mathbb{R}^d}\phi''(\rho)\nabla\rho^\top\big((D-R)K^{-1}D_0 + D_0K^{-1}(D+R)\big)\nabla\rho u_\infty\mathrm{d}x \\ &-2\int_{\mathbb{R}^d}\mathrm{tr}(XY)u_\infty\mathrm{d}x,\end{aligned}$$

where X is defined in (2.31) and Y is similar to the matrix (2.32). Both matrices are positive semidefinite implying that $\mathrm{tr}(XY) \geq 0$. Furthermore, we find that

$$(D - R)K^{-1}D_0 + D_0K^{-1}(D + R) = SD_0 + D_0S^\top \geq 2\mu D_0,$$

recalling that $D = \frac{1}{2}(CK + KC^\top)$ and $R = \frac{1}{2}(CK - KC^\top)$. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{P}^*[u] \leq -2\mu \mathrm{P}^*[u] \quad \text{or} \quad \mathrm{P}^*[u(t)] \leq \mathrm{P}^*[u(s)]\mathrm{e}^{-2\mu(t-s)} \quad \text{for } t \geq s \geq 0.$$

Then, using a convex Sobolev inequality, which relates $\mathrm{H}[u]$ and $\mathrm{P}^*[u]$,

$$\mathrm{H}_\phi[u(t)] \leq \frac{1}{2\lambda_P}\mathrm{P}^*[u(t)] \leq \frac{1}{2\lambda_P}\mathrm{P}^*[u(\delta)]\mathrm{e}^{-2\mu(t-\delta)}, \quad t > 0. \tag{2.40}$$

Setting $\delta = 0$, we conclude the exponential time decay, but we need that $\mathrm{P}^*[u(0)]$ is finite which is not optimal. We wish to replace this factor by $\mathrm{H}_\phi[u(0)]$. The idea, which goes back to Hérau [25], is to exploit a (nontrivial) regularization property of hypoelliptic operators [5, Theorem 4.8]:

$$\mathrm{P}^*[u(t)] \leq c_1 t^{-\kappa}\mathrm{H}_\phi[u_0], \quad t > 0,$$

where $c_1 > 0$ and $\kappa > 1$. We infer that

$$\mathrm{P}^*[u(\delta)]\mathrm{e}^{-2\mu(t-\delta)} \leq c_1\mathrm{e}^{2\mu\delta}\delta^{-\kappa}\,\mathrm{e}^{-2\mu t}\mathrm{H}_\phi[u_0] \quad \text{for } t \geq \delta$$

and $\mathrm{H}_\phi[u(t)] \leq \mathrm{H}_\phi[u_0]$ for $t \leq \delta$ (since $t \mapsto \mathrm{H}_\phi[u(t)]$ is monotone). Then, setting $c(\delta) = \mathrm{e}^{2\mu\delta}\max\{1, c_1/(2\delta^\kappa\lambda_P)\}$ and employing (2.40), we infer that $\mathrm{H}_\phi[u(t)] \leq c(\delta)\mathrm{H}_\phi[u_0]\mathrm{e}^{-2\mu t}$ for *all* $t \geq 0$, which concludes the proof. □

Remark 2.12 The property $SD_0 + D_0S^\top \geq 2\mu D_0$ is the key ingredient of the proof. It generalizes the Bakry–Emery condition from Sect. 2.1. Indeed, we obtain a symmetric Fokker–Planck equation by choosing $D = \mathbb{I}$ and $C = C^\top \geq \mu\mathbb{I}$. Then $K^{-1} = S = C$ and we may take $D_0 = \mathbb{I}$. Consequently, $\frac{1}{2}(SD_0 + D_0S^\top) = C = \nabla^2 V \geq \mu\mathbb{I}$, where $V(x) = \frac{1}{2}x^\top K^{-1}x$, and we recover the usual Bakry–Emery condition. □

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