Chapter 2
Explicit Solutions, Special Transformations, and Further Examples

Few mean-field games can be solved explicitly. However, examples for which closed solutions are known illustrate essential features of the theory. Moreover, explicit solutions to MFGs are a key ingredient in the continuation method discussed in Chap. 11.

2.1 Explicit Solutions

We begin our study of explicit solutions by considering a first-order quadratic MFG with a logarithmic nonlinearity. While logarithmic nonlinearities pose several technical challenges (see Chaps. 7 and 9), the model considered here can be solved by elementary methods. This game is given by

$$\begin{cases} \frac{|u_x|^2}{2} + V(x) + b(x)u_x = \ln m + \overline{H}, \\ -(m(Du + b(x)))_x = 0, \end{cases} \tag{2.1}$$

with $u, m : \mathbb{T} \to \mathbb{R}$, $m \geq 0$,

$$\int_{\mathbb{T}} m dx = 1,$$

and, for definiteness,

$$\int_{\mathbb{T}} u \, \mathrm{d}x = 0. \tag{2.2}$$

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Moreover, we suppose that

$$\int_{\mathbb{T}} b(y)\mathrm{d}y = 0.$$

If $Du + b = 0$, the second equation in (2.1) holds immediately. This suggests that we set

$$u(x) = -\int_0^x b(y)\,\mathrm{d}y + \int_{\mathbb{T}}\int_0^z b(y)\,\mathrm{d}y\,\mathrm{d}z.$$

Using the previous formula in the first equation, we get

$$m(x) = \frac{e^{V(x)-\frac{b^2(x)}{2}}}{\int_{\mathbb{T}} e^{V(y)-\frac{b^2(y)}{2}}\,\mathrm{d}y}.$$

In particular, let $\psi : \mathbb{T}^d \to \mathbb{R}$ be a periodic C^∞ function with $\int_{\mathbb{T}} \psi\,\mathrm{d}x = 0$. Suppose that $b(x) = \psi_x(x)$. Then,

$$u(x) = -\psi(x), \qquad m(x) = \frac{e^{V(x)-\frac{\psi_x^2(x)}{2}}}{\int_{\mathbb{T}} e^{V(y)-\frac{\psi_x^2(y)}{2}}\,\mathrm{d}y}, \quad \text{and} \quad \overline{H} = \ln\left[\int_{\mathbb{T}} e^{V(y)-\frac{\psi_x^2(y)}{2}}\,\mathrm{d}y\right]$$

solves (2.1).

A related problem is the congestion model:

$$\begin{cases} \frac{u_x^2}{2m^{1/2}} + V(x) = \ln m + \overline{H}, \\ -(m^{1/2}u_x)_x = 0. \end{cases} \tag{2.3}$$

It is easy to see that $u(x) = 0$, $m(x) = \frac{e^{V(x)}}{\int_{\mathbb{T}} e^{V(y)}\,\mathrm{d}y}$ and $\overline{H} = \ln \int_{\mathbb{T}} e^{V(x)}dx$ solve (2.3).

2.2 The Hopf–Cole Transform

The Hopf–Cole transform is a well-known technique to convert certain nonlinear equations into linear equations. Here, we illustrate an application to MFGs. For $P \in \mathbb{R}^d$, consider the system

$$\begin{cases} -\Delta u + \frac{1}{2}|P + Du|^2 + V(x) = \ln m \\ -\Delta m - \operatorname{div}((P + Du)m) = 0. \end{cases} \tag{2.4}$$

Define m by the Hopf–Cole transform

$$m = e^{\frac{v-u}{2}}, \tag{2.5}$$

where u and v solve

$$\begin{cases} -\Delta u + \frac{1}{2}|P + Du|^2 + V(x) & = \frac{v-u}{2} \\ \Delta v + \frac{1}{2}|P + Dv|^2 + V(x) & = \frac{v-u}{2}. \end{cases} \tag{2.6}$$

By a direct computation, the function, m, given by (2.5) solves

$$-\Delta m - \operatorname{div}((P + Du)m) = 0. \tag{2.7}$$

To check this, it is enough to observe that

$$\begin{aligned} -\Delta m &= m\left[\frac{1}{2}\Delta u - \frac{1}{2}\Delta v - \frac{|Du - Dv|^2}{4}\right] \\ &= m\Delta u - \frac{m}{4}\left[|P + Du|^2 - |P + Dv|^2 + |Du - Dv|^2\right] \\ &= (P + Du) \cdot Dm + m\Delta u = \operatorname{div}((P + Du)m). \end{aligned}$$

2.3 Gaussian-Quadratic Solutions

Gaussian-quadratic solutions to MFGs are relevant in several applications. In dimension $d \geq 1$, we consider the MFG in $\mathbb{R}^d$ given by

$$\begin{cases} -\Delta u + \frac{1}{2}|Du|^2 + \beta|x|^2 = \ln m + \overline{H} \\ -\Delta m - \operatorname{div}(mDu) = 0. \end{cases} \tag{2.8}$$

We set $m = \mu e^{-u}$ so that the second equation holds trivially. Next, we select

$$u = \alpha|x|^2.$$

Using the ansatz in the first equation of (2.8) gives that α solves

$$2\alpha^2 + \alpha + \beta = 0.$$

If $\beta < 0$, the preceding equation has a solution, $\alpha > 0$. Finally, we determine μ by the normalization condition, $\int_{\mathbb{R}} m dx = 1$. To find $\overline{H}$, we use the expressions for u and m in the first equation of (2.8).

2.4 Interface Formation

In this last example, we describe the formation of interfaces and the breakdown of regularity. For $\lambda \in \mathbb{R}$, we consider the MFG

$$\begin{cases} \frac{|u_x|^2}{2} + \lambda V(x) = m + \overline{H}(\lambda), \\ -(mu_x)_x = 0, \end{cases} \tag{2.9}$$

with periodic conditions; that is, $u, m : \mathbb{T} \to \mathbb{R}$, $m \geq 0$, and $\int_{\mathbb{T}} m \mathrm{d}x = 1$.

First, we attempt to solve (2.9). The second equation in (2.9) implies that $mu_x = c$, for some constant c. If $c \neq 0$, then $u_x = \frac{c}{m}$. This is not possible because $\int_{\mathbb{T}} u_x \mathrm{d}x = 0$ and $m > 0$. Therefore, $mu_x = 0$. Accordingly, u is constant in the set $m > 0$. Hence, the second equation holds trivially. Moreover, we gather

$$m(x, \lambda) = \lambda V(x) - \overline{H}(\lambda)$$

on the set $m > 0$. In addition, on the set $m = 0$, the first equation gives

$$\lambda V(x) - \overline{H}(\lambda) \leq 0.$$

Thus,

$$m(x, \lambda) = \left(\lambda V(x) - \overline{H}(\lambda)\right)^+ .$$

The map

$$h \mapsto \int_{\mathbb{T}} (\lambda V(x) - h)^+ \, \mathrm{d}x$$

is monotone decreasing. Hence, there is a unique value, $\overline{H}(\lambda)$, for which

$$\int_{\mathbb{T}} \left(\lambda V(x) - \overline{H}(\lambda)\right)^+ \mathrm{d}x = 1.$$

If λ is small, the condition $\int_{\mathbb{T}} m = 1$ gives

$$\overline{H}(\lambda) = \lambda \int_{\mathbb{T}} V - 1.$$

Thus,

$$m(x, \lambda) = 1 + \lambda \left(V(x) - \int_{\mathbb{T}} V \right).$$

In contrast, for large $|\lambda|$, the condition $m > 0$ fails.

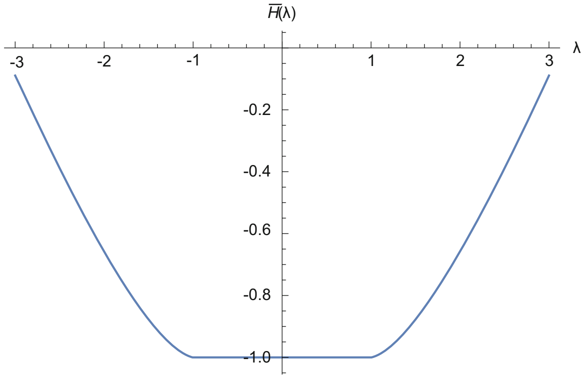

Fig. 2.1 $\overline{H}(\lambda)$

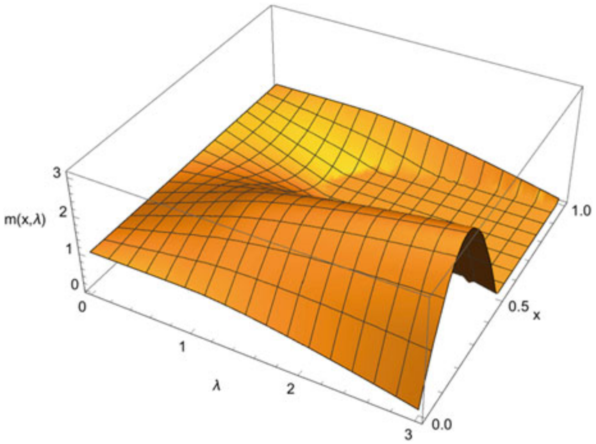

Fig. 2.2 $m(x, \lambda)$

Because $m(x, \lambda) + \overline{H}(\lambda) - \lambda V(x) = (\overline{H}(\lambda) - \lambda V(x))^+$, we have

$$\frac{|u_x|^2}{2} = (\overline{H}(\lambda) - \lambda V(x))^+.$$

The solution u to the preceding equation can fail to be a classical solution. Furthermore, as we show next, it may admit multiple solutions.

Figure 2.1 illustrates the behavior of $\overline{H}(\lambda)$ for $V(x) = \sin(2\pi x)$, and Fig. 2.2 depicts $m(x, \lambda)$. In general, the solution, u, is not unique and may not be differentiable. In Figs. 2.3 and 2.4, we plot two two-periodic solutions (m, u) for $\lambda = 2$.

2.5 Bibliographical Notes

The explicit solution in Sect. 2.1 appeared in [8]. The Hopf–Cole transform was introduced in the context of MFGs in [174]. Similar ideas were used in [138] to develop numerical methods and in [66] to show the existence of classical solutions.

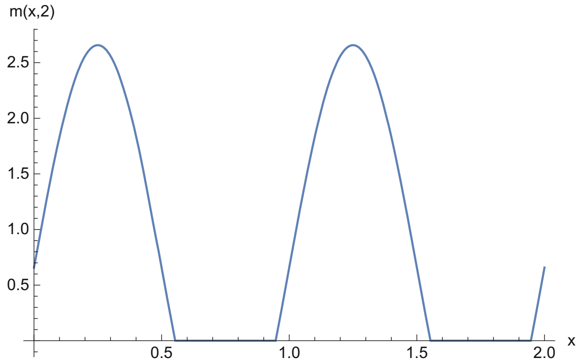

Fig. 2.3 $m(x, 2)$

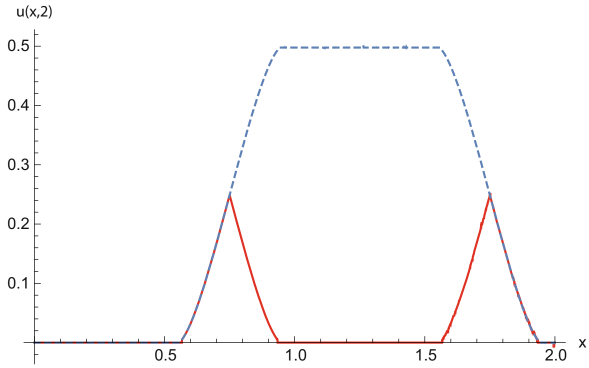

Fig. 2.4 $u(x, 2)$—two distinct solutions

A remarkable extension of the Hopf–Cole transform was presented in [78]. The Hopf–Cole transform was used in [203] to convert an MFG into a system of Schrödinger equations. Gaussian-quadratic solutions were discussed in [137] and, with more generality, in [15]. Moreover, they have applications in machine learning, in particular, in clustering and non-supervised learning [189, 190]. The N-player linear-quadratic counterpart was considered in [17, 18, 197]. Some applications of linear-quadratic MFGs are presented in [22, 31, 32, 145, 184]. The discussion in Sect. 2.3 is inspired by [130]. Explicit examples where MFG partial differential equations are converted into ordinary differential equations were examined in [25, 182]. A further explicit example was studied in [204].

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